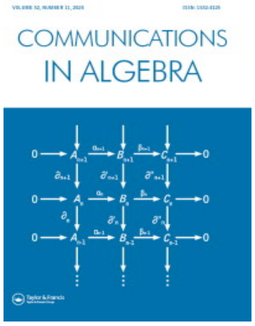

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Generalized derivations of current Lie algebras

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ABSTRACT

Let L be a Lie algebra and let A be an associative commutative algebra with unity, both over the same field F. We consider the following question. Is every generalized derivation (resp. quasiderivation) of $L \otimes A$ the sum of a derivation and a map from the centroid of $L \otimes A$, if the same holds true for L?

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1. Introduction

Let L be a Lie algebra over a field F. A linear map $d : L \to L$ is called a *derivation* if $d([x,y]) = [d(x),y]+[x,d(y)]$ for all $x, y \in L$. As usual, we denote the set of all derivations of L by $\mathrm{Der}(L)$. Obviously, $\mathrm{Der}(L)$ is a Lie subalgebra of the general linear algebra $\mathfrak{gl}(L)$. There are several generalizations of the notion of a derivation. In this paper we consider generalized derivations and quasiderivations of Lie algebras as defined by Leger and Luks in [6]. Let $f : L \to L$ be a linear map. If there exist linear maps $g, h : L \to L$ such that

$$[f(x),y] + [x,g(y)] = h([x,y])$$

for all $x, y \in L$, then f is called a *generalized derivation*. In case there exists a linear map $h : L \to L$ such that

$$[f(x),y] + [x,f(y)] = h([x,y])$$

for all $x, y \in L$, then f is said to be a *quasiderivation*. By $\mathrm{GDer}(L)$ we shall denote the set of all generalized derivations of L and by $\mathrm{QDer}(L)$ the set of all quasiderivations of L. Obviously, $\mathrm{QDer}(L)$ and $\mathrm{GDer}(L)$ are Lie subalgebras of $\mathfrak{gl}(L)$ such that

$$\mathrm{Der}(L) \subseteq \mathrm{QDer}(L) \subseteq \mathrm{GDer}(L) \subseteq \mathfrak{gl}(L).$$

Yet another Lie subalgebra of $\mathfrak{gl}(L)$ is the centroid of L, which is defined as

$$\mathrm{Cent}(L) = \left\{\gamma \in \mathfrak{gl}(L) \mid \gamma([x,y]) = [x,\gamma(y)] \text{ for all } x,y \in L\right\}.$$

For each map $\gamma \in \mathrm{Cent}(L)$ we have

$$[\gamma(x),y] + [x,\gamma(y)] = 2\gamma([x,y])$$

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for all $x, y \in L$. Thus, $\mathrm{Cent}(L) \subseteq \mathrm{QDer}(L)$ and so

$$\mathrm{Der}(L) + \mathrm{Cent}(L) \subseteq \mathrm{QDer}(L).$$

In several cases this is a strict inclusion. However, for some Lie algebras we have

$$\mathrm{Der}(L) + \mathrm{Cent}(L) = \mathrm{QDer}(L) \tag{1.1}$$

or even

$$\mathrm{Der}(L) + \mathrm{Cent}(L) = \mathrm{GDer}(L). \tag{1.2}$$

Let us mention that Leger and Luks [6, Corollary 4.16] proved that (1.1) holds true for each centerless Lie algebra L generated by special weight spaces. Examples of Lie algebras satisfying (1.2) can be found in Brešar's paper [1], where the structure of near-derivations was described for certain Lie algebras arising from associative ones. Note that the notion of a near-derivation, which was introduced in [1], is even more general than the notion of a generalized derivation.

Suppose that L and A are algebras over a field F, where L is a Lie algebra and A is an associative commutative algebra with unity. The tensor product algebra $L \otimes_F A$ (or shortly $L \otimes A$) is also a Lie algebra over F, which is called a *current Lie algebra*. Recall that the Lie product on $L \otimes A$ is defined as a bilinear map such that

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for any simple tensors $x \otimes a, y \otimes b \in L \otimes A$.

The aim of this paper is to consider the following two questions.

(a) Does $L \otimes A$ satisfy (1.1), if L satisfies (1.1)?
(b) Does $L \otimes A$ satisfy (1.2), if L satisfies (1.2)?

Our research was motivated by [6] and by Brešar's papers [2, 3], where the study of functional identities on tensor products of algebras was initiated.

