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Note

On the diameter of a super-order-commuting graph

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ABSTRACT

We answer a question about the diameter of an order-super-commuting graph on a symmetric group by studying the number-theoretical concept of d-complete sequences of primes in arithmetic progression.

1. Introduction

In a recent paper [6], the authors studied the properties of super graphs on finite groups. These graphs were introduced in [2] and are based on already well-studied commuting, power, and enhanced power graphs (we refer to [4] for a nice survey and the current developments), but with an equivalence relation thrown in. More precisely, if $\sim$ is an equivalence relation on a graph Γ, then $\sim$-super-Γ has the same vertex set as Γ; however, its edge set is enlarged, whereby distinct x, y form an edge if there exist $u \sim x$ and $v \sim y$ with either $u = v$ or $(u, v) \in E(\Gamma)$. In particular, with the order relation on a finite group $\mathcal{G}$ (i.e., $x \sim y$ if x and y have the same order), the order-super-commuting graph, $\Delta^o(\mathcal{G})$, of a group $\mathcal{G}$ is a simple graph with the vertex set equal to $\mathcal{G}$ and where two disjoint vertices x, y form an edge if there exist commuting u, v with $|u| = |x|$ and $|v| = |y|$ (here we also allow $u = v$ so, in particular, each conjugacy class forms an induced complete graph). We should caution that, as in [2] and [6], the central elements and, in particular, the identity also belong to the vertex set of the commuting graph, but we do not allow the loops. Notice that this contrasts with a similar definition in some of the existing literature [1,8], where the central elements are removed. It was shown in [6] that, for $n \geq 4$, the only dominant vertex of $\Delta^o(\mathcal{S}_n)$, the order-super-commuting graph of the symmetric group $\mathcal{S}_n$ on n elements, is the identity, that is, the center of the group. By deleting all the dominant vertices, one obtains the reduced graph, $\Delta^o(\mathcal{S}_n)^*$. This is connected if and only if neither $n-1$ nor n is a prime number; moreover, if it is disconnected, then it has exactly two components, and if it is connected, then its diameter is bounded above by 3, see [6, Proposition 4.9 and Theorem 4.11]. Whether its diameter is 3 or smaller was not determined for all values of $n \geq 4$, but it was shown [6, Proposition 4.12] that the following holds.

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Proposition 1.1. *Let $n \geq 4$. If neither n nor $n-1$ is a prime number, then* $\operatorname{diam} \Delta^o(\mathcal{S}_n)^* = 3$ *if and only if there exist nonempty disjoint subsets $\mathscr{T}_1$, $\mathscr{T}_2$ consisting of primes smaller or equal to n, such that, for some positive integers α_p and β_q we have*

$$M^{\alpha}_{\mathscr{T}_1} := \sum_{p \in \mathscr{T}_1} p^{\alpha_p} \in \{n-1, n\}, \qquad M^{\beta}_{\mathscr{T}_2} := \sum_{q \in \mathscr{T}_2} q^{\beta_q} \leq n,$$

and $p + M^{\beta}_{\mathscr{T}_2} > n$, for every $p \in \mathscr{T}_1$. $\square$

With this proposition, the authors proved that the diameter is 3 if n or $n-1$ is either a nontrivial power of a prime or a sum of two prime powers, where both primes are distinct and greater than or equal to 5. The latter, assuming the strong Goldbach conjecture holds, immediately yields that $\operatorname{diam} \Delta^o(\mathcal{S}_n)^* = 3$, for every even integer $n \geq 4$. This paper aims to give a complete solution, without relying on the strong Goldbach conjecture and proves that $\operatorname{diam} \Delta^o(\mathcal{S}_n)^* = 3$ for every $n \geq 4$ such that neither n nor $n-1$ is a prime number. Our main ingredient is the fact that the sequences consisting of primes congruent to ± 1 modulo 4 are complete (see Theorem 2.5 and its consequences).

