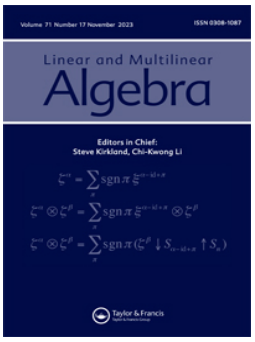

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Inequalities and equalities on the joint and generalized spectral and essential spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators

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ABSTRACT
We prove new inequalities and equalities for the generalized and the joint spectral radius (and their essential versions) of Hadamard (Schur) geometric means of bounded sets of positive kernel operators on Banach function spaces. In the case of non-negative matrices that define operators on Banach sequences, we obtain additional results. Our results extend the results of several authors that appeared relatively recently.

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1. Introduction

In [1], X. Zhan conjectured that, for non-negative $N \times N$ matrices A and B, the spectral radius $\rho(A \circ B)$ of the Hadamard product satisfies

$$\rho(A \circ B) \leq \rho(AB), \tag{1}$$

where AB denotes the usual matrix product of A and B. This conjecture was confirmed by K.M.R. Audenaert in [2] by proving

$$\rho(A \circ B) \leq \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \leq \rho(AB). \tag{2}$$

These inequalities were established via a trace description of the spectral radius. Soon after, inequality (1) was reproved, generalized and refined in different ways by several

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authors ([3–12]). Using the fact that the Hadamard product is a principal submatrix of the Kronecker product, R.A. Horn and F. Zhang proved in [6], the inequalities

$$\rho(A \circ B) \le \rho(AB \circ BA)^{\frac{1}{2}} \le \rho(AB). \tag{3}$$

Applying the techniques of [6], Z. Huang proved that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \le \rho(A_1 A_2 \cdots A_m) \tag{4}$$

for $n \times n$ non-negative matrices $A_1, A_2, \ldots, A_m$ (see [7]). A.R. Schep was the first one to observe that the results from [13,14] are applicable in this context (see [11,12]). He extended inequalities (2) and (3) to non-negative matrices that define bounded operators on sequence spaces (in particular on l^p spaces, $1 \le p < \infty$) and proved in [11, Theorem 2.7] that

$$\rho(A \circ B) \le \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \le \rho(AB \circ AB)^{\frac{1}{2}} \le \rho(AB) \tag{5}$$

(note that there was an error in the statement of [11, Theorem 2.7], which was corrected in [8,12]). In [8], the second author of the current paper extended the inequality (4) to non-negative matrices that define bounded operators on Banach sequence spaces (see below for the exact definitions) and proved that the inequalities

$$\rho(A \circ B) \le \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \le \rho(AB \circ AB)^{\frac{\beta}{2}} \rho(BA \circ BA)^{\frac{1-\beta}{2}} \le \rho(AB) \tag{6}$$

and

$$\rho(A \circ B) \le \rho(AB \circ BA)^{\frac{1}{2}} \le \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB). \tag{7}$$

hold, where $\beta \in [0, 1]$. Moreover, he generalized these inequalities to the setting of the generalized and the joint spectral radius of bounded sets of such non-negative matrices.

In [11, Theorem 2.8], A.R. Schep proved that the inequality

$$\rho\left(A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}\right) \le \rho(AB)^{\frac{1}{2}} \tag{8}$$

holds for positive kernel operators on L^p spaces. Here $A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}$ denotes the Hadamard geometric mean of operators A and B. In [5, Theorem 3.1], R. Drnovšek and the second author, generalized this inequality and proved that the inequality

$$\rho\left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)}\right) \le \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}} \tag{9}$$

holds for positive kernel operators $A_1, \ldots, A_m$ on an arbitrary Banach function space L. In [10], the second author refined (9) and showed that the inequalities

$$\rho\left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)}\right)$$

$$\le \rho\left(P_1^{\left(\frac{1}{m}\right)} \circ P_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ P_m^{\left(\frac{1}{m}\right)}\right)^{\frac{1}{m}} \le \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}}. \tag{10}$$

hold, where $P_j = A_j \ldots A_m A_1 \ldots A_{j-1}$ for $j = 1, \ldots, m$. In [15, Theorem 3.2], the second author showed that (10) holds also for the essential radius ρ_{ess} under the additional condition that L and its Banach dual L^* have to order continuous norms. Formally, here and throughout the article $A_{j-1} = I$ for $j = 1$ (eventhough I might not be a well-defined kernel operator). In particular, the following kernel version of (3) holds:

$$\rho\left(A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\right) \le \rho\left((AB)^{(\frac{1}{2})} \circ (BA)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \le \rho(AB)^{\frac{1}{2}}. \tag{11}$$

Several additional closely related results, generalizations and refinements of the above results were obtained in [3,15–17].

In [9, Theorem 3.4] and [15, Theorem 3.5], the second author generalized inequalities (9) and (11) and their essential version to the setting of the generalized and the joint spectral radius (and their essential versions) of bounded sets of positive kernel operators on a Banach function space (see also Theorems 2.3 and 2.4).

The rest of the article is organized in the following way. In Section 2, we recall definitions and results that we will use in our proofs. In Section 3, we extend the main results of [15] by proving new inequalities and equalities for the generalized and the joint spectral radius (and their essential versions) of Hadamard (Schur) geometric means of bounded sets of positive kernel operators on Banach function spaces (Theorems 3.1(i), 3.2(i), 3.3, 3.4 and 3.6(i)). In the case of non-negative matrices that define operators on Banach sequences we prove further new inequalities that extend the main results of [8] (Theorems 3.2(ii), 3.5 and 3.6(ii)). All the inequalities mentioned above are very special instances of our results. In Section 4, we prove new results on geometric symmetrization of bounded sets of positive kernel operators on $L^2(X, \mu)$ and on weighted geometric symmetrization of bounded sets of non-negative matrices that define operators on $l^2(R)$, which extend some results from [3,16,18].

2. Preliminaries

Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ of subsets of a non-void set X. Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X. A Banach space $L \subseteq M(X, \mu)$ is called a *Banach function space* if $f \in L, g \in M(X, \mu)$, and $|g| \le |f|$ imply that $g \in L$ and $\|g\| \le \|f\|$. Throughout the article, it is assumed that X is the carrier of L, that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$ (see [19]).

Let R denote the set $\{1, \dots, N\}$ for some $N \in \mathbb{N}$ or the set $\mathbb{N}$ of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_n)_{n\in R}$. A Banach space $L \subseteq S(R)$ is called a *Banach sequence space* if $x \in S(R)$, $y \in L$ and $|x| \le |y|$ imply that $x \in L$ and $\|x\|_L \le \|y\|_L$. Observe that a Banach sequence space is a Banach function space over a measure space (R, μ), where μ denotes the counting measure on R. Denote by $\mathcal{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_n = \chi_{\{n\}} \in L$ and $\|e_n\|_L = 1$ for all $n \in R$. For $L \in \mathcal{L}$ the set R is the carrier of L.