2. The results

Let L be a Lie algebra over a field F. Recall that the center

$$Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$$

and the derived algebra

$$[L, L] := \mathrm{Span}(\{[x, y] \mid x, y \in L\})$$

are ideals of L. If $Z(L) = \{0\}$, we say that L is *centerless*. For any subset S of L the set

$$Z_L(S) := \{x \in L \mid [x, s] = 0 \text{ for all } s \in S\}$$

is called the *annihilator* of S in L. If I is an ideal of L then $Z_L(I)$ is also an ideal of L. Thus, $Z_L([L, L])$ is an ideal of L and

$$Z(L) = Z_L(L) \subseteq Z_L([L, L]).$$

Note that for any centerless Lie algebra L the sum $\mathrm{Der}(L) + \mathrm{Cent}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$ is a direct sum of vector spaces. Recall that a Lie algebra L is *prime*, if L has no nonzero ideals I, J such that $[I, J] = 0$. Clearly, all prime Lie algebras are centerless. A Lie algebra is said to be *perfect* if $[L, L] = L$. Let us state our main result on the form of quasiderivations of a current Lie algebra $L \otimes A$.

Theorem 2.1. *Let $L \otimes A$ be a current Lie algebra over a field F, where L is centerless and $\mathrm{char}(F) \neq 2$. Suppose that L is either perfect or prime. If $\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$, then $\mathrm{QDer}(L \otimes A) = \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.*

Another aim of this paper is to obtain a similar result for generalized derivations. Recall that the notion of a *quasicentroid* $\mathrm{QCent}(L)$ of a Lie algebra L was defined in [6] as

$$\mathrm{QCent}(L) = \left\{ f \in \mathfrak{gl}(L) \mid [f(x), y] = [x, f(y)] \text{ for all } x, y \in L \right\}.$$

Obviously, $\mathrm{Cent}(L) \subseteq \mathrm{QCent}(L)$ and

$$\mathrm{GDer}(L) = \mathrm{QDer}(L) + \mathrm{QCent}(L) \tag{2.1}$$

(see [6, Proposition 3.3]). Note that each commuting linear map $f : L \to L$ (i.e. $\left[f(x), x\right] = 0$ for all $x \in L$) belongs to $\mathrm{QCent}(L)$. Moreover, if $\mathrm{char}(F) \neq 2$ then $\mathrm{QCent}(L)$ coincides with the set of all commuting linear maps of L.

Let L be a centerless Lie algebra over a field F with $\mathrm{char}(F) \neq 2$. Suppose that L is either perfect or prime. Then $Z_L([L, L]) = 0$ and hence the result of Brešar and Zhao [4, Corollary 3.3] implies that the set of all commuting linear maps of L coincides with $\mathrm{Cent}(L)$. Thus, $\mathrm{QCent}(L) = \mathrm{Cent}(L) \subseteq \mathrm{QDer}(L)$ and hence (2.1) implies $\mathrm{GDer}(L) = \mathrm{QDer}(L)$. Since $Z_L([L, L]) = 0$ it follows that $Z_{L\otimes A}([L \otimes A, L \otimes A]) = 0$ and so $\mathrm{GDer}(L \otimes A) = \mathrm{QDer}(L \otimes A)$ as well. Hence, Theorem 2.1 implies the following corollary.

Corollary 2.2. *Let $L \otimes A$ be a current Lie algebra over a field F, where L is centerless and $\mathrm{char}(F) \neq 2$. Suppose that L is either perfect or prime. Then $\mathrm{GDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$ implies $\mathrm{GDer}(L \otimes A) = \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.*

If $L \otimes A$ is a current Lie algebra, where A is finite dimensional, then we obtain the same conclusion assuming only that L is centerless.

Theorem 2.3. *Let $L \otimes A$ be a current Lie algebra over a field F with $\mathrm{char}(F) \neq 2$. Suppose that L is centerless and A is finite dimensional.*

(i) If $\mathrm{GenDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$, then $\mathrm{GenDer}(L \otimes A) = \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.
(ii) If $\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$, then $\mathrm{QDer}(L \otimes A) = \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.

The proofs of Theorems 2.1 and 2.3 are given in the next section.

3. The proofs

Let L and A be algebras over a field F, where L is a Lie algebra and A is an associative commutative algebra with unity. Pick a basis $\mathcal{B} = \{b_i | i \in I\}$ of A. Hence every element in $L \otimes A$ can be written uniquely in the form $x_{i_1} \otimes b_{i_1} + x_{i_2} \otimes b_{i_2} + \cdots + x_{i_n} \otimes b_{i_n}$ where $n \geq 1$ and $x_i \in L$.

