Fast winning strategies for Staller in the Maker–Breaker domination game

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ARTICLE INFO

ABSTRACT

The Maker–Breaker domination game is played on a graph G by two players, called Dominator and Staller, who alternately choose a vertex that has not been played so far. Dominator wins the game if his moves form a dominating set. Staller wins if she plays all vertices from a closed neighborhood of a vertex $v \in V(G)$. Dominator's fast winning strategies were studied earlier. In this work, we concentrate on the cases when Staller has a winning strategy in the game. We introduce the invariant $\gamma'_{\mathrm{SMB}}(G)$ (resp., $\gamma_{\mathrm{SMB}}(G)$) which is the smallest integer k such that, under any strategy of Dominator, Staller can win the game by playing at most k vertices, if Staller (resp., Dominator) plays first on the graph G.

We prove some basic properties of $\gamma_{\mathrm{SMB}}(G)$ and $\gamma'_{\mathrm{SMB}}(G)$ and study the parameters' changes under some operators as taking the disjoint union of graphs or deleting a cut vertex. We show that the inequality $\delta(G) + 1 \leq \gamma'_{\mathrm{SMB}}(G) \leq \gamma_{\mathrm{SMB}}(G)$ always holds and that for every three integers r, s, t with $2 \leq r \leq s \leq t$, there exists a graph G such that $\delta(G) + 1 = r$, $\gamma'_{\mathrm{SMB}}(G) = s$, and $\gamma_{\mathrm{SMB}}(G) = t$. We prove exact formulas for $\gamma'_{\mathrm{SMB}}(G)$ where G is a path or it is a tadpole graph which is obtained from the disjoint union of a cycle and a path by adding one edge between them.

1. Introduction

A general frame for studying Maker–Breaker games was introduced by Erdős and Selfridge [12] nearly 50 years ago. Since then, several variants have been defined and studied extensively [16]. In this work, we are interested in the Maker–Breaker domination game [11]. In particular, we focus on the cases when Breaker (i.e., Staller) wins the game and consider Breaker's fast winning strategies. Along this way, we translate the problem to the Maker–Breaker game played on the closed neighborhood hypergraph of a graph. We establish sharp lower and upper bounds on the corresponding graph invariants and determine the exact values for paths and tadpole graphs.

1.1. Standard definitions

Throughout the paper, we denote by $V(G)$ and $E(G)$, respectively, the vertex and edge set of a graph G. For a vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ contains the neighbors of v and the *closed neighborhood* of v is defined as

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$N_G[v] = N_G(v) \cup \{v\}$. Then, $\deg_G(v) = |N_G(v)|$. For a set $S \subseteq V(G)$ we use the analogous notation $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$. If the graph is clear from the context, we omit the index and simply write $N(v)$, $N[v]$, $N(S)$, and $N[S]$. A vertex v is a *leaf* if $\deg_G(v) = 1$, and the neighbor of a leaf is a *support vertex*. A support vertex is said to be a *strong support vertex* if it is adjacent to at least two leaves. Otherwise, we call it a *weak support vertex*.

A vertex dominates itself and its neighbors. A set $D \subseteq V(G)$ is a *dominating set* in G if $N[D] = V(G)$. The minimum cardinality of such a set is the *domination number* of G and denoted by $\gamma(G)$. A dominating set is *minimal* if it does not contain any dominating set as a proper subset.

A hypergraph $\mathcal{H}$ is a set system over the underlying vertex set $V(\mathcal{H})$. Equivalently, the set $E(\mathcal{H})$ of its (hyper)edges is a subset of $2^{V(\mathcal{H})}$. In general, we assume that each $e \in E(\mathcal{H})$ contains at least one vertex, but sometimes we do not exclude the case of $e = \emptyset$. For the latter situation, we will always mention this possibility directly. A hypergraph is *k-uniform* if $|e| = k$ for every $e \in E(\mathcal{H})$ and it is *simple* if $e \not\subseteq f$ holds for every two different edges. Note that a simple graph is just a simple 2-uniform hypergraph. We say that $\mathcal{H}_1$ is a *subhypergraph* of $\mathcal{H}_2$, if $E(\mathcal{H}_1) \subseteq E(\mathcal{H}_2)$.

A vertex set $T \subseteq V(\mathcal{H})$ is a *transversal* (or vertex cover) in a hypergraph $\mathcal{H}$ if it meets every edge of $\mathcal{H}$. Assuming $\emptyset \notin E(\mathcal{H})$, the minimum cardinality of a transversal is the *transversal number* $\tau(\mathcal{H})$ of the hypergraph. A *minimal transversal* of $\mathcal{H}$ is a transversal that does not contain any other transversal as a proper subset. Following Berge's notation, we denote by $Tr(\mathcal{H})$ the hypergraph on the vertex set $V(\mathcal{H})$ that contains all minimal transversals of $\mathcal{H}$ as hyperedges. We cite a related result of Berge.

Proposition 1.1 (*Corollary 1, pp. 44 [2]*). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two simple hypergraphs on the same vertex set. Then, $Tr(\mathcal{H}_1) = \mathcal{H}_2$ if and only if $Tr(\mathcal{H}_2) = \mathcal{H}_1$.*

We also introduce an operator named *simplification* on hypergraphs that transforms $\mathcal{H}$ into $\widehat{\mathcal{H}}$ by sequentially removing, while it is possible, an edge $e \in E(H)$ if it contains another edge as a subset. It is clear by definition that $\widehat{\mathcal{H}}$ is a simple hypergraph and that T is a (minimal) transversal in $\mathcal{H}$ if and only if it is a (minimal) transversal in $\widehat{\mathcal{H}}$.

Given a graph G, we define two hypergraphs on the same vertex set $V(G)$. Let $\mathcal{D}_G$ be the hypergraph that contains all the minimal dominating sets of G as hyperedges and let $\mathcal{H}_G$ be the hypergraph with edge set $E(\mathcal{H}_G) = \{N_G[v] : v \in V(G)\}$. According to the standard terminology, we say that $\mathcal{H}_G$ is the *closed neighborhood hypergraph* of G. By definition, $D \subseteq V(G)$ is a minimal dominating set in G if and only if it is a minimal transversal in the closed neighborhood hypergraph $\mathcal{H}_G$. Observe that $\mathcal{D}_G$ is a simple hypergraph for every G. These observations together with Proposition 1.1 imply the following statement.

Proposition 1.2. *It holds for every graph G that $Tr(\mathcal{H}_G) = \mathcal{D}_G$ and $Tr(\mathcal{D}_G) = \widehat{\mathcal{H}_G}$.*

1.2. Winning number in a Maker–Breaker game

The *Maker–Breaker game* is played by two persons, namely Maker and Breaker, on a hypergraph $\mathcal{H}$. The vertex set $V(\mathcal{H})$ is called the *board* of the game while the edges of $\mathcal{H}$ are interpreted as *winning sets*. The two players alternately choose (i.e., play) an unplayed vertex from the board. We say that a player *claims* the set $S \subseteq V(\mathcal{H})$ if he (or she) plays all vertices of S. It is a *Maker-start game* (resp., *Breaker-start game*) if Maker (resp., Breaker) is the first to play. *Maker wins* the game if he claims a winning set while *Breaker wins* if she can prevent Maker from doing this. The latter equivalently means that Breaker plays at least one vertex from each winning set; that is, she claims a transversal of $\mathcal{H}$. We may conclude the following.

Observation 1.3. *A Maker–Breaker game on $\mathcal{H}$ with a player A as Maker and a player B as Breaker is the same as the Maker–Breaker game on $Tr(\mathcal{H})$ where the roles of A and B are switched.*

In the last decades, several variants of the Maker–Breaker game were specified and intensively studied. These include the connectivity game [13,17], the hamiltonicity game [1,23,24], the Maker–Breaker resolving [21], domination [11,15] and total domination [14] games. For other versions and references see [16].

Fast winning strategies were also studied for some types of Maker–Breaker games [9–11,14,15,21]. We introduce here four hypergraph invariants to measure how fast a player can win in a (general) Maker–Breaker game. Assume first that a Maker–Breaker game is played on $\mathcal{H}$ and Maker can win the game when he is the first player. We say that Maker wins in his ith move if he claims a winning set (i.e., an edge of $\mathcal{H}$) with this move. We suppose that Maker's goal is to win the game as soon as possible and that Breaker's goal is just the opposite. The *winning number of Maker*, denoted by $w_M^M(\mathcal{H})$, is the minimum number of his moves needed to win the game if both players play optimally. If Maker has no winning strategy as a first player, we set $w_M^M(\mathcal{H}) = \infty$. In the Breaker-start game, $w_M^B(\mathcal{H})$ is defined similarly.

Now, consider the problem from Breaker's point of view. We say that the game is over and Breaker wins in her ith move if she claims a minimal transversal of $\mathcal{H}$ with this move. Clearly, this situation makes it impossible for Maker to claim a winning set even if the game would be continued. Analogously to the previous case, we assume that Breaker's goal is to win the game by playing as few vertices as possible, while Maker's goal is the opposite. Then, if Maker starts the game and Breaker has a winning strategy, the *winning number of Breaker*, $w_B^M(\mathcal{H})$, is the minimum number of her moves

needed to win the game if both players play optimally. Further, $w_B^M(\mathcal{H}) = \infty$, if Breaker cannot win the game.[1] For the game when Breaker is the first player, $w_B^B(\mathcal{H})$ is defined analogously. Concerning the notation w_X^Y in general, for every pair (X, Y) with $X, Y \in \{M, B\}$ the superscript Y denotes the player who starts the game while the subscript X indicates the player whose fast winning strategies are considered. By Observation 1.3, $w_M^M(\mathcal{H}) = w_B^B(Tr(\mathcal{H}))$ and $w_M^B(\mathcal{H}) = w_B^M(Tr(\mathcal{H}))$.