Standard examples of Banach sequence spaces are Euclidean spaces, l^p spaces for $1 \le p \le \infty$, the space $c_0 \in \mathcal{L}$ of all null convergent sequences (equipped with the usual norms and the counting measure), while standard examples of Banach function spaces are the well-known spaces $L^p(X, \mu)$ ($1 \le p \le \infty$) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. [20–22] and the references cited there), which are important, e.g. in interpolation theory and in the theory of partial differential equations. Recall that the cartesian product $L = E \times F$

of Banach function spaces is again a Banach function space, equipped with the norm $\|(f,g)\|_L = \max\{\|f\|_E, \|g\|_F\}$.

If $\{f_n\}_{n\in\mathbb{N}} \subset M(X,\mu)$ is a decreasing sequence and $f = \inf\{f_n \in M(X,\mu) : n \in \mathbb{N}\}$, then we write $f_n \downarrow f$. A Banach function space L has an *order continuous norm*, if $0 \le f_n \downarrow 0$ implies $\|f_n\|_L \to 0$ as $n \to \infty$. It is well known that spaces $L^p(X,\mu)$, $1 \le p < \infty$, have order continuous norm. Moreover, the norm of any reflexive Banach function space is order continuous. In particular, we will be interested in Banach function spaces L such that L and its Banach dual space L^* have order continuous norms. Examples of such spaces are $L^p(X,\mu)$, $1 < p < \infty$, while the space $L = c_0$ is an example of a non-reflexive Banach sequence space, such that L and $L^* = l^1$ have order continuous norms.

By an *operator* on a Banach function space L, we always mean a linear operator on L. An operator A on L is said to be *positive* if it maps non-negative functions to non-negative ones, i.e. $AL_+ \subset L_+$, where L_+ denotes the positive cone $L_+ = \{f \in L : f \ge 0 \text{ a.e.}\}$. Given operators A and B on L, we write $A \ge B$ if the operator $A-B$ is positive.

Recall that a positive operator A is always bounded, i.e. its operator norm

$$\|A\| = \sup\{\|Ax\|_L : x \in L, \|x\|_L \le 1\} = \sup\{\|Ax\|_L : x \in L_+, \|x\|_L \le 1\} \tag{12}$$

is finite. Also, its spectral radius $\rho(A)$ is always contained in the spectrum.

An operator A on a Banach function space L is called a *kernel operator* if there exists a $\mu \times \mu$-measurable function $a(x,y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X |a(x,y)f(y)|\, d\mu(y) < \infty \quad \text{and} \quad (Af)(x) = \int_X a(x,y)f(y)\, d\mu(y).$$

One can check that a kernel operator A is positive iff its kernel a is non-negative almost everywhere.

Let L be a Banach function space such that L and L^* have order continuous norms and let A and B be positive kernel operators on L. By $\gamma(A)$ we denote the Hausdorff measure of non-compactness of A, i.e.

$$\gamma(A) = \inf\{\delta > 0 : \text{ there is a finite } M \subset L \text{ such that } A(D_L) \subset M + \delta D_L\},$$

where $D_L = \{f \in L : \|f\|_L \le 1\}$. Then $\gamma(A) \le \|A\|$, $\gamma(A+B) \le \gamma(A) + \gamma(B)$, $\gamma(AB) \le \gamma(A)\gamma(B)$ and $\gamma(\alpha A) = \alpha\gamma(A)$ for $\alpha \ge 0$. Also $0 \le A \le B$ implies $\gamma(A) \le \gamma(B)$ (see e.g. [23, Corollary 4.3.7 and Corollary 3.7.3]). Let $\rho_{ess}(A)$ denote the essential spectral radius of A, i.e. the spectral radius of the Calkin image of A in the Calkin algebra. Then

$$\rho_{ess}(A) = \lim_{j\to\infty} \gamma(A^j)^{1/j} = \inf_{j\in\mathbb{N}} \gamma(A^j)^{1/j} \tag{13}$$

and $\rho_{ess}(A) \le \gamma(A)$. Recall that if $L = L^2(X,\mu)$, then $\gamma(A^*) = \gamma(A)$ and $\rho_{ess}(A^*) = \rho_{ess}(A)$, where A^* denotes the adjoint of A. Note that equalities (13) and $\rho_{ess}(A^*) = \rho_{ess}(A)$ are valid for any bounded operator A on a given complex Banach space L (see e.g. [23, Theorem 4.3.13 and Proposition 4.3.11]).

Observe that (finite or infinite) non-negative matrices, that define operators on Banach sequence spaces, are a special case of positive kernel operators (see e.g. [3,5,8,24,25], and the references cited there).

It is well known that kernel operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [26–29] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [19,20,23,30,31].

Let A and B be positive kernel operators on a Banach function space L with kernels a and b respectively, and $\alpha \geq 0$. The *Hadamard (or Schur) product* $A \circ B$ of A and B is the kernel operator with kernel equal to $a(x,y)b(x,y)$ at point $(x,y) \in X \times X$ which can be defined (in general) only on some order ideal of L. Similarly, the *Hadamard (or Schur) power* $A^{(\alpha)}$ of A is the kernel operator with kernel equal to $(a(x,y))^{\alpha}$ at point $(x,y) \in X \times X$ which can be defined only on some order ideal of L.

Let $A_1, \ldots, A_m$ be positive kernel operators on a Banach function space L, and $\alpha_1, \ldots, \alpha_m$ positive numbers such that $\sum_{j=1}^{m} \alpha_j = 1$. Then the *Hadamard weighted geometric mean* $A = A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}$ of the operators $A_1, \ldots, A_m$ is a positive kernel operator defined on the whole space L, since $A \leq \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_m A_m$ by the inequality between the weighted arithmetic and geometric means.

A matrix $A = [a_{ij}]_{i,j \in R}$ is called *non-negative* if $a_{ij} \geq 0$ for all $i, j \in R$. For notational convenience, we sometimes write $a(i,j)$ instead of a_{ij}.

We say that a non-negative matrix A defines an operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j \in R} a_{ij} x_j$. Then $Ax \in L_+$ for all $x \in L_+$ and so A defines a positive kernel operator on L.

Let us recall the following result which was proved in [13, Theorem 2.2] and [14, Theorem 5.1 and Example 3.7] (see also e.g. [9, Theorem 2.1]).

