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AKCE International Journal of Graphs and Combinatorics

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/uakc20

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To cite this article: S. Prabhu, D. Sagaya Rani Jeba, M. Arulperumjothi & Sandi Klavžar (2024) Metric Dimension of Irregular Convex Triangular Networks, AKCE International Journal of Graphs and Combinatorics, 21:3, 225-231, DOI: 10.1080/09728600.2023.2280799

To link to this article: https://doi.org/10.1080/09728600.2023.2280799

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Published online: 07 Dec 2023.

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2024, VOL. 21, NO. 3, 225–231
https://doi.org/10.1080/09728600.2023.2280799

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Metric Dimension of Irregular Convex Triangular Networks

S. Prabhu[a], D. Sagaya Rani Jeba[b], M. Arulperumjothi[c], and Sandi Klavžar[d,e,f]

[a]Department of Mathematics, Rajalakshmi Engineering College, Chennai, India; [b]Department of Mathematics, Panimalar Engineering College, Chennai, India; [c]Department of Mathematics, St. Joseph's College of Engineering, Chennai, India; [d]Faculty of Mathematics and Physics, University of Ljubljana, Slovenia; [e]Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia; [f]Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

ABSTRACT
Irregular convex triangular networks consist of the interior of a 6-sided convex polygon drawn on the infinite triangular network. Formal description of these applicable networks is provided. In the main result it is proved that the metric dimension of an irregular convex triangular network is either 2 or 3 and the determination of which value is the right one is specified. Metric dimensions of various graph classes and interconnection networks are also collected.

ARTICLE HISTORY
Received 18 October 2023
Accepted 3 November 2023

KEYWORDS
Graph networks; metric dimension; irregular convex triangular network

AMS SUBJECT CLASSIFICATION
05C12; 05C90

1. Introduction

Network science has risen as a multidisciplinary effort to estimate the structural features of complex systems due to the requirement to define interconnections mathematically [5]. Due to the development of big data, the concept of a network (a collective of components, alias "nodes," linked together by the connections, alias "edges," among them) has emerged as a practical way to represent a wide variety of empirical knowledge in physical, biological, and social phenomena [37]. Network science acquired its basic principles via graph theory, mathematically preoccupied with pairwise relationships between features [33], and was inspired by landmark works in sociology and economics [9, 53]. Graph theory and network science, on the other hand, have gone their own ways since then, with so little convergence in their academic buildings and research community. This subject is discussed in [12, 47]. The challenge of coming up with a unique representation for each vertex in a connected graph arises in many contexts including robot navigation [27], coin weighing problems [49], connected joins in graphs [44], hierarchical data structures [38], network discovery and verification [7] and mastermind game strategy [18].

The tessellation of the plane into triangles is the foundation of hexagonal networks. Networks with a regular square separation are termed mesh networks, and those with a regular hexagonal partition are called honeycomb networks. There is a degree of inconsistency in the choice of the name, which arises from the duality between graphs that correspond to hexagonal networks and honeycomb networks. These networks are applied in many areas, say in mathematical chemistry, in image processing, in wireless networks, in computer graphics, and in interconnection networks, cf. [39]. As a generalization of the planar hexagonal networks, higher dimensional hexagonal networks have been defined in [14, 39], and the honeycomb design has been presented in [50]. For a network with hexagonal nodes, an addressing method for the processors and accompanying routing and broadcasting algorithms have been presented [13]. Cellular networks have employed hexagonal networks for user tracking and connection rerouting [39].

In this paper we consider the metric dimension of irregular convex triangular networks and proceed as follows. In the next section we recall the metric dimension and make a small survey of known results related to it, where we concentrate on determining the metric dimension of graph classes. In Section 3 we introduce the networks of interest in this paper and prepare the mathematical formalities on them that we need for the following. Then, in Section 4 we determine their metric dimension.

2. The metric dimension

Visual identification of a distinctive landmark by a robot moving in Euclidean space offers information about the landmark's direction and helps the robot to pinpoint its position. However, a graph lacks any discernible direction or transparency. Moreover, it is generally accepted that a robot moving along a graph can gauge how far apart landmarks are as it travels them. Clearly, if the robot knows the distances to a big enough selection of landmarks, its position on the graph may be computed uniquely. A metric basis is a set of landmarks that uniquely determines the robot's position, and the metric dimension of the graph

CONTACT Sandi Klavžar sandi.klavzar@fmf.uni-lj.si Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia.