1.3. Maker-Breaker domination game

The *Maker–Breaker domination game* (*MBD game* for short) was recently introduced in [11] and further studied in [15]. The MBD game on a graph G is a Maker–Breaker game played on $\mathcal{D}_G$. Following the terminology introduced for the earlier versions of the domination game [3–7,18,19,22], Maker and Breaker are respectively called Dominator and Staller in an MBD game. An MBD game is called *D-game* if Dominator starts the game and it is an *S-game* if Staller starts. By Observation 1.3 and Proposition 1.2, a Maker–Breaker domination game on a graph G can also be considered as a Maker–Breaker game on the closed neighborhood hypergraph $\mathcal{H}_G$ (or on its simplification $\widehat{\mathcal{H}_G}$) where Staller is the Maker and Dominator is the Breaker.

Now suppose that Dominator has a winning strategy on G in the D-game. As it was introduced in [15], the minimum number of moves of Dominator to win the game is the *Maker–Breaker domination number* (*MBD-number*) $\gamma_{\mathrm{MB}}(G)$. If Dominator does not have a winning strategy, we set $\gamma_{\mathrm{MB}}(G) = \infty$. In terms of the winning number defined for the general case, $\gamma_{\mathrm{MB}}(G) = w_M^M(\mathcal{D}_G)$. Similarly, for the S-game, $\gamma'_{\mathrm{MB}}(G) = w_M^B(\mathcal{D}_G)$.

In this paper, as proposed in [15], we consider the problem from Staller's point of view. Suppose that Staller has a strategy to win the (MBD) D-game on a graph G. Then, Staller wins in her ith move if she claims a minimal transversal of $\mathcal{D}_G$ with this move. It equivalently means that she can claim a closed neighborhood $N_G[v]$ of a vertex v. The *Staller–Maker–Breaker domination number* (*SMBD-number*) $\gamma_{\mathrm{SMB}}(G)$ of G is the minimum number of Staller's moves she needs to win the D-game if both players play optimally. If Staller has no winning strategy in the D-game, we set $\gamma_{\mathrm{SMB}}(G) = \infty$. This definition gives $\gamma_{\mathrm{SMB}}(G) = w_B^M(\mathcal{D}_G) = w_M^B(\mathcal{H}_G)$. For the S-game, the invariant $\gamma'_{\mathrm{SMB}}(G)$ is defined analogously and we have $\gamma'_{\mathrm{SMB}}(G) = w_B^B(\mathcal{D}_G) = w_M^M(\mathcal{H}_G)$.

By definition, a Maker–Breaker game is a finite game and there is always a winner. Therefore, for each graph G either both $\gamma_{\mathrm{MB}}(G) < \infty$ and $\gamma_{\mathrm{SMB}}(G) = \infty$ hold, or $\gamma_{\mathrm{MB}}(G) = \infty$ and $\gamma_{\mathrm{SMB}}(G) < \infty$. The analogous statement is true for the invariants related to the S-game. Further, as it was shown in [17, Proposition 2.1.6, pp. 15] for the Maker–Breaker game in general, if Staller (Dominator) can win as the second player, then she (he) can win as the first player. Then, $\gamma_{\mathrm{SMB}}(G) < \infty$ implies $\gamma'_{\mathrm{SMB}}(G) < \infty$.

In this work, we concentrate on the cases where Staller wins in at least one of the D- and S-game that is if $\gamma'_{\mathrm{SMB}}(G) < \infty$. Typically, we will consider the Maker–Breaker game on the closed neighborhood hypergraph $\mathcal{H}_G$.

Structure of the paper. In Section 2.1 we study the behavior of the winning numbers under some changes in the hypergraph such as taking disjoint union, deleting or shrinking edges. Then, in Section 2.2, we state similar results for the SMBD-numbers of a graph and prove the inequality $\delta(G) + 1 \le \gamma'_{\mathrm{SMB}}(G) \le \gamma_{\mathrm{SMB}}(G)$ and the No-Skip Lemma. Section 3 is devoted to a construction which shows that for every three integers satisfying $2 \le r \le s \le t$ there exists a graph G with $\delta(G) + 1 = r$, $\gamma_{\mathrm{SMB}}(G) = s$, and $\gamma'_{\mathrm{SMB}}(G) = t$. In Section 4, we consider the changes of SMBD-numbers if a leaf and its support vertex is removed from a graph. Applying the previous propositions, it is proved in Section 5 that $\gamma'_{\mathrm{SMB}}(P_n) = \lfloor \log_2 n \rfloor + 1$ holds for every path of odd order. Together with the corresponding theorem proved in [11], our results complete the study of the MBD games on paths. The SMBD-numbers of tadpole graphs (that can be obtained from a cycle and a path by adding an edge between a vertex of the cycle and an endvertex of the path) are determined in Section 6. Some additional observations and conjectures are provided in Section 7.

2. Basic properties

In this section we state several basic properties of the Maker–Breaker games in general, of the MBD games, and their hypergraph representation.

2.1. Basics for Maker–Breaker games

We introduce two operators on hypergraphs that help us modeling Maker's and Breaker's moves. Let $\mathcal{H}$ be a hypergraph and $X \subseteq V(\mathcal{H})$. The hypergraph $\mathcal{H} - X$ is obtained from $\mathcal{H}$ by deleting the vertices contained in X and also deleting all incident edges. Formally,

$$V(\mathcal{H} - X) = V(\mathcal{H}) \setminus X \qquad \text{and} \qquad E(\mathcal{H} - X) = \{e : e \in E(\mathcal{H}) \text{ and } e \cap X = \emptyset\}.$$

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[1] If $w_X^Y(\mathcal{H})$ equals ∞ (for $X, Y \in \{M, B\}$) by definition, we consider it as an infinite cardinality number (e.g., $\aleph_0$) and refer to it in calculations or relations accordingly. For instance, for every hypergraph $\mathcal{H}$ and $X \in \{M, B\}$, $w_M^X(\mathcal{H}) + w_B^X(\mathcal{H}) = \infty$ always holds, while $w_M^X(\mathcal{H}) < w_B^X(\mathcal{H})$ means that Maker has a winning strategy in the game if X plays first.

The other operator is named *shrinking* and the resulted hypergraph is denoted by $\mathcal{H} \mid X$. It means that the vertices in X are deleted from $\mathcal{H}$ but we keep the remaining parts of the incident edges. That is,

$$V(\mathcal{H} \mid X) = V(\mathcal{H}) \setminus X \qquad \text{and} \qquad E(\mathcal{H} \mid X) = \{e \setminus X : e \in E(\mathcal{H})\}.$$

Note that shrinking may transform a simple hypergraph into a non-simple one. In particular, it may create some empty edges. If X contains only one vertex v, we may write $\mathcal{H} - v$ and $\mathcal{H} \mid v$ instead of $\mathcal{H} - \{v\}$ and $\mathcal{H} \mid \{v\}$, respectively.

In a Maker–Breaker game, the winning sets can be modified after each move of the players as follows.

Proposition 2.1. *Suppose that a Maker–Breaker game is played on a hypergraph $\mathcal{H}$ and $v \in V(\mathcal{H})$.*

(i) *If Breaker is the first player and she plays v, the continuation of the game corresponds to a Maker-start game on $\mathcal{H} - v$. In particular, if v is an optimal first move of Breaker, then $\mathrm{w}_M^B(\mathcal{H}) = \mathrm{w}_M^M(\mathcal{H} - v)$.*

(ii) *If Maker is the first player and he plays v, the continuation of the game corresponds to a Breaker-start game on $\mathcal{H} \mid v$. In particular, if v is an optimal first move of Maker, then $\mathrm{w}_M^M(\mathcal{H}) = \mathrm{w}_M^B(\mathcal{H} \mid v) + 1$. Further, Maker wins the game with this move v if and only if $\mathcal{H} \mid v$ contains an empty hyperedge but $\mathcal{H}$ does not.*

Proof. (*i*) After Breaker plays v, this vertex cannot be played anymore, and none of the incident edges can be claimed by Maker. As the other winning sets remain untouched, Maker may imagine that he starts a Maker–Breaker game on $\mathcal{H} - v$ with his next move. Concerning the equality $\mathrm{w}_M^B(\mathcal{H}) = \mathrm{w}_M^M(\mathcal{H} - v)$, recall that only Maker's moves are counted for $\mathrm{w}_M^B(\mathcal{H})$.

(*ii*) If Maker plays v on $\mathcal{H}$, this vertex cannot be played in the continuation. Further, if v belongs to a winning set $e \in E(\mathcal{H})$ then, in the continuation, it is enough to claim $e \setminus \{v\}$ for Maker to win the game. Thus, the players may imagine that the continuation is a Breaker-start game on $\mathcal{H} \mid v$. It is also clear that Maker claims a winning set with the move v if and only if $\{v\} \in E(\mathcal{H})$ that is equivalent to $\emptyset \in E(\mathcal{H} \mid v)$. □

Next we show that adding some new winning sets or shrinking some of the winning sets is never disadvantageous for Maker.

