

LOCAL CONTROL AND BOGOMOLOV MULTIPLIERS OF FINITE GROUPS

by PRIMOŽ MORAVEC[†] 

Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Ljubljana 1000, Slovenia

[†]Corresponding author. E-mail: primoz.moravec@fmf.uni-lj.si

ABSTRACT

We show that if a Sylow p -subgroup of a finite group G is nilpotent of class at most p , then the p -part of the Bogomolov multiplier of G is locally controlled.

1. INTRODUCTION

In this note we prove the following result:

THEOREM 1.1 *Let G be a finite group, and P a Sylow p -subgroup of G . If the nilpotency class of P does not exceed p , then the Bogomolov multipliers of G and $N_G(P)$ have isomorphic Sylow p -subgroups.*

This theorem is related to the following result of Holt [3]:

THEOREM 1.2 ([3]) *Let G be a finite group, and P a Sylow p -subgroup of G . If the nilpotency class of P does not exceed $p/2$, then the Schur multipliers of G and $N_G(P)$ have isomorphic Sylow p -subgroups.*

In fact, the majority of the proof of Theorem 1.1 is an adaptation of Holt's argument. The crucial difference is the step where the bound on the nilpotency class of the Sylow p -subgroup is improved from Holt's $p/2$ in the Schur multiplier case to p in the Bogomolov multiplier case. Note that this does not improve the bound in Theorem 1.2; it merely shows that one can relax it when passing to Bogomolov multipliers.

The outline of the paper is as follows. We first provide some preliminaries in Section 2. Then we proceed to the proof of Theorem 1.1. To keep the exposition short, we only include the details where our argument differs from [3] and refer to *loc. cit.* for the rest.

2. PRELIMINARIES

Most of the notations follow [2]. The maps and actions are always written from the right.

Let G be a finite group. The second homology group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is the *Schur multiplier* of G . If $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ is a free presentation of G , then $H^2(G, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to $\text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$, where $M(G) = (F' \cap R)/[R, F] \cong H_2(G, \mathbb{Z})$, see, for example, [2, pp. 42, 145].

Given a G -module M , denote

$$\text{III}^n(G, M) = \bigcap_{\substack{A \leq G, \\ A \text{ abelian}}} \ker(\text{res}_A^G : H^n(G, M) \rightarrow H^n(A, M)).$$

Bogomolov [1] studied the group $\text{III}^2(G, \mathbb{Q}/\mathbb{Z})$ in relationship with Noether's problem from invariant theory. This group is nowadays called the *Bogomolov multiplier* of G . It is shown in [4, Proposition 3.8] that if $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ is a free presentation of G , then $\text{III}^2(G, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to $\text{Hom}(B_0(G), \mathbb{Q}/\mathbb{Z})$, where $B_0(G) = (F' \cap R)/\langle K(F) \cap R \rangle$. Here $K(F)$ stands for the set of all commutators $[x, y]$, where $x, y \in F$.

Let a finite group G that acts on a finite group Q . Then G also acts on $H^2(Q, \mathbb{Q}/\mathbb{Z})$ via the rule $(c + B^2(Q, \mathbb{Q}/\mathbb{Z}))g = c' + B^2(Q, \mathbb{Q}/\mathbb{Z})$, where the cocycle $c' : Q \times Q \rightarrow \mathbb{Q}/\mathbb{Z}$ is given by the rule $c'(q_1, q_2) = c(q_1g^{-1}, q_2g^{-1})$. Via a free presentation $Q \cong F/R$ of Q , we have an action of G on $M(Q)$, given as follows. Let F be free on X . Take an isomorphism $\phi : F/R \rightarrow Q$, and let $g \in G$ and $x \in X$. Pick $y_x \in F$ with the property that $(xR)\phi g\phi^{-1} = y_xR$. This gives rise to an endomorphism $\psi : F \rightarrow F$ that sends x to y_x . Note that R is ψ -invariant. Thus ψ induces an action of G on $M(Q)$ via $(r[R, F])g = (r\psi)[R, F]$, where $r \in F' \cap R$. It is not difficult to show that $H^2(Q, \mathbb{Q}/\mathbb{Z})$ and $\text{Hom}(M(Q), \mathbb{Q}/\mathbb{Z})$ become isomorphic as G -modules.

The Bogomolov multiplier $\text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$ is a subgroup of $H^2(Q, \mathbb{Q}/\mathbb{Z})$. Take $g \in G$ and $\alpha \in \text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$. Take an arbitrary abelian subgroup A of Q . Then Ag^{-1} is abelian, and hence $\alpha \text{res}_{Ag^{-1}}^G = 0$. By definition, $(\alpha g) \text{res}_A^G = 0$. Thus $\text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$ is a submodule of the G -module $H^2(Q, \mathbb{Q}/\mathbb{Z})$.