Theorem 2.1: *Let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be positive kernel operators on a Banach function space L and let α_1, α_2,..., α_m are positive numbers.*

(i) If $\sum_{j=1}^{m} \alpha_j = 1$, then the positive kernel operator

$$A := \left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \ldots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right) \tag{14}$$

satisfies the following inequalities

$$A \leq (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)}, \tag{15}$$

$$\begin{aligned} \|A\| &\leq \left\| (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)} \right\| \\ &\leq \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m} \end{aligned} \tag{16}$$

$$\begin{aligned} \rho(A) &\leq \rho\left((A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)} \right) \\ &\leq \rho(A_{11} \cdots A_{k1})^{\alpha_1} \cdots \rho(A_{1m} \cdots A_{km})^{\alpha_m}. \end{aligned} \tag{17}$$

If, in addition, L and L have order continuous norms, then*

$$\gamma(A) \le \gamma\left((A_{11}\cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m}\cdots A_{km})^{(\alpha_m)}\right)$$
$$\le \gamma(A_{11}\cdots A_{k1})^{\alpha_1} \cdots \gamma(A_{1m}\cdots A_{km})^{\alpha_m}, \quad (18)$$
$$\rho_{ess}(A) \le \rho_{ess}\left((A_{11}\cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m}\cdots A_{km})^{(\alpha_m)}\right)$$
$$\le \rho_{ess}(A_{11}\cdots A_{k1})^{\alpha_1} \cdots \rho_{ess}(A_{1m}\cdots A_{km})^{\alpha_m}. \quad (19)$$

(ii) *If $L \in \mathcal{L}$, $\sum_{j=1}^m \alpha_j \ge 1$ and $\{A_{ij}\}_{i=1,j=1}^{k,m}$ are non-negative matrices that define positive operators on L, then A from (14) defines a positive operator on L and the inequalities (15), (16) and (17) hold.*

The following result is a special case of Theorem 2.1.

Theorem 2.2: *Let $A_1, \dots, A_m$ be positive kernel operators on a Banach function space L and $\alpha_1, \dots, \alpha_m$ positive numbers.*

(i) *If $\sum_{j=1}^m \alpha_j = 1$, then*

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \le \|A_1\|^{\alpha_1}\|A_2\|^{\alpha_2}\cdots\|A_m\|^{\alpha_m} \quad (20)$$

and

$$\rho(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \le \rho(A_1)^{\alpha_1}\rho(A_2)^{\alpha_2}\cdots\rho(A_m)^{\alpha_m}. \quad (21)$$

If, in addition, L and L have order continuous norms, then*

$$\gamma(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \le \gamma(A_1)^{\alpha_1}\gamma(A_2)^{\alpha_2}\cdots\gamma(A_m)^{\alpha_m} \quad (22)$$

and

$$\rho_{ess}(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \le \rho_{ess}(A_1)^{\alpha_1}\rho_{ess}(A_2)^{\alpha_2}\cdots\rho_{ess}(A_m)^{\alpha_m}. \quad (23)$$

(ii) *If $L \in \mathcal{L}$, $\sum_{j=1}^m \alpha_j \ge 1$ and if $A_1, \dots, A_m$ are non-negative matrices that define positive operators on L, then $A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}$ defines a positive operator on L and (20) and (21) hold.*

(iii) *If $L \in \mathcal{L}$, $t \ge 1$ and if $A, A_1, \dots, A_m$ are non-negative matrices that define operators on L, then $A^{(t)}$ defines an operator on L and the following inequalities hold*

$$A_1^{(t)}\cdots A_m^{(t)} \le (A_1\cdots A_m)^{(t)}, \quad (24)$$
$$\rho(A_1^{(t)}\cdots A_m^{(t)}) \le \rho(A_1\cdots A_m)^t, \quad (25)$$
$$\|A_1^{(t)}\cdots A_m^{(t)}\| \le \|A_1\cdots A_m\|^t. \quad (26)$$

Let Σ be a bounded set of bounded operators on a complex Banach space L. For $m \geq 1$, let

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m} \tag{27}$$

and is equal to

$$\rho(\Sigma) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat{\rho}(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}. \tag{28}$$

Similarly, the generalized essential spectral radius of Σ is defined by

$$\rho_{ess}(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho_{ess}(A)]^{1/m} \tag{29}$$

and is equal to

$$\rho_{ess}(\Sigma) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Sigma^m} \rho_{ess}(A)]^{1/m}.$$

The joint essential spectral radius of Σ is defined by

$$\hat{\rho}_{ess}(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \gamma(A)]^{1/m}. \tag{30}$$

It is well known that $\rho(\Sigma) = \hat{\rho}(\Sigma)$ for a precompact nonempty set Σ of compact operators on L (see e.g. [32–34]), in particular for a bounded set of complex $n \times n$ matrices (see e.g. [35–39,40]). This equality is called the Berger–Wang formula or also the generalized spectral radius theorem (for an elegant proof in the finite-dimensional case see [36]). It is known that also the generalized Berger-Wang formula holds, i.e, that for any precompact nonempty set Σ of bounded operators on L we have

$$\hat{\rho}(\Sigma) = \max\{\rho(\Sigma), \hat{\rho}_{ess}(\Sigma)\}$$

(see e.g. [32–34]). Observe also that it was proved in [32] that in the definition of $\hat{\rho}_{ess}(\Sigma)$ one may replace the Haussdorf measure of non-compactness by several other seminorms, for instance, it may be replaced by the essential norm.

In general, $\rho(\Sigma)$ and $\hat{\rho}(\Sigma)$ may differ even in the case of a bounded set Σ of compact positive operators on L (see [39] or also [9]). Also, in [41] the reader can find an example of two positive non-compact weighted shifts A and B on $L = l^2$ such that $\rho(\{A, B\}) = 0 < \hat{\rho}(\{A, B\})$. As already noted in [33] also $\rho_{ess}(\Sigma)$ and $\hat{\rho}_{ess}(\Sigma)$ may in general be different.

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [33–36,42] and the references cited there). In particular, $\hat{\rho}(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions.

In this theory, the quantity $\log \hat{\rho}(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [42]).

We will use the following well-known facts that hold for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$:

$$r(\Sigma^m) = r(\Sigma)^m \quad \text{and} \quad r(\Psi\Sigma) = r(\Sigma\Psi) \tag{31}$$

where $\Psi\Sigma = \{AB : A \in \Psi, B \in \Sigma\}$ and $m \in \mathbb{N}$.

Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then the bounded set of positive kernel operators on L, defined by

$$\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \ldots, A_m \in \Psi_m\},$$

is called the *weighted Hadamard (Schur) geometric mean* of sets $\Psi_1, \ldots, \Psi_m$. The set $\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}$ is called the *Hadamard (Schur) geometric mean* of sets $\Psi_1, \ldots, \Psi_m$.

The following result that follows from Theorem 2.1(i) was established in ([9, Theorem 3.3] and [15, Theorems 3.1 and 3.8].

