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Covering the edges of a graph with triangles

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A B S T R A C T

In a graph G, let $\rho_\triangle(G)$ denote the minimum size of a set of edges and triangles that cover all edges of G, and let $\alpha_1(G)$ be the maximum size of an edge set that contains at most one edge from each triangle. Motivated by a question of Erdős, Gallai, and Tuza, we study the relationship between $\rho_\triangle(G)$ and $\alpha_1(G)$ and establish a sharp upper bound on $\rho_\triangle(G)$. We also prove Nordhaus-Gaddum-type inequalities for the considered invariants.

1. Introduction

Throughout the paper, all graphs are simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A set of pairwise adjacent vertices of G is called a *clique* of G. The *size* of a clique is the number of its vertices.

A triangle is a clique on three vertices, and we call G *triangular* if each edge is contained in a triangle in G. Throughout this paper, we will use the following triangle-related graph invariants:

- $\rho_\triangle(G) :=$ the minimum cardinality of a set consisting of edges and triangles that together cover $E(G)$.
- $\tau_i(G) :=$ the minimum cardinality of an edge set that contains at least i edges from each triangle in G ($i \in \{1, 2\}$).

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- $\alpha_i(G) :=$ the maximum cardinality of an edge set that contains at most i edges from each triangle in G ($i \in \{1,2\}$). In particular, $\alpha_1(G)$ is called the *triangle-independence number* of G.
- $\nu_\triangle(G) :=$ the maximum number of pairwise edge-disjoint triangles in G.

For a vertex $v \in V(G)$, its *(open) neighborhood* $N(v)$ is the set of all neighbors of v in G, while its *closed neighborhood* is $N[v] := N(v) \cup \{v\}$. As usual, $\overline{G}$ stands for the complement of G, while $e(G)$, $\Delta(G)$ and $\delta(G)$ denote the number of edges, the maximum and the minimum vertex degree in G, respectively.

The subgraph induced by a set $X \subset V(G)$ will be referred to as $G[X]$. Given two disjoint vertex sets X and Y in G, the set of edges between X and Y is denoted by $E(X,Y)$. If the edge set $E(X,Y)$ contains all possible edges between X and Y, it is a *complete join* $X \vee Y$. The analogous notation is used for the complete join between two vertex-disjoint graphs. A *clique covering* $\mathcal{C}$ of $E(G)$ is a collection of cliques of G such that every edge of G appears in at least one clique from $\mathcal{C}$.

By definitions, $\alpha_i(G) + \tau_{3-i}(G) = |E(G)|$ holds if $i \in \{1,2\}$. In [8], Erdős, Gallai and Tuza deeply investigated the different relationships between $\alpha_1(G)$ and $\tau_i(G)$, for $i \in \{1,2\}$, and raised many open questions including the following conjecture:

Conjecture 1 *([8]). It holds for every triangular graph G that*

$$\alpha_1(G) + \tau_1(G) \le \frac{|V(G)|^2}{4}.$$

Remark. The original conjecture is about triangular graphs, but it is equivalent to the conjecture for arbitrary graphs and that version is similarly popular.

In [12], Norin and Sun confirmed this conjecture to be true. Actually, they proved a stronger conjecture posed by Lehel [6].[6] In the statement, $\tau_B(G)$ is the minimum cardinality of an edge set $F \subseteq E(G)$ such that $G - F$ is bipartite. Note that $\tau_B(G)$ is always an upper bound on $\tau_1(G)$.

Theorem 2 *([12]). It holds for every graph G that*

$$\alpha_1(G) + \tau_B(G) \le \frac{|V(G)|^2}{4}. \tag{1}$$

Moreover, the authors of [12] characterized the graphs that attain equality in (1).

Concerning triangle coverings, Lehel and Tuza [10] proved that every graph satisfies

$$\alpha_1(G) \le \rho_\triangle(G) \le \alpha_2(G) = e(G) - \tau_1(G).$$

In [8], the authors also asked whether there exists some exact relation between $\rho_\triangle(G)$ and $\alpha_1(G)$. In this paper, our goal is to study this problem. Among other results, we prove that the following is a tight upper bound on $\rho_\triangle(G)$:

$$\rho_\triangle(G) \le \left\lfloor \frac{1}{2}\big(e(G) + \alpha_1(G) - \nu_\triangle(G)\big) \right\rfloor \tag{2}$$

In Section 2, we prove (2) and show that for every $n \ge 1$ there exists a graph of order n that satisfies (2) with equality. Here we show that the coefficient $\frac{1}{2}$ of $e(G)$ in (2) is also tight in a more general sense.