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[27] is the lowest number of landmarks. As a result, a graph-theoretic understanding of this problem is to construct codes for the vertices of a graph in such a way that each vertices has a unique code. The recent publications [1, 25, 29, 31, 32, 41, 52] all reflect on this.

Let $G = (V(G), E(G))$ be a connected graph. The distance between vertices u and v is the length of a shortest u, v-path and denoted by $d_G(u, v)$. If G will be clear from the context, we may also write $d(u, v)$. For an ordered set of vertices $R = \{r_1, r_2, \ldots, r_l\}$ and a vertex $u \in V(G)$, the representation of u with respect to R is the vector $(d_G(u, r_1), d_G(u, r_2), \ldots, d_G(u, r_l))$. If the vertices of G have pairwise different representations with respect to R, then R is a *resolving set* for G. That is, R is a resolving set if for each pair of different vertices $x, y \in V(G)$ there exsits a vertex $r \in R$ such that $d_G(x, r) \neq d_G(y, r)$. The *metric dimension* $\dim(G)$ of G is the cardinality of a smallest resolving set for G, such a set is called a *metric basis*.

These concepts were first suggested by Slater [47, 48], who was inspired by applications to the placement of a small number of sonar detecting devices in a network such that the position of each vertex in the network may be uniquely identified in terms of its distance from the set of devices. Harary and Melter [23] independently introduced this concept. Two decades later, Khuller et al. [27] also (independently) found these ideas and coined the term metric dimension. While attempting to establish a capability for enormous data sets of chemical graphs, these notions were rediscovered once more in [12, 26].

Determination of the metric dimension is NP-hard for general graphs [27], and, moreover, the problem remains NP-hard on bipartite graphs [34]. This makes it interesting to determine the metric dimension for specific classes of graphs, and so far this has been very intensively investigated. In Table 1 we have gathered the work on the metric dimension of various graph classes and interconnection networks.

Table 1. Metric dimension of various classes.

Structure	Reference
Trees, multi-dimensional grids	Khuller et al. [27]
Line graphs, para-line graphs	Klein et al. [28]
Torus networks	Manuel et al. [35]
Some families of graphs	Cáceres et al. [11]
Cayley digraphs	Fehr et al. [15]
Unicyclic graphs	Poisson et al. [40]
Petersen graphs	Shao et al. [45]
Circulant graphs	Rajan et al. [19]; Imran et al. [24]
Harary graphs	Rajan et al. [19]
Illiac networks	Rajan et al. [43]
Jahangir graph	Tomescu et al. [51]
Enhanced hypercubes	Rajan et al. [42]
Random graphs	Bollobas et al. [8]
Benes networks	Manuel et al. [34]
Honeycomb networks	Manuel et al. [36]
Distance-regular graphs	Guo et al. [21]
Infinite graphs	Cáceres et al. [10]
Hypercubes	Beardon [6]
Wheel-related graphs	Siddiqui et al. [46]
Bilinear forms graphs	Feng et al. [16]
Grassmann graphs	Bailey et al. [4]
Symplectic dual polar graphs	Guo et al. [20]
Permutation graphs	Hallaway et al. [22]
Kneser and Johnson graphs	Bailey et al. [3]
$TiO_2[m, n]$	Prabhu et al. [41]
Chain graphs	Fernau et al. [17]
Convex polytopes	Imran et al. [25]
Incidence graphs	Bailey [2]

To conclude this section we recall the following result needed later on.

Theorem 2.1. [27] If $\{s, t\}$ is a metric basis of a graph G, then $\deg_G(s) \leq 3$, $\deg_G(t) \leq 3$, and there is exactly one shortest s, t-path P. Moreover, if w is an internal vertex of P, then $\deg_G(w) \leq 5$.

3. Irregular convex triangular networks

In this section we introduce irregular convex triangular networks and provide their formal description. Intuitively, they consist from the interior of a 6-sided convex polygon drawn on the infinite triangular network.

If $r < s$ are integers, then we will denote the set $\{r, r+1, \ldots, s\}$ by $[\![r, s]\!]$. For given parameters n, m, p, and q, let $\{H_0, H_1, \ldots, H_{2n+m-p-q}\}$, $\{A_0, A_1, \ldots, A_n\}$ and $\{O_0, O_1, \ldots, O_{m+n}\}$ be three respective sets of lines corresponding to horizontal, acute, and obtuse lines fulfilling the following conditions:

1. $\angle H_i A_j = \pi/3$ for $i \in [\![0, m+2n-p-q]\!]$ and $j \in [\![0, n]\!]$;
2. $\angle H_i O_k = 2\pi/3$ for $i \in [\![0, m+2n-p-q]\!]$ and $k \in [\![0, m+n]\!]$;
3. $\angle A_j O_k = \pi/3$ for $j \in [\![0, n]\!]$ and $k \in [\![0, m+n]\!]$.

