ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 25 (2025) #P2.01
https://doi.org/10.26493/1855-3974.3085.3ea
(Also available at http://amc-journal.eu)

On regular graphs with Šoltés vertices*

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Received 13 March 2023, accepted 4 March 2024, published online 10 March 2025

Abstract

Let $W(G)$ be the Wiener index of a graph G. We say that a vertex $v \in V(G)$ is a *Šoltés vertex* in G if $W(G - v) = W(G)$, i.e. the Wiener index does not change if the vertex v is removed. In 1991, Šoltés posed the problem of identifying all connected graphs G with the property that all vertices of G are Šoltés vertices. The only such graph known to this day is C_{11}. As the original problem appears to be too challenging, several relaxations were studied: one may look for graphs with at least k Šoltes vertices; or one may look for *α-Šoltés graphs*, i.e. graphs where the ratio between the number of Šoltés vertices and the order of the graph is at least α. Note that the original problem is, in fact, to find all 1-Šoltés graphs. We intuitively believe that every 1-Šoltés graph has to be regular and has to possess a high degree of symmetry. Therefore, we are interested in *regular* graphs that contain one or more Šoltés vertices. In this paper, we present several partial results. For every $r \geq 1$ we describe a construction of an infinite family of cubic 2-connected graphs with at least 2^r Šoltés vertices. Moreover, we report that a computer search on publicly

*The second and third authors acknowledge partial support of the Slovenian research agency ARRS; program P1-0383 and ARRS project J1-3002, and the annual work program of Rudolfovo.

†Corresponding author. The author is supported in part by the Slovenian Research Agency (research program P1-0294 and research projects J1-1691, N1-0140, and J1-2481).

‡The author acknowledges partial support by Slovak research grants VEGA 1/0069/23, VEGA 1/0011/25, APVV-23-0076 and APVV-22-0005.

available collections of vertex-transitive graphs did not reveal any 1-Šoltés graph. We are only able to provide examples of large $\frac{1}{3}$-Šoltés graphs that are obtained by truncating certain cubic vertex-transitive graphs. This leads us to believe that no 1-Šoltés graph other than C_{11} exists.

Keywords: Šoltés problem, Wiener index, regular graph, cubic graph, Cayley graph, Šoltés vertex.

Math. Subj. Class. (2020): 05C12, 05C90, 20B25

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1 Introduction

All graphs under consideration in this paper are simple and undirected. The *Wiener index* of a graph G, denoted by $W(G)$, is defined as

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u, v) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v), \tag{1.1}$$

where $d_G(u, v)$ is the distance between vertices u and v (i.e. the length of a shortest path between u and v). If the graph G is disconnected, we may take $W(G) = \infty$. The Wiener index was introduced in 1947 [16] and has been extensively studied ever since. For a recent survey on the Wiener index see [9]. The *transmission* of a vertex v in a graph G, denoted by $w_G(v)$, is defined as $w_G(v) = \sum_{u \in V(G)} d_G(u, v)$. Note that (1.1) can also be expressed as $W(G) = \frac{1}{2} \sum_{u \in V(G)} w_G(u)$.

Let $v \in V(G)$. The graph obtaned from G by removing a vertex v is denoted by $G - v$. If we remove a vertex v from a graph G, any of the following scenarios may occur:

(a) the Wiener index *decreases* (e.g. $W(K_7) = 21$ and $W(K_7 - v) = W(K_6) = 15$);

(b) the Wiener index *increases* (e.g. $W(\mathrm{Wh}_9) = 56$ and $W(\mathrm{Wh}_9 - v_0) = W(C_8) = 64$, where Wh_n denotes the wheel graph on n vertices and v_0 is the central vertex of the wheel);

(c) the Wiener index *does not change* (e.g. $W(\mathrm{Wh}_8) = 42$ and $W(\mathrm{Wh}_8 - v_0) = W(C_7) = 42$, where v_0 is the central vertex of Wh_8).

We say that a vertex $v \in V(G)$ is a *Šoltés vertex of* G if $W(G) = W(G - v)$, i.e. the Wiener index of G does not change if the vertex v is removed. If G is disconnected, with two non-trivial components or with at least three components, then every vertex is a Šoltés vertex. Therefore, it is natural to require that G is connected. Let

$$S(G) = \{v \in V(G) \mid W(G) = W(G - v)\}$$

and let $0 \leq \alpha \leq 1$. We say that a graph G is an *α-Šoltés graph* if $|S(G)| \geq \alpha |V(G)|$, i.e. the ratio between the number of Šoltés vertices of G and the order of G is at least α. For example, the graph in Figure 1 is the smallest cubic $\frac{1}{3}$-Šoltés graph. Note that G is an 1-Šoltés graph if every vertex in G is a Šoltés vertex. The only 1-Šoltés graph known to this day is the cycle on 11 vertices, C_{11}. In this paper, Šoltés graph is simply the synonym for 1-Šoltés graph.

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The Šoltés problem [15] was forgotten for nearly three decades. It was revived and popularised in 2018 by Knor *et al.* [7]. They considered a relaxation of the original problem: one may look for graphs with a prescribed number of Šoltés vertices. They showed that there exists a unicyclic graph on n vertices with *at least one* Šoltés vertex for every $n \geq 9$. They also showed that there exists a unicyclic graph with a cycle of length c and at least one Šoltés vertex for every $c \geq 5$, and that every graph is an induced subgraph of some larger graph with a Šoltés vertex. They have further shown that a Šoltés vertex in a graph may have a prescribed degree [8]. Namely, they proved that for any $d \geq 3$ there exist infinitely many graphs with a Šoltés vertex of degree d. Necessary conditions for the existence of Šoltés vertices in Cartesian products of graphs were also considered [8]. In 2021, Bok *et al.* [3] showed that for every $k \geq 1$ there exist infinitely many *cactus graphs* with exactly k distinct Šoltés vertices. In 2021, Hu *et al.* [6] studied a variation of the problem and showed that there exist infinitely many graphs where the Wiener index remains the same even if $r \geq 2$ distinct vertices are removed from the graph.

Akhmejanova *et al.* [1] considered another possible relaxation of the problem: Do there exist graphs with a given percentage of Šoltés vertices? They constructed two infinite families of graphs with a relatively high proportion of Šoltés vertices. Their first family comprises graphs $B(k)$, $k \geq 2$, where $B(k)$ is a $\frac{2k}{5k+6}$-Šoltés graph on $5k + 6$ vertices. Two vertices of $B(k)$ are of degree $k + 1$, while the remaining vertices are of degree 2. The percentage of Šoltés vertices is below $\frac{2}{5}$, but tends to $\frac{2}{5}$ as k goes to infinity. They also introduced a two-parametric infinite family $L(k, m)$, $m \geq 7$ and $k \geq \frac{m-3}{m-6}$. Here, the percentage of Šoltés vertices is below $\frac{1}{2}$, but tends to $\frac{1}{2}$ as k goes to infinity for a fixed m. These graphs contain at least one leaf, at least km vertices of degree 2, and a vertex of degree $km + 1$.

In the present paper we focus on *regular graphs*. Our intuition lead us to believe that the solutions to the original Šoltés problem should be graphs having all vertices of the same degree.

Conjecture 1.1. *If G is a Šoltés graph, then G is regular.*

For a general regular graph G, the values $W(G - u)$ and $W(G - v)$ might be significantly different for two different vertices $u, v \in V(G)$; it may happen that removal of one vertex increases the Wiener index, while removal of the other vertex descreases the Wiener index. However, $W(G-u)$ and $W(G-v)$ are equal if vertices u and v belong to the same vertex orbit. Therefore, we believe that a Šoltés graph is likely to be vertex transitive.