Proposition 2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypergraphs on the same vertex set.*

(i) *If $E(\mathcal{H}_1) \subseteq E(\mathcal{H}_2)$, then $\mathrm{w}_M^M(\mathcal{H}_1) \geq \mathrm{w}_M^M(\mathcal{H}_2)$ and $\mathrm{w}_M^B(\mathcal{H}_1) \geq \mathrm{w}_M^B(\mathcal{H}_2)$ holds.*

(ii) *Suppose that for each $e \in E(\mathcal{H}_1)$ there exists an edge $e' \in E(\mathcal{H}_2)$ such that $e' \subseteq e$. Then, $\mathrm{w}_M^M(\mathcal{H}_1) \geq \mathrm{w}_M^M(\mathcal{H}_2)$ and $\mathrm{w}_M^B(\mathcal{H}_1) \geq \mathrm{w}_M^B(\mathcal{H}_2)$ holds.*

Proof. As (*i*) can be considered as a special case of (*ii*), it is enough to prove the second statement. If $\mathrm{w}_M^M(\mathcal{H}_1) = \infty$ or $\mathrm{w}_M^B(\mathcal{H}_1) = \infty$, the corresponding inequality clearly holds. Otherwise, suppose that Maker plays on $\mathcal{H}_2$ (as the first player) following a strategy that is optimal for $\mathcal{H}_1$ while Breaker plays optimally on $\mathcal{H}_2$. Maker's strategy ensures that, under any strategy of Breaker, he claims a winning set from $E(\mathcal{H}_1)$ in at most $\mathrm{w}_M^M(\mathcal{H}_1)$ moves. Let this winning set be $e \in E(\mathcal{H}_1)$ in the current game. Since there is a winning set e' in $\mathcal{H}_2$ so that $e' \subseteq e$, Maker wins on $\mathcal{H}_2$ either with this move or earlier. This proves $\mathrm{w}_M^M(\mathcal{H}_1) \geq \mathrm{w}_M^M(\mathcal{H}_2)$. The argument is similar if Breaker starts the game. □

Propositions 2.1 and 2.2 together imply that, during a Maker–Breaker game, skipping a move or playing an isolated vertex (i.e., a vertex of degree 0 in $\mathcal{H}$) is never advantageous for Maker. The same is true for Breaker concerning her moves.

Proposition 2.3. *Let $\mathcal{H}$ be a disconnected hypergraph that consists of the components $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$ such that $\mathrm{w}_M^M(\mathcal{H}_1) \leq \cdots \leq \mathrm{w}_M^M(\mathcal{H}_\ell)$. Then, the following holds:*

$$\mathrm{w}_M^M(\mathcal{H}) = \mathrm{w}_M^M(\mathcal{H}_1) \qquad \text{and} \qquad \mathrm{w}_M^M(\mathcal{H}_1) \leq \mathrm{w}_M^B(\mathcal{H}) \leq \mathrm{w}_M^M(\mathcal{H}_2).$$

Proof. Since $E(\mathcal{H}_1) \subseteq E(\mathcal{H})$, Proposition 2.2(i) implies $\mathrm{w}_M^M(\mathcal{H}) \leq \mathrm{w}_M^M(\mathcal{H}_1)$. On the other hand, Breaker may reply to each move of Maker by playing an optimal move in the same component. This strategy ensures the game on $\mathcal{H}$ not being finished before Maker's $\mathrm{w}_M^M(\mathcal{H}_1)^{\text{th}}$ move. We infer of $\mathrm{w}_M^M(\mathcal{H}) = \mathrm{w}_M^M(\mathcal{H}_1)$.

By Proposition 2.1(i) and Proposition 2.2(i), $\mathrm{w}_M^M(\mathcal{H}) \leq \mathrm{w}_M^B(\mathcal{H})$ always holds. Thus, $\mathrm{w}_M^M(\mathcal{H}) = \mathrm{w}_M^M(\mathcal{H}_1)$ implies $\mathrm{w}_M^M(\mathcal{H}_1) \leq \mathrm{w}_M^B(\mathcal{H})$. For the upper bound, if Breaker starts the game on $\mathcal{H}$ by selecting a vertex v, consider the following strategy of Maker. If $v \in V(\mathcal{H}_1)$, Maker plays according to an optimal strategy on $\mathcal{H}_2$ from the next move on. This ensures that the game finishes in his $\mathrm{w}_M^M(\mathcal{H}_2)^{\text{th}}$ move or earlier. If $v \notin V(\mathcal{H}_1)$ then, from the next move, Maker plays according to an optimal strategy on $\mathcal{H}_1$ and ensures that the game finishes in his $\mathrm{w}_M^M(\mathcal{H}_1)^{\text{th}}$ move or earlier. As $\mathrm{w}_M^M(\mathcal{H}_1) \leq \mathrm{w}_M^M(\mathcal{H}_2)$, we may conclude $\mathrm{w}_M^B(\mathcal{H}) \leq \mathrm{w}_M^M(\mathcal{H}_2)$. □

2.2. Basics for Maker–Breaker domination games

Recall that an MBD game on a graph G corresponds to a Maker–Breaker game on the closed neighborhood hypergraph $\mathcal{H}_G$ with $\gamma'_{\text{SMB}}(G) = \mathrm{w}_M^M(\mathcal{H}_G)$ and $\gamma_{\text{SMB}}(G) = \mathrm{w}_M^B(\mathcal{H}_G)$. When we consider an MBD game in which Staller has already played

the vertices from S and Dominator has played the vertices from D, we will often refer to the hypergraph of current winning sets. By Proposition 2.1, this hypergraph is $(\mathcal{H}_G \mid S) - D$. To prove lower bounds on $\gamma_{\text{SMB}}(G)$ or $\gamma'_{\text{SMB}}(G)$ we often modify the hypergraph $\mathcal{H}$ of current winning sets by adding new winning sets or replacing an $e \in E(\mathcal{H})$ with an $e' \subseteq e$. By Proposition 2.2, this makes it easier for Staller to win. In the other case, when we want to prove an upper bound on $\gamma_{\text{SMB}}(G)$ or $\gamma'_{\text{SMB}}(G)$, we can modify $\mathcal{H}$ by deleting some winning sets or replacing an $e \in E(\mathcal{H})$ with an $e' \supseteq e$.

If G consists of the components $G_1, \ldots, G_\ell$, then $\mathcal{H}_G$ consists of the components $\mathcal{H}_{G_1}, \ldots \mathcal{H}_{G_\ell}$. This fact and Proposition 2.3 readily imply the following statement.

Proposition 2.4. *If a disconnected graph G consists of the components $G_1, \ldots, G_\ell$ and $\gamma'_{\text{SMB}}(G_1) \leq \cdots \leq \gamma'_{\text{SMB}}(G_\ell)$, then the following statements hold:*

(i) $\gamma'_{\text{SMB}}(G) = \gamma'_{\text{SMB}}(G_1)$;
(ii) $\gamma'_{\text{SMB}}(G_1) \leq \gamma_{\text{SMB}}(G) \leq \gamma'_{\text{SMB}}(G_2)$.

Proposition 2.5. *Let v be a cut vertex in a connected graph G. If $G_1, \ldots, G_\ell$ are the components of $G - v$ indexed so that $\gamma'_{\text{SMB}}(G_1) \leq \cdots \leq \gamma'_{\text{SMB}}(G_\ell)$, then*

$$\gamma'_{\text{SMB}}(G) \leq \gamma'_{\text{SMB}}(G_2) + 1.$$

Proof. If $\gamma'_{\text{SMB}}(G_2) = \infty$, the inequality is clearly valid. Otherwise, consider the strategy of Staller when she plays v as the first move in the game on G. By Proposition 2.1, the continuation of the game is the same as it would be on $\mathcal{H}_G \mid v$. As v is a cut vertex in G, the hypergraph $\mathcal{H}_G \mid v$ corresponds to the disjoint union of the closed neighborhood hypergraphs of $G_1, \ldots, G_\ell$ supplemented with an extra hyperedge $e = N_G(v)$. Deleting e from $\mathcal{H}_G \mid v$, we obtain the hypergraph $\mathcal{H}'$. By Proposition 2.2, we have $\text{w}_M^B(\mathcal{H}_G \mid v) \leq \text{w}_M^B(\mathcal{H}')$. Observe that $\mathcal{H}'$ contains ℓ components, namely $\mathcal{H}_{G_1}, \ldots, \mathcal{H}_{G_\ell}$. By Proposition 2.4, $\text{w}_M^M(\mathcal{H}') \leq \gamma'_{\text{SMB}}(G_2)$ and, therefore, Staller can ensure that she wins after playing at most $1 + \gamma'_{\text{SMB}}(G_2)$ vertices (together with her first move). $\square$

As a further consequence of Propositions 2.1 and 2.2, we get the following statement which is analogous to the No-Skip Lemma proved in [15] for the MBD games where Dominator's fast winning strategies and the parameters $\gamma_{\text{MB}}(G)$ and $\gamma'_{\text{MB}}(G)$ were studied.

Lemma 2.6 (No-Skip Lemma). *In an optimal strategy of Staller to achieve $\gamma_{\text{SMB}}(G)$ or $\gamma'_{\text{SMB}}(G)$ it is never an advantage for her to skip a move. Moreover, if Dominator skips a move it can never disadvantage Staller.*

We close this section with a simple but important relation between the minimum degree $\delta(G)$ and the invariants $\gamma_{\text{SMB}}(G)$, $\gamma'_{\text{SMB}}(G)$.