Proposition 3. *For every real $\epsilon > 0$ and arbitrarily large $\beta > 0$, there exist infinitely many graphs G with*

$$\rho_\triangle(G) > \beta\alpha_1(G) + \left(\frac{1}{2} - \epsilon\right)e(G).$$

Proof. Let n, d, k be positive integers such that $n = 2kd$, and denote by $G = G(A, B, n, d)$ the graph defined on the vertex set $A \cup B$ such that $G[A] = \frac{n}{2}K_1$, $G[B] = kK_d$, and $G(A,B,n,d) = G[A] \vee G[B]$. Note that $|A| = |B| = \frac{n}{2}$.

Then, for any $x \in A$ and any copy of K_d in B, we need at least $\frac{d}{2}$ triangles to cover the d edges between them. So, we have

$$\rho_\triangle(G) \ge \frac{d}{2} \cdot \frac{n}{2} \cdot \frac{n}{2d} = \frac{n^2}{8}.$$

Let $T \subseteq E(G)$ be a triangle-independent set of size $\alpha_1(G)$; that is, no two edges of T are contained in a triangle of G. For any $x \in A$ and any copy K_d in B, we have $|T \cap E(\{x\}, K_d)| \le 1$ and $|T \cap E(K_d)| \le \frac{d}{2}$. So we have $\alpha_1(G) = |T| \le \frac{n}{2} \cdot \frac{n}{2d} + \frac{d}{2} \cdot \frac{n}{2d}$.

[6] Being unaware of Lehel's conjecture, the same inequality was proposed also by Puleo several decades later in [13].

By simple calculation, we get $e(G) = \frac{n^2}{4} + \frac{(d-1)n}{4}$. For every positive ϵ' there is a real ϵ such that $\epsilon' > \epsilon > 0$ and $\frac{2\beta}{\epsilon}$ is an integer. Therefore, it is enough to consider the cases when the latter property holds and we can choose d so that $d = \frac{2\beta}{\epsilon}$. Then the above inequalities yield

$$\begin{aligned}\beta\alpha_1(G) + \left(\frac{1}{2} - \epsilon\right)e(G) <\ & \beta\left(\frac{n^2}{4d} + \frac{n}{4}\right) + \left(\frac{1}{2} - \epsilon\right)\left(\frac{n^2}{4} + \frac{dn}{4}\right) \\ =\ & \frac{n^2}{8} - \frac{\epsilon n^2}{8} - \frac{\beta n}{4} + \frac{\beta n}{4\epsilon}.\end{aligned}$$

On the other hand, if n is sufficiently large, e.g. $n > \frac{2\beta}{\epsilon^2}$, then we have

$$\rho_\triangle(G) \geq \frac{n^2}{8} > \frac{n^2}{8} - \frac{\epsilon n^2}{8} - \frac{\beta n}{4} + \frac{\beta n}{4\epsilon} \geq \beta\alpha_1(G) + \left(\frac{1}{2} - \epsilon\right)e(G)$$

as required. □

Section 3 of this paper is devoted to the so-called Nordhaus–Gaddum-type problems. Nordhaus and Gaddum [11] proved that the chromatic number of a graph G and of its complement $\overline{G}$ always satisfy

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1,$$

where n denotes the order of G. Later on, similar results for other graph parameters have been found, known as Nordhaus–Gaddum-type theorems, including those related to clique covering parameters (see, for example, [2,4,5]). In the literature, there are hundreds of papers considering inequalities of this type, for many other graph invariants. For a survey on this subject we refer the reader to [1].

In Section 3, we consider analogous problems for the invariants α_1 and $\rho_\triangle$. Our main result on α_1 shows that

$$\alpha_1(G) + \alpha_1(\overline{G}) \leq \frac{n^2}{4} + O\left(\frac{n^2}{\ln n}\right)$$

holds as $n \to \infty$. For $\rho_\triangle$, with a somewhat weaker error term, we obtain

$$\rho_\triangle(G) + \rho_\triangle(\overline{G}) \leq \frac{n^2}{3} + o\left(n^2\right).$$

Both upper bounds are asymptotically tight.

