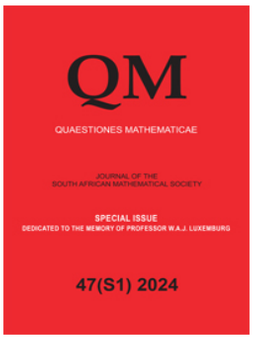

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ON THE DIAGONAL OF RIESZ OPERATORS ON BANACH LATTICES

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Dedicated to the memory of Wim Luxemburg

Abstract. This paper extends the well-known Ringrose theory for compact operators to polynomially Riesz operators on Banach spaces. The particular case of an ideal-triangularizable Riesz operator on an order continuous Banach lattice yields that the spectrum of such operator lies on its diagonal, which motivates the systematic study of an abstract diagonal of a regular operator on an order complete vector lattice E. We prove that the class $\mathscr{D}$ of regular operators for which the diagonal coincides with the atomic diagonal is always a band in $\mathcal{L}_r(E)$, which contains the band of abstract integral operators. If E is also a Banach lattice, then $\mathscr{D}$ contains positive Riesz and positive AM-compact operators.

Mathematics Subject Classification (2020): 46A40, 46B42, 47B06, 47B07.
Key words: Vector lattices, Banach lattices, Riesz operators, diagonal of an operator.

1. Introduction. Although the spectral theory of Riesz operators is the same as that of compact operators, they behave differently. For example, the sum and the product of two Riesz operators can be something other than a Riesz operator, which is not the case for compact operators. The abundance of invariant subspaces for compact operators and Ringrose's theorem guarantee that every compact operator admits an abstract upper-triangular form which is a generalization of Schur's form from linear algebra locating the spectrum of a matrix on its diagonal. Since Riesz operators possibly do not have non-trivial invariant subspaces, they generally do

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not admit such forms. This paper aims first to develop Ringrose's theory (Section 3) for polynomially Riesz operators and then apply it to study some aspects of Riesz operators on Banach lattices. As a result, we obtain that the spectrum of an ideal-triangularizable Riesz operator can be realized as the set of its diagonal coefficients, which motivates the systematic study (Section 4) of the abstract diagonal of a given regular operator on an order complete vector lattice E. We prove that the set of all regular operators for which the diagonal agrees with the atomic diagonal is a band in the order complete vector lattice of all regular operators $\mathcal{L}_r(E)$, which always contains the band of all abstract integral operators. If, in addition, E is a Banach lattice, this set contains every positive Riesz operator and every positive AM-compact operator.

2. Preliminaries. Throughout this paper, X will denote a complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The closed ideal of all compact operators on X will be denoted by $\mathcal{K}(X)$. The spectrum of an element a in a complex Banach algebra $\mathcal{A}$ is denoted by $\sigma(a)$. The spectrum of a bounded linear operator $T \in \mathcal{B}(X)$ is the spectrum of T considered as an element of the Banach algebra $\mathcal{B}(X)$. The spectral radius of T is denoted by $\rho(T)$. By $\pi\colon \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$ we denote the canonical projection onto the *Calkin algebra* $\mathcal{B}(X)/\mathcal{K}(X)$. The *essential spectrum* of $T \in \mathcal{B}(X)$ is defined by $\sigma_e(T) := \sigma(\pi(T))$.

Let $T \in \mathcal{B}(X)$ be a bounded linear operator on X. If $\pi(T)$ is a quasinilpotent element of the Calkin algebra, then T is called a *Riesz operator*. Hence, T is a Riesz operator if and only if $\sigma_e(T) = \{0\}$. It should be noted that Dowson [Dow78, Definition 3.4] defines Riesz operators as those operators which share the spectral theory of compact operators. Aforementioned definitions of a Riesz operator are equivalent by [Dow78, Theorem 3.12]. Hence, the class of Riesz operators on X always contains the ideal $\mathcal{K}(X)$ of compact operators. Since the image of every power compact operator in the Calkin algebra is nilpotent, the class of Riesz operators also contains every power compact operator. If there exists a non-zero complex polynomial p such that $p(T)$ is a Riesz operator, then T is called a *polynomially Riesz operator*. We refer the reader to [AA02] for more details about operators on Banach spaces.

The letter E shall stand for an Archimedean vector lattice. By $E^\sim$ and $E_n^\sim$, we denote the order dual and the order continuous dual of E, respectively. If E is a normed lattice, then the norm dual E^* of E is contained in $E^\sim$. If, in addition, E is a Banach lattice, then $E^\sim = E^*$. Let $T\colon E \to F$ be a positive operator between vector lattices. The *absolute kernel* of the operator T is defined by $N_T = \{x \in E : T|x| = 0\}$. It should be noted that N_T is always an ideal of E. If T is also order continuous, then N_T is a band in E. The disjoint complement $C_T := N_T^d$ of N_T is called the *carrier* of T. A band B in E is a *projection band* if $E = B \oplus B^d$. The band projection P_B of E onto the projection band B is always an order continuous positive operator. An atom a of a vector lattice E is a positive vector a with the following property: if $0 \le x, y \le a$ and $x \perp y$, then $x = 0$ or $y = 0$. Since E is Archimedean, the notion of an atom is equivalent to the notion of a discrete vector saying that the linear span of a is a projection band in E (see

[LZ71, Lemma 26.2 and Theorem 26.4]). The *atomic part* A of E is the band generated by the set of all atoms in E. Its disjoint complement $C := A^d$ is called the *non-atomic part* or the *continuous part* of E. If E is order complete, then every band is a projection band which yields the order decomposition $E = A \oplus C$ of the vector lattice E.

