Results Math (2024) 79:260
Online First

https://doi.org/10.1007/s00025-024-02291-4

Results in Mathematics

A New Approach to Universal F-inverse Monoids in Enriched Signature

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Abstract. We show that the universal X-generated F-inverse monoid $F(G)$, where G is an X-generated group, introduced by Auinger, Szendrei and the first-named author, arises as a quotient inverse monoid of the Margolis-Meakin expansion $M(G, X \cup \overline{G})$ of G, with respect to the extended generating set $X \cup \overline{G}$, where $\overline{G}$ is a bijective copy of G which encodes the m-operation in $F(G)$. The construction relies on a certain dual-closure operator on the semilattice of all finite and connected subgraphs containing the origin of the Cayley graph $\mathrm{Cay}(G, X \cup \overline{G})$ and leads to a new and simpler proof of the universal property of $F(G)$.

Mathematics Subject Classification. 20M18, 20M10, 20F05, 05E18.

Keywords. Inverse monoid, F-inverse monoid, Margolis-Meakin expansion, group presentation, Cayley graph of a group, closure operator, dual-closure operator, partial action, partial action product.

1. Introduction

An F-inverse monoid is an inverse monoid such that every σ-class, where σ is the minimum group congruence, has a maximum element, with respect to the natural partial order. These monoids appear naturally and are useful in various mathematical contexts, see [2] and references therein; for a solution of the finite F-inverse cover problem, see [1].

F-inverse monoids possess the additional unary operation $s \mapsto m(s)$ assigning to each element s the maximum element $m(s)$ in its σ-class. It was observed by Kinyon [7] that F-inverse monoids in the enriched signature $(\cdot, {}^{-1}, m, 1)$ form a variety of algebras. In [2], Auinger, Szendrei and the first-named author found a model for the F-inverse expansion $F(G)$ (in this paper denoted

Published online: 12 October 2024 Birkhäuser

$F(G, X)$) of an X-generated group G, which is an upgrade of the Margolis-Meakin expansion $M(G, X)$ of G [9] (for definitions and properties of $M(G, X)$ and $F(G, X)$, see Subsection 2.5). A special case of this construction, with G being the free X-generated group $FG(X)$, is a model of the free X-generated F-inverse monoid $FFI(X)$. The construction of $F(G, X)$, just as in the case with $M(G, X)$, involves certain subgraphs of the Cayley graph $\mathrm{Cay}(G, X)$ of the X-generated group G. Its key novel feature is that the requirement of the connectedness of subgraphs under consideration is dropped. The appropriate analogues of paths in $\mathrm{Cay}(G, X)$ are *journeys* where, along with traversing edges, it is allowed to jump between vertices, the jumps being captured by the m-operation of $F(G, X)$. The proof of the universal property of $F(G, X)$ from [2] relies on assigning journeys in $\mathrm{Cay}(G, X)$ to terms of a suitable term algebra and evaluating them in $F(G, X)$. The proof is independent of the universal property of $M(G, X)$ and implies the latter, along with the universal property of the Birget-Rhodes expansion $B(G)$ [3,15] of G (see [2, Remark 4.8]).

In this paper, we show that $F(G, X)$ arises as the canonical quotient inverse monoid $M^\wedge(G, Y)$ of the Margolis-Meakin expansion $M(G, Y)$ of G, with respect to the extended set of generators $Y = X \cup \overline{G}$, where $\overline{G}$ is a set in a bijection with G and encodes the m-operation in $F(G, X)$. This quotient arises from a suitable G-invariant dual-closure operator $j\colon \mathcal{X}_Y \to \mathcal{X}_Y$ on the semilattice $\mathcal{X}_Y$ of all finite and connected subgraphs of $\mathrm{Cay}(G, Y)$, which contain the origin. Note that the underlying order of $\mathcal{X}_Y$ is the anti-inclusion order (see Proposition 2.1) and upon reversing this order our dual-closure operator can be equivalently looked at as a closure operator. We show that $M^\wedge(G, Y)$ is an X-generated F-inverse monoid and is canonically isomorphic to $F(G, X)$ (Propositions 4.2 and 4.3), the latter being essentially due to the fact that the gaps in finite and not necessarily connected subgraphs containing the origin of $\mathrm{Cay}(G, X)$ are determined by the edges labeled by $\overline{G}$ in the corresponding *closed* connected subgraphs of $\mathrm{Cay}(G, Y)$. Applying the universal property of $M(G, Y)$, we show in Theorem 4.4 that $M^\wedge(G, Y)$ has the same universal property as $F(G, X)$, which yields a new and simpler proof of the universal property of $F(G, X)$. Our arguments rely on the structure result for E-unitary inverse semigroups in terms of partial actions which is recalled in Subsection 2.3.