An infinite sequence of distinct positive integers $\{a_n;\ n \in \mathbb{N}\}$ is called complete (see [9]) if every sufficiently large positive integer is a sum of distinct a_i (sometimes such sequences are called weakly complete, while the term complete sequence is reserved for the case when every integer is a sum of distinct a_i). Erdős and Lewin [7] call a complete sequence d-complete if every sufficiently large integer is a sum of distinct a_i such that no one divides the other. In [7], there are given several examples of d-complete sequences. For instance, it is proved that, for positive integers p and q, the sequence $\{p^a q^b;\ a, b \in \mathbb{N}\}$ is d-complete if and only if $\{p, q\} = \{2, 3\}$. Bruckman [3] proved that the sequence $\mathscr{P} = \{2, 3, 5, \ldots\}$ of all prime numbers is d-complete (while $\{1\} \cup \mathscr{P}$ is complete, see [9, p. 127]). We adapt Bruckman's proof and show that the sequences of all prime numbers congruent to 1 modulo 4, and those congruent to 3 modulo 4, are d-complete.

2. Results

2.1. Generating polynomials

Let $1 \leq q_1 < q_2 < \ldots$ be a sequence of integers. For every $n \in \mathbb{N}$, let

$$f_n(x) = \left(1 + x^{q_1}\right) \cdots \left(1 + x^{q_n}\right).$$

It is clear that $f_n(x)$ is a polynomial of degree $S_n = q_1 + \cdots + q_n$. Denote the coefficient of $f_n(x)$ at power x^m by $\gamma_m(n)$; we also let $\gamma_m(n) = 0$ if $m \geq S_n + 1$. Then

$$f_n(x) = \sum_{m=0}^{S_n} \gamma_m(n) x^m.$$

Since

$$\begin{aligned}\sum_{m=0}^{S_{n+1}} \gamma_m(n+1) x^m = f_{n+1}(x) = f_n(x) \cdot (1 + x^{q_{n+1}}) &= \left(\sum_{m=0}^{S_n} \gamma_m(n) x^m\right)\left(1 + x^{q_{n+1}}\right) \\ &= \sum_{m=0}^{S_n} \gamma_m(n) x^m + \sum_{m=0}^{S_n} \gamma_m(n) x^{m+q_{n+1}}\end{aligned}$$

the comparison of the coefficients gives

$$\gamma_m(n+1) = \begin{cases} \gamma_m(n), & 0 \leq m < q_{n+1}; \\ \gamma_m(n) + \gamma_{m-q_{n+1}}(n), & q_{n+1} \leq m \leq S_n; \\ \gamma_{m-q_{n+1}}(n), & S_n < m \leq S_{n+1}. \end{cases} \tag{2.1}$$

It follows from (2.1) that $\gamma_m(n+1) \geq \gamma_m(n) \geq 0$. On the other hand, let $m \geq 0$ be arbitrary but fixed. Let $n \in \mathbb{N}$ be such that $m < q_{n+1}$. Then, by (2.1), $\gamma_m(n+1) = \gamma_m(n)$. Since $m < q_{n+1} < q_{n+2}$ we also have $\gamma_m(n+2) = \gamma_m(n+1)$ and therefore $\gamma_m(n+2) = \gamma_m(n)$. By induction, $\gamma_m(k) = \gamma_m(n)$, for all $k > n$. Thus, we may define $\Gamma_m = \max\{\gamma_m(n);\ n \in \mathbb{N}\}$. Note that $\Gamma_m > 0$ if and only if there exist n such that $\gamma_m(n) > 0$ which is equivalent to the fact that there exist distinct sequence members $q_{j_1}, \ldots, q_{j_\ell}$, where $\ell \geq 1$, such that $m = q_{j_1} + \cdots + q_{j_\ell}$.