Proposition 2.7. *If G is a graph with minimum degree $\delta(G)$, then*

$$\delta(G) + 1 \leq \gamma'_{\text{SMB}}(G) \leq \gamma_{\text{SMB}}(G). \tag{1}$$

Proof. Clearly, $\delta(G) + 1 \leq \gamma'_{\text{SMB}}(G)$ as Staller can win the game only by claiming a closed neighborhood a vertex. To show $\gamma'_{\text{SMB}}(G) \leq \gamma_{\text{SMB}}(G)$, we consider the D-game as an S-game in which Staller skips the first move. By Lemma 2.6, $\gamma'_{\text{SMB}}(G) \leq \gamma_{\text{SMB}}(G)$. $\square$

It is clear that the presence of an isolated vertex implies $\gamma'_{\text{SMB}}(G) = 1$ and we may easily construct a graph G_t for each $t \geq 1$ such that $\delta(G) = 0$, $\gamma'_{\text{SMB}}(G) = 1$ and $\gamma_{\text{SMB}}(G) = t$. In Section 3, we will show that for every triple of integers (r, s, t) with $2 \leq r \leq s \leq t$ there exists a graph G satisfying $\delta(G) + 1 = r$, $\gamma_{\text{SMB}}(G) = s$, and $\gamma'_{\text{SMB}}(G) = t$.

3. A realization theorem for δ, γ'_{SMB}, and γ_{SMB}

We start with the definition of two families of graphs. Let F_1 be isomorphic to P_3 and $X_1 \subseteq V(F_1)$ contain the two leaves and let z_1 denote the support (central) vertex. For $k \geq 2$, we construct F_k by using two copies of F_{k-1}, namely F_{k-1}^1 and F_{k-1}^2, and adding a new vertex z_k which is adjacent to every vertex in $X_{k-1}^1 \cup X_{k-1}^2$. Let us define $X_k = X_{k-1}^1 \cup X_{k-1}^2$ and $Y_k = V(F_k) \setminus (X_k \cup \{z_k\})$. (See Fig. 1.) It is clear by the recursive definition that $|X_k| = 2^k$ and each vertex from X_k is of degree k. Moreover, Y_k contains 2^{k-i} vertices of degree 2^i for all $i \in [k-1]$ and, therefore, $|Y_k| = 2^k - 2$.

We define F'_k by modifying F_k as follows. For $i \in [2]$, let $F'_i \cong F_i$. If $k \geq 3$, F'_k is obtained from F_k by adding a complete graph of order $k - 1$. We denote by Y_k^+ the set of these $k - 1$ new vertices and supplement the graph with a join between Y_k^+ and Y_k. Observe that $\deg_{F'_k}(v) > k$ holds for all $v \in Y_k \cup Y_k^+$ and therefore, $\delta(F'_k) = k$.

Lemma 3.1. *For every positive integer k, $\gamma'_{\text{SMB}}(F'_k) = k + 1$ and $\gamma_{\text{SMB}}(F'_k) = \infty$.*
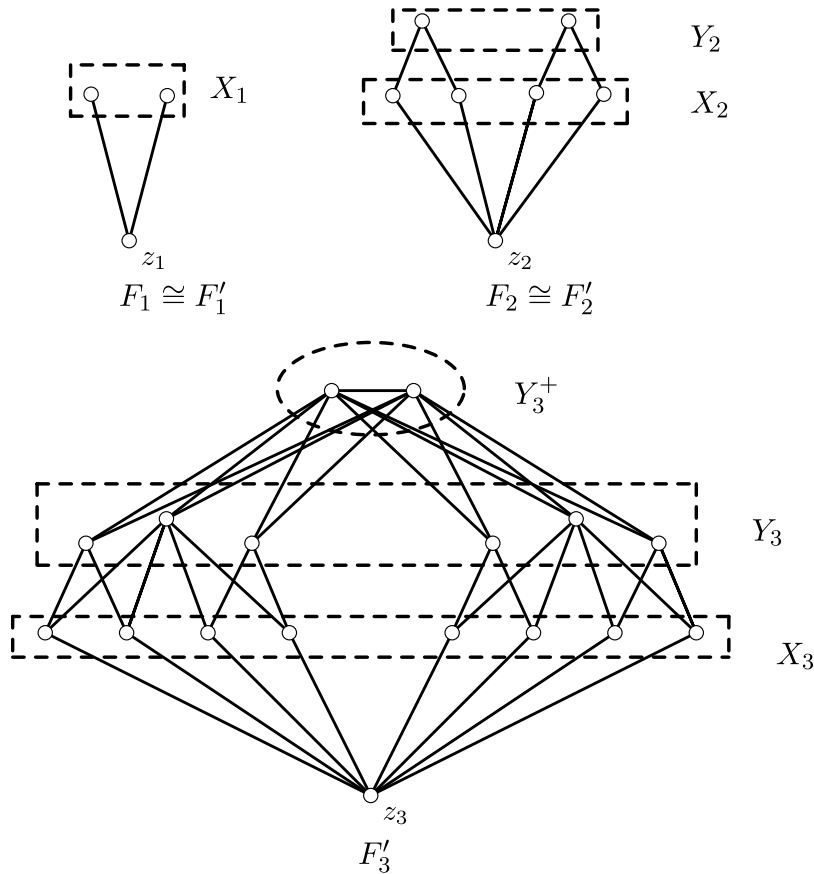
Fig. 1. Graphs $F_1' \cong F_1$, $F_2' \cong F_2$, and F_3'. In F_3' every vertex in Y_3^+ is adjacent to every vertex in Y_3.

Proof. Let k be a positive integer. Since $\delta(F_k') = k$, Proposition 2.7 shows that $\gamma_{SMB}'(F_k') \geq k+1$. To prove $\gamma_{SMB}'(F_k') \leq k+1$, we describe an appropriate strategy for Staller in the S-game. Staller plays z_k in her first move. According to the construction, $F_k' - z_k$ contains two disjoint copies of F_{k-1}. Thus, after Dominator's first move, there remains a subgraph F_{k-1}^i such that none of its vertices are played and Staller may claim its center z_{k-1}^i in her second move. Then, $F_{k-1}^i - z_{k-1}^i$ contains two copies of F_{k-2} and at least one of them remains unplayed after the next move of Dominator. From this unplayed F_{k-2}^j subgraph, Staller plays the center z_{k-2}^j. Continuing this way, in her ℓ^{th} move, $\ell \in [k]$, Staller plays the center of a copy of $F_{k-\ell+1}$. As the described strategy ensures, after Staller's kth move, there remain two vertices, say x_1 and x_2, such that all vertices from $N_{F_k'}(x_1) = N_{F_k'}(x_2)$ have been already claimed by Staller. After Dominator's kth move, Staller may play one of these vertices, say x_1, and win the game by claiming $N_{F_k'}[x_1]$. This strategy shows that $\gamma_{SMB}'(F_k') \leq k + 1$ and then $\gamma_{SMB}'(F_k') = k + 1$ follows.

We next show that $\gamma_{SMB}(F_k') = \infty$ for every positive integer k. Dominator wins the D-game in F_1' with his first move by playing z_1. In F_2', Dominator first plays z_2 and then he can claim one (unplayed) vertex from each component of $F_2' - z_2$ as his second and third move. This way Dominator can always claim a dominating set in F_2'. If $k \geq 3$, Dominator may start the game by playing z_k. We remark that there remains at least one unplayed vertex in Y_k^+ after the first move of Staller. If Dominator responds by playing such a vertex, he claims a dominating set with this move. We therefore conclude $\gamma_{SMB}(F_k') = \infty$. □

By Proposition 2.7, $\delta(G) + 1 \leq \gamma_{SMB}'(G) \leq \gamma_{SMB}(G)$ holds for every graph G. The following realization theorem implies the sharpness of this inequality (1), and also that the differences $\gamma_{SMB}(G) - \gamma_{SMB}'(G)$ and $\gamma_{SMB}'(G) - \delta(G)$ can be arbitrarily large.

Theorem 3.2. *For every three integers r, s, t with $2 \leq r \leq s \leq t$, there exists a graph G such that $\delta(G) + 1 = r$, $\gamma_{SMB}'(G) = s$, and $\gamma_{SMB}(G) = t$.*

Proof. Let r, s, t be integers such that $2 \leq r \leq s \leq t$ and let $G_{r,s,t}$ be the disjoint union of a copy of K_r, a copy of F_{s-1}', and a copy of F_{t-1}'. We show that $\delta(G_{r,s,t}) + 1 = r$, $\gamma_{SMB}'(G_{r,s,t}) = s$, and $\gamma_{SMB}(G_{r,s,t}) = t$ hold.

Notice first that $\delta(K_r) = r - 1$, $\delta(F'_{s-1}) = s - 1$, and $\delta(F'_{t-1}) = t - 1$. Thus, $\delta(G_{r,s,t}) + 1 = r$. As $r \geq 2$ is supposed, Dominator wins in K_r both in a D-game and S-game; that is, $\gamma_{\mathrm{SMB}}(K_r) = \gamma'_{\mathrm{SMB}}(K_r) = \infty$. By Lemma 3.1, $\gamma'_{\mathrm{SMB}}(F'_{s-1}) = s$, $\gamma'_{\mathrm{SMB}}(F'_{t-1}) = t$, and $\gamma_{\mathrm{SMB}}(F'_{s-1}) = \gamma_{\mathrm{SMB}}(F'_{t-1}) = \infty$. Proposition 2.4(i) immediately implies $\gamma'_{\mathrm{SMB}}(G) = s$.