Conjecture 1.2. *If G is a Šoltés graph, then G is vertex transitive.*

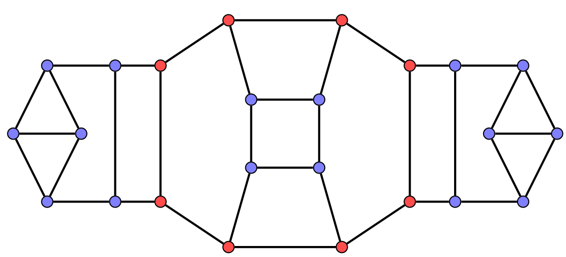

Figure 1: The smallest cubic $\frac{1}{3}$-Šoltés graph has 24 vertices. Its Šoltés vertices are coloured red.

Among truncations of cubic vertex-transitive graphs we found several $\frac{1}{3}$-Šoltés graphs; see Section 4. Interestingly, all our examples are in fact Cayley graphs and this leads us to pose the following conjecture.

Conjecture 1.3. *If G is a Šoltés graph, then G is a Cayley graph.*

It is not hard to obtain small examples of regular graphs with Šoltés vertices. We used the `geng` [11] software to generate small k-regular graphs (for $k = 3, 4$ and 5). Let $\mathcal{R}^r$ denote the class of all r-regular graphs and let $\mathcal{R}^r_n$ denote the set of r-regular graphs on n vertices. Let $N(\mathcal{G}, k)$ be the number of graphs in the class $\mathcal{G}$ that contain exactly k Šoltés vertices.

Table 1 shows the numbers of (non-isomorphic) cubic graphs of orders $n \leq 24$ that contain Šoltés vertices. We can see that cubic graphs of order 12 or less do not contain Šoltés vertices. There are plenty of examples with one Šoltés vertex. Cubic graphs with two Šoltés vertices first appear at order $n = 14$; there are three such graphs (see Figure 2(a)–(c)). Examples with three and four Šoltés vertices appear at order $n = 16$; there is one cubic graph with three and two cubic graphs with four Šoltés vertices (see Figure 2(d)–(f)). At order $n = 18$, there are no graphs with three Šoltés vertices, however there is only one graph with four Šoltés vertices (see Figure 2(g)). Numbers of 4-regular and 5-regular graphs with respect to their number of Šoltés vertices are given in Tables 2 and 3, respectively.

n	$\lvert\mathcal{R}^3_n\rvert$	$N(\mathcal{R}^3_n, k)$							
		$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
≤ 12	112	-	-	-	-	-	-	-	-
14	509	4	**3**	-	-	-	-	-	-
16	4060	108	37	**1**	**2**	-	-	-	-
18	41301	1014	200	-	**1**	-	-	-	-
20	510489	13460	1076	6	13	-	-	-	-
22	7319447	194432	9610	151	52	-	1	-	-
24	117940535	3161124	130087	2596	333	2	3	-	**1**

Table 1: The numbers of (non-isomorphic) cubic graphs with Šoltés vertices. There are no graphs with Šoltés vertices for orders up to 12. The column labeled $\lvert\mathcal{R}^3_n\rvert$ gives the total number of cubic graphs of order n. Symbol '-' is a replacement for 0 (i.e. no such graph exists). Each of the next columns gives the numbers of graphs with k Šoltés vertices for $k = 1, 2, \ldots, 8$. A blue-coloured number means that we have provided drawings of these graphs (see Figures 1 and 2).

In the next two sections we construct an infinite family of cubic 2-connected graphs with at least 2^r, $r \geq 1$, Šoltés vertices. It recently came to our attention that Dobrynin independently found an infinite family of cubic graphs with four Šoltés vertices [4]. However, our method is more general.

n	$\lvert\mathcal{R}_n^4\rvert$	$N(\mathcal{R}_n^4, k)$			
		$k = 1$	$k = 2$	$k = 3$	$k = 4$
≤ 12	1894	-	-	-	-
13	10778	-	**1**	-	-
14	88168	30	6	-	-
15	805491	265	85	-	5
16	8037418	2191	472	-	-
17	86221634	14430	2097	4	1

Table 2: The numbers of (non-isomorphic) quartic graphs of orders up to 17 with Šoltés vertices. Naming conventions used in Table 1 also apply here.

n	$\lvert\mathcal{R}_n^5\rvert$	$N(\mathcal{R}_n^5, k)$			
		$k = 1$	$k = 2$	$k = 3$	$k = 4$
≤ 12	7912	-	-	-	-
14	3459383	8	3	-	-

Table 3: The numbers of (non-isomorphic) quintic graphs of order up to 14 with Šoltés vertices. Naming conventions used in Table 1 also apply here.

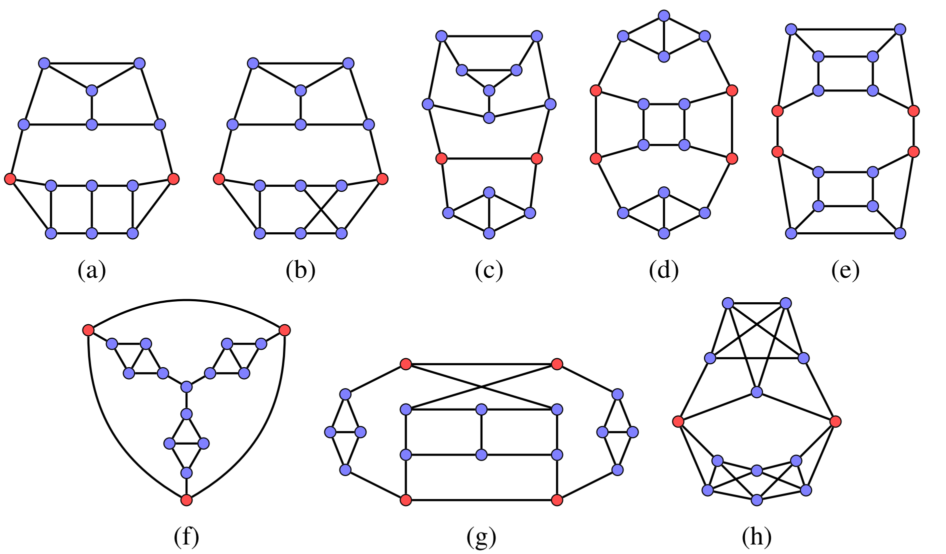

Figure 2: Examples of small regular graphs with two or more Šoltés vertices: (a) to (c) are the three cubic graphs of order 14 with two Šoltés vertices; (d) and (e) are the two cubic graphs of order 16 with four Šoltés vertices; (f) is the only cubic graph of order 16 with three Šoltés vertices; (g) is the only cubic graph of order 18 with four Šoltés vertices; (h) is the only quartic graph of order 13 with two Šoltés vertices. Šoltés vertices are coloured red.

2 Cubic 2-connected graphs with two Šoltés vertices

In the present section we prove the following result.

Theorem 2.1. *There exist infinitely many cubic* 2*-connected graphs* G *which contain at least two Šoltés vertices.*

We prove Theorem 2.1 by a sequence of lemmas. We start by giving several definitions. First, we define a graph G_t on $8t + 8$ vertices, where $t \geq 1$. Take $2t$ copies of the diamond graph (i.e. $K_4 - e$) and connect their degree-2 vertices, so that a ring of $2t$ copies of $K_4 - e$ is formed. Add a disjoint 4-cycle to that graph. Then subdivide one of the edges that connects two consecutive diamonds by two vertices, denote them by z_1 and z_2, and connect z_1 and z_2 with two opposite vertices of the 4-cycle. Add a leaf to each remaning degree-2 vertex of the 4-cycle. Denote the resulting graph by G_t. For an illustration, see G_3 in Figure 3. Note that G_t has exactly two leaves, denote them by v_1 and v_2, and all the remaining vertices have degree 3. Denote by u_1 and u_2 the two vertices at the longest distance from v_1. This distance is $d_G(v_1, u_1) = d_G(v_1, u_2) = 3t + 3$ and also $d_G(v_2, u_1) = d_G(v_2, u_2) = 3t + 3$. Observe that G_t has an automorphism (a symmetry) fixing both v_1 and v_2, while interchanging u_1 with u_2.