Given $F \subseteq E^\sim$ a subspace, the symbol $E \otimes F$ denotes the vector space of all finite-rank operators which can be expressed as a finite sum of rank one operators of the form $x \otimes \varphi$ with $x \in E$ and $\varphi \in F$. If E is order complete, then the band generated by $E \otimes E_n^\sim$ in $\mathcal{L}_r(E)$ is called the band of *abstract integral operators*. If every band of E is invariant under $T\colon E \to E$, then T is a *band preserving* operator. Order bounded band preserving operators are called *orthomorphisms*. In general, band preserving operators are not order bounded. Abramovich, Veksler and Koldunov proved in [AVK79] (see also [AB06, Theorem 4.76]) that every band preserving operator on a Banach lattice is order bounded, and hence, an orthomorphism. Since E is an Archimedean vector lattice, the set of orthomorphisms $\mathrm{Orth}(E)$ on E is a vector lattice under pointwise algebraic and lattice operations (see [BK69, CD71]). In fact, for each $T \in \mathrm{Orth}(E)$ and for each vector $x \in E$ we have

$$|T|(|x|) = \big|T|x|\big| = |Tx|. \tag{1}$$

By [AB06, Theorem 2.44] (see also [LS78, Theorem 1.3]) every orthomorphism on an Archimedean vector lattice is order continuous, and therefore, order bounded by [AS05, Theorem 2.1]. If, in addition, E is order complete, then by [AB06, Theorem 2.45] the vector lattice $\mathrm{Orth}(E)$ is a band in $\mathcal{L}_r(E)$ generated by the identity operator. Since $\mathcal{L}_r(E)$ is also order complete, the band of orthomorphisms is a projection band in $\mathcal{L}_r(E)$. A bounded operator between Banach lattices is *AM-compact* if it maps order bounded sets onto relatively compact sets. We refer the reader to [AA02, AB06, LZ71] for more details about vector and Banach lattices and operators acting on them.

We say that a chain $\mathcal{C}$ of closed subspaces of a Banach space X is a *complete* chain if it contains arbitrary intersections and closed linear spans of its members. If a closed subspace $\mathcal{M}$ is in a complete chain $\mathcal{C}$, then the *predecessor* $\mathcal{M}_-$ of $\mathcal{M}$ in $\mathcal{C}$ is defined as the closed linear span of all proper subspaces of $\mathcal{M}$ belonging to $\mathcal{C}$. Every maximal chain $\mathcal{C}$ of closed subspaces of X is complete and, for each subspace $\mathcal{M}$ in $\mathcal{C}$, the dimension of the space quotient space $\mathcal{M}/\mathcal{M}_-$ is at most one.

A family $\mathcal{F}$ of operators on a Banach space X is *reducible* if there exists a non-trivial closed subspace of X that is invariant under every operator from $\mathcal{F}$. If there exists a maximal chain $\mathcal{C}$ of closed subspaces of X which all are invariant under every operator from $\mathcal{F}$, then $\mathcal{F}$ is said to be *triangularizable*, and $\mathcal{C}$ is called a *triangularizing chain* for $\mathcal{F}$. A family $\mathcal{F}$ of operators on a Banach lattice E is said to be *ideal-reducible* if there exists a non-trivial closed ideal of E that is invariant under every operator from $\mathcal{F}$. Otherwise, we say that $\mathcal{F}$ is *ideal-irreducible*. A family $\mathcal{F}$ of operators on a Banach lattice is said to be *ideal-triangularizable* if it is triangularizable and at least one of (possibly many) triangularizing chains of $\mathcal{F}$ consists of closed ideals of E. For a more detailed exposition on triangularizability we refer the reader to [RR00].

3. Ringrose type theorems for polynomially Riesz operators. It is a well-known fact from linear algebra that every complex $n \times n$ matrix is similar to an upper-triangular matrix. Ringrose [Rin62] proved that the spectrum of a given compact operator T on a complex Banach space could also be realized in terms of something resembling diagonal coefficients as follows. By [Rin62, Theorem 1], the operator T is triangularizable. Pick any triangularizing chain $\mathcal{C}$ for T. Then for each $\mathcal{M} \in \mathcal{C}$, we have either $\mathcal{M} = \mathcal{M}_-$ or the quotient Banach space $\mathcal{M}/\mathcal{M}_-$ is one-dimensional. If $\mathcal{M} \neq \mathcal{M}_-$, then T induces the operator $T_{\mathcal{M}}$ on the one-dimensional Banach space $\mathcal{M}/\mathcal{M}_-$ given as $T_{\mathcal{M}}(x+\mathcal{M}_-) = \lambda_{\mathcal{M}}(x+\mathcal{M}_-)$ for any $x \in \mathcal{M} \setminus \mathcal{M}_-$. Hence, $T_{\mathcal{M}}$ acts on $\mathcal{M}/\mathcal{M}_-$ as a multiplication by the uniquely determined number $\lambda_{\mathcal{M}}$ which is called the *diagonal coefficient* of T with respect to $\mathcal{M}$. The cardinality of the set $\{\mathcal{M} \in \mathcal{C} : \ \lambda = \lambda_{\mathcal{M}}\}$ is called the *diagonal multiplicity* of λ (with respect to the chain $\mathcal{C}$). Ringrose proved (see [Rin62, Theorem 2]) that

$$\sigma(T) \setminus \{0\} = \{\lambda_{\mathcal{M}} : \ \mathcal{M} \neq \mathcal{M}_-\} \setminus \{0\}.$$

Furthermore, the diagonal multiplicity of $\lambda \in \sigma(T) \setminus \{0\}$ agrees with its algebraic multiplicity. Konvalinka extended Ringrose's theorem (see [Kon05, Theorem 2.7]) to polynomially compact operators. It should be noted that Konvalinka [Kon05] did not consider the part of Ringrose's theorem regarding multiplicities.

In this section, we first extend Ringrose's theorem to the class of polynomially Riesz operators and then obtain a version for ideal-triangularizable Riesz operators on order continuous Banach lattices. Before we proceed to the main results concerning Ringrose's theory for polynomially Riesz operators, we extend the well-known spectral theory of Riesz operators (see [AA02]) to polynomially Riesz operators, which will be needed in the paper. Although the results should be known, we include our proofs for the sake of the reader.