Let $\phi : F/R \rightarrow Q$ be a free presentation of Q , and let ψ be as above. Denote $M_0(Q) = \langle K(F) \cap R \rangle/[R, F]$. Take $g \in G$ and $x, y \in F$, with $[x, y] \in R$. Then $([x, y][R, F])g = ([x, y]\psi)[R, F] = [x\psi, y\psi][R, F]$. As $R\psi \subseteq R$, it follows that $[x, y]\psi \in K(F) \cap R$. This shows that $M_0(Q)$ is a submodule of the G -module $M(Q)$, and hence $B_0(Q) = M(Q)/M_0(Q)$ becomes a G -module. Similarly as above, $\text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$ and $\text{Hom}(B_0(Q), \mathbb{Q}/\mathbb{Z})$ are isomorphic as G -modules.

We will also need a couple of auxiliary results on commutator subgroups. The notations follow [5, p. 119].

LEMMA 2.1 *Let A and B be normal subgroups of a group G .*

- 1 $[A, A, {}_n B] \leq \prod_{i=1}^n [[A, {}_i B], [A, {}_{n-i} B]]$ for all $n \geq 1$.
- 2 $[\gamma_n(B), A] \leq [A, {}_n B]$ for all $n \geq 1$.

Proof The item (1) is proved in [3, Lemma 6]. We prove (2) by induction on n . The case $n = 1$ is clear. Suppose the claim holds for some $n \geq 1$ and all normal subgroups A and B of G .

Then the three subgroups lemma [5, 5.1.10] implies $[\gamma_{n+1}(B), A] = [\gamma_n(B), B, A] \leq [B, A, \gamma_n(B)][A, \gamma_n(B), B]$. By induction assumption we have $[B, A, \gamma_n(B)] = [[B, A], \gamma_n(B)] \leq [[B, A], {}_n B] = [A, {}_{n+1} B]$. In addition to that, $[A, \gamma_n(B), B] = [[A, \gamma_n(B)], B] \leq [A, {}_{n+1} B]$, again by induction assumption. This proves the result. \square

3. PROOF OF THEOREM 1.1

Let H be a subgroup of a finite group G . Let M be a G -module. Given $g \in G$, the conjugation map $H \rightarrow H^g$ induces an isomorphism $\text{conj}_H^g : H^n(H, M) \rightarrow H^n(H^g, M)$. According to [3], we say that $\alpha \in H^n(H, M)$ is *stable* with respect to G (also *G -invariant* according to [2]) if

$$\alpha \text{res}_{H \cap H^g}^H = \alpha \text{conj}_H^g \text{res}_{H \cap H^g}^{H^g}$$

for every $g \in G$. The following is a crucial property of stable elements:

LEMMA 3.1 ([3], Lemma 2) *Let p be a prime, and let H be a subgroup of G of index coprime to p . Let P_G and P_H be the Sylow p -subgroups of $\text{III}^n(G, M)$ and $\text{III}^n(H, M)$, respectively. Then P_G is isomorphic to the group of stable elements of P_H , which is a direct factor of P_H .*

We first show that an analogous result holds for III^n . Let G and H be as in Lemma 3.1. Let S_G and S_H be the Sylow p -subgroups of $\text{III}^n(G, M)$ and $\text{III}^n(H, M)$, respectively. Then $S_G \leq P_G$ and $S_H \leq P_H$. Let ρ be the restriction of the map res_H^G to S_G , and let σ be the restriction of the map cor_H^G to S_H . Pick $\alpha \in \text{III}^n(G, M)$, and let A be an abelian subgroup of H . Then $\alpha \text{res}_H^G \text{res}_A^H = \alpha \text{res}_A^G = 0$, and hence $\alpha \text{res}_H^G \in \text{III}^n(H, M)$. This shows that ρ maps S_G into S_H . Similarly, if $\beta \in \text{III}^n(H, M)$ and if A is an abelian subgroup of G , then [2, Proposition 9.5, Chapter III] gives

$$\beta \text{cor}_H^G \text{res}_A^G = \sum_{s \in \mathcal{S}} \beta \text{conj}_H^s \text{res}_{H^s \cap A}^{H^s} \text{cor}_{H^s \cap A}^A,$$

where \mathcal{S} is a complete set of representatives of double cosets HgA , where $g \in G$. Note that $\beta \text{conj}_H^s \in \text{III}^n(H^s, M)$, and, as $H^s \cap A$ is an abelian subgroup of H^s , we get $\beta \text{conj}_H^s \text{res}_{H^s \cap A}^{H^s} = 0$. Thus the above formula implies $\beta \text{cor}_H^G \in \text{III}^n(G, M)$. Hence σ maps S_H into S_G . As $|G : H|$ is not divisible by p ; it follows from [2, Proposition 9.5, Chapter III] that $\rho\sigma = |G : H| \cdot 1$ is an automorphism of $\text{III}^n(G, M)$. Similarly as in [3] we can now show the following:

LEMMA 3.2 *Let p be a prime, and let H be a subgroup of G of index coprime to p . Let S_G and S_H be the Sylow p -subgroups of $\text{III}^n(G, M)$ and $\text{III}^n(H, M)$, respectively. Then S_G is isomorphic to the group of stable elements of S_H , which is a direct factor of S_H .*

Lemma 3.2 can be used to prove the following counterpart of [3, Theorem 1], with proof being a straightforward adaptation of Holt's argument:

COROLLARY 3.3 *Let G be a finite group, and P a Sylow p -subgroup of G . Let M be a trivial G -module. Let W be a characteristic p -functor which strongly controls fusion in G . Then the Sylow p -subgroups of $\text{III}^n(G, M)$ and $\text{III}^n(N_G(W(P)), M)$ are isomorphic.*

Our next result is a key step in proving Theorem 1.1.

PROPOSITION 3.4 *Let Q be a normal subgroup of a finite group G . Suppose that $[Q, {}_c G] = 1$ for some $c \geq 1$. Then $[\text{III}^2(Q, \mathbb{Q}/\mathbb{Z}), {}_{c-1} G] = 1$.*

Proof It suffices to show that $[B_0(Q), {}_{c-1} G] = 1$. Let $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ be a free presentation of G . Then Q has a free presentation of the form $1 \rightarrow R \rightarrow F_1 \rightarrow Q \rightarrow 1$, where $F_1 \trianglelefteq F$. By assumption we have that $[F_1, {}_c F] \leq R$. Now Lemma 2.1 (1) gives

$$\begin{aligned} [F'_1 \cap R, {}_{c-1} F] &\leq [F_1, F_1, {}_{c-1} F] \\ &\leq \prod_{i=1}^{c-1} [[F_1, {}_i F], [F_1, {}_{c-i-1} F]]. \end{aligned}$$

Consider a commutator of the form

$$\omega = [[x, a_1, \dots, a_i], [y, b_1, \dots, b_{c-i-1}]],$$

where $x, y \in F_1$, $a_1, \dots, a_i, b_1, b_{c-i-1} \in F$, $1 \leq i \leq c-1$. As F_1 is a normal subgroup of F , we have that $\omega \in K(F_1)$. On the other hand, Lemma 2.1 yields that $[[F_1, {}_iF], [F_1, {}_{c-i-1}F]] \leq [F_1, {}_iF, \gamma_{c-i}F] \leq [F_1, {}_cF] \leq R$, and therefore $\omega \in R$. This shows that $[F_1' \cap R, {}_{c-1}F] \leq \langle K(F_1) \cap R \rangle$ and hence the result. \square

The following result can be proved by essentially repeating the argument of the second part of the proof of [3, Lemma 7]:

COROLLARY 3.5 Let G be a finite group and P a Sylow p -subgroup of G . Suppose that the nilpotency class of P does not exceed p . Let Q be a normal subgroup of G . If Q is a p -group, then $[\text{III}^2(Q, \mathbb{Q}/\mathbb{Z}), G] = [\text{III}_2(Q, \mathbb{Q}/\mathbb{Z}), N_G(P)]$.

Proof Form $H = G \ltimes \text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$. Then $S = P \ltimes \text{III}^2(Q, \mathbb{Q}/\mathbb{Z})$ is a Sylow p -subgroup of H .

As P is nilpotent of class $\leq p$, Proposition 3.4 implies $[\text{III}^2(Q, \mathbb{Q}/\mathbb{Z}), {}_{p-1}P] = 1$. The rest of the proof now follows the lines of [3, Proof of Lemma 7]. \square

Having Corollary 3.5 at hand, we finish the proof of Theorem 1.1 by applying [3, Lemma 8] and repeating the argument following it.

ACKNOWLEDGEMENTS

The author acknowledges the financial support from the Slovenian Research Agency, research core funding No. P1-0222, and projects No. J1-3004, N1-0217, J1-2453, J1-1691.

REFERENCES

1. F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, *Izv. Akad. Nauk SSSR Ser. Mat.* **51** no. 3 (1987), 485–516.
2. K. S. Brown, *Cohomology of groups* Springer-Verlag, New York, NY, Heidelberg, Berlin, 1982.
3. D. F. Holt, *On the local control of Schur multipliers*, *The Quarterly Journal of Mathematics* **28** no. 2 (1977), 495–508, [10.1093/qmath/28.4.495](https://doi.org/10.1093/qmath/28.4.495).
4. P. Moravec, *Unramified Brauer groups of finite and infinite groups*, *American Journal of Mathematics* **134** no. 6 (2012), 1679–1704, [10.1353/ajm.2012.0046](https://doi.org/10.1353/ajm.2012.0046).
5. D. J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, NY, 1982.