Let $f : L \otimes A \to L \otimes A$ be a linear map. For any element $x \in L$ there exist unique elements $f_i(x) \in L$, $i \in I$, such that

$$f(x \otimes 1) = \sum_{i \in I} f_i(x) \otimes b_i, \tag{3.1}$$

where $f_i(x) = 0$ for all but finitely many $i \in I$. For each $i \in I$ the map $f_i : L \to L$ defined by $f_i : x \mapsto f_i(x)$ is obviously linear. Let $f_{\mathcal{B}} : L \otimes A \to L \otimes A$ be a linear map such that

$$f_{\mathcal{B}}(x \otimes a) = \sum_{i \in I} f_i(x) \otimes ab_i \tag{3.2}$$

for each simple tensor $x \otimes a \in L \otimes A$. Obviously, $f_{\mathcal{B}}$ is well-defined since for each $x \in L$ we have $f_i(x) \neq 0$ for only finitely many elements $i \in I$. Note that $f(x \otimes 1) = f_{\mathcal{B}}(x \otimes 1)$ for all $x \in L$.

The following proposition shows that certain properties of a linear map f are inherited to the map $f_{\mathcal{B}}$.

Proposition 3.1. *Let $L \otimes A$ be a current Lie algebra over a field F. For any basis $\mathcal{B}$ of A the following assertions hold true:*

(i) If $f \in \mathrm{GenDer}(L \otimes A)$*, then* $f_{\mathcal{B}} \in \mathrm{GenDer}(L \otimes A)$*.*
(ii) If $f \in \mathrm{QDer}(L \otimes A)$*, then* $f_{\mathcal{B}} \in \mathrm{QDer}(L \otimes A)$*.*

Proof. First, suppose that $f \in \mathrm{GenDer}(L \otimes A)$. Then there exist linear maps $g, h : L \otimes A \to L \otimes A$ such that

$$\left[f(x), y\right] + \left[x, g(y)\right] = h\left(\left[x, y\right]\right) \tag{3.3}$$

for all $x, y \in L \otimes A$. Pick any basis $\mathcal{B} = \{b_i | i \in I\}$ of A. Then according to (3.2) the linear maps $g_{\mathcal{B}}, h_{\mathcal{B}} : L \otimes A \to L \otimes A$ are given by

$$g_{\mathcal{B}}(x \otimes a) = \sum_{i \in I} g_i(x) \otimes ab_i \quad \text{and} \quad h_{\mathcal{B}}(x \otimes a) = \sum_{i \in I} h_i(x) \otimes ab_i \tag{3.4}$$

for any simple tensor $x \otimes a \in L \otimes A$. In order to prove that $f_{\mathcal{B}} \in \mathrm{GenDer}(L \otimes A)$, let us show that

$$\left[f_{\mathcal{B}}(x), y\right] + \left[x, g_{\mathcal{B}}(y)\right] = h_{\mathcal{B}}\left(\left[x, y\right]\right) \tag{3.5}$$

for all $x, y \in L \otimes A$. Setting simple tensors $x \otimes 1, y \otimes 1$ in (3.3) and using $\left[x \otimes 1, y \otimes 1\right] = \left[x, y\right] \otimes 1$, we get

$$h\left(\left[x, y\right] \otimes 1\right) = \left[f(x \otimes 1), y \otimes 1\right] + \left[x \otimes 1, g(y \otimes 1)\right].$$

According to (3.2) this identity can be rewritten as

$$\begin{aligned}\sum_{i \in I} h_i(\left[x, y\right]) \otimes b_i &= \left[\sum_{i \in I} f_i(x) \otimes b_i, y \otimes 1\right] + \left[x \otimes 1, \sum_{i \in I} g_i(y) \otimes b_i\right] \\ &= \sum_{i \in I} \left[f_i(x), y\right] \otimes b_i + \sum_{i \in I} \left[x, g_i(y)\right] \otimes b_i\end{aligned}$$

and consequently

$$\sum_{i \in I} \left(\left[f_i(x), y\right] + \left[x, g_i(y)\right] - h_i(\left[x, y\right])\right) \otimes b_i = 0.$$

for all $x, y \in L$. Thus, for any $i \in I$ we have

$$\left[f_i(x), y\right] + \left[x, g_i(y)\right] = h_i(\left[x, y\right]) \tag{3.6}$$

for all $x, y \in L$. Using (3.6) we obtain

$$\begin{aligned}h_{\mathcal{B}}\left(\left[x \otimes a, y \otimes b\right]\right) = h_{\mathcal{B}}\left(\left[x, y\right] \otimes ab\right) &= \sum_{i \in I} h_i(\left[x, y\right]) \otimes abb_i \\ &= \sum_{i \in I} (\left[f_i(x), y\right] + \left[x, g_i(y)\right]) \otimes abb_i \\ &= \sum_{i \in I} \left[f_i(x), y\right] \otimes abb_i + \sum_{i \in I} \left[x, g_i(y)\right] \otimes abb_i \\ &= \sum_{i \in I} \left[f_i(x) \otimes ab_i, y \otimes b\right] + \sum_{i \in I} \left[x \otimes a, g_i(y) \otimes bb_i\right] \\ &= \left[\sum_{i \in I} f_i(x) \otimes ab_i, y \otimes b\right] + \left[x \otimes a, \sum_{i \in I} g_i(y) \otimes bb_i\right] \\ &= \left[f_{\mathcal{B}}(x \otimes a), y \otimes b\right] + \left[x \otimes a, g_{\mathcal{B}}(y \otimes b)\right].\end{aligned}$$