Assuming the above conditions, a *convex triangular grid* $\Delta_{2n+m-p-q,n,m+n}$ is a graph derived by treating all triple intersection points as vertices and the line joining all triple intersection points as edges. (Many non-isomorphic convex triangular grids have the same parameters [30].) If the boundary of the triangular grid $\Delta_{h,a,o}$ is a trapezium, we say that the grid is a *triangular trapezium*. In this case we have $a = o$ in $\Delta_{h,a,o}$. The height and the length of the base are respectively denoted by $h-1$ and $a-1$, the graph is denoted by $T_{h-1,a-1}$. This triangular trapezium is reduced to a *triangular triangle* T_{h-1} when $h = a$. By I_{2n-1}, $n \geq 1$, we denote a chain of $(2n-1)-$ triangles. It is easy to observe that $I_{2n-1} \cong T_{1,n}$. In $\Delta_{h,a,o}$, if the boundary of $\Delta_{h,a,o}$ forms a parallelogram, then $\Delta_{h,a,o}$ is a *triangular parallelogram*. For this case we have $h-1 = |o-a|$ and $\Delta_{h,a,o} \cong \Delta_{h,o,a}$. For a triangular parallelogram we always assume that $a < o$ and denote $\Delta_{h,a,h+a-1}$ by $P_{h-1,a-1}$. Also for $n \geq 1$, we have $P_{1,n} \cong I_{2n}$. Several examples of these networks can be seen in Figure 1.

Now, an *irregular convex triangular hexagon* $H_{p,q,m,n}$ is formed by merging the base of $T_{n-q,n}$ and $T_{n-p,n}$ with two bases of $P_{m,n}$, where $p \in [\![0, n]\!]$, $q \in [\![0, n]\!]$, and $m \geq 0$. By symmetry we have $H_{p,q,m,n} \cong H_{q,p,m,n}$, hence we may assume in the rest that always $p \leq q$ holds. Also note that $T_{n,n-q} \cong T_{n-q,n} \cong H_{q,0,0,n}$; $P_{m,n} \cong P_{n,m} \cong H_{0,0,m,n}$; $H_n \cong H_{p,p,0,2p}$; $I_{2n-1} \cong H_{1,0,0,n}$ and $I_{2n} \cong H_{0,0,1,n}$. The structure and the construction of $H_{p,q,m,n}$ are illustrated in Figure 2.

To complete the description of the irregular convex triangular networks, we introduce the coordinates of its vertices in a natural way. That is, based on the horizontal, acute, and obtuse lines, a vertex of $H_{p,q,m,n}$ is assigned the natural coordinates (h, a, o), see Figure 3.