2. Upper bounds for $\rho_\triangle$

In this section we prove sharp upper bounds for $\rho_\triangle$.

Theorem 4. *It holds for every graph G that*

$$\rho_\triangle(G) \leq \left\lfloor \frac{1}{2}\big(e(G) + \alpha_1(G) - \nu_\triangle(G)\big) \right\rfloor. \tag{3}$$

Moreover, there exist triangular connected graphs of order n for every $n \geq 6$, as well as non-triangular connected graphs for every $n \geq 1$, that satisfy (3) with equality.

Proof. Given a graph G, we construct a set $\mathcal{C}$ of triangles and edges that together cover $E(G)$. First, take $\nu_\triangle(G)$ edge-disjoint triangles and put them into $\mathcal{C}$. In this way, we have already covered $3\nu_\triangle(G)$ edges of G. If there is a triangle with two uncovered edges, we add this triangle to $\mathcal{C}$. We continue this procedure till there remains no such triangle in G. Suppose that we added r_2 new triangles to $\mathcal{C}$ in this step and there remain r_1 uncovered edges. Adding these r_1 edges to $\mathcal{C}$, the procedure finishes with a clique covering $\mathcal{C}$. Clearly,

$$e(G) = 3\nu_\triangle(G) + 2r_2 + r_1 \tag{4}$$

and

$$\rho_\triangle(G) \leq |\mathcal{C}| = \nu_\triangle(G) + r_2 + r_1. \tag{5}$$

By equation (4), we have

$$\frac{e(G)}{2} - \frac{\nu_\Delta(G)}{2} + \frac{r_1}{2} = \nu_\Delta(G) + r_2 + r_1. \tag{6}$$

Moreover, $r_1 \leq \alpha_1(G)$ holds by our construction. Then, (5) and (6) imply the desired upper bound.

Trivial non-triangular examples for equality are the trees, or more generally the triangle-free graphs, because then we always have $\rho_\Delta(G) = \alpha_1(G) = e(G)$.

To see sharpness with triangular graphs, for $k \geq 3$, we consider the graph $H = K_3 \vee \overline{K_k}$ which is constructed from an independent vertex set $\{v_1, \ldots, v_k\}$ and a triangle $\{u_1, u_2, u_3\}$ by adding a complete join between them. Clearly, $|V(H)| = k+3$ and $e(H) = 3k+3$. For each v_i, we need at least two cliques of size at most three to cover the three edges incident to it. Consequently, $\rho_\Delta(H) \geq 2k$. On the other hand, $\{u_1u_2v_1, u_2u_3v_2\} \cup \{u_3u_1v_i : 3 \leq i \leq k\}$ along with those edges which are not covered by these triangles form a clique covering $\mathcal{C}$ for $E(H)$ with $|\mathcal{C}| = 2k$. It follows that $\rho_\Delta(H) = 2k$. As $\{u_2u_3\} \cup \{u_1v_i : 1 \leq i \leq k\}$ is a triangle-independent set, we have $\alpha_1(H) \geq k+1$. Consider now a maximum triangle-independent edge set S in G. By definition, at most one of the edges of the triangle $\{u_1u_2u_3\}$ belongs to S and, moreover, every vertex v_i is incident to at most one edge from S. Therefore, $|S| \leq k+1$ and $\alpha_1(H) = k+1$. Since $\nu_\Delta(H) = 3$, we have $\rho_\Delta(H) = \left\lfloor \frac{1}{2}\big(e(H) + \alpha_1(H) - \nu_\Delta(H)\big) \right\rfloor = \left\lfloor \frac{1}{2}(4k+1) \right\rfloor = 2k = \rho_\Delta(H)$. The above construction, therefore, gives sharp examples of order n for every $n \geq 6$. □

The following lemma is about maximum matchings in graphs.

Lemma 5. *Let G be a connected graph that is not a complete graph of odd order. Then there exists a maximum matching M and a vertex $v \in V(M)$ in G so that $\{v\} \cup (V(G) \backslash V(M))$ is an independent set in G.*

Proof. Suppose that $M = \{x_1y_1, \ldots, x_ky_k\}$ is a maximum matching in G and the statement is not true for M. Then, every vertex from $V(M)$ is adjacent to a vertex from $Z = V(G) \backslash V(M)$. In particular, $Z \neq \emptyset$.