Theorem 2.3: *Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. If $r \in \{\rho, \hat{\rho}\}$ and $n \in \mathbb{N}$, then*

$$r(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \le r((\Psi_1^n)^{(\alpha_1)} \circ \cdots \circ (\Psi_m^n)^{(\alpha_m)})^{\frac{1}{n}} \le r(\Psi_1)^{\alpha_1} \ldots r(\Psi_m)^{\alpha_m} \tag{32}$$

and

$$r\left(\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}\right) \le r(\Psi_1\Psi_2 \cdots \Psi_m)^{\frac{1}{m}}. \tag{33}$$

If, in addition, L and L^* have order continuous norms, then (32) and (33) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

The following theorem [15, Theorem 3.5] was one of the main results in [15].

Theorem 2.4: *Let Ψ and Σ be bounded sets of positive kernel operators on a Banach function space L. If $r \in \{\rho, \hat{\rho}\}$ and $\beta \in [0, 1]$, then we have*

$$\begin{aligned} r\left(\Psi^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})}\right) &\le r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \\ &\le r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^{\frac{1}{4}} r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{\frac{1}{4}} \le r(\Psi\Sigma)^{\frac{1}{2}}, \end{aligned} \tag{34}$$

$$\begin{aligned} r\left(\Psi^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})}\right) &\le r\left(\left(\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})}\right)\left(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})}\right)\right)^{\frac{1}{2}} \\ &\le r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^{\frac{\beta}{2}} r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{\frac{1-\beta}{2}} \le r(\Psi\Sigma)^{\frac{1}{2}}. \end{aligned} \tag{35}$$

If, in addition, L and L^* have order continuous norms, then (34) and (35) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Given $L \in \mathcal{L}$, let $\Psi_1, \ldots, \Psi_m$ be bounded sets of non-negative matrices that define operators on L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$. Then the set

$$\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \ldots, A_m \in \Psi_m\}$$

is a bounded set of non-negative matrices that define operators on L by Theorem 2.2(ii). By applying Theorem 2.1(ii), one can also prove the following result in a similar way as [15, Theorem 3.8]. We omit the details of the proof.

Theorem 2.5: *Given $L \in \mathcal{L}$, let $\Psi, \Psi_1, \ldots, \Psi_m$ be bounded sets of non-negative matrices that define operators on L. Let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{j=1}^m \alpha_j \geq 1$, $n \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}\}$. Then Inequalities (32) hold.*

In particular, if $t \geq 1$, then

$$r(\Psi^{(t)}) \leq r((\Psi^n)^{(t)})^{\frac{1}{n}} \leq r(\Psi)^t. \tag{36}$$

3. Further inequalities and equalities

In [15] and later it remained unnoticed that several inequalities in Theorem 2.4 are in fact equalities, which are established in the following result.

Theorem 3.1: *Let Ψ and Σ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $\sum_{j=1}^m \alpha_j = 1$.*

(i) If $r \in \{\rho, \hat{\rho}\}$ and $\beta \in [0, 1]$, then

$$r(\Psi) = r(\Psi^{(\alpha_1)} \circ \cdots \circ \Psi^{(\alpha_m)}) \tag{37}$$

and

$$\begin{aligned} r(\Psi\Sigma) &= r((\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})) \\ &= r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^{\beta} r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{1-\beta}. \end{aligned} \tag{38}$$

If, in addition, L and L^* have order continuous norms, then (37) and (38) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

(ii) If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $m, n \in \mathbb{N}$, $\alpha \geq 1$ and if Ψ is a bounded set of non-negative matrices that define operators on L, then

$$r(\Psi^{(m)}) \leq r(\Psi \circ \cdots \circ \Psi) \leq r(\Psi^n \circ \cdots \circ \Psi^n)^{\frac{1}{n}} \leq r(\Psi)^m, \tag{39}$$

where in (39) the Hadamard products in $\Psi \circ \cdots \circ \Psi$ and in $\Psi^n \circ \cdots \circ \Psi^n$ are taken m times, and

$$r(\Psi^{(\alpha)}) \leq r(\Psi^{(\alpha-1)} \circ \Psi) \leq r((\Psi^n)^{(\alpha-1)} \circ \Psi^n)^{\frac{1}{n}} \leq r(\Psi)^{\alpha}. \tag{40}$$

Proof: (i) To prove (37) first observe that $\Psi \subset \Psi^{(\alpha_1)} \circ \cdots \circ \Psi^{(\alpha_m)}$, since $A = A^{(\alpha_1)} \circ \cdots \circ A^{(\alpha_m)}$ for all $A \in \Psi$. It follows that

$$r(\Psi) \leq r(\Psi^{(\alpha_1)} \circ \cdots \circ \Psi^{(\alpha_m)}) \leq r(\Psi)^{\alpha_1} \cdots r(\Psi)^{\alpha_m} = r(\Psi)$$

by Theorem 2.3 and so $r(\Psi) = r(\Psi^{(\alpha_1)} \circ \cdots \circ \Psi^{(\alpha_m)})$.

Similary, to prove (38) observe that $\Psi\Sigma \subset (\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})$, since $AB = (A^{(\frac{1}{2})} \circ A^{(\frac{1}{2})})(B^{(\frac{1}{2})} \circ B^{(\frac{1}{2})})$ for all $A \in \Psi$ and $B \in \Sigma$. It follows that

$$\begin{aligned} r(\Psi\Sigma) &\leq r((\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})) \\ &\leq r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^{\beta} r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{1-\beta} \leq r(\Psi\Sigma) \end{aligned}$$

by (35), which proves (38). It is proved similarly that (37) and (38) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ in the case when L and L^* have order continuous norms.

(ii) For the proof of (39) observe that $\Psi^{(m)} \subset \Psi \circ \cdots \circ \Psi$, since $A^{(m)} = A \circ \cdots \circ A$ for all $A \in \Psi$. By Theorem 2.5, Inequalities (39) follow. Inequalities (40) are proved in a similar way. ■

Remark 1: Equalities (38) show that the third inequality in (34) and the second and third inequality in (35) are in fact equalities.

This also implies that (only) [15, Remark 3.6] is false. Indeed, [8, Example 3.11] is not an example that would support the claim stated in [15, Remark 3.6]. The second author of this article regrets for stating this false remark in [15].

The following result extends Inequalities (17) and (32) and Theorem 2.5.