Proposition 3.1. *If $\mathcal{A}$ is a unital complex Banach algebra and the spectrum of $a \in \mathcal{A}$ is finite, then there exists a unique monic polynomial m of smallest degree such that $m(a)$ is quasinilpotent and such that m divides every polynomial p for which $p(a)$ is quasinilpotent.*

Proof. If $\sigma(a) = \{\lambda_1, \dots, \lambda_n\}$, then for the monic polynomial f given by $f(z) = (z-\lambda_1)\cdots(z-\lambda_n)$ the element $f(a)$ is quasinilpotent. Let m be such a polynomial with the smallest possible degree. Let p be any polynomial for which $p(a)$ is quasinilpotent. Then there exist polynomials k and r such that the degree of r is strictly smaller than that of m and that $p = km + r$. Suppose that r is non-zero. Since $k(a)$ and $m(a)$ commute, the elements $p(a)$ and $k(a)m(a)$ are quasinilpotent. Furthermore, since $p(a)$ and $k(a)m(a)$ commute, the equality $r(a) = p(a) - k(a)m(a)$ together with the same argument as above shows that $r(a)$ is quasinilpotent. This is in contradiction with the minimality of the degree of m. $\square$

An application of Proposition 3.1 immediately yields the following corollary.

Corollary 3.2. *Let T be a polynomially Riesz operator on an infinite-dimensional Banach space. Then there exists the unique monic polynomial m of the smallest degree such that $m(T)$ is a Riesz operator and that m divides every polynomial p for which $p(T)$ is a Riesz operator.*

The polynomial m from Corollary 3.2 is called the *minimal polynomial* of T.

Proposition 3.3. *Let T be a polynomially Riesz operator on a Banach space X. Let $m(z) = (z-\lambda_1)^{j_1}\cdots(z-\lambda_k)^{j_k}$ be the minimal polynomial of T. Then*

$$\sigma_e(T) = m^{-1}(\{0\}) = \{\lambda_1, \dots, \lambda_k\}.$$

Each $\lambda \in \sigma(T)\setminus\sigma_e(T)$ is an isolated point of $\sigma(T)$, it is an eigenvalue of T and is a pole of the resolvent of T. Furthermore, the spectral projection $P_\lambda(T)$ is of finite rank, and the operator $\lambda - T$ has finite ascent and descent.

A point $\lambda \in \sigma(T)\setminus\sigma_e(T)$ which satisfies the conclusion of Proposition 3.3 is sometimes called a *Riesz point* of T.

Proof. The essential spectrum of the operator T is the spectrum of the element $T+\mathcal{K}(X)$ in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$. Since the operator $m(T)$ is a Riesz operator, the element $m(T)+\mathcal{K}(X)$ in the Calkin algebra is quasinilpotent. From the spectral mapping for polynomials, we conclude that

$$m(\sigma(T+\mathcal{K}(X)) = \sigma(m(T+\mathcal{K}(X)) = \sigma(m(T)+\mathcal{K}(X)) = \{0\}$$

from where it follows that

$$\sigma_e(T) = \sigma(T+\mathcal{K}(X)) \subseteq m^{-1}(\{0\}).$$

Suppose that $\sigma_e(T)$ is a proper subset of $m^{-1}(\{0\})$. Then there exists a root λ of the polynomial m such that the image of the operator $T-\lambda I$ is invertible in the Calkin algebra. If we write $m(z)-m(\lambda) = (z-\lambda)q(z)$, we have

$$m(T) = m(T) - m(\lambda)I = (T-\lambda I)q(T)$$

from where we conclude $\pi(q(T)) = (\pi(T-\lambda I))^{-1}\pi(m(T))$. Since $\pi((T-\lambda I))^{-1}$ and $\pi(m(T))$ commute, and since the image of $\pi(m(T))$ is quasinilpotent, the image of the operator $q(T)$ is quasinilpotent in Calkin algebra as well. Hence, $q(T)$ is a Riesz operator. This is in contradiction with the fact that m is the minimal polynomial of T. This proves that $\sigma_e(T) = m^{-1}(\{0\})$.

Now, take $\lambda \in \sigma(T)\setminus\sigma_e(T)$. By [AA02, Lemma 7.43], λ is an isolated point of $\sigma(T)$ since it can be joined with any point of the resolvent set of T by a path lying outside a finite set $\sigma_e(T)$. The remaining assertions follow from [AA02, Theorem 7.44]. □

The following corollary follows from Proposition 3.3 in a similar way as [AA02, Theorem 7.48] gives [AA02, Corollaries 7.50 and 7.51].

COROLLARY 3.4. *The spectrum of a polynomially Riesz operator is at most countable, and its resolvent set is connected.*

In the proof of the next theorem we will need the following proposition.

PROPOSITION 3.5. *Let V be a closed subspace of a Banach space X that is invariant under the operator T, and let the complex number λ be an element of the unbounded connected component of the resolvent set of T. Then λ is also in the resolvent sets of the restriction $T|_V$ and of the operator induced by T on X/V.*

Proof. For all complex numbers μ with large modulus it follows from the Neumann series expansion that V is invariant under the resolvent $R(\mu, T) = (\mu I - T)^{-1}$. Due to the identity theorem for analytic functions, V is also invariant under $R(\lambda, T)$. This easily gives both claims. □

Now we are finally able to extend the well-known Ringrose's theorem for block triangularizations (see [RR00, Theorem 7.2.7]) to the class of polynomially Riesz operators.

THEOREM 3.6. *Let T be a polynomially Riesz operator on a Banach space X. Then for every complete chain $\mathcal{C}$ of closed invariant subspaces of the operator T, we have*

$$\sigma(T) = \sigma_e(T) \cup \bigcup \{\sigma(T_{\mathcal{M}}) : \ \mathcal{M} \in \mathcal{C}, \mathcal{M}_- \neq \mathcal{M}\}.$$

Proof. We use Proposition 3.5 to prove the inclusion "$\supseteq$". Let $\lambda \notin \sigma(T)$. Let $\mathcal{M} \in \mathcal{C}$ satisfying $\mathcal{M} \neq \mathcal{M}_-$. Since the resolvent set of T is connected, Proposition 3.5 yields that the resolvent set of the restriction $T|_{\mathcal{M}}$ contains the resolvent set of T. Therefore, the resolvent set of $T|_{\mathcal{M}}$ contains λ and is also connected (as its complement is countable). Hence, again by Proposition 3.5, λ is also contained in the resolvent set of $T_{\mathcal{M}}$, that is $\lambda \notin \sigma(T_{\mathcal{M}})$.