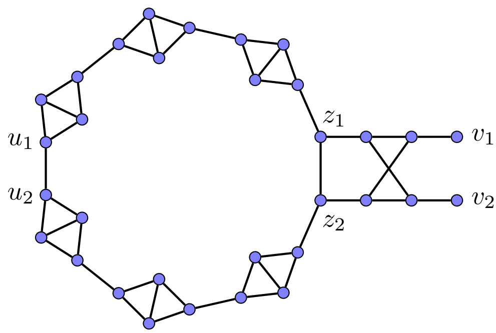

Figure 3: The graph G_3.

Now, we determine $f(t) = W(G_t - u_1) - W(G_t)$. We compute $f(t)$ by summing the contributions of all vertices (first the contribution of quadruples of vertices of all copies of $K_4 - e$, and then the contribution of the vertices which are not in any copy of $K_4 - e$). As the calculation is long and tedious, we present just the result

$$f(t) = 16t^3 - 8t^2 - 26t - 14, \tag{2.1}$$

which was checked by a computer. Actually, the exact value of $f(t)$ is not important here. The crucial property is that for t big enough, $f(t)$ is positive. In fact, $\lim_{t\to\infty} f(t) = \infty$; see also Section 3, where a lower bound for $f(t)$ is given.

To motivate the above definition, we briefly describe the main idea of the proof. We attach trees T_1 and T_2 to vertices v_1 and v_2 of G_t, and then we add edges to them so that the resulting graph H will be cubic and 2-connected. See Figure 4 for an example. This will be done in three phases.

P1) In the first phase, we will construct trees T_1 and T_2 to guarantee that vertices u_1 and u_2 are indeed Šoltés vertices. The vertices of T_1 and T_2 can be partitioned into several layers based on their distance to v_1 and v_2, respectively. The resulting graph will be denoted by Q.

P2) In the second phase, we will add the 'red' edges, whose endpoints are in two consecutive layers, to graph Q. The resulting graph will be denoted by R. The purpose of the second phase is to make sure that the sum of free valencies is even within each layer, making the next phase possible. Note that although the final graph H is cubic, the intermediate graphs, Q and R, are subcubic. The *free valency* of a vertex v in a subcubic graph G is $3 - \deg_G(v)$.

P3) In the last phase, we will add blue edges to R in order to to obtain cycles and paths, so that the resulting graph H will be cubic and 2-connected. The endpoints of 'blue' edges will reside in the same layer of the forest $T_1 \cup T_2$. Adding 'red' and 'blue' edges has no influence on Šoltésness of u_1 and u_2.

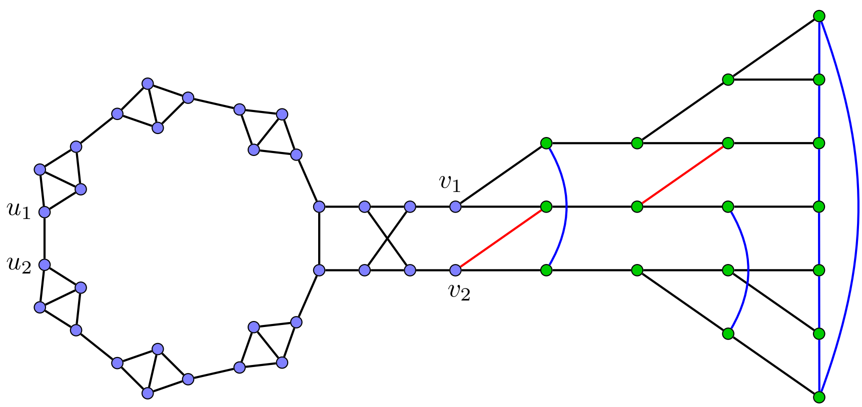

Figure 4: A graph H with two Šoltés vertices, namely u_1 and u_2, that contains G_3.

Let us consider the resulting graph H. Observe that if $x \in V(H) \setminus V(G_t)$, then we have $w_H(x) - w_{H-u_1}(x) = d_H(u_1, x)$. Clearly, every (u_1, x)-path contains one of the vertices from $\{v_1, v_2\}$. Hence, for calculating $W(H) - W(H - u_1)$, only the distance of the new vertices x to $\{v_1, v_2\}$ matters.

We need to find suitable trees T_1 and T_2 rooted at v_1 and v_2, respectively. Each of these trees will have q vertices and their depth, say d, will be determined later. What next properties do T_1 and T_2 need to have? Let ℓ_i be the number of vertices of the forest $T_1 \cup T_2$ at distance i, $1 \leq i \leq d$, from $\{v_1, v_2\}$. Since the resulting graph will be cubic and 2-connected, we have $2 \leq \ell_1 \leq 4, 2 \leq \ell_2 \leq 8, 2 \leq \ell_3 \leq 2^4, \ldots$ and for the last value ℓ_d we have $1 \leq \ell_d \leq 2^{d+1}$. The trees attached to v_1 and v_2 may be paths in which case we get $\ell_1 = \ell_2 = \cdots = \ell_q = 2$ (since each of T_1 and T_2 have exactly q vertices). In this case the transmission of v_j in T_j is biggest possible, $1 \leq j \leq 2$. In the other extremal situation the transmission of v_j in T_j is smallest possible, $1 \leq j \leq 2$, which results in $\ell_i = 2^{i+1}$, $1 \leq i < d$. If $x \in V(H) \setminus V(G_t)$ is at distance i from $\{v_1, v_2\}$, then $d_H(u_1, x) = 3t+3+i$.

Denote by D the sum of distances from all vertices of $V(H) \setminus V(G_t)$ to u_1. Then

$$D = \sum_{i=1}^{d} (3t + 3 + i)\ell_i. \tag{2.2}$$

Observe that we need to find a finite sequence $(\ell_1, \ell_2, \dots, \ell_d)$ so that $f(t) = D$. As the resulting graph H has to be cubic, we need to add an even number of vertices, $2q$, to the graph G_t. What are the bounds for D?

First, we determine the lower bound; let us denote it by D_m. This bound will be obtained when ℓ_i attains the maximum possible value of 2^{i+1} for every $1 \le i \le d-1$. In other words, we are attaching complete binary trees to v_1 and v_2. Let $a = \lfloor \log_2(2q+3) \rfloor$. Recall that the depth of a complete binary tree with n vertices is $\lfloor \log_2(n) \rfloor$ and note that in our case $a - 1 = d$. Then

$$\begin{aligned}
\ell_1 &= 4, \\
\ell_2 &= 8, \\
&\vdots \\
\ell_{a-2} &= 2^{a-1}, \\
\ell_{a-1} &= 2q - \sum_{i=1}^{a-2} \ell_i = 2q - 2^a + 4.
\end{aligned}$$

The above sequence will be called the *short sequence* and denoted by L_m. Using the formula $\sum_{i=1}^{a} i x^{i-1} = (a x^{a+1} - (a+1)x^a + 1)/(x-1)^2$, we get

$$\begin{aligned}
D_m &= \sum_{i=1}^{a-2} (3t + 3 + i)2^{i+1} + (3t + a + 2)(2q - 2^a + 4) \\
&= (3t+1)2q + \sum_{i=1}^{a-2} (i+2)2^{i+1} + (a+1)(2q - 2^a + 4) + 2 \cdot 2^1 + 1 \cdot 2^0 - 5 \\
&= (3t+1)2q + \sum_{i=1}^{a} i 2^{i-1} + (a+1)(2q - 2^a + 4) - 5 \\
&= (3t+1)2q + a2^{a+1} - (a+1)2^a + 1 + (a+1)2q - (a+1)2^a + 4(a+1) - 5 \\
&= (3t+1)2q + a2^{a+1} - (a+1)2^{a+1} + (a+1)2q + 4a \\
&= (3t+1)2q - 2^{a+1} + (a+1)2q + 4a.
\end{aligned} \tag{2.3}$$