Now consider a D-game played on $G_{r,s,t}$. By Proposition 2.4 (ii), $\gamma_{\mathrm{SMB}}(G_{r,s,t}) \leq t$ holds. To prove the other direction, we present a strategy for Dominator which guarantees that Staller needs to play at least t vertices to win. Dominator first plays z_{s-1} in the component F'_{s-1}. Then, if Staller plays a vertex from a component, in the next turn Dominator replies with an optimal move in the same component. This ensures that Staller cannot win by claiming a closed neighborhood of a vertex from $V(K_r) \cup V(F'_{s-1})$. As $\delta(F'_{t-1}) = t - 1$, it needs at least t moves to claim a closed neighborhood of a vertex from F'_{t-1}. It follows that $\gamma_{\mathrm{SMB}}(G_{r,s,t}) \geq t$ and we infer $\gamma_{\mathrm{SMB}}(G_{r,s,t}) = t$. □

The disconnected graph $G_{r,s,t}$ in the above proof can be replaced with a connected graph that still satisfies the required properties. Indeed, fix a vertex $v \in V(K_r)$ and add the edges vz_{s-1} and vz_{t-1} to $G_{r,s,t}$. As $r \geq 2$, the obtained graph $G'_{r,s,t}$ is of minimum degree $r - 1$, and it can be shown that both $\gamma'_{\mathrm{SMB}}(G'_{r,s,t}) = s$ and $\gamma_{\mathrm{SMB}}(G'_{r,s,t}) = t$ remain true.

We note, related to Theorem 3.2, that the statement remains true if the cases $t = \infty$ and $s = t = \infty$ are allowed. Indeed, deleting the component F'_{t-1} from $G_{r,s,t}$, we obtain a graph with $\delta + 1 = r$, $\gamma'_{\mathrm{SMB}} = s$ and $\gamma_{\mathrm{SMB}} = \infty$, while the deletion of both F'_{s-1} and F'_{t-1} leaves K_r with $\delta(K_r) + 1 = r$ and $\gamma'_{\mathrm{SMB}}(K_r) = \gamma_{\mathrm{SMB}}(K_r) = \infty$.

4. Deleting a leaf and a support vertex

The deletion of a strong support vertex and all the adjacent leaves from a graph G might increase $\gamma'_{\mathrm{SMB}}(G)$ and $\gamma_{\mathrm{SMB}}(G)$ drastically.[2] In this section we consider the possible changes of the SMBD-numbers when a weak support vertex and its leaf are removed from a graph.

Proposition 4.1. *Let G' be a graph obtained from G by removing a weak support vertex and the adjacent leaf. Then, the following inequalities hold:*

$$\gamma'_{\mathrm{SMB}}(G) - 1 \leq \gamma'_{\mathrm{SMB}}(G') \leq \gamma'_{\mathrm{SMB}}(G) \qquad \text{and} \qquad \gamma_{\mathrm{SMB}}(G') \leq \gamma_{\mathrm{SMB}}(G).$$

Proof. Let u be a weak support vertex in G that is adjacent to the leaf v. Then u is a cut vertex in G and $G - u$ is isomorphic to the disjoint union of $G' = G - \{u, v\}$ and the isolated vertex v. By Proposition 2.5, $\gamma'_{\mathrm{SMB}}(G) - 1 \leq \gamma'_{\mathrm{SMB}}(G')$.

To prove the inequalities $\gamma'_{\mathrm{SMB}}(G') \leq \gamma'_{\mathrm{SMB}}(G)$ and $\gamma_{\mathrm{SMB}}(G') \leq \gamma_{\mathrm{SMB}}(G)$, we use the imagination strategy. Let Game 1 be an S-game on G' where Dominator plays optimally, and let Game 2 be an S-game on G in which Staller applies an optimal strategy. Whenever Dominator plays a vertex in Game 1, he copies this move to Game 2. There Staller replies optimally on G, say by playing s_i, and copies this move to Game 1 if it is possible. In particular, whenever $s_i \in V(G) \setminus \{u, v\}$, Staller can copy the move to Game 1. Observe that if a winning set in Game 2 contains v, it also contains u. We may therefore assume that Staller never plays v while u is also available. If $s_i = u$, then Staller will not play any vertex in Game 1. To continue Game 2, Staller imagines that Dominator replies by playing v in G. Then, if Game 2 is not yet over, Staller selects an optimal move s_{i+1} in Game 2 and copies it to Game 1. From this moment, the process continues by simply copying the players' optimal moves to the other game.

For $i \in [2]$, let t_i be the number of Staller's moves until she wins in Game i and set $t_i = \infty$ if she does not win. Observe that $N_{G'}[x] \subseteq N_G[x]$ holds for every vertex $x \in V(G) \setminus \{u, v\}$ and, by the described strategy, Staller cannot win in G by claiming the entire $N_G[u]$ or $N_G[v]$. Thus, when Game 2 ends by the move s_{t_2} and it is copied to Game 1, Staller has already claimed a closed neighborhood in Game 1. We may therefore conclude that $t_1 \leq t_2$. Suppose first that Staller starts in both Game 1 and Game 2. Since Dominator plays optimally in Game 1 but Staller might not, we infer $\gamma'_{\mathrm{SMB}}(G') \leq t_1$. Similarly, as Dominator might not follow an optimal strategy in Game 2, but Staller plays optimally, we have $t_2 \leq \gamma'_{\mathrm{SMB}}(G)$. This proves $\gamma'_{\mathrm{SMB}}(G') \leq \gamma'_{\mathrm{SMB}}(G)$. If Dominator starts in both Game 1 and Game 2, then the same argumentation yields $\gamma_{\mathrm{SMB}}(G') \leq \gamma_{\mathrm{SMB}}(G)$. □

It is not hard to see that $\gamma_{\mathrm{SMB}}(G) - 1 \leq \gamma_{\mathrm{SMB}}(G')$ is not always true for G and G' under the conditions of Proposition 4.1. As an example, we may consider a graph G such that a weak support vertex u is adjacent to all vertices. In this graph Dominator may win the D-game with his first move $d_1 = u$ that implies $\gamma_{\mathrm{SMB}}(G) = \infty$ even if $\gamma_{\mathrm{SMB}}(G')$ is finite. However we can prove that $\gamma_{\mathrm{SMB}}(G) - 1 \leq \gamma_{\mathrm{SMB}}(G')$ holds under stronger conditions. Note that the removal of a weak support vertex of degree 2 and its leaf, had a central role when Dominator's fast winning strategies were studied in [15] and also where the outcome of the game were considered in [11].

Proposition 4.2. *Let G be a graph with a weak support vertex of degree 2 and G' a graph obtained from G by removing this weak support vertex and the adjacent leaf. Then, $\gamma_{\mathrm{SMB}}(G) - 1 \leq \gamma_{\mathrm{SMB}}(G')$ holds.*

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[2] For example, let G be the graph obtained from a $P_2 : v_1v_2$ and a $P_3 : u_1u_2u_3$ component by adding the edge v_1u_2. It is easy to see that $\gamma'_{\mathrm{SMB}}(G) = 2$, but Staller cannot win if we remove u_2 and the leaves u_1, u_2. For the D-game, consider the graph G' which is the disjoint union of G and an isolated vertex. Observe that $\gamma_{\mathrm{SMB}}(G') = 2$ and $\gamma_{\mathrm{SMB}}(G' - \{u_2, u_1, u_3\}) = \infty$.

Proof. Let u be a weak support vertex in G adjacent to the leaf v and to the non-leaf vertex u'. We set $G' = G - \{u, v\}$ and note that the closed neighborhoods of vertices in $V(G) \backslash \{u, u', v\}$ are the same in G as in G'. We prove $\gamma_{\text{SMB}}(G) - 1 \leq \gamma_{\text{SMB}}(G')$ by using the imagination strategy and comparing the winning sets in the two games after a vertex from $\{u, v, u'\}$ was played. Let Game 1 be a D-game on G where Dominator plays optimally, and Game 2 be a D-game on G' where Staller plays optimally. If neither of u, u' and v has been played until a moment in the game, then the moves of the players are simply copied to the another game. That is, whenever Dominator plays a vertex d_i in Game 1, he copies this move to Game 2. There Staller replies optimally on G', say by playing s_i, and copies this move to Game 1. Under the given conditions, it is always possible. If a vertex from $\{u, v, u'\}$ is played, the corresponding move(s) in the other game will be specified in Case 1 and 2. For $i \in [2]$, let t_i be the number of Staller's moves until she wins in Game i and set $t_i = \infty$ if she does not win. We prove $t_1 \leq t_2 + 1$ by considering three cases.

Case 1. If Dominator plays first a vertex d_i from $\{u, v, u'\}$, this move belongs to Game 1. We may suppose that Dominator's corresponding move in Game 2 is u'. After these moves, each remaining winning set of Staller in Game 2 is also a winning set in Game 1. By Proposition 2.2(i), this proves $t_1 \leq t_2$.

Case 2. If Staller plays first a vertex s_i from $\{u, v, u'\}$, then $s_i = u'$ as this move belongs to Game 2. Note that Case 1 excludes Case 2 in a game and, therefore, u, u' and v all are unplayed in Game 1. We may set $s_i = u$ in Game 1. Dominator, who plays optimally there, has to reply by claiming v as otherwise Staller could win in her next move by playing v. These two moves u and v are not copied to Game 2. Instead, we set $s_{i+1} = u'$ in Game 1 and observe that the winning sets in Game 1 and Game 2 are the same, while the number of moves of Staller in Game 1 is one more than the number of her moves in Game 2. We may infer $t_1 = t_2 + 1$. Remark that this argumentation is also true if Staller finishes Game 2 with the move $s_{t_2} = u'$.

Case 3. If neither of u, u' and v is played during the games, then the moves are just copied from one game to the other. Further, under this condition, Staller cannot win by claiming the entire $N_{G'}[u']$, $N_G[u']$, $N_G[u]$, or $N_G[v]$. For the remaining vertices, we have $N_G[x] = N_{G'}[x]$ and thus, the two games end together. This shows $t_1 = t_2$.

