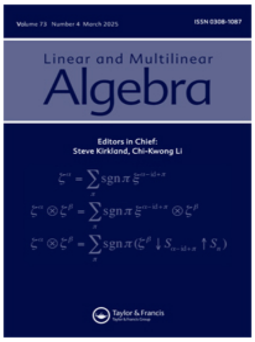

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On commutators of idempotents

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ABSTRACT

Let T be an operator on a Banach space X that is similar to $-T$ via an involution U. Then, U decomposes the Banach space X as $X = X_1 \oplus X_2$ with respect to which decomposition we have $U = \begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix}$, where I_i is the identity operator on the closed subspace X_i ($i = 1, 2$). Furthermore, T has necessarily the form $T = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ with respect to the same decomposition. In this note, we consider the question when T is a commutator of the idempotent $P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$ and some idempotent Q on X. We also determine which scalar multiples of unilateral shifts on l^p spaces ($1 \leq p < \infty$) are commutators of idempotent operators.

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1. Introduction

A (bounded linear) operator C on a Banach space is said to be the *commutator* of the operators A and B if $C = AB - BA = [A, B]$. The operators on a separable infinite-dimensional Hilbert space that arise as commutators have been characterized by Brown and Pearcy [1] as the operators that are not the sum of a compact operator and a non-zero scalar multiple of the identity.

It is natural to ask which operators are commutators of operators of given forms. For example, commutators of self-adjoint operators have been characterized in [2], and commutators of idempotent matrices have been characterized in [3]. In this paper, we first improve the ring-theoretic characterization of commutators of idempotents from [3], and then we characterize commutators of the idempotent operators on a Banach space. Motivated with the case $p = 2$ in [4, Corollary 5.9] we also determine which scalar multiples of unilateral shifts on l^p spaces ($1 \leqslant p < \infty$) are commutators of idempotent operators.

We now recall some definitions. Let R be a unital ring with identity 1. An *idempotent* is any $p \in R$ such that $p^2 = p$, while $u \in R$ is called an *involution* if $u^2 = 1$. Of course, elements in R of the form $[a, b] := ab - ba$ are called *commutators*.

2. A characterization of commutators of idempotents

We first complement the ring-theoretic characterization of commutators of idempotents as given in [3, Theorem 1]. This sheds new light on the proof of [3, Theorem 1].

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Theorem 2.1: *Let R be a unital ring with identity 1 in which the element 2 is invertible, and let $t \in R$. The following assertions are equivalent:*

(i) *t is a commutator of a pair of idempotents in R;*
(ii) *$4t$ is a commutator of a pair of involutions in R;*
(iii) *There exist $u \in R$ and $s \in R$ such that $u^2 = 1$, $ut + tu = 0$, $us = su$, $st = ts$ and $s^2 = t^2 + 1/4$.*

Proof: Assume that $t = pq - qp$ for some idempotents p and q. Define $u := 2p - 1$ and $v := 2q - 1$. Then $u^2 = v^2 = 1$ and

$$4t = [2p, 2q] = [2p - 1, 2q] = [u, 2q - 1] = [u, v].$$

This proves the implication (i) ⇒ (ii). Since the proof of the converse implication is similar, we omit it.

Now assume (ii), that is, $4t = uv - vu$ for some involutions u and v. Then

$$4ut + 4tu = (v - uvu) + (uvu - v) = 0.$$

Define

$$s := \frac{1}{4}(uv + vu).$$

Then

$$4us = v + uvu = 4su,$$
$$16st = (uv + vu)(uv - vu) = (uv)^2 - 1 + 1 - (vu)^2 = 16ts,$$

and

$$16s^2 - 16t^2 = ((uv)^2 + 1 + 1 + (vu)^2) - ((uv)^2 - 1 - 1 + (vu)^2) = 4.$$

This completes the proof of the implication (ii) ⇒ (iii).

Now assume (iii). Define $v := 2u(s + t) = 2(s - t)u$. Then

$$v^2 = 2(s - t)u \cdot 2u(s + t) = 4(s^2 - t^2) = 1,$$

and

$$uv - vu = 2(s + t) - 2(s - t) = 4t.$$

This proves (ii), and it completes the proof of the theorem. ■

Let T be an operator on a complex Banach space X. Suppose that T is similar to $-T$ via an involution U, so that $TU + UT = 0$. Then (it is well-known that) U decomposes the

Banach space X as $X = X_1 \oplus X_2$ with respect to which decomposition we have

$$U = \begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I_i is the identity operator on the closed subspace $X_i (i = 1, 2)$. Write

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with respect to the same decomposition. Since $TU + UT = 0$, we have $A = 0$ and $D = 0$. Now Theorem 2.1 gives the following characterization.