Now we find the upper bound D^m for D. In this case $\ell_1 = \ell_2 = \cdots = \ell_q = 2$; this sequence will be called the *long sequence* and denoted by L^m. Therefore,

$$\begin{aligned}
D^m &= 2(3t+4) + 2(3t+5) + \cdots + 2(3t+3+q) \\
&= (3t+3)2q + 2\sum_{i=1}^{q} i \\
&= (3t+1)2q + q^2 + 5q.
\end{aligned} \tag{2.4}$$

Note that D_m and D^m are functions of q and t.

t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
q_{min}	9	17	27	38	50	64	78	94	110	128	146	165	185	205	227	249
q_{max}	9	21	38	59	85	116	152	192	238	288	344	405	471	542	617	698

t	19	20	21	22	23	24	25	26	27	28	29	30	31
q_{min}	272	295	319	344	370	396	422	450	478	506	535	565	595
q_{max}	785	876	973	1075	1182	1294	1411	1533	1661	1795	1933	2077	2225

Table 4: The minimum and maximum value of q which satisfy the condition of Lemma 2.2.

Lemma 2.2. *Let $t \geq 3$. Then there exists q, such that $D_m \leq f(t) \leq D^m$.*

Proof. Observe that if $q \sim 4t\sqrt{t}$ then we have $D_m \leq f(t) \leq D^m$ for large enough t. For small values of t we computed the minimum and maximum value of q that satisfies the condition of the lemma; see Table 4. □

Note that the value q in Lemma 2.2 is uniquely determined only for $t = 3$. For larger values of t we get a range of options. Any q between q_{min} and q_{max} can be used. Moreover, different values of q lead to non-isomorphic graphs H.

Now, we show that for every integer D, $D_m \leq D \leq D^m$, there exists a finite sequence $(\ell_1, \ell_2, \ldots, \ell_d)$ which realises D. Moreover, the graph G_t can be extended to H by attaching trees T_1 and T_2 that have ℓ_i vertices at distance i from $\{v_1, v_2\}$, such that H is a 2-connected cubic graph.

Let us define an operation $\mathcal{M}$ that modifies one such sequence $L = (\ell_1, \ell_2, \ldots, \ell_d)$.

Definition 2.3. Let $L = (\ell_1, \ell_2, \ldots, \ell_d)$. Set $\ell_{d+1} = 0$. Let i be the smallest value such that $\ell_i \geq 3$ and either

(i) $2(\ell_i - 1) > \ell_{i+1} + 1$ or

(ii) $\ell_i = \ell_{i+1} = 3$.

We say that $\mathcal{M}(L) = (\ell'_1, \ell'_2, \ldots, \ell'_d)$ is a *modification* of sequence L if $\ell'_j = \ell_j$ for all j, $1 \leq j \leq d+1$, except for $j \in \{i, i+1\}$, for which $\ell'_i = \ell_i - 1$ and $\ell'_{i+1} = \ell_{i+1} + 1$.

Lemma 2.4. *Let $t \geq 1$ and $q \geq 0$. For every D, $D_m \leq D \leq D^m$, there exists a finite sequence $L = (\ell_1, \ell_2, \ldots, \ell_d)$, such that*

(i) $\sum_{i=1}^{d} \ell_i = 2q$;

(ii) $2 \leq \ell_i \leq 2^{i+1}$ *for* $i < d$ *and* $1 \leq \ell_d \leq 2^{d+1}$;

(iii) $\ell_{i+1} \leq 2\ell_i$.

Namely, $L = \mathcal{M}^{D-D_m}(L_m)$.

Proof. We start with the short sequence $L_m = (4, 8, 16, \ldots)$. It clearly satisfies conditions (i) to (iii). We have already seen that L_m realises D_m. This established the base of induction.

Every other D can be realised by a sequence that is obtained from L_m by iteratively applying operation $\mathcal{M}$. Assume that after $(D-1) - D_m$ steps we obtained the sequence $L = (\ell_1, \ell_2, \ldots, \ell_d)$ which realises $D-1$. Obviously, $\mathcal{M}(L)$ realises D. It is easy to check that conditions (i) to (iii) are satisfied for $\mathcal{M}(L)$. □

Next, we prove two additional properties that hold when operation $\mathcal{M}$ is iteratively applied on L_m.

Lemma 2.5. *Let $t \geq 1$ and $q \geq 0$. Let $L = (\ell_1, \ell_2, \dots, \ell_d)$ be obtained from L_m by iteratively applying operation $\mathcal{M}$. Then the following holds:*

(i) *If* (ii) *of* Definition 2.3 *applies, then $\ell_j = 2$ for all $j < i$.*

(ii) *If index i in* Definition 2.3 *is such that $\ell_{i+1} = 0$, then $\ell_i \geq 4$.*

Proof. (i): Observe that if part (ii) of Definition 2.3 applies and $i \geq 2$, then $\ell_{i-1} = 2$, since otherwise $i - 1$ satisfies the assumption of the definition, thus i is not the smallest such value. Moreover, if there is k, $k < i$, with $\ell_k \geq 3$, then choose the largest possible k with this property. Then $\ell_{k+1} = \ell_{k+2} = \cdots = \ell_{i-1} = 2$. Hence, k satisfies the assumption of the definition, a contradiction. It means that if part (ii) of the definition applies, then $\ell_{i-1} = \ell_{i-2} = \cdots = \ell_1 = 2$.

(ii): If $\ell_{i+1} = 0$ then clearly $\ell_i \geq 3$. Suppose that $\ell_i = 3$. Then $\ell_{i-1} = 2$, since otherwise the assumptions apply to $i - 1$. As $2q$ is even, there must be $k < i$ such that $\ell_k \geq 3$. Let k be the largest possible value with this property. Then the assumptions apply to k, a contradiction. Thus, if $\ell_{i+1} = 0$ then $\ell_i \geq 4$. □

Note that part (ii) of Lemma 2.5 means that in the sequence $L' = \mathcal{M}(L)$, $\ell'_i \geq 3$ and $\ell'_{i+1} = 1$. The corresponding graph H will thus have a single vertex of degree 3 in the final layer.

Example 2.6. For an illustration, the sequence of sequences

$$L_m,\ \mathcal{M}(L_m),\ \mathcal{M}^2(L_m),\ \mathcal{M}^3(L_m),\ \dots$$

for $2q = 20$ is

$(4,8,8)$,	$(4,5,6,5)$,	$(3,4,5,6,2)$,	$(2,3,4,7,4)$,	$(2,2,3,5,7,1)$,
$(4,7,9)$,	$(4,4,7,5)$,	$(3,4,4,7,2)$,	$(2,3,4,6,5)$,	$(2,2,3,5,6,2)$,
$(4,6,10)$,	$(3,5,7,5)$,	$(3,3,5,7,2)$,	$(2,3,4,5,6)$,	$(2,2,3,4,7,2)$,
$(4,6,9,1)$,	$(3,5,6,6)$,	$(2,4,5,7,2)$,	$(2,3,4,4,7)$,	$(2,2,3,4,6,3)$,
$(4,6,8,2)$,	$(3,4,7,6)$,	$(2,4,5,6,3)$,	$(2,3,3,5,7)$,	$(2,2,3,4,5,4)$,
$(4,5,9,2)$,	$(3,4,6,7)$,	$(2,4,4,7,3)$,	$(2,2,4,5,7)$,	etc.
$(4,5,8,3)$,	$(3,4,5,8)$,	$(2,3,5,7,3)$,	$(2,2,4,5,6,1)$,	
$(4,5,7,4)$,	$(3,4,5,7,1)$,	$(2,3,5,6,4)$,	$(2,2,4,4,7,1)$,	

There are altogether 67 sequences since $D^m - D_m = 66$ when $2q = 20$.