We may therefore conclude that $t_1 \leq t_2 + 1$. Our argumentation is also valid for the cases when Staller does not win. For this situation, we suppose that every vertex is played in G and G' but no winning set is claimed by Staller.

Since Dominator plays optimally all along Game 1, we infer $\gamma_{\text{SMB}}(G) \leq t_1$. Similarly, as Staller follows an optimal strategy in Game 2, we have $t_2 \leq \gamma_{\text{SMB}}(G')$. Together with $t_1 \leq t_2 + 1$ we conclude $\gamma_{\text{SMB}}(G) \leq \gamma_{\text{SMB}}(G') + 1$. $\square$

5. Paths

The "pairing strategy" of Maker [16] ensures that he can win in the Maker–Breaker game if an appropriate set of vertex pairs can be defined. Its immediate consequence for the MBD game shows that Dominator has a winning strategy in both the D-game and S-game if the graph admits a perfect matching. Here we state the lemma in a more general form.

Lemma 5.1. *Consider an MBD game on G and let X and Y be the set of vertices played by Dominator and Staller, respectively, until a moment during the game. If there exists a matching M in $G - (X \cup Y)$ such that $V(G) \setminus V(M) \subseteq N_G[X]$, then Dominator has a strategy to win the continuation of the game, no matter who plays the next vertex.*

Proof. By the conditions of the lemma,

$$(X \cup Y) \subseteq (V(G) \setminus V(M)) \subseteq N_G[X]$$

that is all vertices from $X \cup Y$ are dominated by the first $|X|$ moves of Dominator. In the continuation Dominator can win the game by applying the following strategy. If Staller plays a vertex v such that $uv \in E(M)$ and u has not been played yet, then Dominator plays u in his next turn. Otherwise, Dominator plays an arbitrary unplayed vertex. As $V(M)$ does not contain any vertices from Y, this strategy makes sure that Dominator plays at least one vertex from each edge of the matching (unless the game finishes earlier as the entire $V(M)$ is dominated by Dominator's moves). We conclude that Dominator's moves dominate every vertex from $V(G) = V(M) \cup (V(G) \setminus V(M))$ and hence, he wins the game. $\square$

As it was shown in [11], Dominator can win the MBD game on all paths and cycles if he is the first player; that is, $\gamma_{\text{SMB}}(P_n) = \gamma_{\text{SMB}}(C_n) = \infty$. Moreover, Dominator also wins the S-game if it is played on a cycle or a path of an even order; that is, $\gamma'_{\text{SMB}}(P_n) = \gamma'_{\text{SMB}}(C_n) = \infty$ if n is even. On the other hand, if n is odd, $\gamma'_{\text{SMB}}(P_n) < \infty$.

By Proposition 4.1, $\gamma'_{\text{SMB}}(P_{n+2}) \leq \gamma'_{\text{SMB}}(P_n) + 1$. Starting with $\gamma'_{\text{SMB}}(P_1) = 1$ and $\gamma'_{\text{SMB}}(P_3) = 2$, this yields $\gamma'_{\text{SMB}}(P_{2k+1}) \leq k + 1$. For paths of even order the corresponding invariant satisfies $\gamma'_{\text{MB}}(P_{2k}) = k$ (see [15]). These facts might suggest that $\gamma'_{\text{SMB}}(P_{2k+1})$ is equal to or not far from $k + 1$. However, as we will show, the value of $\gamma'_{\text{SMB}}(P_{2k+1})$ is significantly different from $k + 1$. The exact formula is established in the following theorem. If $n \geq 3$ and n is odd, we may also cite the statement of the theorem in the form of $\gamma'_{\text{SMB}}(P_n) = \lceil \log_2 n \rceil$.

Theorem 5.2. *If n is an odd positive integer, then*

$$\gamma'_{\text{SMB}}(P_n) = \lfloor \log_2 n \rfloor + 1. \tag{2}$$

Moreover, Staller has an optimal strategy in the S-game on P_n such that she wins on the closed neighborhood of a vertex v that is at an even distance from the ends of the path.

Proof. First we observe that $\gamma'_{\text{SMB}}(P_1) = 1$ and $\gamma'_{\text{SMB}}(P_3) = 2$ and thus, the formula is valid for $n = 1$ and $n = 3$. Then we proceed by induction on n. Suppose that $n \geq 5$ and (2) is true for every odd integer which is smaller than n. We first prove $\gamma'_{\text{SMB}}(P_n) \leq \lfloor \log_2 n \rfloor + 1$ by considering two cases.

Upper bound, Case 1. Assume that $n = 4k + 3$ for an integer $k \geq 1$ and consider the path $P_n : v_1 \ldots v_{4k+3}$. Observe that both components of $G - v_{2k+2}$ are isomorphic to P_{2k+1} where $2k + 1 = \frac{n-1}{2}$. Proposition 2.5 and our hypothesis imply

$$\gamma'_{\text{SMB}}(P_n) \leq \gamma'_{\text{SMB}}\left(P_{\frac{n-1}{2}}\right) + 1 = \left\lfloor \log_2 \frac{n-1}{2} \right\rfloor + 2 = \lfloor \log_2(n-1) \rfloor + 1.$$

To finish the proof we add that, since $n = 4k + 3$ is not a power of 2, the equality $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$ necessarily holds.

Upper bound, Case 2. Assume that $n = 4k + 1$ for an integer $k \geq 1$ and consider the path $P_n : v_1 \ldots v_{4k+1}$. Now, we choose the cut vertex v_{2k} and consider the components $P^1 : v_1 \ldots v_{2k-1}$ and $P^2 : v_{2k+1} \ldots v_{4k+1}$ of $P_n - v_{2k}$. Remark that P^2 is a path of order $\frac{n+1}{2}$. As $\gamma'_{\text{SMB}}(P^1) \leq \gamma'_{\text{SMB}}(P^2)$, Proposition 2.5 and our hypothesis together imply

$$\gamma'_{\text{SMB}}(P_n) \leq \gamma'_{\text{SMB}}\left(P_{\frac{n+1}{2}}\right) + 1 = \left\lfloor \log_2 \frac{n+1}{2} \right\rfloor + 2 = \lfloor \log_2(n+1) \rfloor + 1.$$

As $n + 1 = 4k + 2$ and $n + 1 \geq 6$, it cannot be a power of 2 and therefore, we get $\lfloor \log_2(n+1) \rfloor = \lfloor \log_2 n \rfloor$. It finishes the proof for the upper bound.

Lower bound. We now prove that Dominator can ensure that Staller does not win in her first $\lfloor \log_2 n \rfloor$ moves on the path $P_n : v_1 \ldots v_n$. We analyze different cases according to Staller's first move s_1.

- Assume that $s_1 = v_i$ for an odd integer $i \in \{1, 3, \ldots, n\}$. By symmetry and the condition $n \geq 5$, we may suppose $i \geq 3$. Dominator then replies by playing $d_1 = v_{i-1}$. The remaining vertices, except v_{i-2}, are covered by the following matching in $G - \{v_i, v_{i-1}\}$:

  $$v_1 v_2, \ldots, v_{i-4} v_{i-3}, \ v_{i+1} v_{i+2}, \ldots, v_{n-1} v_n.$$

  As it satisfies the conditions in Lemma 5.1, we may conclude, by Lemma 5.1, that Dominator can win the game if Staller's first move is v_i with an odd index i.

- Suppose now that $s_1 = v_i$ and i is even. We may also assume, without loss of generality, that $i \leq \frac{n+1}{2}$. Dominator's response is the move $d_1 = v_{i-1}$. After these two moves, by Proposition 2.1, the winning sets for Staller are exactly the edges of the hypergraph $\mathcal{H}' = (\mathcal{H}_{P_n} \mid v_i) - v_{i-1}$. If $i = 2$, then $\mathcal{H}'$ is isomorphic to the closed neighborhood hypergraph of the path P_{n-2} and, by the induction hypothesis, Staller needs at least $\lfloor \log_2(n-2) \rfloor + 1$ further moves there to win. Note that, as $n \geq 5$, the inequality $\lfloor \log_2(n-2) \rfloor + 2 \geq \lfloor \log_2 n \rfloor + 1$ is valid and completes the proof for the case of $i = 2$. If $i \geq 4$, then $\mathcal{H}'$ contains two components. One of them is exactly the closed neighborhood hypergraph of the odd path $P^2 : v_{i+1} \ldots v_n$. Adding the extra (hyper)edge $\{v_{i-2}, v_{i-3}\}$ to the other component of $\mathcal{H}'$, we obtain the closed neighborhood hypergraph of the even path $P^1 : v_1 \ldots v_{i-2}$. Since $\gamma'_{\text{SMB}}(P^1) = \infty$, Proposition 2.4 and the induction hypothesis together imply that Staller needs at least $\gamma'_{\text{SMB}}(P^2) = \lfloor \log_2(n-i) \rfloor + 1$ moves to win the S-game on $P^1 \cup P^2$. Since $\mathcal{H}'$ can be obtained from the closed neighborhood hypergraph $\mathcal{H}_{P^1 \cup P^2}$ by deleting the edge $\{v_{i-2}, v_{i-3}\}$, Staller needs at least as many moves to win on $\mathcal{H}'$ as on $\mathcal{H}_{P^1 \cup P^2}$. We may conclude that, if Dominator plays optimally and Staller wins with her tth move, then

  $$t \geq \lfloor \log_2(n-i) \rfloor + 2 \geq \left\lfloor \log_2 \frac{n-1}{2} \right\rfloor + 2 = \lfloor \log_2(n-1) \rfloor + 1 = \lfloor \log_2 n \rfloor + 1,$$

  where the last equality is true because n cannot be a power of 2.