Theorem 2.1. *Let $1 \leq q_1 < q_2 < \cdots$ be a sequence of integers with partial sums $S_n = q_1 + \cdots + q_n$. Consider*

$$f_n(x) = (1 + x^{q_1}) \cdots (1 + x^{q_n}) = \sum_{m=0}^{S_n} \gamma_m(n) x^m.$$

Table 1
Small primes congruent to 1 and 3 modulo 4, along with their partial sums, $S_{4,1}(n)$ and $S_{4,3}(n)$, respectively.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$q_{4,1}(n)$	5	13	17	29	37	41	53	61	73	89	97	101	109	113	137
$S_{4,1}(n)$	5	18	35	64	101	142	195	256	329	418	515	616	725	838	975
$q_{4,3}(n)$	3	7	11	19	23	31	43	47	59	67	71	79	83	103	107
$S_{4,3}(n)$	3	10	21	40	63	94	137	184	243	310	381	460	543	646	753

If there exist positive integers k_0 and n_0 such that

$$2k_0 + q_{n+1} \le S_n, \quad \text{for } n \ge n_0, \qquad \text{and} \qquad \gamma_m(n_0) \ge 1, \quad \text{for } k_0 \le m \le S_{n_0} - k_0,$$

then every integer $m \ge k_0$ is a sum of (one or more) different members of $\{q_1, q_2, \dots\}$.

Proof. We will prove by induction that, for $n \ge n_0$, we have

$$\gamma_m(n) \ge 1 \quad \text{if } k_0 \le m \le S_n - k_0.$$

It is obvious that this will imply $\Gamma_m \ge 1$, for all $m \ge k_0$, and the statement will follow.

Let $\mathscr{N} \subseteq \mathbb{N}$ denote the set of all integers $n \ge n_0$ such that $\gamma_m(n) \ge 1$ if $k_0 \le m \le S_n - k_0$. By the hypothesis, $n_0 \in \mathscr{N}$. Assume that $n \in \mathscr{N}$. If $k_0 \le m \le S_n - k_0$, then $\gamma_m(n+1) \ge \gamma_m(n) \ge 1$, by the inductive hypothesis. Similarly, if m is such that $k_0 \le m - q_{n+1} \le S_n - k_0$, then $\gamma_{m-q_{n+1}}(n) \ge 1$, again by the inductive hypothesis, and therefore $\gamma_m(n+1) \ge 1$, by (2.1). Clearly, $k_0 \le m - q_{n+1} \le S_n - k_0$ is equivalent to $k_0 + q_{n+1} \le m \le S_n + q_{n+1} - k_0 = S_{n+1} - k_0$. Also, by the assumptions, $k_0 + q_{n+1} \le S_n - k_0$ so that the intersection of the intervals $[k_0, S_n - k_0]$ and $[k_0 + q_{n+1}, S_{n+1} - k_0]$ is nonempty. We conclude that $\gamma_m(n+1) \ge 1$, for all m such that $k_0 \le m \le S_{n+1} - k_0$. Hence, $n+1 \in \mathscr{N}$. □

We next show that there always exists k_0 which satisfies the first condition in Theorem 2.1, provided that the sequence of integers grows at most exponentially.

Lemma 2.2. *Let $1 \le q_1 < q_2 < \dots$ be a sequence of integers satisfying $q_{n+1} < 2q_n$ and let $k_0 \ge 0$ be a given integer. Then, $S_n - q_{n+1} \ge 2k_0$, for every $n \ge q_1 + 2k_0 + 1$.*

Proof. Notice first that $q_i < q_{i+1} \le 2q_i - 1$, so that $q_i - q_{i+1} \ge q_i - (2q_i - 1) = -q_i + 1$. Then, proceeding backward, we get

$$\begin{aligned} S_i - q_{i+1} &= q_1 + q_2 + \dots + (q_i - q_{i+1}) \ge q_1 + q_2 + \dots + (q_{i-1} - q_i) + 1 \ge \\ &\ge q_1 + q_2 + \dots + (q_{i-2} - q_{i-1}) + 1 + 1 \ge \dots \ge q_1 - q_2 + (i-2) \ge -q_1 + (i-1). \end{aligned}$$

Hence, with $i \ge q_1 + 2k_0 + 1$ we get that $S_i - q_{i+1} \ge 2k_0$. □

2.2. Prime numbers congruent to 1*, respectively* 3*, modulo* 4

Let d be a positive integer and let $1 \le r < d$ be such that $\gcd(d, r) = 1$. The celebrated Dirichlet's Theorem says that there are infinitely many prime numbers congruent to r modulo d. Let $\mathscr{P}(d, r) = \{q_{d,r}(1) < q_{d,r}(2) < q_{d,r}(3) < \cdots\}$ be the sequence of all prime numbers congruent to r modulo d and let $S_{d,r}(n)$ denote the sum of the first n prime numbers in $\mathscr{P}(d, r)$. In what follows, we are interested in $\mathscr{P}(4, 1)$ and $\mathscr{P}(4, 3)$.