When this work was nearly complete, we learned of the preprint version of the paper [14] by Nora Szakács, which also treats quotients of the Margolis-Meakin expansions arising from closure operators, but with a different purpose. While we use dual-closure operators to give a new perspective on $F(G, X)$, [14] shows an equivalence of categories between certain closure operators and suitable E-unitary or F-inverse monoids. The work [14] separately considers closure operators on *not necessarily connected* subgraphs of $\mathrm{Cay}(G, X)$ (and also an appropriate analogue of Stephen's procedure in inverse monoids), in order to study presentations of F-inverse monoids in enriched signature. Our results, together with those of [14], suggest that this setting can be alternatively handled using *connected* subgraphs of $\mathrm{Cay}(G, Y)$ (see Remark 4.5).

For the undefined notions in inverse semigroups we refer the reader to [8,11], and in universal algebra to [4].

2. Preliminaries

2.1. X-Generated Algebraic Structures

We say that a group (or an involutive monoid, or an inverse monoid, or an F-inverse monoid) is X-*generated* via the *assignment map* $\iota_S\colon X \to S$ if S is generated by $\iota_S(X)$. A map $\varphi\colon S \to T$ between X-generated groups (or involutive monoids, or inverse monoids, or F-inverse monoids) is called *canonical*, if $\varphi\iota_S = \iota_T$. Let $(X \cup X^{-1})^*$ be the free involutive monoid on $X \cup X^{-1}$ and S an X-generated inverse monoid (in particular a group). For each $u \in (X \cup X^{-1})^*$ by $[u]_S$ (or simply $[u]$ when S is understood) we denote the *value* of u in S, that is, the image of u under the canonical morphism $(X \cup X^{-1})^* \to S$; if $x \in X$, we have $[x]_S = \iota_S(x)$.

2.2. Partial Group Actions and Premorphisms

Let $\leq$ denote the natural partial order on an inverse monoid. A *premorphism* from a group G to a inverse monoid S is a map $\varphi : G \to S$, such that the following conditions hold:

(PM1) $\varphi(1) = 1$,
(PM2) $\varphi(g)\varphi(h) \leq \varphi(gh)$, for all $g, h \in G$,
(PM3) $\varphi(g^{-1}) = \varphi(g)^{-1}$, for all $g \in G$.

If S is the symmetric inverse monoid $\mathcal{I}(X)$, we will denote $\varphi(g)(x)$ by $\varphi_g(x)$. By a *partial map* $f\colon A \to B$ from a set A to a set B we mean a map $f\colon C \to B$ where $C \subseteq A$. For $a \in A$ we say that $f(a)$ is *defined* if $a \in C$.

Let G be a group and X a (non-empty) set. We say that G *acts partially* on X if there exists a partial map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, which satisfies the following conditions:

(PA1) $1 \cdot x$ is defined and equals x, for all $x \in X$,
(PA2) if $g \cdot x$ and $g \cdot (h \cdot x)$ are defined, then $gh \cdot x$ is defined and $g \cdot (h \cdot x) = gh \cdot x$, for all $g, h \in G$ and $x \in X$,
(PA3) if $g \cdot x$ is defined, then $g^{-1} \cdot (g \cdot x)$ is defined and equals x, for all $g \in G$ and $x \in X$.

A partial action $G \times X \to X$, $(g, x) \mapsto g \cdot x$, gives rise to a premorhpism $\varphi\colon G \to \mathcal{I}(X)$ given by $\varphi_g(x) = g \cdot x$. The notions of a partial action of G on X and of a premorphism $G \to \mathcal{I}(X)$ are easily seen to be equivalent. For more background on partial group actions, we refer the reader to [6]; for a comprehensive survey on partial actions to [5].

2.3. The Structure of E-Unitary Inverse Semigroups in Terms of Partial Actions

Let $\mathcal{P}$ be a poset. A non-empty subset I of $\mathcal{P}$ is said to be an *order ideal*, if $x \le y$ and $y \in I$ imply that $x \in I$, for all $x, y \in \mathcal{P}$. A map $f\colon \mathcal{P} \to \mathcal{Q}$ between posets is called an *order isomorphism*, provided that $x \le y$ if and only if $f(x) \le f(y)$, for all $x, y \in \mathcal{P}$.

For a semilattice Y, by $\Sigma(Y)$ we denote the inverse monoid of all order-isomorphisms between order ideals of Y. Partial actions of a group G on Y by order isomorphisms between order ideals correspond to premoprhisms $G \to \Sigma(Y)$.