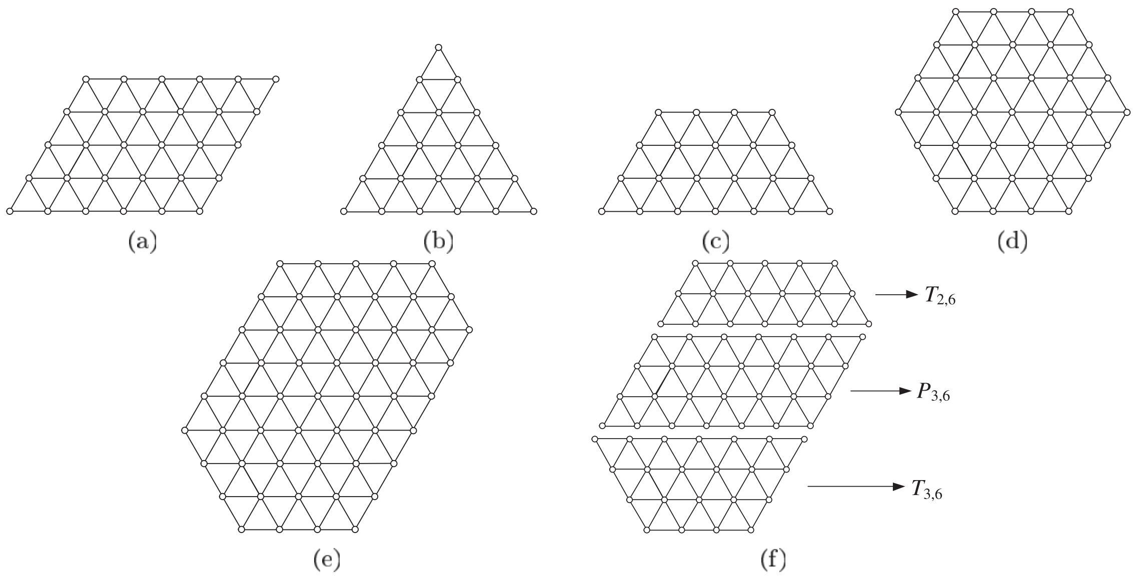

Figure 1. (a) $P_{4,5}$; (b) T_5; (c) $T_{3,6}$; (d) H_3; (e) $H_{3,4,3,6}$; (f) Convex parts of $H_{3,4,3,6}$.

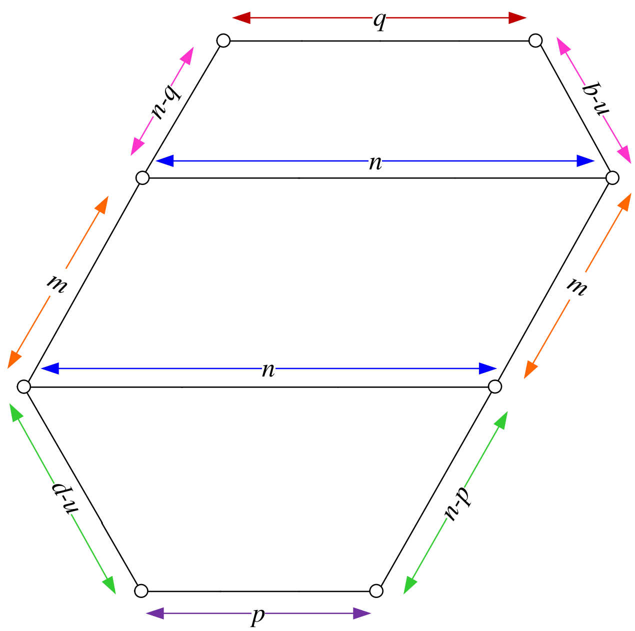

Figure 2. Construction of irregular convex triangular networks.

4. The metric dimension of irregular convex hexagonal networks

The main result of this paper reads as follows.

Theorem 4.1. If $H_{p,q,m,n}$ is an irregular convex hexagonal net, then

$$\dim(H_{p,q,m,n}) = \begin{cases} 2; & m \geq p, \\ 3; & m < p. \end{cases}$$

In the rest of the section we prove Theorem 4.1. We first show:

Lemma 4.2. If $m < p$, then $\dim(H_{p,q,m,n}) > 2$.

Proof. Suppose on the contrary that $\dim(H_{p,q,m,n}) = 2$ and let $\{u, v\}$ be a resolving set of $H_{p,q,m,n}$. By Theorem 2.1, $\deg_G(u) \leq 3$, $\deg_G(v) \leq 3$, and there is a unique shortest u, v-path. Therefore, we need to consider the following six cases.

Case 1: $u = (0, n-p, n+m)$ and $v = (0, n, m+n-p)$.
We have $d(u, (p+1, n-p-1, m+n-p)) = d(u, (p+1, n-p, m+n-p-1))$ and $d(v, (p+1, n-p-1, m+n-p)) = d(v, (p+1, n-p, m+n-p-1))$, a contradiction.

Case 2: $u = (0, n, m+n-p)$ and $v = (n+m-p, n, 0)$.
We have $d(u, (n-p+m, p-m-1, n-p+m+1)) = d(u, (n-p+m+1, p-m-1, n-p+m))$ and $d(v, (n-p+m, p-m-1, n-p+m+1)) = d(v, (n-p+m+1, p-m-1, n-p+m))$, a contradiction.

Case 3: $u = (n+m-p, n, 0)$ and $v = (2n+m-p-q, q, 0)$.
We have $d(u, (n-p+m, q-1, n-q+1)) = d(u, (n-p+m-1, q, n-q+1))$ and $d(v, (n-p+m, q-1, n-q+1)) = d(v, (n-p+m-1, q, n-q+1))$, a contradiction.

Case 4: $u = (2n+m-p-q, q, 0)$ and $v = (2n+m-p-q, 0, q)$.
We have $d(u, (2n+m-p-2q-1, q, q+1)) = d(u, (2n+m-p-2q-1, q+1, q))$ and $d(v, (2n+m-p-2q-1, q, q+1)) = d(v, (2n+m-p-2q-1, q+1, q))$, a contradiction.