Lemma 2.3. *For $x \ge 7$, the interval $(x, 2x]$ contains a prime number congruent to* 1 *modulo* 4 *as well as a prime number congruent to* 3 *modulo* 4*.*

Proof. By [5, Theorem 1], for every $x \ge 887$, the sets $(x, 1.048x] \cap \mathscr{P}(4, 1)$ and $(x, 1.048x] \cap \mathscr{P}(4, 3)$ are nonempty, that is, there exist prime numbers congruent to 1 and to 3 modulo 4 in the interval $(x, 1.048x]$. This proves the lemma when $x \ge 887$. For smaller values we note that the list $(13, 17, 29, 53, 101, 197, 389, 773, 929)$ consists of primes congruent to 1 modulo 4 while $(7, 11, 19, 31, 59, 107, 211, 419, 827, 887)$ consists of primes congruent to 3, modulo 4. Also, one easily verifies that if $7 \le x < 887$, then the interval $(x, 2x]$ intersects both lists. □

In the following table (Table 1) we list the first 15 prime numbers congruent to 1 and 3 modulo 4, respectively, along with their partial sums.

Lemma 2.4. *(a) If $n \ge 10$, then $S_{4,1}(n) - q_{4,1}(n+1) \ge 244$.*
(b) If $n \ge 8$, then $S_{4,3}(n) - q_{4,3}(n+1) \ge 112$.

Proof. (a) Let $\mathcal{N}$ denote the set of all integers $n \geq 10$ such that the statement (a) of the lemma is valid. The table (Table 1) shows that $S_{4,1}(10) - q_{4,1}(11) = 418 - 97 = 321$, so $10 \in \mathcal{N}$. To prove the inductive step, let $n \in \mathcal{N}$. Then, $n \geq 10 > 7$ so by Lemma 2.3,

$$S_{4,1}(n+1) - q_{4,1}(n+2) = S_{4,1}(n) - q_{4,1}(n+1) + 2q_{4,1}(n+1) - q_{4,1}(n+2) \geq S_{4,1}(n) - q_{4,1}(n+1) \geq 244,$$

and therefore $n+1 \in \mathcal{N}$.

(b) Let now $\mathcal{N}$ denote the set of all integers $n \geq 8$ such that the statement (b) of the lemma is valid. The table shows that $S_{4,3}(8) - q_{4,3}(9) = 184 - 59 = 125$, so $8 \in \mathcal{N}$. The rest proceeds as above. □

Theorem 2.5. *(a) For every integer $m \geq 122$, there exist distinct prime numbers $p_{j_1}, \dots, p_{j_k} \in \mathscr{P}(4,1)$ $(k \geq 1)$ such that $m = p_{n_{j_1}} + \cdots + p_{j_k}$.*

(b) For every integer $m \geq 56$, there exist distinct prime numbers $q_{j_1}, \dots, q_{j_\ell} \in \mathscr{P}(4,3)$ $(\ell \geq 1)$ such that $m = q_{j_1} + \cdots + q_{j_\ell}$.

Proof. (a) Let $(k_0, n_0) = (122, 13)$. By Lemma 2.4, $S_{4,1}(n) - q_{4,1}(n+1) \geq 2k_0 = 244$ for all $n \geq n_0$. Also, a direct, though tedious, computation shows that the generating polynomial

$$\prod_{i=1}^{13}(1 + x^{q_{4,1}(i)}) = (1+x^5)(1+x^{13})\cdots(1+x^{101})(1+x^{109}) = \sum_{i=0}^{S_{4,1}(13)} \gamma_m(13)x^m$$

satisfies $\gamma_m(13) \geq 1$ for all m in interval $[122, 603] = [122, 725 - 122] = [k_0, S_{4,1}(n_0) - k_0]$ (we remark that $\gamma_{121}(13) = \gamma_{604}(13) = 0$). The rest follows from Theorem 2.1.