This establishes the lower bound $\gamma'_{\text{SMB}}(P_n) \geq \lfloor \log_2 n \rfloor + 1$ and, together with the first part of the proof, we obtain $\gamma'_{\text{SMB}}(P_n) = \lfloor \log_2 n \rfloor + 1$ for every odd integer n. Analyzing Staller's strategy described in the proof of the upper bound, we also infer that the claimed winning set is always one of $N[v_1], N[v_3], \ldots, N[v_n]$. $\quad\square$

6. Tadpole graphs

The *tadpole graph* $T(n, k)$ with parameters $n \geq 3$ and $k \geq 1$ is obtained from an n-cycle and a (vertex disjoint) path of order k such that one vertex from the cycle and one end vertex of the path are made adjacent (see Fig. 2). These graphs have been recently studied in [8] concerning Rall's 1/2-conjecture [20] on the domination game.

First we prove a short technical lemma.

Lemma 6.1. *For every two integers a and b with $a \geq 1$ and $b \geq 2$, the following is true:*

$$\max\{\lfloor \log_2 a \rfloor + 1, \lceil \log_2(b-1) \rceil\} \geq \lceil \log_2(a+b) \rceil - 1.$$
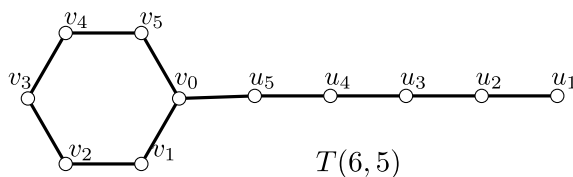
Fig. 2. The tadpole graph $T(6, 5)$.

Proof. Let $\lceil \log_2(a+b) \rceil = s$ and note that $s \geq 2$. Suppose for a contradiction that $\lfloor \log_2 a \rfloor + 1 \leq s-2$ and $\lceil \log_2(b-1) \rceil \leq s-2$ simultaneously hold. These assumptions imply $a < 2^{s-2}$ and $b \leq 2^{s-2} + 1$. We infer $a + b < 2^{s-1} + 1$ and, as $a + b$ is an integer, we get $a + b \leq 2^{s-1}$ that contradicts the definition of s. $\square$

To state the main theorem of this section, we introduce the notation

$$\sigma(n) = \frac{2^{\lceil \log_2 n \rceil} - n}{2} + 1.$$

Theorem 6.2. *For every tadpole graph $T(n, k)$ it is true that $\gamma_{\mathrm{SMB}}(T(n, k)) = \infty$ and*

$$\gamma'_{\mathrm{SMB}}(T(n, k)) = \begin{cases} \lceil \log_2(n + k + \sigma(n)) \rceil & \text{if } n \text{ is even and } k \text{ is odd;} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $v_0 v_1 \ldots v_{n-1} v_0$ and $u_1 \ldots u_k$ be the cycle and the path of the tadpole graph, respectively, such that v_0 and u_k are adjacent.

We first prove that Dominator can win the D-game that is $\gamma_{\mathrm{SMB}}(T(n, k)) = \infty$ holds for every pair (n, k) of positive integers with $n \geq 3$. Let Dominator play $d_1 = v_0$ as his first move. The edges

$$v_1 v_2, \ldots, v_{2\lceil n/2 \rceil - 3} v_{2\lceil n/2 \rceil - 2}, \ u_1 u_2, \ldots, u_{2\lfloor k/2 \rfloor - 1} u_{2\lfloor k/2 \rfloor}$$

form a matching in $T(n, k) - v_0$ that covers all vertices which are not dominated by v_0. Lemma 5.1 thus implies $\gamma_{\mathrm{SMB}}(T(n, k)) = \infty$.

Next, we consider the S-game. If both n and k are odd or both are even, then $T(n, k)$ has a perfect matching and Dominator can win by applying the pairing strategy. If n is odd and k is even, we may iteratively remove a weak support vertex and its leaf until a cycle C_n remains. Thus, by Proposition 4.1, $\gamma'_{\mathrm{SMB}}(C_n) \leq \gamma'_{\mathrm{SMB}}(T(n, k))$ is valid and then, $\gamma'_{\mathrm{SMB}}(C_n) = \infty$ [11] implies $\gamma'_{\mathrm{SMB}}(T(n, k)) = \infty$.

In the rest of the proof, we suppose that n is even, k is odd, and an S-game is played on $T(n, k)$. From now on, ℓ denotes $\lceil \log_2 n \rceil$. Note that $\ell \geq 3$ unless $n = 4$.

Upper bound We consider three cases and show that Staller can win by playing at most $\lceil \log_2(n+k+\sigma(n)) \rceil$ vertices. The first two cases together settle the proof for every tadpole graph $T(n, 1)$, $n \geq 4$. Then, in Case 3, we proceed by induction on k for every fixed n.

Case 1. $n + k + \sigma(n) \leq 2^\ell$.

To prove $\gamma'_{\mathrm{SMB}}(T(n, k)) \leq \ell$, consider the following strategy of Staller. First, Staller plays $s_1 = v_i$ with $i = 2\lceil \frac{n}{4} \rceil$. Observe that v_i and v_0 split the cycle into two nearly equal odd paths that are $P^1 : v_1 \ldots v_{i-1}$ and $P^2 : v_{i+1} \ldots v_{n-1}$. If $n = 4p$, both paths are of order $\frac{n}{2} - 1$, and if $n = 4p + 2$, then P^1 is of order $\frac{n}{2}$ and the order of P^2 is $\frac{n}{2} - 2$. Note that $2 \leq i \leq n - 2$.

In the continuation, if Dominator's first move d_1 is not a neighbor of v_i, Staller selects her next move s_2 from $\{v_{i-1}, v_{i+1}\}$ such that the distance between s_2 and d_1 is at least 3. It can be done as our present condition $n + k + \sigma(n) \leq 2^\ell$ excludes $n = 4$. Without loss of generality, we may suppose that $s_2 = v_{i-1}$. Then, after Dominator's move d_2, one of v_{i-2} and v_{i+1} remains unplayed. Playing this vertex as s_3, Staller can win the game by claiming $N[v_i]$ or $N[v_{i-1}]$. In this case, Staller wins the game in $3 \leq \ell$ moves.

If Dominator plays a neighbor of s_1 as his first move d_1, the game continues with the winning sets in $(\mathcal{H}_{T(n,k)} \mid s_1) - d_1$. Suppose that $d_1 = v_{i+1}$. Deleting the winning sets $N[v_{i+3}], \ldots, N[v_{n-1}]$ from $(\mathcal{H}_{T(n,k)} \mid s_1) - d_1$, we obtain the hypergraph $\mathcal{H}'$. By Proposition 2.2, $\mathrm{w}_M^M(\mathcal{H}') \geq \mathrm{w}_M^M((\mathcal{H}_{T(n,k)} \mid s_1) - d_1)$. Note that $\mathcal{H}'$ can be obtained from the closed neighborhood hypergraph of the path $P' : v_{i-1} \ldots v_0 u_k \ldots u_1$ by replacing the edge $\{v_1, v_0, u_k\}$ with $\{v_1, v_0, u_k, v_{n-1}\}$. Then, Staller continues playing according to the optimal strategy on the path P' that makes sure that the claimed winning set is either v_j or u_j with an odd index j. By Theorem 5.2, such a winning strategy exists and it can be applied for $\mathcal{H}'$. The order of P' is at most $\frac{n}{2} + 1 + k$, and therefore, Staller can win the game in at most $1 + \lceil \log_2(\frac{n}{2} + 1 + k) \rceil$ moves. We observe that

$$1 + \left\lceil \log_2 \left(\frac{n}{2} + 1 + k \right) \right\rceil = 1 + \lceil \log_2(n + k + \sigma(n) - 2^{\ell-1}) \rceil \leq 1 + \lceil \log_2(2^\ell - 2^{\ell-1}) \rceil = \ell,$$

where the inequality follows from the condition $n + k + \sigma(n) \leq 2^\ell$.

If $d_1 = v_{i-1}$, the argumentation is similar. Here we consider the path $P' : v_{i+1} \ldots v_{n-1} v_0 u_k \ldots u_1$ which is of order $n - i + k \leq \frac{n}{2} + k$.

We conclude that, under the present condition, Staller has a strategy to win the S-game by playing at most ℓ vertices. Therefore, $\gamma'_{\text{SMB}}(T(n, k)) \leq \lceil \log_2(n + k + \sigma(n)) \rceil$ holds if the parameters n and k satisfy the condition in Case 1.

Case 2. $n \geq 2^\ell - 2$ and $k = 1$.

As $\sigma(n) = 1$ if $n = 2^\ell$, and $\sigma(n) = 2$ if $n = 2^\ell - 2$, we have $\lceil \log_2(n + k + \sigma(n)) \rceil = \ell + 1$ under the present condition. To show $\gamma'_{\text{SMB}}(T(n, 1)) \leq \ell + 1$, we apply Proposition 2.5 for the cut vertex v_0. The components of $T(n, 1) - v_0$ are an isolated vertex with $\gamma'_{\text{SMB}}(P_1) = 1$ and an $(n - 1)$-path with $\gamma'_{\text{SMB}}(P_{n-1}) = \lceil \log_2(n - 1) \rceil = \ell$. Proposition 2.5 therefore implies $\gamma'_{\text{SMB}}(T(n, 1)) \leq \ell + 1$.

Case 3. $2^{s-1} < n + k + \sigma(n) \leq 2^s$ and $s \geq \ell + 1$.