Consider an arbitrary edge $x_iy_i \in M$. If there exist two different vertices $z, z' \in Z$ such that zx_i and $z'y_i$ are edges in G, then $(M \backslash \{x_iy_i\}) \cup \{zx_i, z'y_i\}$ would be a matching of cardinality $|M|+1$. This contradicts the maximality of M. Thus, for each $x_iy_i \in M$ there exists a vertex z such that $N(x_i) \cap Z = N(y_i) \cap Z = \{z\}$. This also implies $N(z) \cap N(z') = \emptyset$ for any two distinct $z, z' \in Z$.

Now, suppose that Z contains more than one vertex, say $Z = \{z_1, \ldots, z_\ell\}$ and $\ell \geq 2$. There must be an edge between $N(z_1)$ and $V(M) \backslash N(z_1)$ because G is connected, Z is independent, and $\{N(z_1), \ldots, N(z_\ell)\}$ gives a partition of $V(M)$. We may suppose, without loss of generality, that $x_1x_j \in E(G)$ where $x_1y_1, x_jy_j \in M$, and $x_1, y_1 \in N(z_1)$, $x_j, y_j \in N(z_j)$. This assumption yields a contradiction again as $(M \backslash \{x_1y_1, x_jy_j\}) \cup \{x_1x_j, y_1z_1, y_jz_j\}$ would be a matching containing $|M|+1$ edges. It follows that $Z = \{z_1\}$, and therefore, the number of vertices is odd, and z_1 is adjacent to every vertex of G.

By our condition, G is not a complete graph, hence there exist two nonadjacent vertices, say $y_1y_2 \notin E(G)$. Let $M' = (M \backslash \{x_1y_1\}) \cup \{x_1z_1\}$. Since M' is also a maximum matching and $(V(G) \backslash V(M')) \cup \{y_2\} = \{y_1, y_2\}$ is an independent set, M' satisfies the property stated in the lemma. □

Having Lemma 5 at hand, we now are able to prove another upper bound on $\rho_\Delta(G)$ under some conditions on the structure of the graph G. The requirement in the theorem is met by, for instance, all maximal planar graphs with minimum degree 4 or 5.

Theorem 6. *Let G be an n-vertex graph with no isolated vertex, such that for every vertex $v \in V(G)$, the induced subgraph $G[N(v)]$ contains no component that is a complete graph of odd order. Then it holds that*

$$\rho_\Delta(G) \leq \left\lfloor \frac{1}{2}\big(e(G) + \alpha_1(G)\big) - \frac{n}{6} \right\rfloor.$$

Proof. It suffices to prove the statement for connected graphs. Let G be a connected graph that satisfies the conditions in the theorem, and choose a largest set T of edge-disjoint triangles. Further, let us set $S = V(G) \backslash V(T)$. Clearly, $G[S]$ is triangle-free.

Claim A. *S is an independent set.*

Proof. Suppose to the contrary that u_1 and u_2 are two adjacent vertices in S. The maximality of T implies $N(u_1) \cap N(u_2) = \emptyset$ and, therefore, $G[N(u_1)]$ contains a K_1 component $\{u_2\}$, a contradiction. (□)

Claim B. *For each $u \in S$, T covers all edges of $G[N(u)]$.*

Proof. If an edge xy from $G[N(u)]$ is not covered by T, then we can add the triangle uxy to the set T of edge-independent triangles, that contradicts the maximality of T. (□)

Construction of clique covering $\mathcal{C}$. First, we add the $\nu_\triangle(G)$ triangles from T to $\mathcal{C}$. Then, for each $u \in S$, we take a maximum matching M_u in $G[N(u)]$ and define the following sets:

$$R_u^2 = \{uxy : xy \in M_u\} \quad \text{and} \quad R_u^1 = \{uz : z \in N(u) \setminus V(M_u)\}.$$

We also add the triangles in R_u^2 and edges in R_u^1 to $\mathcal{C}$. Note that when a triangle $uxy \in R_u^2$ is added, it always covers two new edges from $E(G)$. Finally, we consider the subgraph $G[V(T)]$. While there exists a triangle t that has two uncovered edges, we add t to R_T^2 and $\mathcal{C}$. Finally, when there is no such triangle, we add the uncovered edges to the set R_T^1 and $\mathcal{C}$.