We now recall the variation of the McAlister structure result [10] on E-unitary inverse semigroups in terms of partial actions [6,12]. Suppose that a group G acts partially on a semilattice $Y = (Y, \wedge)$ by order isomorphisms between order ideals and that $\varphi\colon G \to \Sigma(Y)$ is the associated premorphism. On the set

$$Y \rtimes_{\varphi} G = \{(e, g) \in Y \times G\colon e \in \operatorname{ran} \varphi_g\}$$

define the following operations:

$$(e, g)(f, h) = (\varphi_g(\varphi_{g^{-1}}(e) \wedge f), gh),\ (e, g)^{-1} = (\varphi_{g^{-1}}(e), g^{-1}).$$

When φ is understood, we suppress the index φ and denote $Y \rtimes_{\varphi} G$ by $Y \rtimes G$. Then $Y \rtimes G$ is an inverse semigroup with $Y \simeq E(Y \rtimes G)$ via the map $y \mapsto (y, 1)$. The natural partial order on it is given by $(e, g) \le (f, h)$ if and only if $g = h$ and $e \le f$. For $(e, g), (f, h) \in Y \rtimes G$ we have $(e, g)\ \sigma\ (f, h)$ if and only if $g = h$, so that $Y \rtimes G$ is E-unitary and $(Y \rtimes G)/\sigma \simeq G$ via the map $(y, g) \mapsto g$. It is easy to see that $Y \rtimes G$ is a monoid if and only if Y has a top element, 1_Y, in which case the identity element of $Y \rtimes G$ is $(1_Y, 1)$. Furthermore, for each E-unitary inverse semigroup S we have that $S \simeq E(S) \rtimes_{\varphi} S/\sigma$ via the map $s \mapsto (ss^{-1}, [s]_\sigma)$ where φ is the *underlying premorphism* of S, defined by setting, for all $g \in S/\sigma$,

$$\operatorname{dom} \varphi_g = \{e \in E(S)\colon \text{there exists } s \in S \text{ with } [s]_\sigma = g \text{ such that } e \le s^{-1}s\},$$

$\varphi_g(e) = ses^{-1}$, where $e \in \operatorname{dom} \varphi_g$ and s is such that $[s]_\sigma = g$ and $e \le s^{-1}s$, where $[s]_\sigma$ denotes the σ-class of s.

2.4. Cayley Graphs of Groups

The *Cayley graph* $\operatorname{Cay}(G, X)$ of an X-generated group G is defined as the oriented graph $\mathrm{V} \sqcup \mathrm{E}^+ \sqcup \mathrm{E}^-$, where $\mathrm{V} = G$ is the set of vertices, $\mathrm{E}^+ = G \times X$ is the set of *positive edges* and $\mathrm{E}^- = G \times X^{-1}$ is the set of *negative edges*. We set $\mathrm{E} = \mathrm{E}^+ \sqcup \mathrm{E}^-$.

For convenience, we will denote an edge (g, x) by $(g, x, g[x])$. We let $\alpha(g, x, g[x]) = g$, $\omega(g, x, g[x]) = g[x]$ and $l(g, x, g[x]) = x$ be the *beginning*, the *end* and the *label* of the edge $(g, x, g[x])$. There is the involution $^{-1}\colon \mathrm{E} \to \mathrm{E}$, defined by $(g, x, g[x])^{-1} = (g[x], x^{-1}, g)$. The edge $(g[x], x^{-1}, g)$ should be thought of as 'the same edge' as $(g, x, g[x])$ but 'traversed in the opposite direction'.

A *non-empty path* in $\mathrm{Cay}(G, X)$ is a sequence $e_1 e_2 \cdots e_n$ $(n \geq 1)$ of edges, for which $\omega(e_i) = \alpha(e_{i+1})$ for all $i \in \{1, \dots, n-1\}$. For the path $p = e_1 \cdots e_n$ we set $\alpha(p) = \alpha(e_1)$ and $\omega(p) = \omega(e_n)$. The *inverse path* of p is the path $p^{-1} = e_n^{-1} \cdots e_1^{-1}$. The *empty path* at a vertex g is denoted by ε_g and we set $\alpha(\varepsilon_g) = \omega(\varepsilon_g) = g$. Two paths, p in q, in $\mathrm{Cay}(G, X)$ are called *coterminal* if $\alpha(p) = \alpha(q)$ and $\omega(p) = \omega(q)$. The *label* of the path $p = e_1 \cdots e_n$, where $n \geq 1$, is defined by $l(p) = l(e_1) \cdots l(e_n) \in (X \cup X^{-1})^+$ while $l(\varepsilon_g) = 1$ for all $g \in G$. A *subgraph* Γ of $\mathrm{Cay}(G, X)$ is a subset $\Gamma \subseteq \mathrm{Cay}(G, X)$, which is closed under α, ω and $^{-1}$. By $\mathrm{V}(\Gamma)$ and $\mathrm{E}(\Gamma)$ we denote the sets of vertices and edges of the subgraph Γ. Any subset $P \subseteq \mathrm{Cay}(G, X)$ yields a unique subgraph of $\mathrm{Cay}(G, X)$ called the *subgraph spanned by* P and denoted by $\langle P \rangle$. If p is a path in $\mathrm{Cay}(G, X)$, then the graph $\langle p \rangle$ spanned by p is defined as the graph spanned by the edges of p. Note that if P is finite, so is $\langle P \rangle$. A subgraph Γ is *connected*, if for any two vertices $u, v \in \Gamma$ there exists a path p in Γ which begins in u and ends in v.