for all simple tensors $x \otimes a, y \otimes b \in L \otimes A$. Since $f_{\mathcal{B}}, g_{\mathcal{B}}$, and $h_{\mathcal{B}}$ are linear maps it follows that (3.5) holds true. Thus, $f_{\mathcal{B}} \in \mathrm{GenDer}(L \otimes A)$ and so the proof of (i) is complete.

Note that (ii) can be proved analogously by setting $f = g$ in the arguments above. □

Lemma 3.2. Let $L \otimes A$ be a current Lie algebra over a field F and let $\mathcal{B} = \{b_i | i \in I\}$ be a basis of A. Suppose that $\{f_i : L \to L | i \in I\}$ is a family of linear maps such that for any $x \in L$ we have $f_i(x) \neq 0$ for only finitely many elements $i \in I$. Let a linear map $f_{\mathcal{B}} : L \otimes A \to L \otimes A$ be defined as in (3.2).

(i) If $f_i \in \mathrm{Der}(L)$ for all $i \in I$, then $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A)$.
(ii) If $f_i \in \mathrm{Cent}(L)$ for all $i \in I$, then $f_{\mathcal{B}} \in \mathrm{Cent}(L \otimes A)$.

Proof. (i) Suppose that $f_i \in \mathrm{Der}(L)$ for all $i \in I$. Thus, for each $i \in I$

$$[f_i(x), y] + [x, f_i(y)] = f_i([x, y])$$

for all $x, y \in L$. For any simple tensors $x \otimes a, y \otimes b \in L \otimes A$ we have

$$\begin{aligned}
f_{\mathcal{B}}([x \otimes a, y \otimes b]) = f_{\mathcal{B}}([x, y] \otimes ab) &= \sum_{i\in I} f_i([x, y]) \otimes abb_i \\
&= \sum_{i\in I} \left([f_i(x), y] + [x, f_i(y)] \right) \otimes abb_i \\
&= \sum_{i\in I} [f_i(x), y] \otimes abb_i + \sum_{i\in I} [x, f_i(y)] \otimes abb_i \\
&= \sum_{i\in I} [f_i(x) \otimes ab_i, y \otimes b] + \sum_{i\in I} [x \otimes a, f_i(y) \otimes bb_i] \\
&= \left[\sum_{i\in I} f_i(x) \otimes ab_i, y \otimes b \right] + \left[x \otimes a, \sum_{i\in I} f_i(y) \otimes bb_i \right] \\
&= [f_{\mathcal{B}}(x \otimes a), y \otimes b] + [x \otimes a, f_{\mathcal{B}}(y \otimes b)].
\end{aligned}$$

Since $f_{\mathcal{B}}$ is linear it follows that $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A)$.

(ii) Let's assume that $f_i \in \mathrm{Cent}(L)$ for all $i \in I$. Thus, for each $i \in I$

$$[f_i(x), y] = f_i([x, y])$$

for all $x, y \in L$. For any simple tensors $x \otimes a, y \otimes b \in L \otimes A$ we have

$$\begin{aligned}
f_{\mathcal{B}}([x \otimes a, y \otimes b]) = f_{\mathcal{B}}([x, y] \otimes ab) &= \sum_{i\in I} f_i([x, y]) \otimes abb_i \\
&= \sum_{i\in I} [f_i(x), y] \otimes abb_i \\
&= \sum_{i\in I} [f_i(x) \otimes ab_i, y \otimes b] \\
&= \left[\sum_{i\in I} f_i(x) \otimes ab_i, y \otimes b \right] \\
&= [f_{\mathcal{B}}(x \otimes a), y \otimes b].
\end{aligned}$$

Since $f_{\mathcal{B}}$ is linear it follows that $f_{\mathcal{B}} \in \mathrm{Cent}(L \otimes A)$. □

Lemma 3.3. Let $L \otimes A$ be a current Lie algebra over a field F and let $\mathcal{B} = \{b_i | i \in I\}$ be a basis of A. Suppose that L is perfect or L is prime. Furthermore, assume that $\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$. If $f \in \mathrm{QDer}(L \otimes A)$, then $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.