Theorem 3.2: *Let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers.*

(i) If $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^{m} \alpha_i = 1$ and $n \in \mathbb{N}$, then

$$\begin{aligned} &r\left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) \cdots \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right) \\ &\leq r\left((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}\right) \\ &\leq r\left(((\Psi_{11} \cdots \Psi_{k1})^n)^{(\alpha_1)} \circ \cdots \circ ((\Psi_{1m} \cdots \Psi_{km})^n)^{(\alpha_m)}\right)^{\frac{1}{n}} \\ &\leq r(\Psi_{11} \cdots \Psi_{k1})^{\alpha_1} \cdots r(\Psi_{1m} \cdots \Psi_{km})^{\alpha_m}. \end{aligned} \tag{41}$$

If, in addition, L and L^* have order continuous norms, then Inequalities (41) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

(ii) If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^{m} \alpha_j \geq 1$ and $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ are bounded sets of non-negative matrices that define positive operators on L, then Inequalities (41) hold.

In particular, if $\Psi_1, \ldots, \Psi_k$ are bounded sets of non-negative matrices that define positive operators on L and $t \geq 1$, then

$$r(\Psi_1^{(t)} \cdots \Psi_k^{(t)}) \leq r((\Psi_1 \cdots \Psi_k)^{(t)}) \leq r(((\Psi_1 \cdots \Psi_k)^n)^{(t)})^{\frac{1}{n}} \leq r(\Psi_1 \cdots \Psi_k)^t. \tag{42}$$

Proof: (i) Let $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^m \alpha_i = 1$ and $n \in \mathbb{N}$. To prove the first inequality in (41), let $l \in \mathbb{N}$ and

$$A \in \left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) \ldots \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right)^l.$$

Then $A = A_1 \cdots A_l$, where for each $i = 1, \ldots, l$, we have

$$A_i = \left(A_{i11}^{(\alpha_1)} \circ \cdots \circ A_{i1m}^{(\alpha_m)}\right) \ldots \left(A_{ik1}^{(\alpha_1)} \circ \cdots \circ A_{ikm}^{(\alpha_m)}\right),$$

where $A_{i11} \in \Psi_{11}, \ldots, A_{i1m} \in \Psi_{1m}, \ldots, A_{ik1} \in \Psi_{k1}, \ldots, A_{ikm} \in \Psi_{km}$. Then by (15) for each $i = 1, \ldots, l$, we have

$$A_i \le C_i := (A_{i11}A_{i21} \cdots A_{ik1})^{(\alpha_1)} \circ \cdots \circ (A_{i1m}A_{i2m} \cdots A_{ikm})^{(\alpha_m)},$$

where $C_i \in (\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}$. Therefore

$$A \le C := C_1 \cdots C_l \in \left((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}\right)^l,$$

$\rho(A)^{1/l} \le \rho(C)^{1/l}$ and $\|A\|^{1/l} \le \|C\|^{1/l}$, which implies the first inequality in (41). The second and third inequality in (41) follow from (32).

If, in addition, L and L^* have order continuous norms and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, then Inequalities (41) are proved similarly. Under the assumptions of (ii) Inequalities (41) are proved in a similar way by applying Theorems 2.1(ii) and 2.5. ■

Next, we extend Theorem 2.4 by refining (33).

Theorem 3.3: *Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\Phi_j = \Psi_j \ldots \Psi_m \Psi_1 \ldots \Psi_{j-1}$ for $j = 1, \ldots, m$. If $r \in \{\rho, \hat{\rho}\}$, then*

$$r\left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}\right) \le r\left(\Phi_1^{(\frac{1}{m})} \circ \Phi_2^{(\frac{1}{m})} \circ \cdots \circ \Phi_m^{(\frac{1}{m})}\right)^{\frac{1}{m}}$$

$$\le r\left(\left(\Phi_1^n\right)^{(\frac{1}{m})} \circ (\Phi_2^n)^{(\frac{1}{m})} \circ \cdots \circ \left(\Phi_m^n\right)^{(\frac{1}{m})}\right)^{\frac{1}{nm}} \le r(\Psi_1\Psi_2 \cdots \Psi_m)^{\frac{1}{m}}. \tag{43}$$

If, in addition, L and L^* have order continuous norms, then Inequalities (43) are valid also for all $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Proof: Let $r \in \{\rho, \hat{\rho}\}$. Denote

$$\Sigma_1 = \Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}, \quad \Sigma_2 = \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})} \circ \Psi_1^{(\frac{1}{m})}, \ldots,$$
$$\Sigma_m = \Psi_m^{(\frac{1}{m})} \circ \Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_{m-1}^{(\frac{1}{m})}.$$

Then by (31), (41) and commutativity of Hadamard product, we have

$$r\left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}\right)^m = r\left(\left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}\right)^m\right)$$
$$= r(\Sigma_1\Sigma_2 \cdots \Sigma_m) \le r\left(\Phi_1^{(\frac{1}{m})} \circ \Phi_2^{(\frac{1}{m})} \circ \cdots \circ \Phi_m^{(\frac{1}{m})}\right),$$

which proves the first inequality in (43). The second and the third inequality in (43) follow from (32) (or from (41)), since $r(\Phi_1) = r(\Phi_2) = \cdots r(\Phi_m) = r(\Psi_1\Psi_2 \cdots \Psi_m)$ by (31). If, in addition, L and L^* have order continuous norms, then (43) for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ is proved in a similar way. ■

The following result extends (38).

Theorem 3.4: *Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots, \alpha_m$ be non-negative numbers such that $\sum_{j=1}^m \alpha_j = 1$. If $\Phi_j = \Psi_j \ldots \Psi_m \Psi_1 \ldots \Psi_{j-1}$ for $j = 1, \ldots, m$, $\beta \in [0,1]$, then for all $r \in \{\rho, \hat{\rho}\}$ we have*

$$\begin{aligned} r(\Psi_1\Psi_2\cdots\Psi_m) &= r\left(\left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)}\right)\cdots\left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)}\right)\right) \\ &= r\left(\Phi_1^{(\beta)} \circ \Phi_1^{(1-\beta)}\right)^{\alpha_1}\cdots r\left(\Phi_m^{(\beta)} \circ \Phi_m^{(1-\beta)}\right)^{\alpha_m}. \end{aligned} \tag{44}$$

If, in addition, L and L^* have order continuous norms, then Equalities (44) are valid for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Proof: Let $r \in \{\rho, \hat{\rho}\}$. To prove Equalities (44) we use the first inequality in (41) and (31) to obtain that

$$r\left(\left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)}\right)\cdots\left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)}\right)\right) \le r\left(\Phi_i^{(\beta)} \circ \Phi_i^{(1-\beta)}\right) \tag{45}$$

for all $i = 1, \ldots, m$. Indeed, by (31) and the first inequality in (41) we have

$$\begin{aligned} &r((\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)})\cdots(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})) \\ &\quad = r((\Psi_i^{(\beta)} \circ \Psi_i^{(1-\beta)})\cdots(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)})\cdots(\Psi_{i-1}^{(\beta)} \circ \Psi_{i-1}^{(1-\beta)})) \\ &\quad \le r(\Phi_i^{(\beta)} \circ \Phi_i^{(1-\beta)}), \end{aligned}$$

which proves (45). Since $\sum_{j=1}^m \alpha_j = 1$, Inequality (45) implies

$$\begin{aligned} r((\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)})\cdots(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})) &\le r(\Phi_1^{(\beta)} \circ \Phi_1^{(1-\beta)})^{\alpha_1}\cdots r(\Phi_m^{(\beta)} \circ \Phi_m^{(1-\beta)})^{\alpha_m} \\ &\le r(\Psi_1\cdots\Psi_m). \end{aligned} \tag{46}$$