Theorem 2.2: *The operator*

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and some idempotent* Q *on* X *if and only if the operator* $BC + \frac{1}{4}I_1$ *has a square root* S_1 *on* X_1*, the operator* $CB + \frac{1}{4}I_2$ *has a square root* S_2 *on* X_2*, and* $S_1B = BS_2$*,* $S_2C = CS_1$.

Proof: ($\Rightarrow$) By Theorem 2.1 and its proof (or by the proof of [3, Theorem 1]), there exists an operator S on X such that $SU = US$, $ST = TS$ and $S^2 = T^2 + \frac{1}{4}I$. Since $SU = US$, S has the form $S = \left(\begin{smallmatrix} S_1 & 0 \\ 0 & S_2 \end{smallmatrix}\right)$. Since

$$\begin{pmatrix} S_1^2 & 0 \\ 0 & S_2^2 \end{pmatrix} = S^2 = T^2 + \frac{1}{4}I = \begin{pmatrix} BC + \frac{1}{4}I_1 & 0 \\ 0 & CB + \frac{1}{4}I_2 \end{pmatrix},$$

we have $S_1^2 = BC + \frac{1}{4}I_1$ and $S_2^2 = CB + \frac{1}{4}I_2$. It follows from $ST = TS$ that $S_1B = BS_2$ and $S_2C = CS_1$.

($\Leftarrow$) Define $S := \left(\begin{smallmatrix} S_1 & 0 \\ 0 & S_2 \end{smallmatrix}\right)$ on $X = X_1 \oplus X_2$. Then

$$S^2 = \begin{pmatrix} S_1^2 & 0 \\ 0 & S_2^2 \end{pmatrix} = \begin{pmatrix} BC + \frac{1}{4}I_1 & 0 \\ 0 & CB + \frac{1}{4}I_2 \end{pmatrix} = T^2 + \frac{1}{4}I.$$

Since $S_1B = BS_2$, $S_2C = CS_1$, we have $ST = TS$. Clearly, $SU = US$. By Theorem 2.1 and its proof (or by the proof of [3, Theorem 1]), $T = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and some idempotent Q on the Banach space X. ■

We say that a compact subset K of the complex plane *does not separate* 0 *from* ∞ if 0 lies in the unbounded component of the complement of K.

Corollary 2.3: *Let*

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

be an operator on the Banach space $X = X_1 \oplus X_2$. *Suppose that* $\sigma(BC + \frac{1}{4}I_1)$ *does not separate* 0 *from* ∞. *Then* T *is a commutator of the idempotent* $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and some idempotent* Q *on* X.

Proof: By the Riesz functional calculus, the assumption implies that $BC + \frac{1}{4}I_1$ admits a square root S_1 which lies in the norm-closed algebra generated by $BC + \frac{1}{4}I_1$. More precisely, the hypothesis implies that there is a function f, holomorphic in a simply connected open set Ω containing the closed set $\sigma(BC + \frac{1}{4}I_1) \cup \{\frac{1}{4}\}$, which satisfies $(f(z))^2 = z$. It is well known that the spectra $\sigma(BC + \frac{1}{4}I_1)$ and $\sigma(CB + \frac{1}{4}I_2)$ differ perhaps only in the point $\frac{1}{4}$. So, we can define $S_1 = f(BC + \frac{1}{4}I_1)$ and $S_2 = f(CB + \frac{1}{4}I_2)$. Since $(BC + \frac{1}{4}I_1)B = B(CB + \frac{1}{4}I_2)$, we have $S_1 B = BS_2$. Similarly, since $C(BC + \frac{1}{4}I_1) = (CB + \frac{1}{4}I_2)C$, it holds that $CS_1 = S_2 C$. By Theorem 2.2, T is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and some idempotent Q on the Banach space X. ■

As a special case of Corollary 2.3 we obtain the following generalization of [4, Lemma 5.6].

Corollary 2.4: *Let*

$$T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

be an operator on the Banach space $X = X_1 \oplus X_2$. *Suppose that* $r(BC) < \frac{1}{4}$, *where* r *denotes the spectral radius function. Then* T *is a commutator of the idempotent* $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and some idempotent* Q *on* X.

3. Multiples of unilateral shifts on l^p spaces

Let us begin with the following simple lemma.

Lemma 3.1: *Let* A *be an operator on a Banach space* X *such that* $\dim(\ker A) = 1$ *and* $\dim(\ker A^2) = 2$. *Then* A *is not a square of some operator* B *on* X.