To prove the inclusion "$\subseteq$", choose $\lambda \in \sigma(T) \setminus \sigma_e(T)$. Since the spectral projection $P_\lambda(T)$ is of finite-rank, the kernel $\ker(T - \lambda I)$ which is contained in the range of $P_\lambda(T)$ is finite-dimensional, so that the set

$$\mathcal{O} = \{x \in X : \ Tx = \lambda x, \|x\| = 1\}$$

is a compact subset of X. By the finite intersection property of compact sets, the subspace

$$\mathcal{M} = \bigcap \{\mathcal{N} : \mathcal{N} \cap \mathcal{O} \neq \emptyset, \mathcal{N} \in \mathcal{C}\}$$

has a non-empty intersection with $\mathcal{O}$, and so it is non-trivial. Since $\mathcal{C}$ is complete, we have $\mathcal{M} \in \mathcal{C}$.

We claim that $\mathcal{M} \neq \mathcal{M}_-$ and $\lambda \in \sigma(T_{\mathcal{M}})$. For every proper subspace $\mathcal{L} \in \mathcal{C}$ of $\mathcal{M}$ we can prove similarly as in [Kon05, Theorem 2.7] that for the descent N of $T - \lambda I$ we have $\mathcal{L} \subseteq \operatorname{im}(\lambda I - T)^N$. Since $\mathcal{M}_-$ is equal to the closure of the span of proper subspaces of $\mathcal{M}$, it follows $\mathcal{M}_- \subseteq \operatorname{im}(\lambda I - T)^N$. Since the intersection of $\mathcal{M}$ with $\ker(\lambda I - T)^N$ is non-trivial, we have $\mathcal{M} \neq \mathcal{M}_-$. If $x \in \mathcal{M} \setminus \mathcal{M}_-$ satisfies

$Tx = \lambda x$, then $T_{\mathcal{M}}(x + \mathcal{M}_-) = \lambda(x + \mathcal{M}_-)$, so that $\lambda \in \sigma(T_{\mathcal{M}})$. □

The following example shows that in Theorem 3.6 we cannot omit the assumption that T is a polynomially Riesz operator.

Example 3.7. Let T be the left shift operator on ℓ^2, that is,

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be the standard basis of ℓ^2. For each $n \in \mathbb{N}$ let $\mathcal{M}_n$ be the linear span of vectors e_1, e_2, …, e_n. Set also $\mathcal{M}_0 = \{0\}$. Then

$$\mathcal{C} = \{\mathcal{M}_n : n \in \mathbb{N} \cup \{0\}\} \cup \{\ell^2\}$$

is a complete chain of closed invariant subspaces for the operator T. Since $T(\mathcal{M}_n) = \mathcal{M}_{n-1}$, we have $T_{\mathcal{M}_n} = 0$. Therefore,

$$\sigma_e(T) \cup \bigcup \{\sigma(T_{\mathcal{M}}) : \mathcal{M} \in \mathcal{C}, \mathcal{M}_- \neq \mathcal{M}\}$$

is equal to $\{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$, while $\sigma(T)$ is equal to the closed unit disk $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.8. (Ringrose's theorem for polynomially Riesz operators.) *Let T be a triangularizable polynomially Riesz operator on a Banach space X. Then for every triangularizing chain $\mathcal{C}$ of the operator T we have*

$$\sigma(T) = \sigma_e(T) \cup \{\lambda_{\mathcal{M}} : \mathcal{M} \in \mathcal{C}, \mathcal{M} \neq \mathcal{M}_-\}.$$

Positive ideal-triangularizable Riesz operators have positive spectra.

Corollary 3.9. *Let T be an ideal-triangularizable positive operator on a Banach lattice E.*

(i) *If T is a polynomially Riesz operator, then $\sigma(T) \setminus \sigma_e(T) \subseteq [0, \rho(T)]$.*

(ii) *If T is a Riesz operator, then $\sigma(T) \subseteq [0, \rho(T)]$.*

Proof. (i) Let $\mathcal{C}$ be an ideal-triangularizing chain for the operator T. Pick any $\lambda \in \sigma(T) \setminus \sigma_e(T)$. By Corollary 3.8, there exists an ideal $\mathcal{J} \in \mathcal{C}$ such that the dimension of $\mathcal{J}/\mathcal{J}_-$ is one and $\lambda = \lambda_{\mathcal{J}}$. Hence, $T_{\mathcal{J}} = \lambda_{\mathcal{J}} I$ on $\mathcal{J}/\mathcal{J}_-$. Since T is positive, the induced operator $T_{\mathcal{J}}$ is also positive, so that $\lambda_{\mathcal{J}} \geq 0$.

(ii) follows from (i) and the fact that the essential spectrum of a Riesz operator equals $\{0\}$. □

A close inspection of the proof of [RR00, Theorem 7.2.9] shows that we have the following sharpening of Ringrose's theorem. To avoid repetition, we omit its proof.

Theorem 3.10. *Let T be a triangularizable Riesz operator on a Banach space X. The diagonal multiplicity of each non-zero eigenvalue of T with respect to any triangularizing chain is equal to its algebraic multiplicity.*

We conclude this section by Theorem 3.11, which roughly says that the spectrum of an ideal-triangularizable Riesz operator lies on its "matrix" diagonal, which we shall explain in the following lines. Let a be an atom in E. Then for each vector $x \in E$ we have the order decomposition $x = \lambda_x a + y$ for uniquely determined $\lambda_x \in \mathbb{R}$ and $y \in \{a\}^\perp$. The positive linear functional $\varphi_a \colon E \to \mathbb{R}$ defined as $\varphi_a(x) = \lambda_x$ is a positive lattice homomorphism called the *coordinate functional* associated with the atom a. For an operator T on E, the number $\varphi_a(Ta)$ is called the *diagonal entry* of T with respect to the atom a. If a' is any other atom in E whose linear span equals the linear span of a, then a and a' are called *equivalent*. Since for equivalent atoms a and a' we have

$$\varphi_a(Ta) = \varphi_{a'}(Ta'),$$

the notion of the diagonal entry is independent of the choice of equivalent atoms. Let $\mathcal{A}$ be any maximal set of pairwise non-equivalent atoms.