2.5. The Universal Inverse Monoid $M(G, X)$ and F-inverse Monoid $F(G, X)$ of an X-Generated Group G

Let G be an X-generated group and $\mathrm{Cay}(G, X)$ its Cayley graph. We introduce the following notation:

- $\mathscr{X}_X$ – the set of all finite connected subgraphs of $\mathrm{Cay}(G, X)$ which contain the origin,
- $\tilde{\mathscr{X}}_X$ – the set of all finite (and not necessarily connected) subgraphs of $\mathrm{Cay}(G, X)$ which contain the origin.

These are semilattices with $A \leq B$ if and only if $A \supseteq B$; their top element is the graph Γ_1 with only one vertex, 1, and no edges. We put:

$$M(G, X) = \{(\Gamma, g) \colon \Gamma \in \mathscr{X}_X \text{ and } g \in \mathrm{V}(\Gamma)\},$$

$$F(G, X) = \{(\Gamma, g) \colon \Gamma \in \tilde{\mathscr{X}}_X \text{ and } g \in \mathrm{V}(\Gamma)\}$$

and define the operations on $M(G, X)$ and $F(G, X)$ by

$$(A, g)(B, h) = (A \cup gB, gh), \ \ (A, g)^{-1} = (g^{-1}A, g^{-1}).$$

Then $M(G, X)$ is an E-unitary inverse monoid called the *Margolis-Meakin expansion* of the X-generated group G [9], and $F(G, X)$ is the F-inverse monoid $F(G)$ introduced by Auinger, Szendrei and the first-named author in [2]. In the following proposition we collect some properties of $M(G, X)$ and $F(G, X)$.

Proposition 2.1 [2,9]. *Let G be an X-generated group.*

(1) *The identity element of each of $M(G, X)$ and $F(G, X)$ is $(\Gamma_1, 1)$.*
(2) *$M(G, X)$ is an X-generated inverse monoid and $F(G, X)$ is an X-generated F-inverse monoid via the assignment map $x \mapsto (\Gamma_x, [x])$,*

where Γ_x *is the graph with two vertices,* 1 *and* $[x]$*, and the positive edge* $(1, x, [x])$*.*

(3) *The natural partial order on each of* $M(G,X)$ *and* $F(G,X)$ *is given by* $(A, g) \leq (B, h)$ *if and only if* $g = h$ *and* $B \subseteq A$*.*

(4) *Let* $(A, g), (B, h)$ *be elements of one of* $M(G,X)$ *or* $F(G,X)$*. Then* $(A, g)\ \sigma\ (B, h)$ *if and only if* $g = h$*, which implies that* $M(G,X)/\sigma \simeq G$ *and* $F(G,X)/\sigma \simeq G$ *via the canonical morphism* $(\Gamma, g) \mapsto g$*.*

(5) *In* $F(G,X)$ *the maximum element of the* σ*-class of* (A, g) *is* $(\{1, g\}, g)$ *where* $\{1, g\}$ *is the graph with vertices* $1, g$ *and no edges.*

(6)
- $M(G,X) = \mathcal{X}_X \rtimes G$*, where the underlying premorphisms* $\varphi\colon G \to \Sigma(\mathcal{X}_X)$ *is given, for each* $g \in G$*, by* $\operatorname{dom}\varphi_g = \{\Gamma \in \mathcal{X}_X : g^{-1} \in V(\Gamma)\}$ *and* $\varphi_g(\Gamma) = g\Gamma$ *for all* $\Gamma \in \operatorname{dom}\varphi_g$*.*
- $F(G,X) = \tilde{\mathcal{X}}_X \rtimes G$*, where the underlying premorphism* $\tilde{\varphi}\colon G \to \Sigma(\tilde{\mathcal{X}}_X)$ *is given similarly as above for* $M(G,X)$*, with* $\mathcal{X}_X$ *replaced by* $\tilde{\mathcal{X}}_X$*.*

(7) *(Universal properties of* $M(G,X)$ *and* $F(G,X)$*) Let* S *be an* X*-generated* E*-unitary inverse monoid (respectively, an* X*-generated* F*-inverse monoid) such that there is a canonical morphism* $\nu\colon G \to S/\sigma$*. Then there is a canonical morphism* $\varphi\colon M(G,X) \to S$ *(respectively,* $F(G,X) \to S$*) such that the following diagram of canonical morphisms of* X*-generated inverse monoids (respectively, of* X*-generated* F*-inverse monoids) commutes:*

$$\begin{array}{ccc} U(G,X) & \xrightarrow{\ \varphi\ } & S \\ \downarrow & & \downarrow \\ G & \xrightarrow{\ \nu\ } & S/\sigma \end{array}$$

where $U(G,X)$ *is one of* $M(G,X)$ *or* $F(G,X)$*.*

(8) *Let* p *be a path from* 1 *to* g *in* $\operatorname{Cay}(G,X)$*. Then* $[l(p)]_{M(G,X)} = (\langle p\rangle, g)$*.*

(9) *Let* p *and* q *be paths in* $\operatorname{Cay}(G,X)$*. Then* $[l(p)]_{M(G,X)}\ \sigma\ [l(q)]_{M(G,X)}$ *if and only if* $[l(p)]_G = [l(q)]_G$*.*

We remark that the results in (6) above are not explicitly stated in [2,9], but easily follow from the results therein and are known.