Proof. Let us pick an arbitrary quasi-derivation $f \in \mathrm{QDer}(L \otimes A)$. Then there exists a linear map $h : L \otimes A \to L \otimes A$ such that

$$[f(x), y] + [x, f(y)] = h([x, y]) \tag{3.7}$$

for all $x, y \in L \otimes A$. Recall that there exist families of linear maps $\{f_i : L \to L | i \in I\}$ and $\{h_i : L \to L | i \in I\}$ such that

$$f(x \otimes 1) = \sum_{i \in I} f_i(x) \otimes b_i \text{ and } h(x \otimes 1) = \sum_{i \in I} h_i(x) \otimes b_i$$

for all $x \in L$, where for each $x \in L$ we have $f_i(x) \neq 0$ for only finitely many elements $i \in I$ and $h_i(x) \neq 0$ for only finitely many elements $i \in I$ (see (3.1)). Similarly as in the proof of Proposition 3.1 we see that for any $i \in I$ we get

$$[f_i(x), y] + [x, f_i(y)] = h_i([x, y]) \tag{3.8}$$

for all $x, y \in L$. Hence, each f_i is a quasi-derivation of L. Consequently, our assumption implies $f_i \in \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$ for all $i \in I$. Thus, for each $i \in I$ there exist maps $d_i \in \mathrm{Der}(L)$ and $\gamma_i \in \mathrm{Cent}(L)$ such that $f_i = d_i + \gamma_i$. Hence, (3.8) can be rewritten as

$$\begin{aligned} h_i([x, y]) &= [d_i(x) + \gamma_i(x), y] + [x, d_i(y) + \gamma_i(y)] \\ &= [d_i(x), y] + [x, d_i(y)] + [\gamma_i(x), y] + [x, \gamma_i(y)] \\ &= d_i([x, y]) + \gamma_i([x, y]) + \gamma_i([x, y]) \\ &= f_i([x, y]) + \gamma_i([x, y]) \end{aligned} \tag{3.9}$$

for all $x, y \in L$ and $i \in I$.

First, suppose that L is perfect. Since $[L, L] = L$ it follows from (3.9) that $\gamma_i(x) = h_i(x) - f_i(x)$ for all $x \in L$ and $i \in I$. Hence, for each $x \in L$ we have $\gamma_i(x) = 0$ for all but finitely many elements $i \in I$ and consequently $d_i(x) = f_i(x) - \gamma_i(x) = 0$ for all but finitely many elements $i \in I$.

Next, suppose that L is prime. According to (3.9) we see that for each pair $x, y \in L$ we have $\gamma_i([x, y]) = 0$ for all but finitely many elements $i \in I$. Without loss of generality, we may assume that L is nonzero. Since L is prime it follows that $[L, L] \neq \{0\}$. Hence, there exist elements $x_0, y_0 \in L$ such that $[x_0, y_0] \neq 0$. Since L is torsion free $\mathrm{Cent}(L)$-module (see [5, Theorem 1.1]) and since $\gamma_i([x_0, y_0]) = 0$ for all but finitely many elements $i \in I$ it follows that $\gamma_i = 0$ for all but finitely many elements $i \in I$. Consequently, for each $x \in L$ also $d_i(x) = f_i(x) - \gamma_i(x) = 0$ for all but finitely many elements $i \in I$.

Let $d_{\mathcal{B}}, \gamma_{\mathcal{B}} : L \otimes A \to L \otimes A$ be linear maps such that

$$d_{\mathcal{B}}(x \otimes a) = \sum_{i \in I} d_i(x) \otimes ab_i \quad \text{and} \quad \gamma_{\mathcal{B}}(x \otimes a) = \sum_{i \in I} \gamma_i(x) \otimes ab_i \tag{3.10}$$

for each simple tensor $x \otimes a \in L \otimes A$. Obviously, $d_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ are well-defined, since in case L is perfect or prime, both sums in (3.10) are finite. Namely, for any $x \in L$ in both cases $d_i(x) = 0$ for all but finitely many elements $i \in I$ and $\gamma_i(x) = 0$ for all but finitely many elements $i \in I$. Now, Lemma 3.2 implies that $d_{\mathcal{B}} \in \mathrm{Der}(L \otimes A)$ and $\gamma_{\mathcal{B}} \in \mathrm{Cent}(L \otimes A)$. We can now conclude that

$$\begin{aligned} f_{\mathcal{B}}(x \otimes a) &= \sum_{i \in I} f_i(x) \otimes ab_i = \sum_{i \in I} (d_i(x) + \gamma_i(x)) \otimes ab_i \\ &= \sum_{i \in I} d_i(x) \otimes ab_i + \sum_{i \in I} \gamma_i(x) \otimes ab_i \\ &= d_{\mathcal{B}}(x \otimes a) + \gamma_{\mathcal{B}}(x \otimes a) \end{aligned}$$

for each simple tensor $x \otimes a \in L \otimes A$. Since $f_{\mathcal{B}}, d_{\mathcal{B}}$, and $\gamma_{\mathcal{B}}$ are linear maps it follows $f_{\mathcal{B}} = d_{\mathcal{B}} + \gamma_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$. □

If we assume that A is a finite dimensional algebra in Lemma 3.3, then we can drop the assumption of L being perfect or prime. Namely, in this case both sums in (3.10) are finite and so the maps $d_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ are well-defined. Thus, using similar arguments as in the proof of Lemma 3.3 we obtain the following proposition.