The second inequality in (46) follows from (32) and the fact that $r(\Phi_1) = \cdots = r(\Phi_m) = r(\Psi_1\cdots\Psi_m)$. Since $\Psi_i \subset \Psi_i^{(\beta)} \circ \Psi_i^{(1-\beta)}$ for all $i = 1, \ldots, m$ and $\beta \in [0,1]$, we obtain

$$r(\Psi_1\cdots\Psi_m) \le r((\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)})\cdots(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})),$$

which together with (46) proves Equalities (44). If, in addition, L and L^* have order continuous norms, then Equalities (44) are proved in a similar way for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$. ■

The following result, that extends the main results from [8], is proved in a similar way as Theorem 3.3 by applying Theorems 2.5 and 3.2(ii) instead of Theorems 2.3 and 3.2(i) in the proofs above.

Theorem 3.5: *Given $L \in \mathcal{L}$, let $\Psi_1, \ldots, \Psi_m$ be bounded sets of non-negative matrices that define operators on L and $\Phi_j = \Psi_j \ldots \Psi_m \Psi_1 \ldots \Psi_{j-1}$ for $j = 1, \ldots, m$. Assume* *that* $\alpha \ge \frac{1}{m}$, $\alpha_j \ge 0$, $j = 1, \ldots, m$, $\sum_{j=1}^{m} \alpha_j \ge 1$ *and* $n \in \mathbb{N}$. *If* $r \in \{\rho, \hat{\rho}\}$ *and* $\Sigma_j = \Psi_j^{(\alpha m)} \ldots \Psi_m^{(\alpha m)} \Psi_1^{(\alpha m)} \ldots \Psi_{j-1}^{(\alpha m)}$ *for* $j = 1, \ldots, m$, *then we have*

$$r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) \le r\left(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)}\right)^{\frac{1}{m}}$$
$$\le r\left((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)}\right)^{\frac{1}{mn}} \le r\left(\Psi_1 \cdots \Psi_m\right)^{\alpha}, \tag{47}$$

$$r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) \le r\left(\Psi_1^{(\alpha m)} \cdots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}}$$
$$\le r\left((\Psi_1 \cdots \Psi_m)^{(\alpha m)}\right)^{\frac{1}{m}} \le r\left(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha m)}\right)^{\frac{1}{nm}} \le r\left(\Psi_1 \cdots \Psi_m\right)^{\alpha}. \tag{48}$$

If, in addition, $\alpha \ge 1$ *then*

$$r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) \le r\left(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)}\right)^{\frac{1}{m}} \le r\left((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)}\right)^{\frac{1}{mn}}$$
$$\le \left(r\left((\Phi_1^n)^{(m)}\right) \cdots r\left((\Phi_m^n)^{(m)}\right)\right)^{\frac{\alpha}{m^2 n}} \le r\left(\Psi_1 \cdots \Psi_m\right)^{\alpha}, \tag{49}$$

$$r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) \le r\left(\Sigma_1^{(\frac{1}{m})} \circ \cdots \circ \Sigma_m^{(\frac{1}{m})}\right)^{\frac{1}{m}}$$
$$\le r\left((\Sigma_1^n)^{(\frac{1}{m})} \circ \cdots \circ (\Sigma_m^n)^{(\frac{1}{m})}\right)^{\frac{1}{mn}} \le r\left(\Psi_1^{(\alpha m)} \cdots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}}$$
$$\le r\left((\Psi_1 \cdots \Psi_m)^{(\alpha m)}\right)^{\frac{1}{m}} \le r\left(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha m)}\right)^{\frac{1}{nm}} \le r\left(\Psi_1 \cdots \Psi_m\right)^{\alpha}. \tag{50}$$

Proof: Inequalities (47) are proved in a similar way as Theorem 3.3 by applying Theorems 2.5 and 3.2(ii) instead of Theorems 2.3 and 3.2(i). For the proof of (48) observe that

$$\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)} = (\Psi_1^{(\alpha m)})^{(\frac{1}{m})} \circ \cdots \circ (\Psi_m^{(\alpha m)})^{(\frac{1}{m})}$$

for $i = 1, \ldots, m$. Now the first inequality in (48) follows from (33) (or from (49)):

$$r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) = r\left((\Psi_1^{(\alpha m)})^{(\frac{1}{m})} \circ \cdots \circ (\Psi_m^{(\alpha m)})^{(\frac{1}{m})}\right) \le r\left(\Psi_1^{(\alpha m)} \cdots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}}.$$

Other inequalities in (48) follow from Theorem 3.2(ii).

Assume $\alpha \ge 1$. The first and second inequality in (49) follow from (47). To prove the third inequality in (49) notice that $(\Phi_i^n)^{(\alpha)} = ((\Phi_i^n)^{(m)})^{(\frac{\alpha}{m})}$, $\frac{\alpha}{m} \ge \frac{1}{m}$ and apply Theorem 2.5. The last inequality in (49) follows again from Theorem 2.5 and the fact that $r(\Phi_1) = \cdots = r(\Phi_m) = r(\Psi_1 \cdots \Psi_m)$.

To prove the first three inequalities in (50), observe that $\Psi_i^{(\alpha)} = (\Psi_i^{(m\alpha)})^{(\frac{1}{m})}$, $\frac{\alpha}{m} \ge \frac{1}{m}$ and apply Theorem 3.3. The remaining three inequalities in (50) follow from (48), which completes the proof. ■

We will need the following well-known inequalities (see e.g. [43]). For non-negative measurable functions and for non-negative numbers α and β such that $\alpha + \beta \ge 1$, we have

$$f_1^\alpha g_1^\beta + \cdots + f_m^\alpha g_m^\beta \le (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^\beta \tag{51}$$

More generally, for non-negative measurable functions $\{f_{ij}\}_{i=1,j=1}^{k,m}$ and for non-negative numbers α_j, $j = 1, \ldots, m$, such that $\sum_{j=1}^m \alpha_j \ge 1$ we have

$$(f_{11}^{\alpha_1} \cdots f_{1m}^{\alpha_m}) + \cdots + (f_{k1}^{\alpha_1} \cdots f_{km}^{\alpha_m}) \le (f_{11} + \cdots + f_{k1})^{\alpha_1} \cdots (f_{1m} + \cdots + f_{km})^{\alpha_m} \tag{52}$$

The sum of bounded sets Ψ and Σ is a bounded set defined by $\Psi + \Sigma = \{A + B : A \in \Psi, B \in \Sigma\}$.