The following result yields that the non-zero part of the spectrum of an ideal-triangularizable Riesz operator on an order continuous Banach lattice is equal to the set of its non-zero diagonal entries counted with multiplicities. It should be noted that Theorem 3.11 is an improvement of [Kan13, Proposition 4] which was proved for power compact operators.

Theorem 3.11. *Let T be an ideal-triangularizable Riesz operator on an order continuous Banach lattice E. Then*

$$\sigma(T) \setminus \{0\} = \{\varphi_a(Ta) : a \in \mathcal{A}\} \setminus \{0\} \tag{2}$$

where non-zero elements of both sets are counted with respect to their multiplicities.

Proof. Let $\mathcal{C}$ be a triangularizing chain of closed ideals for T. Assume that $\lambda \in \sigma(T) \setminus \{0\}$. We first show that $\lambda = \varphi_a(Ta)$ for some atom $a \in E$. By Ringrose's theorem for polynomially Riesz operators (see Corollary 3.8) there is $\mathcal{J} \in \mathcal{C}$ such that $\lambda = \lambda_{\mathcal{J}}$. Since $\mathcal{J}/\mathcal{J}_-$ is one-dimensional and E is order continuous, there exists an atom $a \in \mathcal{J}$ such that $\mathcal{J} = \mathcal{J}_- \oplus \mathbb{R}a$. From $Ta - \lambda_{\mathcal{J}} a \in \mathcal{J}_-$ we conclude $\varphi_a(Ta - \lambda_{\mathcal{J}} a) = 0$ or equivalently $\varphi_a(Ta) = \lambda_{\mathcal{J}} \varphi_a(a) = \lambda_{\mathcal{J}}$.

Assume that the multiplicity of λ is $k > 1$. We must show that $\lambda = \varphi_{a_j}(Ta_j)$ for some atoms $a_1, \dots, a_k \in E$ with norm one. By Theorem 3.10 there is a chain $\mathcal{J}_1 \subsetneq \cdots \subsetneq \mathcal{J}_k$ in $\mathcal{C}$ such that $\lambda = \lambda_{\mathcal{J}_j}$ for each $1 \le j \le k$. By the first part of the proof for each $1 \le j \le k$ there exists an atom a_j of norm one such that $\mathcal{J}_j = (\mathcal{J}_j)_- \oplus \mathbb{R}a_j$ and $\lambda = \varphi_{a_j}(Ta_j)$. If $i < j$, then $a_i \in \mathcal{J}_i \subsetneq \mathcal{J}_j$ yields that $a_i \in (\mathcal{J}_j)_-$, and so $a_i \neq a_j$.

To prove the converse inclusion, let a be an atom of E such that $\varphi_a(Ta) \neq 0$. Define $\mathcal{F} = \{\mathcal{J} \in \mathcal{C} : a \in \mathcal{J}\}$. Let $\mathcal{J}_a$ be the intersection of all members of $\mathcal{F}$. Then $\mathcal{J}_a$ is an ideal from $\mathcal{C}$ and $a \in \mathcal{J}_a$. It is easy to see that $a \notin (\mathcal{J}_a)_-$ and that $\varphi_a(Ta)$ belongs to $\sigma(T) \setminus \{0\}$. $\square$

The preceding theorem can be applied in the case of the classical spaces ℓ^p $(1 \leq p < \infty)$ and c_0 to prove that the non-zero spectrum counted according to multiplicities of eigenvalues of an ideal-triangularizable Riesz operator lies on the diagonal of the associated matrix with respect to the standard basis. Indeed, the standard basis vectors of ℓ^p $(1 \leq p < \infty)$ and c_0 are precisely atoms of norm one.

4. The diagonal and the atomic diagonal. Let T be an ideal-triangularizable Riesz operator on an order continuous Banach lattice E. By Theorem 3.11, we have

$$\sigma(T) \setminus \{0\} = \{\varphi_a(Ta) : a \in \mathcal{A}\} \setminus \{0\} \tag{3}$$

according to multiplicities of non-zero eigenvalues of the operator T. If we consider the 2×2 operator matrix associated with T with respect to the decomposition $E = A \oplus C$, then the spectrum of T is contained on the "diagonal" of T. In this section, we study the abstract diagonal of a regular operator on an order complete vector lattice. We prove that every diagonal splits into its atomic and its continuous part. Moreover, we prove that the continuous part of the diagonal of an abstract integral operator or a Riesz operator is always zero. This motivates the study of those regular operators for which the diagonal is equal to its atomic part.

Throughout this section, we assume that E is an order complete vector lattice. Then the band $\mathrm{Orth}(E)$ is a projection band in $\mathcal{L}_r(E)$ generated by the identity operator, and so we have the following order decomposition

$$\mathcal{L}_r(E) = \mathrm{Orth}(E) \oplus \mathrm{Orth}(E)^d = \{I\}^{dd} \oplus \{I\}^d$$

of the order complete vector lattice $\mathcal{L}_r(E)$.

Let $\mathcal{P}_E$ be the band projection onto the projection band $\mathrm{Orth}(E)$. Given a regular operator $T \in \mathcal{L}_r(E)$, the projection $\mathcal{P}_E(T)$ of T on $\mathrm{Orth}(E)$ is called the *diagonal* of T. If T is positive, then Schep [Sch80, Theorem 1.1] proved that the projection $\mathcal{P}_E(T)$ is given by the formula

$$\mathcal{P}_E(T) = \inf\{\sum_{i=1}^{n} P_i T P_i : \ 0 \leq P_i \leq I, P_i^2 = P_i, \sum_{i=1}^{n} P_i = I\}. \tag{4}$$

Given a band B of E and a regular operator T on E, we write shortly $\mathcal{P}_B(T)$ instead of $\mathcal{P}_B(P_B T P_B)$. The operator $\mathcal{P}_B$ is the projection onto $\mathrm{Orth}(B)$ of the order complete vector lattice B. We call the operator $\mathcal{P}_B(T)$ the diagonal of T with respect to B. The regular operator $\mathcal{P}_A(T)$ is the *atomic diagonal* of the operator T. It is not hard to see that for a positive operator T, we have

$$\mathcal{P}_A(T) = \sup\{\sum_{a \in \mathcal{F}} P_a T P_a : \ \mathcal{F} \subseteq \mathcal{A} \text{ is finite}\},$$

and that for a regular operator T, we have

$$\mathcal{P}_A(T) = \sum_{a \in \mathcal{A}} P_a T P_a$$

where the latter series converges in order. For more details, see [Kan13].