3. Dual-Closure Operators on Semilattices and Quotients of Partial Action Products

Recall that a *dual-closure operator* (or an *interior operator*) on a poset $(\mathcal{P}, \leq)$ is a map $j\colon \mathcal{P} \to \mathcal{P}$, which satisfies the following conditions, for all $x, y \in \mathcal{P}$:

(Cl1) $j(x) \leq x$,
(Cl2) if $x \leq y$ then $j(x) \leq j(y)$,
(Cl3) $j(j(x)) = j(x)$.

From now on let X be a semilattice and $j\colon X \to X$ a dual-closure operator on it.

Lemma 3.1. *For all $x, y \in X$ we have $j(j(x) \wedge j(y)) = j(x \wedge y)$.*

Proof. Since $j(x) \leq x$ and $j(y) \leq y$ by (Cl1), we have $j(x) \wedge j(y) \leq x \wedge y$. From (Cl2), it follows that $j(j(x) \wedge j(y)) \leq j(x \wedge y)$. For the opposite inequality observe that, since $x \wedge y \leq x$, we have $j(x \wedge y) \leq j(x)$, by (Cl2), and similarly $j(x \wedge y) \leq j(y)$. Hence $j(x \wedge y) \leq j(x) \wedge j(y)$. Applying (Cl2), we write $j(j(x \wedge y)) \leq j(j(x) \wedge j(y))$ which, in view of (Cl3), yields $j(x \wedge y) \leq j(j(x) \wedge j(y))$, as required. □

We define the equivalence relation ρ_j on X by

$$x \; \rho_j \; y \iff j(x) = j(y).$$

It is easy to see that ρ_j is a congruence on X. It will be convenient to identify the quotient semilattice X/ρ_j with the semilattice $(j(X), \bar{\wedge})$ with the operation $j(x) \,\bar{\wedge}\, j(y) = j(x \wedge y)$.

Let G be a group acting partially on the semilattice X by order isomorphisms between order ideals. We say that a dual-closure operator j on X is *G-invariant* provided that if $g \cdot x$ is defined then $j(g \cdot x) = g \cdot j(x)$, for all $g \in G$ and $x \in X$. If j is a G-invariant dual-closure operator on X, define the relation $\tilde{\rho}_j$ on the inverse semigroup $X \rtimes G$ by

$$(e, g) \; \tilde{\rho}_j \; (f, h) \iff g = h \text{ and } j(e) = j(f). \tag{3.1}$$

Proposition 3.2. *The relation $\tilde{\rho}_j$ is a congruence on $X \rtimes G$, which is contained in σ. Moreover $(X \rtimes G)/\tilde{\rho}_j \simeq j(X) \rtimes G$.*

Proof. Obviously, $\tilde{\rho}_j$ is an equivalence relation, contained in σ. Suppose that $(e, g) \; \tilde{\rho}_j \; (f, h)$. Then $g = h$ and $j(e) = j(f)$. Let $(d, s) \in X \rtimes G$ and show that $(e, g)(d, s) \; \tilde{\rho}_j \; (f, h)(d, s)$ or, equivalently, $(\varphi_g(\varphi_{g^{-1}}(e) \wedge d), gs) \; \tilde{\rho}_j \; (\varphi_g(\varphi_{g^{-1}}(f) \wedge d), gs)$. It suffices to show that $j(\varphi_g(\varphi_{g^{-1}}(e) \wedge d)) = j(\varphi_g(\varphi_{g^{-1}}(f) \wedge d))$. The left-hand side rewrites to

$$\varphi_g(j(\varphi_{g^{-1}}(e) \wedge d)) = \varphi_g(j(\varphi_{g^{-1}}(e)) \bar{\wedge} j(d)) = \varphi_g(j(\varphi_{g^{-1}}(j(e)) \wedge j(d)))$$

and, similarly, the right-hand side to $\varphi_g(j(\varphi_{g^{-1}}(j(f)) \wedge j(d)))$. Since $j(e) = j(f)$, the two expressions coincide. Likewise, one shows that $(d, s)(e, g) \; \tilde{\rho}_j \; (d, s)(f, h)$. The map $(X \rtimes G)/\tilde{\rho}_j \to j(X) \rtimes G$ given by $[(e, g)]_{\tilde{\rho}_j} \mapsto (j(e), g)$, is obviously well defined, and it is routine to check that it is an isomorphism of semigroups. □

4. A New Approach to the Universal F-Inverse Monoid $F(G, X)$

4.1. The Inverse Monoid $M(G, Y)$.

Let X be a nonempty set and G an X-generated group. In what follows, we will need to consider G also with respect to another generating set, so to distinguish between the assignment maps for different generating sets, we will denote the assignment map $X \to G$ by $\iota_{G,X}$. Recall that we abbreviate $\iota_{G,X}(x)$ by $[x]$. Let, further, $\overline{G}$ be a disjoint copy of G, and we fix the bijection $g \mapsto \overline{g}$ between G and $\overline{G}$.