Theorem 3.6: *Let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers.*

(i) If $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^m \alpha_j = 1$ and $n \in \mathbb{N}$, then

$$\begin{aligned} & r\left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) + \ldots + \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right) \\ & \le r\left((\Psi_{11} + \cdots + \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} + \cdots + \Psi_{km})^{(\alpha_m)}\right) \\ & \le r\left(((\Psi_{11} + \cdots + \Psi_{k1})^n)^{(\alpha_1)} \circ \cdots \circ ((\Psi_{1m} + \cdots + \Psi_{km})^n)^{(\alpha_m)}\right)^{\frac{1}{n}} \\ & \le r(\Psi_{11} + \cdots + \Psi_{k1})^{\alpha_1} \cdots r(\Psi_{1m} + \cdots + \Psi_{km})^{\alpha_m}. \end{aligned} \tag{53}$$

If, in addition, L and L^* have order continuous norms, then Inequalities (53) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

(ii) If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^m \alpha_j \ge 1$ and $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ are bounded sets of non-negative matrices that define positive operators on L, then Inequalities (53) hold.

Proof: (i) Let $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^m \alpha_i = 1$ and $n \in \mathbb{N}$. To prove the first inequality in (53) let $l \in \mathbb{N}$ and

$$A \in \left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) + \ldots + \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right)^l.$$

Then $A = A_1 \cdots A_l$, where for each $i = 1, \ldots, l$ we have

$$A_i = \left(A_{i11}^{(\alpha_1)} \circ \cdots \circ A_{i1m}^{(\alpha_m)}\right) + \ldots + \left(A_{ik1}^{(\alpha_1)} \circ \cdots \circ A_{ikm}^{(\alpha_m)}\right),$$

where $A_{i11} \in \Psi_{11}, \ldots, A_{i1m} \in \Psi_{1m}, \ldots, A_{ik1} \in \Psi_{k1}, \ldots, A_{ikm} \in \Psi_{km}$. Then by (52) for each $i = 1, \ldots, l$ we have

$$A_i \le C_i := (A_{i11} + A_{i21} + \cdots + A_{ik1})^{(\alpha_1)} \circ \cdots \circ (A_{i1m} + A_{i2m} + \cdots + A_{ikm})^{(\alpha_m)},$$

where $C_i \in (\Psi_{11} + \cdots + \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} + \cdots + \Psi_{km})^{(\alpha_m)}$. Therefore

$$A \le C := C_1 \cdots C_l \in \left((\Psi_{11} + \cdots + \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} + \cdots + \Psi_{km})^{(\alpha_m)}\right)^l,$$

$r(A)^{1/l} \le r(C)^{1/l}$ and $\|A\|^{1/l} \le \|C\|^{1/l}$, which implies the first inequality in (53). The second and third inequality in (53) follow from (32).

If, in addition, L and L^* have order continuous norms and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, then Inequalities (53) are proved similarly. Under the assumptions of (ii) Inequalities (53) are proved in a similar way by applying Theorems 2.1(ii) and 2.5. ∎

4. Weighted geometric symmetrizations

Let Ψ and Σ be bounded sets of positive kernel operators on $L^2(X,\mu)$ and $\alpha \in [0,1]$. Denote by Ψ^* and $S_\alpha(\Psi)$ bounded sets of positive kernel operators on $L^2(X,\mu)$ defined by $\Psi^* = \{A^* : A \in \Psi\}$ and

$$S_\alpha(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(1-\alpha)} = \{A^{(\alpha)} \circ (B^*)^{(1-\alpha)} : A, B \in \Psi\}.$$

We denote simply $S(\Psi) = S_{\frac{1}{2}}(\Psi)$, *the geometric symmetrization* of Ψ. Observe that $(\Psi\Sigma)^* = \Sigma^*\Psi^*$ and $(\Psi^m)^* = (\Psi^*)^m$ for all $m \in \mathbb{N}$. By (32) it follows that

$$r(S_\alpha(\Psi)) \le r(S_\alpha(\Psi^m))^{\frac{1}{m}} \le r(\Psi) \tag{54}$$

for all $m \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$, since $r(\Psi) = r(\Psi^*)$. In particular, for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$ and $n \in \mathbb{N} \cup \{0\}$ we have

$$r(S_\alpha(\Psi)) \le r(S_\alpha(\Psi^{2^n}))^{2^{-n}} \le r(\Psi). \tag{55}$$

Consequently,

$$r(S_\alpha(\Psi))^2 \le r(S_\alpha(\Psi^2)) \le r(\Psi)^2 \tag{56}$$

holds for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$.

The following result that follows from (55) extends [18, Theorem 2.2], [16, Theorem 3.5] and [3, Theorem 3.5(i)].

Theorem 4.1: *Let Ψ be a bounded set of positive kernel operators on $L^2(X,\mu)$, $\alpha \in [0,1]$ and let $r_n = r(S_\alpha(\Psi^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N} \cup \{0\}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$. Then for each n*

$$r(S_\alpha(\Psi)) = r_0 \le r_1 \le \cdots \le r_n \le r(\Psi).$$

Proof: By (55) we have $r_n \le r(\Psi)$. Since $r_{n-1} \le r_n$ for all $n \in \mathbb{N}$ by the first inequality in (56), the proof is completed. ∎

The following result extends [3, Proposition 3.2].

Proposition 4.2: *Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on $L^2(X,\mu)$, $\alpha \in [0,1]$, $n \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$. Then we have*

$$\begin{aligned} r(S_\alpha(\Psi_1)\cdots S_\alpha(\Psi_m)) &\le r\left((\Psi_1\cdots\Psi_m)^{(\alpha)} \circ ((\Psi_m\cdots\Psi_1)^*)^{(1-\alpha)}\right) \\ &\le r\left(((\Psi_1\cdots\Psi_m)^n)^{(\alpha)} \circ (((\Psi_m\cdots\Psi_1)^*)^n)^{(1-\alpha)}\right)^{\frac{1}{n}} \\ &\le r(\Psi_1\cdots\Psi_m)^\alpha\, r(\Psi_m\cdots\Psi_1)^{1-\alpha}, \end{aligned} \tag{57}$$

$$\begin{aligned} r(S_\alpha(\Psi_1) + \cdots + S_\alpha(\Psi_m)) &\le r\left(S_\alpha(\Psi_1 + \cdots + \Psi_m)\right) \\ &\le r\left(S_\alpha((\Psi_1 + \cdots + \Psi_m)^n)\right)^{\frac{1}{n}} \le r(\Psi_1 + \cdots + \Psi_m). \end{aligned} \tag{58}$$