By Claims A and B, there are no edges inside S, and the edges of $G[N(u)]$ are covered by T. Moreover, by construction, $R_u^2 \cup R_u^1$ covers all edges incident to u. We may conclude that the constructed set $\mathcal{C} = T \cup R_T^2 \cup R_T^1 \cup \bigcup_{u \in S}(R_u^2 \cup R_u^1)$ covers $E(G)$. Introducing the notations

$$r_2 = |R_T^2| + \sum_{u \in S} |R_u^2| \quad \text{and} \quad r_1 = |R_T^1| + \sum_{u \in S} |R_u^1|,$$

and recalling that $|T| = \nu_\triangle(G)$, we have the same inequalities as in the proof of Theorem 4:

$$e(G) = 3\nu_\triangle(G) + 2r_2 + r_1,$$
$$\rho_\triangle(G) \le \nu_\triangle(G) + r_2 + r_1.$$

These relations give

$$\rho_\triangle(G) \le \frac{e(G)}{2} - \frac{\nu_\triangle(G)}{2} + \frac{r_1}{2}.$$

By construction, the r_1 edges in $\mathcal{C}$ form a triangle-independent set in G. It shows $r_1 \le \alpha_1(G)$, but we can prove a stronger upper bound under the present conditions.

Claim C. $r_1 \le \alpha_1(G) - |S|$.

Proof. Construct a triangle-independent edge set F as follows. First, consider a vertex u from S. Under the conditions given in the theorem, $N(u)$ is not empty, and the induced subgraph $G[N(u)]$ satisfies the conditions in Lemma 5. Then, we can find an independent vertex set Z_u in $G[N(u)]$ such that $|Z_u| = |N(u)| - 2|M_u| + 1 = |R_u^1| + 1$. For every $z \in Z_u$, we add the edge uz to F, and perform the same procedure for every vertex from S. Finally, we add all edges from R_T^1 to F. Since S is an independent set in G, no two edges from F belong to the same triangle. Thus, F is a triangle-independent set and

$$\alpha_1(G) \ge |F| = |R_T^1| + \sum_{u \in S}(|R_u^1| + 1) = r_1 + |S|.$$

This finishes the proof of the claim. (□)

Now, we can estimate $\rho_\triangle(G)$ as

$$\rho_\triangle(G) \le \frac{e(G)}{2} + \frac{\alpha_1(G)}{2} - \frac{\nu_\triangle(G) + |S|}{2}.$$

If $\nu_\triangle(G) \ge \frac{n}{3}$, then we are done. Otherwise, $\nu_\triangle(G) = \frac{n-t}{3}$ for some positive integer t. In this case, we have at most $n-t$ vertices that are covered by $\nu_\triangle(G)$ edge-disjoint triangles. Thus,

$$\nu_\triangle(G) + |S| \ge \frac{n-t}{3} + t > \frac{n}{3}$$

that completes the proof of the theorem. □

3. Nordhaus-Gaddum inequalities

In this section we prove asymptotically tight upper bounds for $\alpha_1(G) + \alpha_1(\overline{G})$ and $\rho_\triangle(G) + \rho_\triangle(\overline{G})$.

Theorem 7. *There exists a constant $C > 0$ such that for all graphs G on n vertices, the triangle-independence number satisfies the inequality*

$$\alpha_1(G) + \alpha_1(\overline{G}) \le \frac{n^2}{4} + C\,\frac{n^2}{\ln n}.$$

Proof. It is well known that every graph of order n contains an independent set or a complete subgraph on at least $\frac{1}{2}\log_2 n$ vertices, as one can select a sequence of this length by picking any vertex and continue the process either among its neighbors or among its non-neighbors, whichever are more. The set of vertices whose neighborhood is kept in the corresponding step induces a complete subgraph, while the other part of the sequence forms an independent set. (The last vertex can be in both.)