We will consider the group G also with respect to the 'extended' generating set $Y = X \cup \overline{G}$ via the asisgnment map $\iota_{G,Y} \colon Y \to G$ given by

$$\begin{aligned} x &\mapsto [x], \quad \text{if } x \in X, \\ \overline{g} &\mapsto g, \quad \text{if } \overline{g} \in \overline{G}. \end{aligned} \tag{4.1}$$

In particular, for all $x \in X$ we have $\iota_{G,Y}(x) = \iota_{G,X}(x) = [x]$.

For $\overline{g} \in \overline{G}$ we define $\Gamma_{\overline{g}}$ to be the graph with two vertices, 1 and g, and one positive edge $(1, \overline{g}, g)$. Then the inverse monoid $M(G,Y) = \mathcal{X}_Y \rtimes G$ is Y-generated via the map $\iota_{M(G,Y)} \colon Y \to M(G,Y)$ given by

$$\begin{aligned} x &\mapsto (\Gamma_x, [x]), \quad \text{if } x \in X, \\ \overline{g} &\mapsto (\Gamma_{\overline{g}}, g), \quad \text{if } \overline{g} \in \overline{G}, \end{aligned} \tag{4.2}$$

and its identity element is $(\Gamma_1, 1)$.

4.2. The F-Inverse Monoid $M^\wedge(G,Y)$

We call a subgraph $\Gamma \in \mathcal{X}_Y$ *closed* provided that it satisfies the following condition:

(C) If $a, b \in G$ are such that $a, b \in \mathrm{V}(\Gamma)$, then $(a, \overline{a^{-1}b}, b) \in \mathrm{E}(\Gamma)$.

For $\Gamma \in \mathcal{X}_Y$ we put $\Gamma^\wedge$ to be smallest closed graph in $\mathcal{X}_Y$ which contains Γ. It is clearly well defined and we have $\mathrm{V}(\Gamma^\wedge) = \mathrm{V}(\Gamma)$ and $\mathrm{E}^+(\Gamma^\wedge) = \mathrm{E}^+(\Gamma) \cup \{(a, \overline{a^{-1}b}, b) \colon a, b \in \mathrm{V}(\Gamma)\}$. It is easy to see that $j \colon \mathcal{X}_Y \to \mathcal{X}_Y$, $\Gamma \mapsto \Gamma^\wedge$, is a G-invariant dual-closure operator. We put $\mathcal{X}_Y^\wedge = j(\mathcal{X}_Y)$.

The congruence $\tilde{\rho}_j$ of (3.1) on $M(G,Y) = \mathcal{X}_Y \rtimes G$ is given by

$$(A, g) \ \tilde{\rho}_j \ (B, h) \iff g = h \text{ and } A^\wedge = B^\wedge. \tag{4.3}$$

By Proposition 3.2 the quotient inverse monoid $M(G,Y)/\tilde{\rho}_j$ is isomorphic to $\mathcal{X}_Y^\wedge \rtimes G$, which we denote by $M^\wedge(G,Y)$. This is a Y-generated inverse monoid via the assignment map

$$\begin{aligned} x &\mapsto (\Gamma_x^\wedge, [x]), \quad \text{if } x \in X, \\ \overline{g} &\mapsto (\Gamma_{\overline{g}}^\wedge, g), \quad \text{if } \overline{g} \in \overline{G}, \end{aligned} \tag{4.4}$$

and its identity element is $(\Gamma_1^\wedge, 1) = (\Gamma_{\overline{1}}, 1)$. The operations on it are given by

$$(A, g)(B, h) = ((A \cup gB)^\wedge, gh), \tag{4.5}$$

$$(A, g)^{-1} = (g^{-1}A, g^{-1}). \tag{4.6}$$

Remark 4.1. It is not hard to show that the congruence $\tilde{\rho}_j$ is generated by the relations $\overline{[x]} \geq x$, $\overline{gh} \geq \overline{g}\,\overline{h}$ and $\overline{g^{-1}} = \overline{g}^{-1}$, where $x \in X$, $g, h \in G$, but we will not use this fact in our arguments.