Proposition 3.4. *Let $L \otimes A$ be a current Lie algebra over a field F and let $\mathcal{B}$ be a basis of A. Suppose that* $\dim_{\mathbb{F}} A < \infty$.

(i) If $f \in \mathrm{GenDer}(L \otimes A)$ and $\mathrm{GenDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$, then $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.
(ii) If $f \in \mathrm{QDer}\,(L \otimes A)$ and $\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$, then $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$.

Recall that a map $f : L \to L$ is commuting if $\left[f(x), x\right] = 0$ for all $x \in L$. The following lemma, which will be used in the proof of Theorem 2.3, follows directly from [6, Proposition 5.26].

Lemma 3.5. Let L be a centerless Lie algebra over a field F with $\mathrm{char}(F) \neq 2$. If a quasi-derivation $f \in \mathrm{QDer}(L)$ is commuting, then $f \in \mathrm{Cent}(L)$.

Now, we can prove our main results, Theorems 2.1 and 2.3.

Proof of Theorem 2.1. Suppose that $\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$. Pick any basis $\mathcal{B} = \{b_i | i \in I\}$ of A. Let $f \in \mathrm{QDer}\,(L \otimes A)$ be an arbitrary quasi-derivation. According to Proposition 3.1 the map $f_{\mathcal{B}}$ is a quasi-derivation of $L \otimes A$. Moreover, Lemma 3.3 implies that $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$. Let $F = f - f_{\mathcal{B}}$. Obviously, $F \in \mathrm{QDer}\,(L \otimes A)$ and $F(x \otimes 1) = f(x \otimes 1) - f_{\mathcal{B}}(x \otimes 1) = 0$ for all $x \in L$. Since F is a quasi-derivation there exists a linear map $H : L \otimes A \to L \otimes A$ such that

$$\left[F(x), y\right] + \left[x, F(y)\right] = H\left(\left[x, y\right]\right) \tag{3.11}$$

for all $x, y \in L \otimes A$. For each simple tensor $x \otimes a \in L \otimes A$ there exist unique elements $F_i\,(x \otimes a) \in L$, $i \in I$, such that

$$F(x \otimes a) = \sum_{i \in I} F_i(x \otimes a) \otimes b_i,$$

where $F_i(x \otimes a) \neq 0$ for only finitely many $i \in I$. For each $a \in A$ and each $i \in I$ we define a map $F_{a,i} : L \to L$ by $F_{a,i} : x \mapsto F_i(x \otimes a)$, which is obviously linear. First, we shall show that $F_{a,i} \in \mathrm{Cent}(L)$ for any $a \in A$ and any $i \in I$. Let us fix an arbitrary $a \in A$. Setting simple tensors $x \otimes a$ and $x \otimes 1$ in (3.11) and using $F(x \otimes 1) = 0$ we obtain

$$[F(x \otimes a), x \otimes 1] = H\left([x \otimes a, x \otimes 1]\right) = H\left([x, x] \otimes a\right) = 0$$

for all $x \in L$. Hence,

$$\begin{aligned} 0 = [F(x \otimes a), x \otimes 1] &= \left[\sum_{i \in I} F_i(x \otimes a) \otimes b_i, x \otimes 1\right] \\ &= \sum_{i \in I} \left[F_{a,i}(x), x\right] \otimes b_i \end{aligned}$$

for all $x \in L$. Consequently, $\left[F_{a,i}(x), x\right] = 0$ for all $x \in L$ and all $i \in I$. Thus, for each $a \in A$ and each $i \in I$ the map $F_{a,i}$ is commuting. According to our assumption L is perfect and centerless or L is prime. In both cases it follows that $Z_L\left([L, L]\right) = \{0\}$. Hence, [4, Corollary 3.3] yields that $F_{a,i} \in \mathrm{Cent}(L)$ for all $i \in I, a \in A$. Next, we claim that $F \in \mathrm{Der}(L \otimes A)$. Namely, setting simple tensors $x \otimes a$ in $y \otimes b$ in (3.11) we get

$$H\left(\left[x \otimes a, y \otimes b\right]\right) = \left[F(x \otimes a), y \otimes b\right] + \left[x \otimes a, F(y \otimes b)\right]. \tag{3.12}$$