The main concern of this section is to study those regular operators T whose diagonal and atomic diagonal coincide, i.e., $\mathcal{P}_E(T) = \mathcal{P}_A(T)$. Schep proved [Sch80, Corollary 1.7] that for a positive compact operator T on an atomless order complete Banach lattice the zero operator is the only positive orthomorphism S which satisfies $0 \leq S \leq T$. If E is an order complete Banach lattice, then for every positive compact operator we have $\mathcal{P}_E(T) = \mathcal{P}_A(T)$. Therefore, positive compact operators on order complete Banach lattices are examples of operators for which the diagonal and the atomic diagonal coincide. As already mentioned, we will extend this list by the class of abstract integral operators on order complete vector lattices (see Theorem 4.4) and positive Riesz operators T on order complete Banach lattices (see Theorem 4.5).

If $E = B_1 \oplus B_2$ is a band decomposition of E, then $\mathrm{Orth}(E) \cong \mathrm{Orth}(B_1) \oplus \mathrm{Orth}(B_2)$. In particular, $\mathrm{Orth}(E) \cong \mathrm{Orth}(A) \oplus \mathrm{Orth}(C)$. The following lemma shows that every band decomposition of a vector lattice yields a decomposition of the diagonal operator.

Lemma 4.1. *If $E = B_1 \oplus B_2$ is a band decomposition of an order complete vector lattice E, then $\mathcal{P}_E = \mathcal{P}_{B_1} \oplus \mathcal{P}_{B_2}$.*

Proof. Let T be any positive operator on E. Suppose that $P_1, \ldots, P_n$ are band projections satisfying $0 \leq P_i \leq I, P_i^2 = P_i, \sum_{i=1}^n P_i = I$. Since band projections commute, we have

$$\begin{aligned}
\sum_{i=1}^n P_i T P_i &= (P_{B_1} + P_{B_2}) \sum_{i=1}^n P_i T P_i (P_{B_1} + P_{B_2}) \\
&\geq \sum_{i=1}^n P_{B_1} P_i T P_i P_{B_1} + \sum_{i=1}^n P_{B_2} P_i T P_i P_{B_2} \\
&= \sum_{i=1}^n (P_{B_1} P_i P_{B_1}) P_{B_1} T P_{B_1} (P_{B_1} P_i P_{B_1}) \\
&\quad + \sum_{i=1}^n (P_{B_2} P_i P_{B_2}) P_{B_2} T P_{B_2} (P_{B_2} P_i P_{B_2}) \\
&\geq \mathcal{P}_{B_1}(T) + \mathcal{P}_{B_2}(T).
\end{aligned}$$

By definition of $\mathcal{P}_E(T)$ we have $\mathcal{P}_E(T) \geq \mathcal{P}_{B_1}(T) + \mathcal{P}_{B_2}(T)$. The opposite inequality can be shown similarly. □

The set of all regular operators for which diagonal operators coincide is always a band.

Proposition 4.2. *For each band B of an order complete vector lattice E, the set*

$$\{T \in \mathcal{L}_r(E) : \ \mathcal{P}_E(T) = \mathcal{P}_B(T)\}$$

is a band in $\mathcal{L}_r(E)$.

Proof. Since $\mathrm{Orth}(E) \cong \mathrm{Orth}(B) \oplus \mathrm{Orth}(B^d)$, the operator $\mathcal{P}_B \colon \mathcal{L}_b(E) \to \mathrm{Orth}(B)$ is an order projection onto the band $\mathrm{Orth}(B)$ in $\mathcal{L}_r(E)$. In particular, $\mathcal{P}_B$ is order continuous. Since $\mathcal{P}_B$ and $\mathcal{P}_E$ satisfy $0 \le \mathcal{P}_B \le \mathcal{P}_E \le \mathcal{I}$ where $\mathcal{I}$ denotes the identity operator on $\mathcal{L}_r(E)$, the difference $\mathcal{P}_E - \mathcal{P}_B$ is an orthomorphism on $\mathcal{L}_r(E)$. Hence, by [AB06, Theorem 2.48], the kernel of $\mathcal{P}_E - \mathcal{P}_B$ is a band in $\mathcal{L}_r(E)$ and the proof is concluded. □

The following corollary follows from Proposition 4.2 and the discussion preceding Lemma 4.1.

Corollary 4.3. *If E is an order complete Banach lattice, then every operator $T \in (E \otimes E^{\sim})^{dd}$ satisfies $\mathcal{P}_E(T) = \mathcal{P}_A(T)$.*

Proof. Since the set of all regular operators T for which $\mathcal{P}_E(T) = \mathcal{P}_A(T)$ is a band in $\mathcal{L}_r(E)$, it suffices to prove that $\mathcal{P}_E(T) = \mathcal{P}_A(T)$ for each positive rank-one operator T. Since the rank of the operator $P_C T P_C$ is at most one, the inequality $0 \le \mathcal{P}_C(P_C T P_C) \le P_C T P_C$ and [Sch80, Corollary 1.7] imply that $\mathcal{P}_E(T) - \mathcal{P}_A(P_A T P_A) = \mathcal{P}_C(P_C T P_C) = 0$. □

Since $E_n^{\sim} \subseteq E^{\sim}$, Corollary 4.3, in particular, yields that every abstract integral operator T on an order complete Banach lattice satisfies $\mathcal{P}_E(T) = \mathcal{P}_A(T)$. By the following theorem, the assumption of a complete norm structure is redundant.

Theorem 4.4. *If E is an order complete vector lattice, then every abstract integral operator T satisfies $\mathcal{P}_E(T) = \mathcal{P}_A(T)$.*

Proof. Since the band of all abstract integral operators on E is generated by the set of all positive order continuous rank one operators in $E \otimes E_n^{\sim}$, by Proposition 4.2 it suffices to prove that $\mathcal{P}_E(T) = \mathcal{P}_A(T)$ holds for every operator $T = x \otimes \varphi$ on E with $0 \le \varphi \in E_n^{\sim}$ and $x \in E^+$.