Lemma 2.7. *Let $t \geq 3$. There exist rooted trees T_1 and T_2 such that vertices u_1 and u_2 are Šoltés vertices in the graph Q obtained from G_t by attaching T_1 and T_2 to vertices v_1 and v_2.*

Proof. By Lemma 2.2, there exists q such that $D_m \leq f(t) \leq D^m$. From Lemma 2.4, we obtain the sequence L, which gives us the appropriate number of vertices in every layer of the forest $T = T_1 \cup T_2$. This ensures that v_1 and v_2 are Šoltés vertices in Q.

Now, we construct a graph Q containing G_t and realising L. Let T_j be the tree rooted at v_j, $1 \leq j \leq 2$. Then T_1 will have $\lceil \ell_i/2 \rceil$ vertices at distance i from v_1 and T_2 will have $\lfloor \ell_i/2 \rfloor$ vertices at distance i from v_2. Observe that it is possible to construct both T_1 and T_2. We have two possibilities.

Case 1: $\ell_i = 2\ell_{i-1}$. Then either $\ell_i = 2^{i+1}, \ell_{i-1} = 2^i, \ldots, \ell_1 = 4$ and on the first i levels both T_1 and T_2 are complete binary trees of height i; or $\ell_1 = \ell_2 = \cdots = \ell_{i-1} = 2$ and $\ell_i = 4$, which means that both T_1 and T_2 contain one vertex at levels $1, 2, \ldots, i-1$ and two vertices at level i.

Case 2: $\ell_i \leq 2\ell_{i-1} - 1$. If ℓ_{i-1} is even then we can construct i-th level of both T_1 and T_2, and at least one vertex of level $i-1$ of T_2 will have degree less than 3. On the other hand, if ℓ_{i-1} is odd then T_2 has only $(\ell_{i-1} - 1)/2$ vertices at level $i-1$. However, it has $\lfloor \ell_i/2 \rfloor$ vertices at level i and $\lfloor \ell_i/2 \rfloor \leq \lfloor (2\ell_{i-1} - 1)/2 \rfloor = \ell_{i-1} - 1 = 2\lfloor \ell_{i-1}/2 \rfloor$, so in T_2, the number of vertices at level i is at most twice the number of vertices at level $i-1$. In this case, at least one vertex at level $i-1$ in T_1 will have degree less than 3. This concludes Case 2. □

We construct the trees T_1 and T_2 so that at each level we minimise the number of vertices of degree 3. Observe that then there is no level in which there are vertices of degree 1 and also vertices of degree 3.

Consider v_1 and v_2 as vertices of level 0, and set $\ell_0 = 2$. We plan to add edges within levels (i.e. 'blue' edges) to create a cubic graph, but sometimes we must also add edges between consecutive levels (i.e. 'red' edges). First, we add necessary edges connecting vertices of different levels. For every $i \geq 1$, if $\sum_{j=0}^{i} \ell_j$ is odd then add an edge joining a vertex (of degree ≤ 2) of $(i-1)$-th level with a vertex of i-th level.

Lemma 2.8. *It is possible to add edges to Q as described above, so that the resulting graph has no parallel edges and it is subcubic.*

Proof. We add the red edges step by step starting with level 1, together with creating the trees T_1 and T_2. And we show that at each level it is possible to add a required edge. We distinguish two cases:

Case 1: *There is no red edge between levels* $i-2$ *and* $i-1$. If $2\ell_{i-1} = \ell_i$, then either (a) $\ell_1 = \ell_2 = \cdots = \ell_{i-1} = 2$ and $\ell_i = 4$, or (b) $\ell_j = 2^{j+1}$ for all $j \leq i$. In both subcases $\sum_{j=0}^{i} \ell_j$ is even, and no red edge is added between levels $i-1$ and i. Suppose that $2\ell_{i-1} > \ell_i$. Then there is a vertex at $(i-1)$-st level, say x, whose degree is less than 3. If $2\ell_i > \ell_{i+1}$, then there is a vertex at i-th level, say y, whose degree is also less than 3, and we can add the edge xy. (Observe that if we add these additional edges together with the construction of trees T_1 and T_2, then we do not create parallel edges. The only problem occures when x is the unique vertex at level $i-1$ in T_j and y is also in T_j. But then either $\ell_{i-1} = 3$ and $j = 2$, in which case x can be chosen in T_1, or $\ell_{i-1} = 2$ and $\ell_i = 3$ in which case x can be chosen in T_2 and y in T_1.) On the other hand, if $2\ell_i = \ell_{i+1}$ then (recall that $2\ell_{i-1} > \ell_i$) $\ell_1 = \cdots = \ell_{i-1} = \ell_i = 2$ and $\ell_{i+1} = 4$, so $\sum_{j=0}^{i} \ell_j$ is even, and no red edge is added between levels $i-1$ and i.

Case 2: *There is a red edge between levels* $i-2$ *and* $i-1$. Then $\sum_{j=0}^{i-1} \ell_j$ is odd. Assume that we also have to add a red edge between levels $i-1$ and i. Then $\sum_{j=0}^{i} \ell_j$ is also odd which means that ℓ_i is even and that $2\ell_{i-1} > \ell_i$, as shown in Case 1. Hence, $2\ell_{i-1} > \ell_i + 1$, so there is a vertex at $(i-1)$-st level, say x, whose degree is less than 3. As $2\ell_i > \ell_{i+1}$, there is a vertex at i-th level, say y, whose degree is also less than 3, and we can add the edge xy. (Multiple edges can be avoided analogously as in Case 1.) □

We remark that we did not precisely specify how to choose the vertices x and y, when a red edge is add between levels $i-1$ and i in case there are several possibilities. Here are a few simple rules to follow when choosing x or y at level i:

(i) if there are at least three leaves (in $T_1 \cup T_2$) at level i, then do not choose these leaves;

(ii) if there is eactly one leaf w at level i, $w \in V(T_j)$, then there must be a protected degree-2 vertex at level i in T_{3-j} (i.e. the protected vertex shall not be chosen);

(iii) if there are two leaves w and w' at level i (one of them is in T_1 and the other in T_2, as we will prove later), then one degree-2 vertex from T_1 and one degree-2 vertex from T_2 has to be protected;

(iv) if there are no leaves at level i, we have no constraints.

The above rules will be fully justified later, when we will consider 2-connectivity of the resulting graph H. Denote by R the graph obtained after adding red edges, as described above. We have the following statement.

Lemma 2.9. *In each level of R, the sum of free valencies is even.*

Proof. Let $1 \le i \le d$. We prove the statement for level i. So denote by $a_1, a_2, \dots, a_{\ell_i}$ the vertices at i-th level. Our task is to show that $\sum_{j=1}^{\ell_i}(3-\deg_R(a_j))$ is even. We distinguish two cases, with two subcases each.

Case 1: *$\sum_{j=0}^{i} \ell_j$ is odd.* Then there are $\ell_i + 1$ edges between levels ℓ_{i-1} and ℓ_i in R. If ℓ_{i+1} is odd, then $\sum_{j=0}^{i+1} \ell_j$ is even, so there are ℓ_{i+1} edges between levels i and $i+1$. Hence, $\sum_{j=1}^{\ell_i}(3-\deg_R(a_j)) = 3\ell_i - \ell_i - 1 - \ell_{i+1}$ is even. On the other hand, if ℓ_{i+1} is even, then $\sum_{j=0}^{i+1} \ell_j$ is odd, so there are $\ell_{i+1}+1$ edges between levels i and $i+1$. Hence, $\sum_{j=1}^{\ell_i}(3-\deg_R(a_j)) = 3\ell_i - \ell_i - 1 - \ell_{i+1} - 1$ is even.