Proposition 4.2. *$M^\wedge(G,Y)$ is an X-generated (in the enriched signature $(\cdot,{}^{-1},m,1)$) F-inverse monoid with $m(\Gamma,g) = (\Gamma_g^\wedge, g)$, for all $(\Gamma,g) \in M^\wedge(G,Y)$.*

Proof. The second claim is clear by the description of σ and the fact that $\Gamma_g^\wedge$ is minimum among all the graphs in $\mathcal{X}_Y^\wedge$, which have 1 and g as vertices. For the first claim, it suffices to show that each $(\Gamma_g^\wedge, g)$, where g runs through $G \setminus \{1\}$, can be written via the generators $(\Gamma_x^\wedge, [x])$ in the enriched signature. We write $g = [x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}]$, where $n \geq 1$, all $x_i \in X$ and $\varepsilon_i \in \{1,-1\}$, then $(\Gamma_g^\wedge, g) = m((\Gamma_{x_1}^\wedge, [x_1])^{\varepsilon_1} \cdots (\Gamma_{x_n}^\wedge, [x_n])^{\varepsilon_n})$. □

Proposition 4.3. *The X-generated F-inverse monoids $M^\wedge(G,Y)$ and $F(G,X)$ are canonically isomorphic.*

Proof. Define the map $\mathbf{f}\colon \mathcal{X}_Y^\wedge \to \tilde{\mathcal{X}}_X$ by $\mathbf{f}(\Gamma) = \Gamma \cap \mathrm{Cay}(G,X)$ where $\Gamma \cap \mathrm{Cay}(G,X)$ is the graph obtained from Γ by erasing all its edges labeled by $\overline{G}$ and their inverse edges labeled by $\overline{G}^{-1}$. This map is injective as Γ can be reconstructed from $\Gamma \cap \mathrm{Cay}(G,X)$ by adding to the latter all the edges of $\mathrm{Cay}(G,Y)$ labeled by $\overline{G}$ and $\overline{G}^{-1}$ between its vertices. It is clearly surjective. Since, in addition,

$$(\Gamma_1 \cap \mathrm{Cay}(G,X)) \cup (\Gamma_2 \cap \mathrm{Cay}(G,X)) = (\Gamma_1 \cup \Gamma_2)^\wedge \cap \mathrm{Cay}(G,X),$$

for all $\Gamma_1, \Gamma_2 \in \mathcal{X}_Y^\wedge$, it is an isomorphism of semilattices. It is immediate that $\mathbf{f}$ respects the partial action of G, that is, for all $g \in G$ and $\Gamma \in \mathcal{X}_Y^\wedge$ we have that $\varphi_g(\Gamma)$ is defined if and only if so is $\tilde{\varphi}_g(\mathbf{f}(\Gamma))$, in which case we have $\mathbf{f}(\varphi_g(\Gamma)) = \tilde{\varphi}_g(\mathbf{f}(\Gamma))$. Here $\varphi\colon G \to \Sigma(\mathcal{X}_Y^\wedge)$ and $\tilde{\varphi}\colon G \to \Sigma(\tilde{\mathcal{X}}_X)$ are the underlying premorphisms of $M^\wedge(G,Y)$ and $F(G,X)$ (see part (6) of Proposition 2.1). It now easily follows that the map $M^\wedge(G,Y) = \mathcal{X}_Y^\wedge \rtimes G \to \tilde{\mathcal{X}}_X \rtimes G = F(G,X)$, given by $(\Gamma, g) \mapsto (\mathbf{f}(\Gamma), g)$, is an isomorphism of F-inverse monoids. That it is canonical is immediate by the construction. □

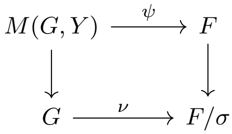
We now prove the universal property of $M^\wedge(G,Y)$.

Theorem 4.4. *For any X-generated group G and any X-generated F-inverse monoid F (in the signature $(\cdot,{}^{-1},m,1)$) such that there is a canonical morphism $\nu\colon G \to F/\sigma$, there is a canonical morphism $\varphi\colon M^\wedge(G,Y) \to F$ such that the diagram of canonical morphisms of X-generated F-inverse monoids*

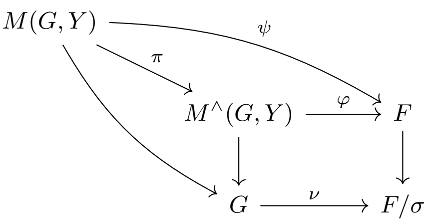
$$\begin{array}{ccc} M^\wedge(G,Y) & \xrightarrow{\ \varphi\ } & F \\ \downarrow & & \downarrow \\ G & \xrightarrow{\ \nu\ } & F/\sigma \end{array}$$

commutes.