Case 5: $u = (2n+m-p-q, 0, q)$ and $v = (n-p, 0, m+n)$.
Then we have $d(u, (n-p, m+n-q+1, q-1)) = d(u, (n-p-1, m+n-q+1, q))$ and $d(v, (n-p, m+n-q+1, q-1)) = d(v, (n-p-1, m+n-q+1, q))$, a contradiction.

Case 6: Let $u = (n-p, 0, m+n)$ and $v = (0, n-p, n+m)$.
We have, $d(u, (n-p, n-p+1, m+p-1)) = d(u, (n-p+1, n-p, m+p-1))$ and $d(v, (n-p, n-p+1, m+p-1)) = d(v, (n-p+1, n-p, m+p-1))$, a contradiction.

We conclude that $\dim(H_{p,q,m,n}) > 2$. □

If u is a vertex of a $H_{p,q,m,n}$ and r is a positive integer, then let $N_r(u)$ denote the set of vertices of $H_{p,q,m,n}$ which are at distance r from u. In addition, we will use the following convention. If $N_r(u)$ consists of consecutive vertices from a horizontal line, then this will be denoted by $N_r(u) = P_H$. If needed, we will

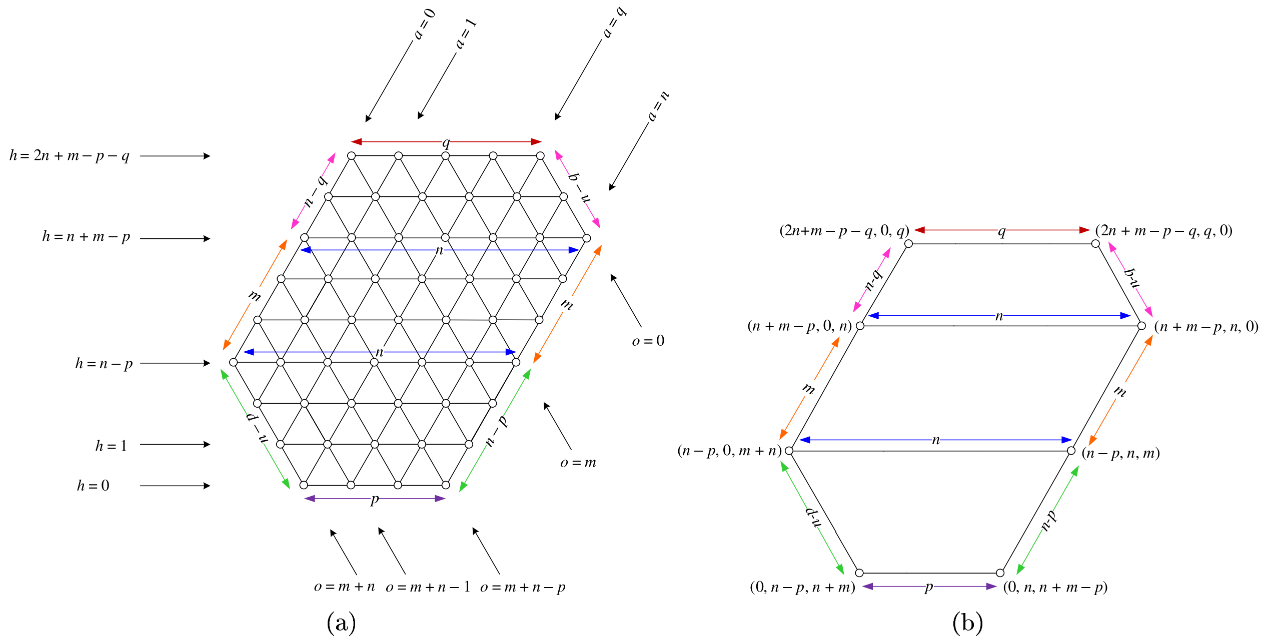

Figure 3. (a) Addressing scheme; (b) Coordinates of corner vertices of $H_{p,q,m,n}$.

also identify P_H with the set of vertices in the horizontal line that are at distance r from u. We will use analogues conventions P_A and P_O when $N_r(u)$ consists of consecutive vertices from an acute or an obtuse line, respectively. Moreover, if $N_r(u)$ consists of, say, some consecutive vertices from a horizontal line followed by some consecutive vertices from an obtuse line, this will be briefly denoted by $N_r(u) = P_H \cup P_O$.