On the other hand, since $\left[x \otimes a, y \otimes b\right] = \left[x \otimes ab, y \otimes 1\right]$ and $F\left(y \otimes 1\right) = 0$, we obtain

$$H\left(\left[x \otimes a, y \otimes b\right]\right) = \left[F\left(x \otimes ab\right), y \otimes 1\right] = \left[\sum_{i \in I} F_{ab,i}(x) \otimes b_i, y \otimes 1\right]$$

for all $x, y \in L$ and $a, b \in A$. Since $F_{ab,i} \in \mathrm{Cent}(L)$ it follows

$$H\left(\left[x \otimes a, y \otimes b\right]\right) = \sum_{i\in I} \left[F_{ab,i}(x), y\right] \otimes b_i = \sum_{i\in I} F_{ab,i}(\left[x, y\right]) \otimes b_i = F\left(\left[x, y\right] \otimes ab\right)$$

for all $x, y \in L$ and $a, b \in A$. Hence,

$$H\left(\left[x \otimes a, y \otimes b\right]\right) = F\left(\left[x \otimes a, y \otimes b\right]\right) \tag{3.13}$$

for all $x \otimes a, y \otimes b \in L \otimes A$. Consequently, (3.12) can be rewritten as

$$F\left(\left[x \otimes a, y \otimes b\right]\right) = \left[F(x \otimes a), y \otimes b\right] + \left[x \otimes a, F(y \otimes b)\right]$$

for all $x \otimes a, y \otimes b \in L \otimes A$. Since F is linear it follows that $F \in \mathrm{Der}(L \otimes A)$. We can now conclude that

$$f = f_{\mathcal{B}} + F,$$

where $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$ and $F \in \mathrm{Der}(L \otimes A)$. Thus, $f \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$ and so the proof is complete. □

Proof of Theorem 2.3. In order to prove (i) let us assume that $\mathrm{GenDer}(L) = \mathrm{Der}(L) \oplus \mathrm{Cent}(L)$. Pick any basis $\mathcal{B} = \{b_i | i \in I\}$ of A. Here, $I = \{1, 2, \ldots, n\}$, since A is finite dimensional. Let $f \in \mathrm{GenDer}(L \otimes A)$. Then there exist linear maps $g, h : L \otimes A \to L \otimes A$ such that

$$\left[f(x), y\right] + \left[x, g(y)\right] = h\left(\left[x, y\right]\right) \tag{3.14}$$

for all $x, y \in L \otimes A$. Proposition 3.1 implies $f_{\mathcal{B}} \in \mathrm{GenDer}\,(L \otimes A)$. Moreover,

$$\left[f_{\mathcal{B}}(x), y\right] + \left[x, g_{\mathcal{B}}(y)\right] = h_{\mathcal{B}}\left(\left[x, y\right]\right) \tag{3.15}$$

for all $x, y \in L \otimes A$ (see the proof of Proposition 3.1). According to Proposition 3.4 we know that $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$. Let us define the following maps: $F = f - f_{\mathcal{B}}$, $G = g - g_{\mathcal{B}}$, and $H = h - h_{\mathcal{B}}$. Obviously, F, G, and H are linear maps. Using (3.14) and (3.15) we get

$$\left[F(x), y\right] + \left[x, G(y)\right] = H\left(\left[x, y\right]\right) \tag{3.16}$$

for all $x, y \in L \otimes A$. Moreover, $F(x \otimes 1) = 0 = G(x \otimes 1)$ for all $x \in L$. For each simple tensor $x \otimes a \in L \otimes A$ there exist unique elements $F_i\,(x \otimes a)\,, G_i\,(x \otimes a)\,, H_i\,(x \otimes a) \in L$, $i \in I$, such that

$$F(x \otimes a) = \sum_{i\in I} F_i(x \otimes a) \otimes b_i, \tag{3.17}$$

$$G(x \otimes a) = \sum_{i\in I} G_i(x \otimes a) \otimes b_i,$$

$$H(x \otimes a) = \sum_{i\in I} H_i(x \otimes a) \otimes b_i.$$

For each $a \in A$ and each $i \in I$ we define maps $F_{a,i}, G_{a,i}, H_{a,i} : L \to L$ by $F_{a,i} : x \mapsto F_i(x \otimes a)$, $G_{a,i} : x \mapsto G_i(x \otimes a)$, and $H_{a,i} : x \mapsto H_i(x \otimes a)$. It is easy to see that $F_{a,i}, G_{a,i}, H_{a,i}$ are linear maps. Our aim is to prove that F is a derivation. First, let us prove that $F_{a,i} \in \mathrm{Cent}(L)$ for all $a \in A$ and $i \in I$. Since $G(y \otimes 1) = 0$ it follows from (3.16) that

$$\left[F(x \otimes a), y \otimes 1\right] = H\left(\left[x \otimes a, y \otimes 1\right]\right) = H\left(\left[x, y\right] \otimes a\right) \tag{3.18}$$