In particular, we have

$$r\left(S_\alpha(\Psi_1)S_\alpha(\Psi_2)\right) \le r\left((\Psi_1\Psi_2)^{(\alpha)} \circ ((\Psi_2\Psi_1)^*)^{(1-\alpha)}\right)$$
$$\le r\left(((\Psi_1\Psi_2)^n)^{(\alpha)} \circ (((\Psi_2\Psi_1)^*)^n)^{(1-\alpha)}\right)^{\frac{1}{n}} \le r(\Psi_1\Psi_2). \quad (59)$$

Proof: By Theorem 3.2(i) we have

$$r\left(S_\alpha(\Psi_1)\cdots S_\alpha(\Psi_m)\right) = r\left((\Psi_1^{(\alpha)} \circ (\Psi_1^*)^{(1-\alpha)})\cdots\left(\Psi_m^{(\alpha)} \circ (\Psi_m^*)^{(1-\alpha)}\right)\right)$$
$$\le r\left((\Psi_1\cdots\Psi_m)^{(\alpha)} \circ ((\Psi_m\cdots\Psi_1)^*)^{(1-\alpha)}\right)$$
$$\le r\left(((\Psi_1\cdots\Psi_m)^n)^{(\alpha)} \circ (((\Psi_m\cdots\Psi_1)^*)^n)^{(1-\alpha)}\right)^{\frac{1}{n}}$$
$$\le r(\Psi_1\cdots\Psi_m)^\alpha\, r((\Psi_m\cdots\Psi_1)^*)^{1-\alpha} = r(\Psi_1\cdots\Psi_m)^\alpha\, r(\Psi_m\cdots\Psi_1)^{1-\alpha},$$

where the last equality follows from the fact that $r(\Psi) = r(\Psi^*)$. The inequalities in (58) are proved in a similar way by applying Theorem 3.6 and (55). The first and second inequalities in (59) are special cases of (57), while the third inequality follows from (57) and the fact that $r(\Psi_1\Psi_2) = r(\Psi_2\Psi_1)$. ∎

Let Ψ be a bounded set of non-negative matrices that define operators on $l^2(R)$ and let α and β be non-negative numbers such that $\alpha+\beta \ge 1$. The set $S_{\alpha,\beta}(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(\beta)} = \{A^{(\alpha)} \circ (B^*)^{(\beta)} : A, B \in \Psi\}$ is a bounded set of non-negative matrices that define operators on $l^2(R)$ by Theorem 2.1(ii).

For $r \in \{\rho, \hat{\rho}\}$, the following result extends Theorem 4.1 in the case of bounded set of non-negative matrices that define operators on $l^2(R)$. It also extends a part of [3, Theorem 3.5(ii)].

Theorem 4.3: *Let Ψ be a bounded set of non-negative matrices that define operators on $l^2(R)$ and $r \in \{\rho, \hat{\rho}\}$. Assume α and β are non-negative numbers such that $\alpha+\beta \ge 1$ and denote $r_n = r(S_{\alpha,\beta}(\Psi^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$r(S_{\alpha,\beta}(\Psi)) = r_0 \le r_1 \le \cdots \le r_n \le r(\Psi)^{\alpha+\beta}. \quad (60)$$

Proof: By Theorem 2.5, we have

$$r(S_{\alpha,\beta}(\Psi)) = r(\Psi^{(\alpha)} \circ (\Psi^*)^{(\beta)}) \le r\left((\Psi^{2^n})^{(\alpha)} \circ ((\Psi^*)^{2^n})^{(\beta)}\right)^{2^{-n}} = r_n \le r(\Psi)^{\alpha+\beta}. \quad (61)$$

In particular, for $n = 1$, we have

$$r(S_{\alpha,\beta}(\Psi))^2 \le r(S_{\alpha,\beta}(\Psi^2)) \le r(\Psi)^{2(\alpha+\beta)}. \quad (62)$$

Since $r_{n-1} \le r_n$ for all $n \in \mathbb{N}\cup\{0\}$ by the first inequality in (62), the proof of (60) is completed. ∎

The following result is proved in similar way as Proposition 4.2 using Theorem 3.2(ii) instead of Theorem 3.2(i).

Proposition 4.4: *Let Ψ, $\Psi_1, \ldots, \Psi_m$ be bounded sets of non-negative matrices that define operators on $l^2(R)$, $n \in \mathbb{N}$ and let α and β be non-negative numbers such that $\alpha + \beta \geq 1$. Then we have*

$$r(S_{\alpha,\beta}(\Psi_1)\cdots S_{\alpha,\beta}(\Psi_m)) \leq r\left((\Psi_1\cdots\Psi_m)^{(\alpha)} \circ ((\Psi_m\cdots\Psi_1)^*)^{(\beta)}\right)$$

$$\leq r\left(((\Psi_1\cdots\Psi_m)^n)^{(\alpha)} \circ (((\Psi_m\cdots\Psi_1)^*)^n)^{(\beta)}\right)^{\frac{1}{n}}$$

$$\leq r(\Psi_1\cdots\Psi_m)^{\alpha}\, r(\Psi_m\cdots\Psi_1)^{\beta}, \tag{63}$$

$$r(S_{\alpha,\beta}(\Psi)) \leq r(S_{\alpha,\beta}(\Psi^n))^{\frac{1}{n}} \leq r(\Psi)^{\alpha+\beta}, \tag{64}$$

$$r(S_{\alpha,\beta}(\Psi_1) + \cdots + S_{\alpha,\beta}(\Psi_m)) \leq r\left(S_{\alpha,\beta}(\Psi_1 + \cdots + \Psi_m)\right)$$

$$\leq r\left(S_{\alpha,\beta}((\Psi_1 + \cdots + \Psi_m)^n)\right)^{\frac{1}{n}} \leq r(\Psi_1 + \cdots + \Psi_m)^{\alpha+\beta}, \tag{65}$$

$$r(S_{\alpha,\beta}(\Psi_1)S_{\alpha,\beta}(\Psi_2)) \leq r\left((\Psi_1\Psi_2)^{(\alpha)} \circ ((\Psi_2\Psi_1)^*)^{(\beta)}\right)$$

$$\leq r\left(((\Psi_1\Psi_2)^n)^{(\alpha)} \circ (((\Psi_2\Psi_1)^*)^n)^{(\beta)}\right)^{\frac{1}{n}} \leq r(\Psi_1\Psi_2)^{\alpha+\beta} \tag{66}$$

for $r \in \{\rho, \hat{\rho}\}$.

Proof: Inequalities (63) and (65) are proved in a similar way as inequalities (57) and (58) by using Theorems 3.2(ii) and 3.6(ii). Inequalities (64) and (66) are special cases of (63). ∎

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