Let us first consider the case when φ is strictly positive (that is, $N_\varphi = \{0\}$) and E is atomless. Let us define $\|x\|_\varphi := \varphi(|x|)$. Then $\|\cdot\|_\varphi$ is a norm on E. If $\widetilde{E}$ is the norm completion of E, then $(\widetilde{E}, \|\cdot\|_\varphi)$ is an AL-space in which E is embedded as a norm dense sublattice. Since φ is order continuous and E is order complete, an application of [GTX17, Theorem 4.1] shows that E is also an order dense ideal in $\widetilde{E}$.

We claim that $\widetilde{E}$ is atomless. If this were not the case, then there would exist an atom $\widetilde{a}$ in $\widetilde{E}$. Since E is order dense in $\widetilde{E}$, there exists a non-zero positive vector a in E such that $0 < a \le \widetilde{a}$. If $0 \le x, y \le a$ for some vectors $x, y \in E$ with $x \perp y = 0$, then $x \perp y = 0$ also in $\widetilde{E}$. Since $0 \le x, y \le \widetilde{a}$, we have $x = 0$ or $y = 0$ proving that a is an atom in E. This contradiction shows that $\widetilde{E}$ is atomless.

If $\widetilde{B}$ is a band in $\widetilde{E}$, then it is easy to see that $B := \widetilde{B} \cap E$ is a band in E. We claim that B is order dense in $\widetilde{B}$. If $\widetilde{x} \in \widetilde{B}$ is a positive vector, then there exists a net (x_α) of positive vectors in E such that $0 \le x_\alpha \uparrow \widetilde{x}$ in $\widetilde{E}$. The inequality $0 \le x_\alpha \le \widetilde{x}$ gives $x_\alpha \in E \cap \widetilde{B} = B$, which proves the claim. From the order continuity of the norm on $\widetilde{E}$, it follows that B is norm dense in $\widetilde{B}$.

To prove that $\mathcal{P}_E(x \otimes \varphi) = 0$, by (4) it suffices to prove that the zero operator is the only positive operator S on E which satisfies

$$0 \leq S \leq \sum_{i=1}^{n} P_i(x \otimes \varphi)P_i$$

for all finite families of band projections P_i which satisfy $\sum_{i=1}^{n} P_i = I$. Let $\widetilde{P}_1, \ldots, \widetilde{P}_n$ be band projections on $\widetilde{E}$ which satisfy $\sum_{i=1}^{n} \widetilde{P}_i = \widetilde{I}$ where $\widetilde{I}$ denotes the identity operator on $\widetilde{E}$. Then $P_i := \widetilde{P}_i|_E$ is the band projection onto the band, which is the intersection of the range of $\widetilde{P}_i$ with E. By assumption, we have $0 \leq S \leq \sum_{i=1}^{n} P_i(x \otimes \varphi)P_i$, so that S is dominated by a bounded operator on E. Hence, S is also bounded, and let us denote by $\widetilde{S}$ its norm extension to $\widetilde{E}$. Let $\widetilde{\varphi}$ be the norm extension of φ to $\widetilde{E}$. Since the norm extension of a positive bounded operator is again positive, it follows that

$$0 \leq \widetilde{S} \leq \sum_{i=1}^{n} \widetilde{P}_i(x \otimes \widetilde{\varphi})\widetilde{P}_i.$$

By (4) it follows that $0 \leq \widetilde{S} \leq \mathcal{P}_{\widetilde{E}}(x \otimes \widetilde{\varphi})$. Since $x \otimes \widetilde{\varphi}$ is a positive rank-one operator on the atomless order complete Banach lattice $\widetilde{E}$, it follows that $\mathcal{P}_{\widetilde{E}}(x \otimes \widetilde{\varphi}) = 0$ by [Sch80, Corollary 1.7]. This gives $\widetilde{S} = 0$ and, therefore, $S = 0$.

For the general case, observe first that by order continuity of φ and order completeness of E, we have the band decomposition $E = C_\varphi \oplus N_\varphi$. The functional φ is strictly positive on C_φ and is zero on N_φ. This implies that $P_{N_\varphi}(x \otimes \varphi)P_{N_\varphi} = 0$, and so $\mathcal{P}_{N_\varphi}(P_{N_\varphi}(x \otimes \varphi)P_{N_\varphi}) = 0$. This implies

$$\begin{aligned}
\mathcal{P}_E(x \otimes \varphi) &= \mathcal{P}_E(P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi} + P_{N_\varphi}(x \otimes \varphi)P_{N_\varphi}) \\
&= \mathcal{P}_E(P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi}) \\
&= \mathcal{P}_A(P_A P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi} P_A) + \mathcal{P}_C(P_C P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi} P_C).
\end{aligned}$$

Since C is atomless and φ is strictly positive on $C_\varphi \cap C$, the proof above shows that $\mathcal{P}_C(P_C P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi} P_C) = 0$. Since $x \otimes \varphi = 0$ on N_φ, we have $P_{A \cap N_\varphi}(x \otimes \varphi)P_{A \cap N_\varphi} = 0$, and so

$$\begin{aligned}
\mathcal{P}_E(x \otimes \varphi) &= \mathcal{P}_A(P_A P_{C_\varphi}(x \otimes \varphi)P_{C_\varphi} P_A) \\
&= \mathcal{P}_A(P_{A \cap C_\varphi}(x \otimes \varphi)P_{A \cap C_\varphi} + P_{A \cap N_\varphi}(x \otimes \varphi)P_{A \cap N_\varphi}) \\
&= \mathcal{P}_A(P_A(x \otimes \varphi)P_A) \\
&= \mathcal{P}_A(x \otimes \varphi).
\end{aligned}$$

□

We complete this paper by proving that $\mathcal{P}_E(T) = \mathcal{P}_A(T)$ for every positive Riesz operator and every positive AM-compact operator on every order complete Banach lattice. Our proof relies on a subclass of the class of orthomorphisms. An operator $T\colon E \to E$ is *central* if there exists some scalar $\lambda > 0$ such that $|Tx| \leq \lambda|x|$

for all $x \in E$. The set of central operators is denoted by $\mathcal{Z}(E)$. In the case when $E = \mathbb{R}^n$, a matrix T is central if and only if it is diagonal.