for all $x, y \in L$ and $a \in A$. Fix an arbitrary element $a \in A$. Using (3.17) we can rewrite (3.18) as

$$\begin{aligned} 0 &= \left[F(x \otimes a), y \otimes 1\right] - H\left(\left[x, y\right] \otimes a\right) \\ &= \left[\sum_{i\in I} F_i(x \otimes a) \otimes b_i, y \otimes 1\right] - \sum_{i\in I} H_i([x, y] \otimes a) \otimes b_i \end{aligned}$$

$$= \sum_{i\in I}\left(\left[F_{i,a}(x), y\right] - H_{i,a}(\left[x,y\right])\right) \otimes b_i$$

for all $x, y \in L$. Hence,

$$\left[F_{a,i}(x), y\right] = H_{a,i}(\left[x,y\right]) \tag{3.19}$$

for all $x, y \in L$. Consequently, $\left[F_{a,i}(x), x\right] = 0$ for all $x \in L$. Thus, $F_{a,i}$ is a commuting linear map for each $a \in A$ and each $i \in I$. By interchanging the roles of x and y in (3.19) and using $\left[x,y\right] = -\left[y,x\right]$ we get $\left[x, F_{a,i}(y)\right] = H_{a,i}(\left[x,y\right])$ and so

$$\left[F_{a,i}(x), y\right] + \left[x, F_{a,i}(y)\right] = 2H_{a,i}(\left[x,y\right])$$

for all $x, y \in L$. This means that $F_{a,i} \in \mathrm{QDer}(L)$ and hence Lemma 3.5 yields that $F_{i,a} \in \mathrm{Cent}(L)$ for each $a \in A$ and each $i \in I$. Next, we shall prove that $F = G$. Namely, since $G(y \otimes 1) = 0 = F(x \otimes 1)$ we see that (3.16) yields

$$\begin{aligned}\left[F(x\otimes a), y\otimes 1\right] &= H\left(\left[x\otimes a, y\otimes 1\right]\right) = H\left(\left[x\otimes 1, y\otimes a\right]\right)\\ &= \left[x\otimes 1, G(y\otimes a)\right]\end{aligned}$$

for all $x, y \in L$ and $a \in A$. Fix an arbitrary $a \in A$. Using (3.17) we can rewrite the last identity as

$$\begin{aligned}0 &= \left[\sum_{i\in I} F_{a,i}(x)\otimes b_i, y\otimes 1\right] - \left[x\otimes 1, \sum_{i\in I} G_{a,i}(y)\otimes b_i\right]\\ &= \sum_{i\in I}\left[F_{a,i}(x), y\right]\otimes b_i - \sum_{i\in I}\left[x, G_{a,i}(y)\right]\otimes b_i\\ &= \sum_{i\in I}\left(\left[F_{a,i}(x), y\right] - \left[x, G_{a,i}(y)\right]\right)\otimes b_i\end{aligned}$$

for all $x, y \in L$. Consequently, for each $i \in I$ we have

$$\left[F_{a,i}(x), y\right] = \left[x, G_{a,i}(y)\right]$$

for all $x, y \in L$. Since $F_{a,i} \in \mathrm{Cent}(L)$ it follows

$$\left[x, G_{a,i}(y)\right] = \left[F_{a,i}(x), y\right] = F_{a,i}(\left[x,y\right]) = \left[x, F_{a,i}(y)\right]$$

and hence $\left[x, G_{a,i}(y) - F_{a,i}(y)\right] = 0$ for all $x, y \in L$ and $i \in I$. Thus, $G_{a,i}(y) - F_{a,i}(y)$ belongs to the center of L for all $y \in L$ and $i \in I$. Since L is centerless it now follows that $G_{a,i} = F_{a,i}$ for all $i \in I$ and all $a \in A$. Accordingly, (3.17) implies $F(x \otimes a) = G(x \otimes a)$ for each simple tensor $x \otimes a \in L \otimes A$. However, since F and G are linear it follows $F = G$. By using the same arguments as in the proof of Theorem 2.1 we can now show that F is a derivation. Thus, since $f = f_{\mathcal{B}} + F$, where $f_{\mathcal{B}} \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$ and $F \in \mathrm{Der}(L \otimes A)$, it follows that $f \in \mathrm{Der}(L \otimes A) \oplus \mathrm{Cent}(L \otimes A)$. The proof of (i) is now complete.

Note that (ii) can be proved analogously by setting $f = g$ in the arguments above. □

Remark. According to Theorem 2.3 one might conjecture that Theorem 2.1 holds true even without the assumption that a Lie algebra L is either perfect or prime. Unfortunately, we were not able to prove nor disprove this conjecture.

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