(b) Let now $(k_0, n_0) = (56, 12)$. By Lemma 2.4, $S_{4,3}(n) - q_{4,3}(n+1) \geq 2k_0 = 112$ for all $n \geq n_0$. Also, a direct computation shows that the generating polynomial

$$\prod_{i=1}^{12}(1 + x^{q_{4,3}(i)}) = (1+x^3)(1+x^7)\cdots(1+x^{71})(1+x^{79}) = \sum_{i=0}^{S_{4,3}(12)} \gamma_m(12)x^m$$

satisfies $\gamma_m(12) \geq 1$ for all m in the interval $[56, 404] = [56, 460 - 56] = [k_0, S_{4,3}(n_0) - k_0]$ (we remark that $\gamma_{55}(12) = \gamma_{405}(12) = 0$). The rest again follows from Theorem 2.1. □

Corollary 2.6. *For every integer $m \geq 122$, there exist $k, l \in \mathbb{N}$ and distinct prime numbers $p_1, \dots, p_k \in \mathscr{P}(4,1)$ and $q_1, \dots, q_l \in \mathscr{P}(4,3)$ such that $m = p_1 + \cdots + p_k = q_1 + \cdots + q_l$.* □

Table 2 shows that, in addition to the above corollary, every integer which does not belong to the union $[1,17] \cup [19,21] \cup [23,28] \cup [31,33] \cup [35,36] \cup [38,40] \cup [43,45] \cup [48,49] \cup [51,52] \cup [55,57] \cup \{60, 62, 65, 68, 69, 77, 80, 81, 85, 93, 121\}$ can also be written as a sum of different primes from the class $\mathscr{P}(4,1)$ as well as the sum of different primes from the class $\mathscr{P}(4,3)$.

By using also other residue classes of primes we can get additional integers that can be expressed as a sum of different primes from two disjoint subsets (Table 3).

Notice that Tables 2 and 3, together with Corollary 2.6, show that all integers except those in $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21, 25, 27\}$ can be written as a sum of distinct odd primes in at least two ways such that no prime appears in two different sums.

Corollary 2.7. *Let $n \geq 4$. If n and $n-1$ are not prime numbers, then* $\operatorname{diam} \Delta^o(\mathcal{S}_n)^* = 3$.

Proof. By Corollary 2.6 and the tables above, we see that for every nonprime number $n \geq 16$, except for $n = 21, 25, 27$, there are disjoint subsets $\mathscr{T}_1$ and $\mathscr{T}_2$ of the sequence of primes $\mathscr{P}$, neither containing the prime 2, such that $n = \sum_{p \in \mathscr{T}_1} p = \sum_{q \in \mathscr{T}_2} q$. For those integers, the claim follows directly from Proposition 1.1. The remaining integers n, for which neither n nor $n-1$ is a prime number, are $n = 9, 10, 15, 21, 25, 27$. Except for $n = 15, 21$, they are all either powers of a prime number or are immediate successors of powers of a prime number and the claim follows from [6, Corollary 4.13] ($n = 15$ is also considered there). If $n = 21$, then $n - 1 = 20 = 3 + 17 = 7 + 13$ and Proposition 1.1 is again applicable. □

Remark 2.8. Every prime, congruent to 1 or to 3 modulo 4, is distinct from 2. Also, every prime inside Tables 2 and 3 is distinct from 2. Therefore, the proof of Corollary 2.7 verifies also the last conjecture of [6] that the diameter of the reduced order-super-commuting graph of an alternating group satisfies

$$\operatorname{diam} \Delta^o(\mathrm{Alt}_n)^* = 3$$

Table 2
All small integers expressible simultaneously as sums of distinct primes congruent to 1 modulo 4 and to 3 modulo 4.