For a fixed small positive real δ, let us write $\lfloor (1/2-\delta/2)\log_2 n \rfloor$ in the form $c\ln n$. According to the above, the vertex set V of any graph G of order n can be partitioned into sets $X, A_1, \dots, A_k, B_1, \dots, B_\ell$ such that $|X| < n^{1-\delta}$ and for each $1 \le i \le k$ and $1 \le j \le \ell$ the following hold: A_i is independent, B_j induces a complete subgraph in G, and $|A_i| = |B_j| = c\ln n$. Let us denote $A = \cup_{i=1}^k A_i$ and $B = \cup_{j=1}^\ell B_j$. (We also allow the cases with $A = \emptyset$ and $B = \emptyset$ that correspond to $k = 0$ and $\ell = 0$, respectively.) We clearly have

$$\begin{aligned} \alpha_1(G) + \alpha_1(\overline{G}) &\le \alpha_1(G[A \cup B]) + \alpha_1(\overline{G}[A \cup B]) + |X|(n-|X|) + \binom{|X|}{2} \\ &< \alpha_1(G[A \cup B]) + \alpha_1(\overline{G}[A \cup B]) + n^{2-\delta} \end{aligned} \tag{7}$$

where the last term is of $o(\frac{n^2}{\ln n})$. For the first term we observe

$$\alpha_1(G[A \cup B]) \le \alpha_1(G[A]) + \alpha_1(G[B]) + \sum_{i=1}^{k}\sum_{j=1}^{\ell} \alpha_1(G\langle A_i, B_j\rangle),$$

where $\alpha_1(G\langle A_i, B_j\rangle)$ means the maximum number of edges e in a triangle-independent set of G such that one end of e is in A_i and the other is in B_j. The analogous inequality is valid for $\alpha_1(\overline{G}\langle A_i, B_j\rangle)$ in the complementary graph as well, to be used for the second term of (7). We are going to estimate the occurring terms as rearranged in a number of groups as follows.

Inside $G[A]$ and $\overline{G}[B]$ we just apply the trivial fact that a triangle-independent edge set surely is triangle-free. Hence, writing $n_A = |A|$ and $n_B = |B|$, by Turán's theorem we obtain

$$\alpha_1(G[A]) + \alpha_1(\overline{G}[B]) \le \frac{n_A^2}{4} + \frac{n_B^2}{4} \le \frac{n^2}{4}. \tag{8}$$

Further, each $\overline{G}[A_i]$ and each $G[B_j]$ is a complete graph, admitting only a matching in a triangle-independent set inside them, and have no edges after complementation, therefore

$$\sum_{i=1}^{k}(\alpha_1(G[A_i]) + \alpha_1(\overline{G}[A_i])) + \sum_{j=1}^{\ell}(\alpha_1(G[B_j]) + \alpha_1(\overline{G}[B_j])) \le \frac{n_A}{2} + \frac{n_B}{2} \le \frac{n}{2}. \tag{9}$$

For the rest of computation we apply the following fact. If H is a graph on vertex set $P \cup Q$, and Q induces a complete subgraph, then each vertex of P is incident with at most one edge with other end in Q, in any triangle-independent set. As a consequence, the contribution of P–Q edges to $\alpha_1(H)$ is at most $|P|$.

This fact can be applied in G between any two sets B_j, and in $\overline{G}$ between any two sets A_i; and also between any A_i and any B_j, both in G and in $\overline{G}$, because one of A_i and B_j induces a complete subgraph in either of G and $\overline{G}$.

Recall that $|A_i| = |B_j| = c\ln n$, and the number of those sets is $k = \frac{n_A}{c\ln n}$ and $\ell = \frac{n_B}{c\ln n}$, respectively. From this we obtain

$$\begin{aligned} \sum_{i=1}^{k}\sum_{j=1}^{\ell} \alpha_1(G\langle A_i, B_j\rangle) + \alpha_1(\overline{G}\langle A_i, B_j\rangle) &\le 2k\ell c\ln n \\ &= 2\,\frac{n_A}{c\ln n}\,\frac{n_B}{c\ln n}\,c\ln n \\ &\le \frac{n^2}{2c\ln n} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \alpha_1(\overline{G}[A]) + \alpha_1(G[B]) &\le \binom{k}{2} c\ln n + \binom{\ell}{2} c\ln n + \frac{n}{2} \\ &< \frac{n_A^2 + n_B^2}{(c\ln n)^2}\, c\ln n + \frac{n}{2} \\ &\le \frac{n^2}{c\ln n} + \frac{n}{2}. \end{aligned} \tag{11}$$

Summing up (8)–(11), the theorem follows. □

Corollary 8. *If $n \to \infty$, then* $\max\{\alpha_1(G)+\alpha_1(\overline{G})\} = \left(\frac{1}{4}+o(1)\right)n^2$, *where the maximum is taken over all graphs with n vertices.*

Proof. The upper bound is a straightforward corollary of Theorem 7. To see the lower bound, we consider the bipartite Turán graph $G^* = K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. Then $\alpha_1(G^*)+\alpha_1(\overline{G^*}) = \lfloor \frac{n^2}{4}\rfloor + \lfloor \frac{n}{2}\rfloor$. □

Concerning the lower bound, we can prove the following tight result.