Lemma 4.3. In any $H_{p,q,m,n}$ we have

1. $N_r((0, n-p, n+m)) \in \{P_H \cup P_O, P_H, P_O\}$.
2. $N_r((0, n, m+n-p)) \in \{P_H \cup P_A, P_H\}$.
3. $N_r((m+n-p, n, 0)) \in \{P_A \cup P_O, P_O\}$.

Proof. Suppose first that $n - p - q \geq 0$. Then

$$N_r((0, n-p, n+m)) = \begin{cases} P_H \cup P_O; & r \in [\![1, m+n]\!], \\ P_H; & r \in [\![1+m+n, m+2n-q-p]\!], \end{cases}$$

where

$$P_H = \begin{cases} \{(r, n-p-r+i, m+n-i) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, n-p]\!], \\ \{(r, i, m+2n-i-p-r) : i \in [\![0, n-p]\!]\}; \\ \quad r \in [\![1+n-p, m+n]\!], \\ \{(r, i, m+2n-i-p-r) : i \in [\![0, m+2n-r-p]\!]\}; \\ \quad r \in [\![1+n+m, m+2n-q-p]\!], \end{cases}$$

and

$$P_O = \begin{cases} \{(r-i, n+i-p, m+n-r) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, p]\!], \\ \{(r-i, n+i-p, m+n-r) : i \in [\![1, p]\!]\}; \\ \quad r \in [\![1+p, m+n]\!]. \end{cases}$$

If $n - p - q < 0$, then

$$N_r((0, n-p, n+m)) = \begin{cases} P_H \cup P_O; & r \in [\![1, m+2n-p-q]\!], \\ P_O; & r \in [\![1+m+2n-p-q, m+n]\!], \end{cases}$$

where

$$P_H = \begin{cases} \{(r, n+i-p-r, m+n-i) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, n-p]\!], \\ \{(r, i, m+2n-i-p-r) : i \in [\![0, n-p]\!]\}; \\ \quad r \in [\![1+n-p, m+2n-p-q]\!], \end{cases}$$

and

$$P_O = \begin{cases} \{(r-i, n+i-p, m+n-r) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, p]\!], \\ \{(r-i, n+i-p, m+n-r) : i \in [\![1, p]\!]\}; \\ \quad r \in [\![1+p, m+2n-p-q]\!], \\ \{(r-i, n+i-p, m+n-r) : i \in [\![r-m-2n+p+q, p]\!]\}; \\ \quad r \in [\![1+m+2n-p-q, m+n]\!]. \end{cases}$$

For $m - p \geq 0$ we have

$$N_r((0, n, m+n-p)) = \begin{cases} P_H \cup P_A; & r \in [\![1, n]\!], \\ P_H; & r \in [\![1+n, m+2n-p-q]\!], \end{cases}$$

where

$$P_H = \begin{cases} \{(r, n-i, m+n+i-p-r) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, n]\!], \\ \{(r, n-i, m+n+i-p-r) : i \in [\![0, n]\!]\}; \\ \quad r \in [\![1+n, m+n-p]\!], \\ \{(r, m+2n-i-p-r, i) : i \in [\![0, m+2n-p-r]\!]\}; \\ \quad r \in [\![1+m+n-p, m+2n-p-q]\!], \end{cases}$$

and

$$P_A = \begin{cases} \{(r-i, n-r, m+n+i-p) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, p]\!], \\ \{(r-i, n-r, m+n+i-p) : i \in [\![1, p]\!]\}; \\ \quad r \in [\![1+p, n]\!]. \end{cases}$$

If $m - p < 0$, then

$$N_r((0, n, m+n-p)) = \begin{cases} P_H \cup P_A; & r \in [\![1, n]\!], \\ P_H; & r \in [\![1+n, m+2n-p-q]\!], \end{cases}$$

where

$$P_H = \begin{cases} \{(r, n-i, m+n+i-p-r) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, m+n-p]\!], \\ \{(r, m+2n-i-p-r, i) : i \in [\![0, m+2n-p-r]\!]\}; \\ \quad r \in [\![1+m+n-p, m+2n-p-q]\!], \end{cases}$$

and

$$P_A = \begin{cases} \{(r-i, n-r, m+n+i-p) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, p]\!], \\ \{(r-i, n-r, m+n+i-p) : i \in [\![1, p]\!]\}; \\ \quad r \in [\![1+p, n]\!]. \end{cases}$$