Proof: Assume that $A = B^2$ for some operator B on X. Then $\ker B \subseteq \ker A$, and so either $\ker B = \{0\}$ or $\ker B = \ker A$. If $\ker B = \{0\}$, then $\ker A = \ker B^2 = \{0\}$ that is not true. Therefore, $\ker B = \ker A = \ker B^2$. Then $\ker B^k = \ker B$ for all positive integers k, by Abramovich and Aliprantis [5, Lemma 2.19]. In particular, $\ker B = \ker B^4 = \ker A^2$. This contradicts the assumption that $\dim(\ker A^2) = 2$. ■

Let S be the unilateral forward shift on either l^p $(1 \leqslant p < \infty)$ or c_0, that is, the operator defined by $S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. In each case, S is an isometry. In [4, Corollary 5.9] it is shown that when $p = 2$ the operator μS is a commutator of idempotents if and only if the complex number μ satisfies $|\mu| \leqslant \frac{1}{2}$. We now consider this in our context.

To end this, we first write S in the block form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Let $\{e_n\}_n$ denote the standard unit vectors in either l^p or c_0, and write $X_1 = \bigvee_n \{e_{2n}\}$ and $X_2 = \bigvee_n \{e_{2n-1}\}$. Then, $X = X_1 \oplus X_2$, and with respect to this decomposition S has the form

$$S_0 := \begin{pmatrix} 0 & I \\ S & 0 \end{pmatrix}.$$

Theorem 3.2: *Let μ be a complex number. Then μS_0 is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and some idempotent Q on either l^p $(1 \leqslant p < \infty)$ or c_0 if and only if $|\mu| \leqslant \frac{1}{2}$.*

Proof: Since S is similar to αS via a diagonal operator whenever $\alpha \in \{z \in \mathbb{C} : |z| = 1\}$ we may assume without loss of generality that $\mu \geqslant 0$. We consider 3 cases.

Case 1. If $0 \leqslant \mu < \frac{1}{2}$, then $r(\mu S \cdot \mu I) < \frac{1}{4}$, and so by Corollary 2.4 μS_0 is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and some idempotent Q.

Case 2. If $\mu = \frac{1}{2}$, then by Theorem 2.2 we must show that the operator $\frac{1}{4}S + \frac{1}{4}I = \frac{1}{4}(S + I)$ has a square root commuting with S. It is well-known that $\sum_{n=0}^{\infty} |\binom{\frac{1}{2}}{n}| < \infty$, and so $R := \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} S^n$ converges absolutely, $RS = SR$ and $R^2 = S + I$.

Case 3. If $\mu > \frac{1}{2}$, then put $\lambda := -\frac{1}{4\mu^2} \in (-1, 0)$, so that $\mu^2 S + \frac{1}{4}I = \mu^2(S - \lambda I)$. In view of Theorem 2.2 it is enough to show that $S - \lambda I$ has no square root on either l^p $(1 \leqslant p < \infty)$ or c_0. Then it is enough to prove that the adjoint operator $A := S^* - \lambda I$ has no square root on either the dual space $(l^p)^* = l^q$, where q is the conjugate exponent of p, or the dual space $c_0^* = l^1$. To end this, we apply Lemma 3.1. It is easy to show that

$$\ker A = \bigvee \{(1, \lambda, \lambda^2, \lambda^3, \ldots)\} \quad \text{and}$$
$$\ker A^2 = \bigvee \{(1, \lambda, \lambda^2, \lambda^3, \ldots), (0, 1, 2\lambda, 3\lambda^2, 4\lambda^3, \ldots)\},$$

so that $\dim(\ker A) = 1$ and $\dim(\ker A^2) = 2$. This completes the proof. ∎

In the similar manner one can prove the corresponding result for the unilateral backward shift B on either l^p $(1 \leqslant p < \infty)$ or c_0, that is, the operator defined by $B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. We will omit its proof.

As before, we first write B in the block form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Let $\{e_n\}_n$ denote the standard unit vectors in either l^p or c_0, and write $X_1 = \bigvee_n \{e_{2n}\}$ and $X_2 = \bigvee_n \{e_{2n-1}\}$. Then $X = X_1 \oplus X_2$, and with respect to this decomposition B has the form

$$B_0 := \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix}.$$

Theorem 3.3: *Let μ be a complex number. Then μB_0 is a commutator of the idempotent $P = \left(\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and some idempotent Q on either l^p $(1 \leqslant p < \infty)$ or c_0 if and only if $|\mu| \leqslant \frac{1}{2}$.*

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