Proposition 9. *It holds for every graph G of order n that*

$$\alpha_1(G)+\alpha_1(\overline{G}) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

and the bound is tight.

Proof. Partition the vertex set of G into $\lfloor \frac{n}{2}\rfloor$ disjoint pairs (and a singleton if n is odd). For every pair, the corresponding edge is present either in G or in $\overline{G}$. The edge sets containing these pairs clearly form triangle-independent sets in G and $\overline{G}$. This verifies the inequality, while its tightness follows from letting G be the complete graph K_n for every n. □

Theorem 10. *If $n \to \infty$, then*

$$\max\{\rho_\triangle(G)+\rho_\triangle(\overline{G})\} = \left(\frac{1}{3}+o(1)\right)n^2\,, \tag{12}$$

where the maximum is taken over all graphs with n vertices.

Proof. To see this, first note that by applying Theorem 4, we have

$$\rho_\triangle(G)+\rho_\triangle(\overline{G}) \leq \frac{\alpha_1(G)+\alpha_1(\overline{G})}{2} + \frac{1}{2}\binom{n}{2} - \frac{\nu_\triangle(G)+\nu_\triangle(\overline{G})}{2}\,. \tag{13}$$

Then, the first term on the right-hand side of (13) is bounded by $\frac{n^2}{8} + O(\frac{n^2}{\ln n})$, using Theorem 7. Moreover, as Erdős, Faudree and Ordman (see [7]) conjectured and Gruslys and Shoham [9] confirmed, every 2-edge coloring of K_n contains at least $\frac{1}{12}n^2+o(n^2)$ mutually edge-disjoint monochromatic triangles, i.e. $\nu_\triangle(G)+\nu_\triangle(\overline{G}) \geq \frac{1}{12}n^2+o(n^2)$. It gives an upper bound for the last term in (13). Hence,

$$\rho_\triangle(G)+\rho_\triangle(\overline{G}) \leq \frac{n^2}{8} + O\left(\frac{n^2}{\ln n}\right) + \frac{n^2-n}{4} - \frac{n^2}{24} + o(n^2) = \frac{n^2}{3} + o(n^2).$$

To show the other direction, consider again the bipartite Turán graph $G^* = K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. It holds that

$$\begin{aligned}\rho_\triangle(G^*)+\rho_\triangle(\overline{G^*}) &\geq \left\lfloor \frac{n^2}{4}\right\rfloor + 2\left(\frac{1}{3}\binom{\lfloor \frac{n}{2}\rfloor}{2}\right) \\ &\geq \frac{n^2}{4} + \frac{n^2}{12} - o(n^2) = \frac{1}{3}n^2 - o(n^2)\,,\end{aligned}$$

which finishes the proof of (12). □

Proposition 11. *For every graph G on n vertices*

$$\rho_\triangle(G)+\rho_\triangle(\overline{G}) \geq \frac{n(n-1)}{6}$$

and the bound is asymptotically tight as $n \to \infty$.

Proof. The lower bound follows from the easy observation that

$$\rho_\triangle(G)+\rho_\triangle(\overline{G}) \geq \rho_\triangle(K_n) \geq \frac{1}{3}\binom{n}{2} = \frac{n(n-1)}{6}.$$

Asymptotic tightness holds for the complete graphs $G = K_n$. To see this, we only need to prove that $\rho_\triangle(K_n) \leq \frac{n^2}{6} + O(n)$. This follows from the classical study of “leave graphs” of largest partial Steiner triple systems. More explicitly, it is known that every K_n admits a packing of edge-disjoint triangles such that at most $n/2+1$ edges remain uncovered; see e.g. Table 40.22 on page 553 of [3]. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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