If $m - p \geq 0$, then we have

$$N_r((m+n-p, n, 0)) = \begin{cases} P_A \cup P_O; & r \in [\![1, n]\!], \\ P_O; & r \in [\![1+n, m+n]\!], \end{cases}$$

where

$$P_A = \begin{cases} \{(m+n-i-p+r, n-r, i) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, n-p]\!], \\ \{(m+2n-i-p-q, n-r, r-n+i+q) : \\ \quad i \in [\![0, n-q]\!]\}; \\ \quad r \in [\![1+n-q, n]\!], \end{cases}$$

and

$$P_O = \begin{cases} \{(m+n-p-i, n+i-r, r) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, n]\!], \\ \{(1+m+2n-i-p-r, i-1, r) : i \in [\![1, 1+n]\!]\}; \\ \quad r \in [\![1+n, m+n-p]\!], \\ \{(1+m+2n-i-p-r, i-1, r) : \\ \quad i \in [\![1, 1+m+2n-p-r]\!]\}; \\ \quad r \in [\![1+m+n-p, m+n]\!]. \end{cases}$$

Finally, if $m - p < 0$, then

$$N_r((m+n-p, n, 0)) = \begin{cases} P_A \cup P_O; & r \in [\![1, n]\!], \\ P_O; & r \in [\![1+n, m+n]\!], \end{cases}$$

where

$$P_A = \begin{cases} \{(m+n-i-p+r, n-r, i) : i \in [\![0, r]\!]\}; \\ \quad r \in [\![1, n-q]\!], \\ \{(m+2n-i-p-q, n-r, r-n+i+q) : \\ \quad i \in [\![0, n-q]\!]\}; \\ \quad r \in [\![1+n-q, n]\!], \end{cases}$$

and

$$P_O = \begin{cases} \{(m+n-i-p, n+i-r, r) : i \in [\![1, r]\!]\}; \\ \quad r \in [\![1, m+n-p]\!], \\ \{(m+n-p-i, n+i-r, r) : i \in [\![1, m+n-q]\!]\}; \\ \quad r \in [\![1+m+n-q, n]\!], \\ \{(1+m+2n-i-p-r, i-1, r) : \\ \quad i \in [\![1, 1+m+2n-p-r]\!]\}; \\ \quad r \in [\![1+n, m+n]\!]. \end{cases}$$

□

Lemma 4.4. In any $H_{p,q,m,n}$, if $r_1, r_2 \in [\![1, m+2n-p-q]\!]$, then $N_{r_1}((0, n-p, n+m)) \cap N_{r_2}((0, n, m+n-p))$ is either empty, or a vertex, or a line segment of a horizontal line.

Proof. From the proof of Lemma 4.3 we recall that if $n-q-p \geq 0$, then

$$N_r((0, n-p, n+m)) = \begin{cases} P_H \cup P_O; & r \in [\![1, m+n]\!], \\ P_H; & r \in [\![1+m+n, m+2n-p-q]\!], \end{cases}$$

and

$$N_r((0, n, m+n-p)) = \begin{cases} P_H \cup P_A; & r \in [\![1, n]\!], \\ P_H; & r \in [\![1+n, m+2n-p-q]\!]. \end{cases}$$

Moreover, if $n-q-p < 0$, then

$$N_r((0, n-p, n+m)) = \begin{cases} P_H \cup P_O; & r \in [\![1, m+2n-p-q]\!], \\ P_O; & r \in [\![1+m+2n-p-q, m+n]\!], \end{cases}$$

and

$$N_r((0, n, m+n-p)) = \begin{cases} P_H \cup P_A; & r \in [\![1, n]\!], \\ P_H; & r \in [\![1+n, m+2n-p-q]\!]. \end{cases}$$

The conclusion then follows from these facts. □

Lemma 4.5. In any $H_{p,q,m,n}$, if $r_1 \in [\![1, 2n+m-p-q]\!]$, $r_2 \in [\![1, n+m]\!]$, and $m-p \geq 0$, then $|N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))| \in \{0, 1\}$.

Proof. From the proof of Lemma 4.3 we recall that if $m-p \geq 0$, then

$$N_r((0, n, m+n-p)) = \begin{cases} P_H \cup P_A; & r \in [\![1, n]\!], \\ P_H; & r \in [\![1+n, m+2n-p-q]\!], \end{cases}$$

and

$$N_r((m+n-p, n, 0)) = \begin{cases} P_A \cup P_O; & r \in [\![1, n]\!], \\ P_O; & r \in [\![1+n, m+n]\!]. \end{cases}$$

Case 1: $n-p-q > 0$.
Assume first that $r_1, r_2 \in [\![1, m+n-p]\!]$. If $r_1+r_2 < m+n-p$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_1+r_2 \geq m+n-p$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is a singleton.