Case 2: *$\sum_{j=0}^{i} \ell_j$ is even.* Then there are ℓ_i edges between levels ℓ_{i-1} and ℓ_i in R. If ℓ_{i+1} is odd, then $\sum_{j=0}^{i+1} \ell_j$ is odd, so there are $\ell_{i+1}+1$ edges between levels i and $i+1$. Hence, $\sum_{j=1}^{\ell_i}(3-\deg_R(a_j)) = 3\ell_i - \ell_i - \ell_{i+1} - 1$ is even. On the other hand, if ℓ_{i+1} is even, then $\sum_{j=0}^{i+1} \ell_j$ is also even, so there are ℓ_{i+1} edges between levels i and $i+1$. Hence, $\sum_{j=1}^{\ell_i}(3-\deg_R(a_j)) = 3\ell_i - \ell_i - \ell_{i+1}$ is even. □

By Lemma 2.9, the sum of free valencies is even at each level of R. This means that, in general, after we add some edges connecting vertices within level i, and when we do that for all i, $1 \le i \le d$, the resulting graph H will be cubic. We now describe how to add these 'blue' edges, so that H will be 2-connected, and how to resolve the cases when a level has small number of remaining degree-2 vertices.

Observation. H will be 2-connected if for every leaf x of T, say $x \in V(T_k)$, where $1 \le k \le 2$ and x is a vertex at level i, there is a path, say P, containing only the vertices of level i and connecting x with a vertex, say y, of T_{3-k}.

We refer to the above as the 2-*connectivity condition*. The reason is that P can be completed to a cycle using a (v_k, x)-path in T_k and a (v_{3-k}, y)-path in T_{3-k}. Since all cycles constructed in this way contain three vertices of G_t, the resulting graph H will be 2-connected. We remark that in one special case the path P will contain vertices of levels i and $i+1$, but it will still be possible to complete P to a cycle containing three vertices of G_t. Thus, our attention will be focused on the leaves of T.

Lemma 2.10. *It is possible to add edges to R so that the resulting graph H will be cubic and 2-connected.*

Proof. First, we consider the d-th (i.e., the last) level. Note that all the vertices at level d are leaves of T. Since $\sum_{j=1}^{d} \ell_j = 2q$ is even, each vertex at level d has degree 1 in R. We distinguish three cases.

Case 1: $\ell_d \geq 3$. In this case, we add to graph R a cycle passing through all the vertices of level d. Then the vertices of level d will have degree 3 and they will satisfy the 2-connectivity condition, since $\lceil \ell_d/2 \rceil$ vertices of level d are in T_1 and $\lfloor \ell_d/d \rfloor$ of them are in T_2.

Case 2: $\ell_d = 1$. Then $\ell_{d-1} \geq 3$, as already shown. In this case, we replace the sequence $L = (\ell_1, \ell_2, \ldots, \ell_{d-1}, 1)$ by $L^* = (\ell_1, \ell_2, \ldots, \ell_{d-1}, 3)$ and we find a 2-connected cubic graph H^* realizing L^*. In this graph, the pendant vertices of level d are connected to three different vertices of level $d-1$ in T, since at each level we minimised the number of degree-3 vertices. Since $\sum_{j=1}^{d} \ell_j$ is even, there are no other edges connecting vertices of level $d-1$ with those of level d in R. Hence, we add a 3-cycle as described in Case 1, and then contract the three vertices at level d to a single vertex. If H^* is cubic and 2-connected, then so is the resulting graph H.

Case 3: $\ell_d = 2$. In this case, we simultaneously resolve the problem for levels d and $d-1$. Since $\sum_{j=1}^{d-1} \ell_j$ is even, all vertices of level $d-1$ have degree 1 except for two, which have degree 2. (Recall that T was constructed so that at each level the number of vertices of degree 3 was minimised.)

If $\ell_{d-1} = 2$, then connect both vertices of level $d-1$ with both vertices of level d and add an edge connecting the vertices of level d. Then the vertices of levels $d-1$ and d have degree 3 and they satisfy the 2-connectivity condition.

If $\ell_{d-1} \geq 3$, then pick a vertex of degree 1 at level $d-1$, say x, and join it to both vertices of level d. Then x has degree 3, but since it is a leaf of T, it does not satisfy the 2-connectivity condition in the strict sense. Nevertheless, there is a cycle in H which contains edges of G_t, a path connecting v_1 with a vertex of level d in T_1, a path connecting v_2 with a vertex of level d in T_2, and the two edges connecting x with the vertices of level d, which is sufficient. Then add to H the edge connecting vertices of level d and add a path passing through all vertices of level $d-1$ except x, and starting/ending in the two degree-2 vertices. This resolves the problem for levels d and $d-1$. This concludes Case 3.

We now turn to level i, $1 \leq i < d$. In case $\ell_d = 2$, we assume $i < d-1$. Then vertices of level i are connected to vertices of levels $i-1$ and $i+1$ using only the edges of R, and we now add only edges connecting vertices within level i, i.e. the blue edges.

In some cases, we specify positions of red edges that were added to T to form R, to justify the four rules for choosing vertices x and y in the process of creating R.

If there is no leaf at level i, then all vertices of this level have degrees 2 and 3 in R. By Lemma 2.9, there is an even number of degree-2 vertices. Thus, we can add a collection of independent edges so that all vertices of level i will have degree 3. Since there were no leaves, the vertices of level i satisfy the 2-connectivity condition.

Now suppose that there are leaves at level i. Since we minimised the number of vertices of degree 3 when constructing T, there are no vertices of degree 3 in level i. Consequently, each vertex of level i is connected to at most one vertex in level $i+1$ in T. Hence, T_1 and T_2 have, respectively, $\lceil \ell_i/2 \rceil - \lceil \ell_{i+1}/2 \rceil$ and $\lfloor \ell_i/2 \rfloor - \lfloor \ell_{i+1}/2 \rfloor$ leaves at level i. Denote $k = (\lceil \ell_i/2 \rceil - \lceil \ell_{i+1}/2 \rceil) - (\lfloor \ell_i/2 \rfloor - \lfloor \ell_{i+1}/2 \rfloor)$. Since $k = (\lceil \ell_i/2 \rceil - \lfloor \ell_i/2 \rfloor) - (\lceil \ell_{i+1}/2 \rceil - \lfloor \ell_{i+1}/2 \rfloor)$, we have

$$-1 \leq k \leq 1. \tag{2.5}$$

This means that the numbers of leaves at level i in T_1 and T_2 differ by at most one, and also that there are no degree-3 vertices at level i in T. Moreover, since $i < d$, level i contains two vertices, say b_1 and b_2, such that $b_1 \in V(T_1)$, $b_2 \in V(T_2)$ and b_1, b_2 are not leaves in T. (Recall that if $i = d-1$ and $\ell_d = 1$, then then we solve this case for L^* where $\ell_d = 3$, and afterwards we provide the contraction of vertices at level d, see Case 2 above.) Then $\deg_T(b_1) = \deg_T(b_2) = 2$. We distinguish three cases.

Case 1: *T has at least 3 leaves at level i.* If $E(R) \setminus E(T)$ contains an edge connecting levels $i-1$ and i, then this edge will terminate at b_1, and if $E(R) \setminus E(T)$ contains an edge connecting levels i and $i+1$, then this edge will start at b_2. (Note that if we create T and R simultaneously, level by level, then we can form b_1 and b_2, so that we do not get parallel edges. In the worst case we relabel b_1 and b_2, so that $b_1 \in V(T_2)$ and $b_2 \in V(T_1)$.) This leaves the leaves untouched. Then we add a cycle passing through all leaves of level i and add a collection of independent edges so that all vertices of level i become degree-3 vertices. Since, at level i, at least one leaf is in T_1 and at least one is in T_2, the vertices at level i satisfy the 2-connectivity condition.