18=5+13=7+11	22=5+17=3+19	29=29=3+7+19
30=13+17=7+23	34=5+29=3+31	37=37=3+11+23
41=41=3+7+31	42=5+37=11+31	46=5+41=3+43
47=5+13+29=47	50=13+37=3+47	53=53=3+7+43
54=13+41=7+47	58=5+53=11+47	59=5+13+41=59
61=61=3+11+47	63=5+17+41=3+7+11+19+23	64=5+13+17+29=3+7+11+43
66=5+61=7+59	67=13+17+37=67	70=17+53=3+67
71=5+13+53=71	72=5+13+17+37=3+7+19+43	73=73=3+11+59
74=13+61=3+71	75=5+17+53=3+7+11+23+31	76=5+13+17+41=3+7+19+47
78=5+73=7+71	79=5+13+61=79	82=29+53=3+79
83=5+17+61=83	84=5+13+29+37=3+7+31+43	86=13+73=3+83
87=5+29+53=3+7+11+19+47	88=5+13+17+53=3+7+11+67	89=89=3+7+79
90=17+73=7+83	91=5+13+73=3+7+11+23+47	92=5+17+29+41=3+7+11+71
94=5+89=11+83	95=5+17+73=3+7+11+31+43	96=5+13+17+61=3+7+19+67
97=97=3+11+83	98=37+61=19+79	99=5+41+53=3+7+11+19+59
100=5+13+29+53=3+7+11+79	101=101=3+19+79	102=5+97=19+83
103=5+37+61=3+7+11+23+59	104=5+17+29+53=3+7+11+83	105=5+13+17+29+41=3+19+83
106=5+101=23+83	107=5+13+89=3+7+11+19+67	108=5+13+17+73=3+7+19+79
109=109=3+23+83	110=13+97=31+79	111=5+17+89=3+7+11+19+71
112=5+13+41+53=3+7+19+83	113=113=3+31+79	114=5+109=31+83
115=5+13+97=3+7+11+23+71	116=5+13+37+61=3+7+23+83	117=5+13+17+29+53=3+31+83
118=5+113=47+71	119=5+13+101=3+7+11+19+79	120=5+13+29+73=3+7+31+79

Table 3
Small integers expressed as sums of distinct odd primes in two non-overlapping ways, without restriction on residue classes.

16=3+13=5+11	19=19=3+5+11	20=3+17=7+13
23=23=3+7+13	24=5+19=7+17	26=3+23=7+19
28=5+23=11+17	31=31=3+5+23	32=3+29=13+19
33=3+7+23=5+11+17	35=3+13+19=5+7+23	36=5+31=7+29
38=7+31=3+5+11+19	39=3+5+31=7+13+19	40=3+37=11+29
43=43=3+11+29	44=3+41=7+37	45=3+11+31=5+17+23
48=5+43=7+41	49=3+5+41=7+11+31	51=3+5+43=7+13+31
52=5+47=11+41	55=3+5+47=7+11+37	56=13+43=19+37
57=3+7+47=5+11+41	60=13+47=17+43	62=19+43=3+5+7+47
65=3+19+43=5+13+47	68=31+37=3+5+13+47	69=3+19+47=5+23+41
77=3+31+43=7+23+47	80=37+43=3+5+31+41	81=3+31+47=11+29+41
85=5+37+43=7+31+47	93=3+43+47=23+29+41	121=31+43+47=3+11+29+37+41

if $n \geq 4$ and none of $n-2$, $n-1$ and n is a prime. The arguments go unchanged when $n \notin \{9, 10, 15, 21, 25, 27\}$ except that instead of Proposition 1.1 one relies on [6, Proposition 4.18]. In the exceptional cases, the three conditions rule out $n = 9$ (because $n-2 = 7$ is a prime), as well as $n = 15, 21, 25$. What remains is $n = 10$ and $n = 27$. By a text preceding Proposition 4.18 in [6] we know that $\operatorname{diam} \Delta^o(\mathrm{Alt}_{10})^* = 3$. Lastly, $n = 27 = 3^3 = 2 + 5 + 7 + 13$ so one can apply [6, Proposition 4.18] on $T_1 = \{3\}$ with $\alpha_1 = 3$ and $T_2 = \{5, 7, 13\}$ to deduce $\operatorname{diam} \Delta^o(\mathrm{Alt}_{27})^* = 3$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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Data availability

No data was used for the research described in the article.

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