Assume second that $r_1 \in [\![1+m+n-p, m+2n-p-q]\!]$ and $r_2 \in [\![1, n]\!]$. If $r_1-m-n+p-r_2 > 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_1-m-n+p-r_2 \leq 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is a singleton.

Assume finally that $r_1 \in [\![1, n]\!]$ and $r_2 \in [\![1+m+n-p, m+n]\!]$. Now, if $r_2-m-n+p-r_1 > 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_2-m-n+p-r_1 \leq 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is a singleton.

Case 2: $n-p-q \leq 0$.
Assume first that $r_1, r_2 \in [\![1, n]\!]$. If $r_1+r_2 < m+n-p$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_1+r_2 \geq m+n-p$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is a singleton.

Assume second that $r_1 \in [\![1+n, m+2n-p-q]\!]$ and $r_2 \in [\![1, n]\!]$. If $r_1 - m - n + p - r_2 > 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_1 - m - n + p - r_2 \le 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is singleton.

Assume finally that $r_1 \in [\![1, n]\!]$ and $r_2 \in [\![1+n, m+n]\!]$. If $r_2 - n - r_1 > 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0)) = \emptyset$, and if $r_2 - n - r_1 \le 0$, then $N_{r_1}((0, n, m+n-p)) \cap N_{r_2}((m+n-p, n, 0))$ is a singleton. □

Corollary 4.6. If $u = (h_1, a_1, o_1), v = (h_2, a_2, o_2)$ are vertices of $H_{p,q,m,n}$ with $h_1 \neq h_2$, $a_1 \neq a_2$, and $o_1 \neq o_2$, then $|N_{r_1}((0, n-p, n+m)) \cap N_{r_2}((0, n, m+n-p)) \cap \{u, v\}| \le 1$.

Proof. Suppose on the contrary that u and v are both in $N_{r_1}((0, n-p, n+m)) \cap N_{r_2}((0, n, m+n-p))$. Then Lemma 4.4 implies that $N_{r_1}((0, n-p, n+m)) \cap N_{r_2}((0, n, m+n-p))$ is a line segment of a horizontal line. But then $h_1 = h_2$, a contradiction. □

To complete the proof of Theorem 4.1 it remains to demonstrate (having Lemma 4.2 in mind) that if $m < p$, then there exists a metric basis of cardinality 3, and if $m \ge p$, then there exists a metric basis of cardinality 2. We establish these facts in the concluding couple of lemmas.

Lemma 4.7. If $m < p$, then $\{(0, n-p, n+m), (0, n, m+n-p), (m+n-p, n, 0)\}$ forms a metric basis of $H_{p,q,m,n}$.

Proof. Let $u = (h_1, a_1, o_1)$ and $v = (h_2, a_2, o_2)$ be two arbitrary vertices of $H_{p,q,m,n}$.

Case 1: $h_1 = h_2$.
Since in this case u and v lie in a segment of a horizontal line we infer that $d(u, (m+n-p, n, 0)) \neq d(v, (m+n-p, n, 0))$.

Case 2: $a_1 = a_2$.
Now we have $d(u, (0, n-p, n+m)) \neq d(v, (0, n-p, n+m))$.

Case 3: $o_1 = o_2$.
As u and v are in a segment of an obtuse line, $d(u, (0, n, m+n-p)) \neq d(v, (0, n, m+n-p))$.

Case 4: $h_1 \neq h_2$, $a_1 \neq a_2$, and $o_1 \neq o_2$.
Suppose that $d(u, (0, n-p, n+m)) = d(v, (0, n-p, n+m)) = r_1$. Then $u, v \in N_{r_1}((0, n-p, n+m))$ and we claim that $d(u, (0, n, m+n-p)) \neq d(v, (0, n, m+n-p))$. Suppose that $d(u, (0, n, m+n-p)) = d(v, (0, n, m+n-p)) = r_2$. Then $u, v \in N_{r_2}((0, n, m+n-p))$ which implies that $\{u, v\} \subseteq N_{r_1}((0, n-p, n+m)) \cap N_{r_2}((0, n, m+n-p))$. This contradiction to Corollary 4.6 yields $d(u, (0, n, m+n-p)) \neq d(v, (0, n, m+n-p))$. □

Lemma 4.8. If $m \ge p$, then $\{(0, n, m+n-p), (m+n-p, n, 0)\}$ forms a metric basis of $H_{p,q,m,n}$.

Proof. Apply Lemma 4.5. □

Funding

Sandi Klavžar was supported by the Slovenian Research Agency (ARRS) under the grants P1-0297, J1-2452, and N1-0285.

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