Case 2: *T has exactly two leaves at level i.* Denote these vertices by a_1 and a_2. As mentioned above, we may assume that $a_1 \in V(T_1)$ and $a_2 \in V(T_2)$. If $E(R) \setminus E(T)$ contains an edge connecting levels $i-1$ and i, then this edge will terminate at a_1, and if $E(R) \setminus E(T)$ contains an edge connecting levels i and $i+1$, then this edge will start at a_2. (Again, not to create parallel edges, the red edge between levels $i-1$ and i may be connected to a_2 instead of a_1, and then possible red edge between levels i and $i+1$ will start at a_1.) Then add edges a_1b_2, a_2b_1, and a collection of independent edges so that all vertices of level i become degree-3 vertices. Due to the presence of edges a_1b_2 and a_2b_1, the vertices at level i satisfy the 2-connectivity condition.

Case 3: *T has exactly one leaf at level i.* Denote this vertex by a. Without loss of generality, assume that $a \in V(T_1)$. If $E(R) \setminus E(T)$ contains an edge connecting levels $i-1$ and i, then this edge will terminate at b_1, and if $E(R) \setminus E(T)$ contains an edge connecting levels i and $i+1$, then this edge will start at a. (Not to create parallel edges, the red edge between levels $i-1$ and i may be connected to a instead of b_1, and then possible red edge between levels i and $i+1$ will start at b_1.) Then add the edge ab_2, and a collection of independent edges so that all vertices of level i become degree-3 vertices. Due to the presence of edge ab_2, vertices at level i satisfy the 2-connectivity condition. □

3 Cubic 2-connected graphs with 2^r Šoltés vertices

Now we generalise Theorem 2.1 to higher amount of Šoltés vertices.

Theorem 3.1. *Let $r \geq 1$. There exist infinitely many cubic 2-connected graphs G which contain at least 2^r Šoltés vertices.*

Proof. We reconsider the graph G_t from the proof of Theorem 2.1. This graph consists of a chain of $2t$ diamonds attached to vertices z_1 and z_2 of a graph on 8 vertices. Denote this graph on 8 vertices by F.

We construct $G_{t,r}$. Take a binary tree B of depth $r-1$. This tree has $2^r - 1$ vertices, out of which 2^{r-1} are leaves. Denote these leaves by $a_1, a_2, \ldots, a_{2^{r-1}}$. Let B' be a copy of B. To distinguish endvertices of B' from those of B, put to the endvertices of B' dashes. Now take 2^{r-1} chains of $2t$ diamonds and identify the ends (the vertices of degree 2) of k-th chain with a_k and a'_k, respectively. Finally, join the roots of B and B' (i.e., the vertices of degree 2) by edges to z_1 and z_2.

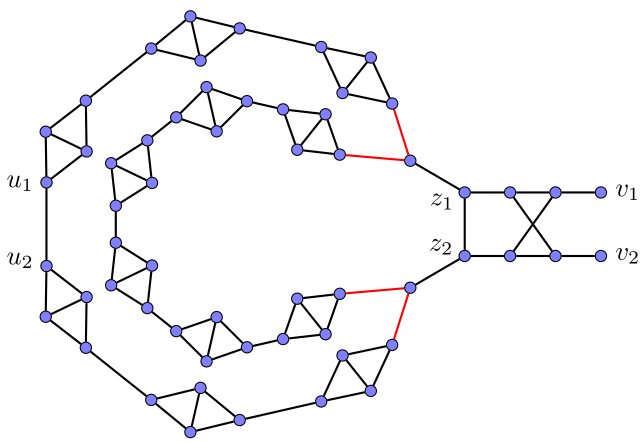

Figure 5: The graph $G_{3,2}$. Edges of binary trees of depth 1 are red.

Denote by $G_{t,r}$ the resulting graph, see Figure 5 for $G_{3,2}$. Then $G_{t,r}$ has $8t2^{r-1} + 2(2^{r-1} - 1) + 8$ vertices. Moreover, all central vertices of 2^{r-1} chains of diamonds belong to the same orbit of $G_{t,r}$. Observe that there are 2^r such vertices. Let u_1 and u_2 be central vertices of one of the chains of diamonds. If we show that $\lim_{t\to\infty}(W(G_{t,r} - u_1) - W(G_{t,r})) = \infty$, we can complete $G_{t,r}$ analogously as G_t was completed to H in the proof of Theorem 2.1, to obtain a cubic $2-$connected graph with at least 2^r Šoltés vertices.

Thus, it remains to show that $W(G_{t,r} - u_1) - W(G_{t,r})$ tends to infinity as $t \to \infty$. Observe that $W(G_{t,r} - u_1) - W(G_{t,r})$ equals

$$\sum_{x,y \in V(G_{t,r})\setminus\{u_1\}} \Big(d_{G_{t,r}-u_1}(x,y) - d_{G_{t,r}}(x,y)\Big) - w_{G_{t,r}}(u_1).$$

We first estimate $w_{G_{t,r}}(u_1)$ from above. For small i, there are at most 4 vertices at distance i from u_1. For bigger i the amount of vertices at distance i grows, but it cannot exceed $4 \cdot 2^r + 8$ since there are 2^{r-1} chains attached to F and F itself has 8 vertices. Thus,

$w_{G_{t,r}}(u_1) \leq (2^{r+2}+8)\sum_{i=1}^{1+3t+2r+3t+3} i$. And if we held r constant, $w_{G_{t,r}}(u_1)$ can be bounded from above by a quadratic polynomial in t.

Now we estimate $\sum_{x,y\in V(G_{t,r})\setminus\{u_1\}} \big(d_{G_{t,r}-u_1}(x,y) - d_{G_{t,r}}(x,y)\big)$ from below. For every $x, y \in V(G_{t,r})\setminus\{u_1\}$ we have $d_{G_{t,r}-u_1}(x,y) \geq d_{G_{t,r}}(x,y)$, since $G_{t,r}$ has all paths which exist in $G_{t,r} - u_1$. However, it suffices to consider only x, y being in the same chain of diamonds as u_1. Observe that the distance from u_2 to a neighbour of u_1 ($\neq u_2$) is 2 in $G_{t,r}$, but it is at least $6t$ in $G_{t,r} - u_1$ (with equality if $r = 1$, i.e. if $G_{t,r} = G_t$). So this distance is increased at least by $6t - 2$. The distance from u_2 to the second neighbour of u_1 is increased at least by $6t - 4$, etc. However, we should consider also a neighbour of u_2 ($\neq u_1$). For this vertex the distances are increased at least by $6t-4, 6t-6, \ldots$ Summing up,

$$D \geq \sum_{j=1}^{3t-1}\sum_{i=1}^{j} 2i = 2\sum_{j=1}^{3t-1}\binom{j+1}{2} = 2\binom{3t+1}{3}.$$

Consequently, $W(G_{t,r} - u_1) - W(G_{t,r})$ is bounded from below by a cubic polynomial (in t) with leading coefficient 9. Thus, $\lim_{t\to\infty}(W(G_{t,r} - u_1) - W(G_{t,r})) = \infty$ as required. □

4 Concluding remarks and further work

We believe that if there exists another Šoltés graph in addition to C_{11}, it is likely to be vertex-transitive or has a low number of vertex orbits. Vertices of the same orbit are either all Šoltés vertices or none of them is.

Holt and Royle [5] have constructed a census of all vertex-transitive graphs with less than 48 vertices; these graphs can be obtained from their Zenodo repository [14] in the `graph6` format [10]. The repository contains 100 720 391 graphs in total, 100 716 591 of which are connected [12]. The computer search revealed that the only Šoltés graph among them is the well-known C_{11}.