By definition it is clear that $\mathcal{Z}(E)$ is a subset of $\mathrm{Orth}(E)$. Since central operators are order bounded and preserve disjointness, by [AB06, Theorem 2.40] the modulus of a central operator T on an Archimedean vector lattice E exists and it satisfies (1) for each $x \in E$. This yields that $|T|$ is also central, and therefore, every central operator on an Archimedean vector lattice is a difference of two positive central operators. Wickstead [Wic77] provided an example when the inclusion of $\mathcal{Z}(E)$ in $\mathrm{Orth}(E)$ is proper. He also proved in [Wic77, Proposition 4.1] that bounded orthomorphisms on normed lattices are automatically central operators. Since every positive operator on a Banach lattice is bounded, for a Banach lattice E we have $\mathcal{Z}(E) = \mathrm{Orth}(E)$.

Theorem 4.5. *Let T be a positive operator on an order complete Banach lattice E. Then $\mathcal{P}_E(T) = \mathcal{P}_A(T)$ in either of the following cases.*

(i) *T is a Riesz operator.*

(ii) *T is an AM-compact operator.*

Proof. (i) Since T is a Riesz operator, by [Dow78, Theorem 3.12], the operator T is an asymptotically quasi-finite-rank operator. This means that

$$\inf\{\|T^n - F\|^{\frac{1}{n}} : F \in E \otimes E^*\} \to 0$$

as n goes to infinity. Pick any $\epsilon > 0$ and find n_ϵ such that

$$\inf\{\|T^n - F\|^{\frac{1}{n}} : F \in E \otimes E^*\} < \epsilon$$

for all $n \geq n_\epsilon$. Hence, for each $n \geq n_\epsilon$ there exists a finite rank operator F_n such that $\|T^n - F_n\| < \epsilon^n$. By [AA02, Theorem 4.14], the finite-rank operator F_n has a compact modulus. An application of [DK14, Proposition 3.2] then gives us that $\mathcal{P}_E(|F_n|) = \mathcal{P}_A(|F_n|)$, so that $\mathcal{P}_C(|F_n|) = 0$. The triangle inequality

$$0 \leq T^n \leq |T^n - F_n| + |F_n|$$

and the fact that $\mathcal{P}_C$ is a lattice homomorphism yield that

$$0 \leq \mathcal{P}_C(T^n) \leq \mathcal{P}_C(|T^n - F_n|).$$

Pick any $n \geq n_\epsilon$. Since the norm and the regular norm of a central operator coincide, we have $\|\mathcal{P}_E(T^n - F_n)\|_r = \|\mathcal{P}_E(T^n - F_n)\|$. Since the diagonal operator $\mathcal{P}_E$ is a contraction with respect to the operator norm by [Voi88, Theorem 1.4], we have

$$\begin{aligned} |\mathcal{P}_C(T^n)\| &= \|\mathcal{P}_C(T^n)\|_r \leq \|\mathcal{P}_C(T^n - F_n)\|_r \leq \|\mathcal{P}_E(T^n - F_n)\|_r \\ &= \|\mathcal{P}_E(T^n - F_n)\| \leq \|T^n - F_n\| < \epsilon^n. \end{aligned}$$

Since the diagonal operator $\mathcal{P}_C$ satisfies $\mathcal{P}_C(AB) \geq \mathcal{P}_C(A)\mathcal{P}_C(B)$ for positive operators A and B, we have $\mathcal{P}_C(T^n) \geq (\mathcal{P}_C(T))^n$, and so

$$\|(\mathcal{P}_C(T))^n\| \leq \|\mathcal{P}_C(T^n)\| \leq \epsilon^n.$$

By [Sch80, Theorem 1.4], the center $\mathcal{Z}(E)$ is an AM-space with a strong unit. Therefore, $\mathcal{Z}(E)$ is algebraically and lattice isometric to the Banach lattice algebra $C(K)$ of continuous functions on some compact Hausdorff space K. In particular, we have $\|(\mathcal{P}_C(T))^n\| = \|\mathcal{P}_C(T)\|^n$, from where it follows $\|\mathcal{P}_C(T)\| < \epsilon$. Since this holds for each $\epsilon > 0$, we finally conclude that $\mathcal{P}_C(T) = 0$.

(ii) As before, we will prove that $\mathcal{P}_C(T) = 0$. Since T is an AM-compact operator, from the inequality $0 \leq \mathcal{P}_C(T) \leq T$ it follows that $\mathcal{P}_C(T)$ is also AM-compact. To reach the contradiction, assume that $\mathcal{P}_C(T) > 0$. Since C is order complete, the operator $\mathcal{P}_C(T)$ is by the Freudenthal spectral theorem (see [AB06, Theorem 2.8]) the (relative) uniform limit of sums of the form $\sum_{k=1}^n \lambda_k P_k$, where $\lambda_k > 0$, P_k is a band projection and $0 \leq \sum_{k=1}^n \lambda_k P_k \leq \mathcal{P}_C(T)$. For each $1 \leq k \leq n$ the inequality $0 \leq P_k \leq (1/\lambda_k)\mathcal{P}_C(T)$ and the fact that $\mathcal{P}_C(T)$ is an AM-compact operator yield that P_k is an AM-compact operator. Hence, the identity operator on the order complete Banach lattice $P_k(C)$ is an AM-compact operator. This implies that order intervals of $P_k(C)$ are compact. Hence, by [Wnu99, Theorem 6.1], $P_k(C)$ is an atomic band in C. Since C is atomless, it follows that $P_k = 0$. This finally gives us that $\mathcal{P}_C(T) = 0$ which concludes the proof. □

The following result can be prove similarly as the second assertion of Theorem 4.5.

Proposition 4.6. *The zero operator is the only positive central AM-compact operator on an atomless order complete Banach lattice.*

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