Proof. Since in this proof, for some algebras A (which are groups, inverse monoids or F-inverse monoids) we work with two generating sets, X and Y, we denote the corresponding assignment maps by $\iota_{A,X}$ and $\iota_{A,Y}$, respectively. Because F is an X-generated F-inverse monoid, it is an $(X \cup \{m(s)\colon s \in F\})$-generated inverse monoid (this is easy to show and known, see [2, Section 3]). Let $\tau_F\colon F/\sigma \to F$ be the map which assigns to each $f \in F/\sigma$ the maximum element $\tau_F(f)$ of the σ-class of F which projects onto f. Then F is a Y-generated inverse monoid via the assignment map $\iota_{F,Y}\colon Y \to F$, such that $\iota_{F,Y}$ coincides with $\iota_{F,X}$ on X and $\iota_{F,Y}(\overline{g}) = \tau_F\nu(g)$ for $\overline{g} \in \overline{G}$. By the universal property of $M(G,Y)$ (see part (7) of Proposition 2.1), there is a canonical morphism of Y-generated inverse monoids $\psi\colon M(G,Y) \to F$, such that the following diagram of canonical morphisms of Y-generated inverse monoids commutes:



We show that $\psi\colon M(G,Y) \to F$ factors through the canonical quotient map $\pi\colon M(G,Y) \to M^\wedge(G,Y)$, as is illustrated below:



In view of (4.3) and since $M^\wedge(G,Y) \simeq M(G,Y)/\tilde{\rho}_j$, it suffices to show that $\psi(A,g) = \psi(A^\wedge,g)$ for all $(A,g) \in M(G,Y)$. Since the graph $A^\wedge$ is obtained from the graph A by adding to it finitely many edges, there is a finite sequence $A = A_0, A_1, \ldots, A_n = A^\wedge$ of graphs in $\mathcal{X}_Y$ such that, for each $i = 0, \ldots, n-1$, the graph A_{i+1} is obtained from the graph A_i by adding to it a single positive edge $(a, \overline{g}, ag)$ (and also its inverse negative edge). It thus suffices to prove that $\psi(B,g) = \psi(C,g)$ where the graph C is obtained from the graph B by adding to it a single positive edge $e = (a, \overline{g}, ag)$ (and also its inverse negative edge) between $a, ag \in \mathrm{V}(B)$. Since B is connected, there is a path, p, in B with $\alpha(p) = \alpha(e)$ and $\omega(p) = \omega(e)$. Let $w'ew''$ be a spanning path in C from the origin to g. Then the path $w = w'pw''$ spans B and the path $\tilde{w} = w'ep^{-1}pw''$ spans C, moreover, w and $\tilde{w}$ are coterminal from the origin to g. Let $s, t, u \in (Y \cup Y^{-1})^*$ be the labels of w', w'' and p, respectively. Then $l = sut$ and $\tilde{l} = s\overline{g}u^{-1}ut$ are the labels of w and $\tilde{w}$, respectively.

Since (B,g) (respectively, (C,g)) equals the value in $M(G,Y)$ of the label of any path in $\mathrm{Cay}(G,Y)$ which spans B (respectively, C) from the origin to g (by part (8) of Proposition 2.1), we have that $\psi(B,g) = [l]_F$ and

$\psi(C,g) = [\tilde{l}]_F$ (the evaluations are taken in the Y-generated inverse monoid F). We then have $\psi(B,g) = [s]_F[u]_F[t]_F$ and $\psi(C,g) = [s]_F[\overline{g}]_F[u]_F^{-1}[u]_F[t]_F$. But $[u]_{M(G,Y)}\ \sigma\ [\overline{g}]_{M(G,Y)}$ as p and e are coterminal (by part (9) of Proposition 2.1), moreover, $[u]_F \leq [\overline{g}]_F$ as $[\overline{g}]_F = \tau_F\nu(g)$ is the maximum element in its σ-class. It follows that $[\overline{g}]_F[u]_F^{-1}[u]_F = [u]_F$, which implies the desired equality $\psi(B,g) = \psi(C,g)$. Therefore, there is a well defined canonical morphism of Y-generated inverse monoids $\varphi\colon M^\wedge(G,Y) \to F$ such that $\varphi\pi = \psi$. Since $[\overline{g}]_F = \varphi(\Gamma^\wedge_{\overline{g}}, g)$, for all $g \in G$, it follows that φ preserves the m-operation, and is thus a canonical morphism of X-generated F-inverse monoids. □

Theorem 4.4 and Propositions 4.2 and 4.3 provide a new proof of the universal property of $F(G,X)$.

Remark 4.5. Combining our results with those of [14], one can show that any X-generated F-inverse monoid F (in the enriched signature $(\cdot, ^{-1}, m, 1)$), looked at as a Y-generated inverse monoid (as in the proof of Theorem 4.4), arises as a canonical quotient of $M(G,Y)$, where $G = F/\sigma$, and generators from $\overline{G}$ are mapped onto respective maximal elements of σ-classes of F. This suggests that presentations of F-inverse monoids in enriched signature can be studied by the usual tools (Stephen's procedure [13]) developed for inverse monoids.

Acknowledgements

We thank the referee for useful comments.

Funding The authors were supported by the Slovenian Research and Innovation Agency grant P1-0288.

Data Availability Our manuscript does not have any associated data.

Declarations

Conflict of interests The authors have no relevant financial or non-financial interests to disclose.

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Received: February 28, 2024.
Accepted: September 27, 2024.