We also examined the census of cubic vertex-transitive graphs by Potočnik, Spiga and Verret [13]. Their census contains all (connected) cubic vertex-transitive graphs on up to 1280 vertices; there are 111 360 such graphs. $\mathrm{CVT}(n,i)$ denotes the i-th graph of order n in the census. No Šoltés graph has been found, but the search revealed that there exist graphs that are $\frac{1}{3}$-Šoltés, i.e. $\frac{1}{3}$ of all vertices are Šoltés vertices. We found 7 cubic $\frac{1}{3}$-Šoltés graphs; all of them are *trunctations* of certain cubic vertex-transitive graphs. In this paper, the truncation of a graph G is denoted by $\mathrm{Tr}(G)$. Note that the truncation of a vertex-transitive graph is not necessarily a vertex-transitive graph; in the case of cubic graphs, there may be up to 3 vertex orbits. When doing the computer search, we have to check the Šoltés property for one vertex from each orbit only. Here is the list of cubic vertex-transitive graphs G, such that $\mathrm{Tr}(G)$ is a $\frac{1}{3}$-Šoltés graph:

$$\mathrm{CVT}(384,805),\quad \mathrm{CVT}(600,259),\quad \mathrm{CVT}(768,3650),\quad \mathrm{CVT}(1000,302),$$
$$\mathrm{CVT}(1056,538),\quad \mathrm{CVT}(1056,511),\quad \mathrm{CVT}(1280,967).$$

Interestingly, all these graphs are *Cayley graphs*. Several properties of these graphs are listed in the Appendix. The graph $\mathrm{CVT}(768,3650)$ is the only non-bipartite example, while the rest are bipartite. Girths of these graph are values from the set $\{4,6,8,10,12\}$. We were able to identify seven such graphs. However, we believe that there could exist many more.

Problem 4.1. Find an infinite family of cubic vertex-trainsitive graphs $\{G_i\}_{i=1}^{\infty}$, such that $\mathrm{Tr}(G_i)$ is a $\frac{1}{3}$-Šoltés graph for all $i \geq 1$.

Moreover, we also found an example of a 4-regular $\frac{1}{3}$-Šoltés graph, namely the graph $L(\mathrm{CVT}(324, 104))$. It has order 486 and is the *line graph* of $\mathrm{CVT}(324, 104)$, which is a Cayley graph. More data can be found in the Appendix.

Of course, $\frac{1}{3}$-Šoltés is still a long way from being Šoltés. The next conjecture is additionally reinforced by the fact that there are no Šoltés graphs among vertex-transitive graphs with less than 48 vertices.

Conjecture 4.2. *The cycle on eleven vertices, C_{11}, is the only Šoltés graph.*

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A Appendix

There are 7 cubic vertex-transitive graphs G on up to 1280 vertices, such that $\mathrm{Tr}(G)$ is a $\frac{1}{3}$-Šoltés graph. Since all these graph are Cayley graphs, we give the generating set for the Cayley graph. Note that the group itself (its permutation representation) is given implicitly by these generators; however, we also give the group's ID from GAP's library of small groups [2]. $\mathrm{SmallGroup}(n, k)$ is the k-th group of order n from that library. We also calculated girth, diameter and tested all graphs for bipartiteness.

- $\mathrm{CVT}(384, 805)$:

$$\begin{aligned}&\mathrm{Group}([(2,4)(6,12,17,19,18,14)(7,10,20)(8,15,16,9,11,13),\\ &\qquad (1,4)(2,3)(5,13)(6,18)(7,17)(8,15)(9,12)(10,14)(11,19)(16,20)])\\ &\quad \cong \mathrm{SmallGroup}(384,5781)\end{aligned}$$

  girth: 6; diameter: 10; bipartite? True

- $\mathrm{CVT}(600, 259)$:

$$\begin{aligned}&\mathrm{Group}([(1,2)(3,4)(5,6)(7,9)(8,10)(13,14)(16,17),\\ &\qquad (1,3)(5,7)(6,8)(9,12)(10,13)(11,14)(15,16),\\ &\qquad (1,4)(2,3)(5,6)(8,11)(9,10)(13,14)(15,17)])\\ &\quad \cong \mathrm{SmallGroup}(600,103)\end{aligned}$$

  girth: 10; diameter: 13; bipartite? True

- $\mathrm{CVT}(768, 3650)$:

$$\begin{aligned}&\mathrm{Group}([(2,3)(4,7)(5,6)(10,12),\\ &\qquad (2,4)(3,5)(6,7)(9,10)(11,12),\\ &\qquad (1,2)(3,6)(4,8)(5,7)])\\ &\quad \cong \mathrm{SmallGroup}(768,1090104)\end{aligned}$$

  girth: 8; diameter: 16; bipartite? False

- $\mathrm{CVT}(1000, 302)$:

$$\begin{aligned}&\mathrm{Group}([(4,5)(6,7)(9,14)(10,18)(11,21)(12,23)(13,25)(15,24)(16,27)(17,29)\\ &\qquad (19,26)(20,31)(22,28)(30,32),\\ &\quad (1,2)(3,4)(5,6)(8,9)(10,15)(11,19)(12,18)(13,21)(14,23)(16,17)\\ &\qquad (20,27)(22,29)(24,31)(25,32)(26,28),\\ &\quad (1,2)(8,17)(9,27)(10,11)(12,26)(13,24)(14,22)(15,18)(16,30)(20,32)\\ &\qquad (21,28)(23,31)(25,29)])\\ &\quad \cong \mathrm{SmallGroup}(1000,105)\end{aligned}$$

  girth: 10; diameter: 15; bipartite? True

- $\mathrm{CVT}(1056, 538)$:

$$\begin{aligned}&\mathrm{Group}([(2,3)(4,5)(6,8)(7,10)(9,11)(13,14)(15,16)(17,18)(19,20)(21,22),\\ &\qquad(1,2)(4,6)(7,9)(12,13)(14,15)(16,17)(18,19)(20,21),\\ &\qquad(4,7)(6,9)(10,11)(12,13)(14,15)(16,17)(18,19)(20,21)])\\ &\cong \mathrm{SmallGroup}(1056, 493)\end{aligned}$$

girth: 4; diameter: 22; bipartite? True

- $\mathrm{CVT}(1056, 511)$:

$$\begin{aligned}&\mathrm{Group}([(2,3)(12,13)(14,15)(16,17)(18,19)(20,21),\\ &\qquad(1,2)(4,5)(6,7)(8,9)(10,11)(12,14)(15,16)(17,18)(19,20)(21,22),\\ &\qquad(5,6)(7,8)(9,10)(12,13)(14,15)(16,17)(18,19)(20,21)])\\ &\cong \mathrm{SmallGroup}(1056, 468)\end{aligned}$$

girth: 4; diameter: 22; bipartite? True

- $\mathrm{CVT}(1280, 967)$:

$$\begin{aligned}&\mathrm{Group}([(1,2)(3,5)(7,9)(8,11)(10,15)(12,19)(13,20)(14,22)(16,25)(17,27)\\ &\qquad\quad(18,29)(21,32)(23,34)(24,30)(26,31)(28,37),\\ &\qquad(1,3)(4,5)(6,7)(8,12)(9,13)(10,16)(11,17)(15,23)(18,30)(20,31)\\ &\qquad\quad(21,27)(22,25)(24,28)(26,36)(29,33)(35,37),\\ &\qquad(1,4)(2,3)(6,8)(7,10)(9,14)(11,18)(12,20)(13,21)(15,24)(16,26)\\ &\qquad\quad(17,28)(19,29)(22,33)(23,35)(25,30)(27,36)(31,34)(32,37)])\\ &\cong \mathrm{SmallGroup}(1280, 81752)\end{aligned}$$

girth: 12; diameter: 16; bipartite? True

There exists one cubic vertex-transitive graph G on up to 1280 vertices, such that $L(G)$ is a $\frac{1}{3}$-Šoltés graph.

- $\mathrm{CVT}(324, 104)$:

$$\begin{aligned}&\mathrm{Group}([(2,9)(3,4)(5,7)(6,8),\\ &\qquad(1,5)(2,6)(3,9)(4,7),\\ &\qquad(2,9)(3,6)(4,8)])\\ &\cong \mathrm{SmallGroup}(324, 39)\end{aligned}$$

girth: 4; diameter: 12; bipartite: True