To prove $\gamma'_{\text{SMB}}(T(n, k)) \leq s$ we identify appropriate cut vertices again. If $k \leq 2^{s-1}$, we consider $T(n, k) - v_0$. The two components are isomorphic to two paths, one is of order $n - 1$ and the other is of order k. by definition of ℓ, we have $n - 1 < 2^\ell \leq 2^{s-1}$ that gives $\max\{n - 1, k\} \leq 2^{s-1}$ in turn and we conclude, by Proposition 2.5 that $\gamma'_{\text{SMB}}(T(n, k)) \leq (s - 1) + 1 = s$.

If $k > 2^{s-1}$, we choose the cut vertex $u_{2^{s-1}}$ and consider the two components obtained after the removal of $u_{2^{s-1}}$. One component is a path with $\gamma'_{\text{SMB}}(P_{2^{s-1}-1}) = s - 1$ while the other component is a tadpole graph $T' = T(n, k - 2^{s-1})$. Since $n + k + \sigma(n) \leq 2^s$ implies $n + k - 2^{s-1} + \sigma(n) \leq 2^{s-1}$, the induction hypothesis yields $\gamma'_{\text{SMB}}(T') \leq s - 1$. By Proposition 2.5, we infer $\gamma'_{\text{SMB}}(T(n, k)) \leq (s - 1) + 1 = s$ again.

Lower bound. In the second part, we show that

$$\gamma'_{\text{SMB}}(T(n, k)) \geq \lceil \log_2(n + k + \sigma(n)) \rceil$$

holds for every pair (n, k) with $n \geq 4$ even and $k \geq 1$ odd. First we eliminate some worst first moves of Staller which make possible Dominator's winning in the continuation. Let s_1 and d_1 denote Staller's and Dominator's first move in the S-game on $T(n, k)$.

Case 1 $s_1 = u_i$ and i is odd or $s_1 = v_j$ and j is odd.

These are Staller's worst first moves. We prove that, under this condition, Dominator has a winning strategy in the continuation of the game.

- If $s_1 = u_1$ and $k = 1$, Dominator replies by playing $d_1 = v_0$. Then, the graph $G - \{u_1, v_0, v_1\}$ admits a perfect matching that satisfies the conditions of Lemma 5.1 with $X = \{d_1\}$, $Y = \{s_1\}$ and hence, Dominator can win the continuation of the game.
- If $s_1 = u_1$ and $k \geq 3$, Dominator plays $d_1 = u_2$. Here, we consider the perfect matching in $G - \{u_1, u_2, u_3\}$ and conclude, by Lemma 5.1, that Dominator can win.
- If $s_1 = u_i$ and i is an odd integer with $i \geq 3$, Dominator replies with $d_1 = u_{i-1}$. The graph $G - \{u_i, u_{i-1}, u_{i-2}\}$ admits a perfect matching and, by applying Lemma 5.1, we infer that Dominator can win in the continuation of the game.
- If $s_1 = v_1$, Dominator plays $d_1 = v_2$. As $G - \{v_1, v_2, v_3\}$ admits a perfect matching, we conclude, by Lemma 5.1, that Dominator can win.
- If $s_1 = v_j$ and j is an odd integer with $j \geq 3$, Dominator plays $d_1 = v_{j-1}$. Since the conditions of Lemma 5.1 are satisfied by the perfect matching in $G - \{v_j, v_{j-1}, v_{j-2}\}$, Dominator can win the game.

Case 2. $s_1 = v_0$.

If $k \leq n - 1$, Dominator replies by playing $d_1 = u_k$ and the continuation of the game reduces to a Maker–Breaker game on $\mathcal{H}' = (\mathcal{H}_{T(n,k)} \mid v_0) - u_k$. If $k = 1$, then $\mathcal{H}'$ is the closed neighborhood hypergraph of the path $v_1, \ldots, v_{n-1}$. If $k \geq 3$ then, by adding $\{u_{k-1}, u_{k-2}\}$ to the winning sets, we obtain a hypergraph $\mathcal{H}''$ that consists of two components. $\mathcal{H}''$ is the closed neighborhood hypergraph of a disjoint union of a path of order $n - 1$ and a path of order $k - 1$. As $k - 1$ is even, $\gamma'_{\text{SMB}}(P_{k-1}) = \infty$. Applying Propositions 2.2, 2.4, and Theorem 5.2 we get

$$w_M^M(\mathcal{H}') \geq w_M^M(\mathcal{H}'') = \gamma'_{\text{SMB}}(P_{n-1}) = \lceil \log(n - 1) \rceil = \ell,$$

no matter whether $k = 1$ or $k \geq 3$. Dominator thus can force Staller to play at least $\ell + 1$ vertices in the game. On the other hand, the condition $k \leq n - 1 \leq 2^\ell - 1$ implies

$$n + k + \sigma(n) = \frac{n}{2} + k + 2^{\ell-1} + 1 \leq 2^{\ell-1} + 2^\ell - 1 + 2^{\ell-1} + 1 = 2^{\ell+1}.$$

Consequently, Dominator can ensure that Staller plays at least $\lceil \log_2(n + k + \sigma(n)) \rceil$ vertices.

Assume now that $k \geq n$. Then, Dominator replies with $d_1 = v_1$ and the game continues on $\mathcal{H}' = (\mathcal{H}_{T(n,k)} \mid v_0) - v_1$. Adding the winning set $\{v_2, v_3\}$ to $\mathcal{H}'$, we obtain $\mathcal{H}''$ that is the closed neighborhood hypergraph of two disjoint paths one of them is an even path of order $n - 2$ and the other one is of order k. By the same reasoning as before, we get that Staller needs to play at least $1 + \lceil \log_2(k) \rceil$ vertices to win. If $\ell + 1 \geq \lceil \log_2(n + k + \sigma(n)) \rceil$, this already finishes the proof as $\lceil \log_2(k) \rceil \geq \lceil \log_2(n) \rceil = \ell$. In the other case, $\lceil \log_2(n + k + \sigma(n)) \rceil = s \geq \ell + 2$ and

$$2^{s-1} < n + k + \sigma(n) = \frac{n}{2} + k + 2^{\ell-1} + 1.$$

This, together with the condition $n/2 \leq 2^{\ell-1}$ implies $\lceil \log_2(k) \rceil \geq s - 1$ and therefore, $1 + \lceil \log_2(k) \rceil \geq s$ as required.

Case 3. $s_1 = v_i$, i is even and $i \geq 2$.

By symmetry, we may suppose that $i \geq n/2$ if $n \equiv 0 \pmod 4$ and $i \geq n/2 + 1$ if $n \equiv 2 \pmod 4$. Let Dominator reply by playing $d_1 = v_{i+1}$. By Proposition 2.1, the game continues on $\mathcal{H} = (\mathcal{H}_{T(n,k)} \mid v_i) - v_{i+1}$. We modify $\mathcal{H}$ by adding the winning set $\{v_{i+2}, v_{i+3}\}$ (if $i + 1 \neq n - 1$) and replacing $N[v_0]$ and $N[v_{n-1}]$ with $N[v_0] \setminus \{v_{n-1}\}$ and $N[v_{n-1}] \setminus \{v_0\}$. The obtained hypergraph is denoted by $\mathcal{H}'$. By Proposition 2.2, $w_M^M(\mathcal{H}') \leq w_M^M(\mathcal{H})$. Observe that $\mathcal{H}'$ is the closed neighborhood hypergraph of the graph G' that consists of two (or only one) path components that are $P^1 : v_{i-1} \ldots v_0 u_k \ldots u_1$ and $P^2 : v_{i+2} \ldots v_{n-1}$. The component P^1 is a path of order $i + k$ and P^2 is a path of even order. By Proposition 2.4, $\gamma'_{\mathrm{SMB}}(G') = \gamma'_{\mathrm{SMB}}(P^1) = \lceil \log_2(i + k) \rceil$. Hence, if t is the number of moves Staller needs to win when $s_1 = v_i$ is fixed, then

$$t \geq 1 + w_M^M(\mathcal{H}') = 1 + \gamma'_{\mathrm{SMB}}(P^1) = 1 + \lceil \log_2(i + k) \rceil. \tag{3}$$

Let $s = \lceil \log_2(n + k + \sigma(n)) \rceil$.

- If $s = \ell$, then (3) together with $i + k \geq \frac{n}{2} + 1 > 2^{\ell-2}$ gives $t \geq s$.
- If $s = \ell + 1$, we consider the inequality

$$\frac{n}{2} + k = n + k + \sigma(n) - 2^{\ell-1} - 1 > 2^{\ell} - 2^{\ell-1} - 1 = 2^{\ell-1} - 1. \tag{4}$$

  If $n \equiv 0 \pmod 4$, then $i \geq n/2$ and, as $n/2 + k$ is odd, the strict inequality in (4) reduces to $\frac{n}{2} + k \geq 2^{\ell-1} + 1$. Then, (3) and (4) together give $t \geq \ell + 1$. If $n \equiv 2 \pmod 4$, then $i \geq 1 + n/2$. By (4), we have

$$i + k \geq \frac{n}{2} + k + 1 > 2^{\ell-1}$$

  and, by using (3), we conclude $t \geq s$ again.
- If $s \geq \ell + 2$, then $i + k > 2^{s-2}$. Then, $1 + \lceil \log_2(i + k) \rceil \geq s$ and (3) imply the desired inequality $t \geq s$.

Remark that Cases 1–3 cover all possibilities if $k = 1$. Hence, we may proceed by induction on k when the last case is considered.

Case 4. $s_1 = u_i$ and i is even.

Applying Lemma 6.1 with $a = i - 1$ and $b = n + k - i + \sigma(n) + 1$ we obtain

$$
\begin{aligned}
m &= \max\{\lfloor \log_2(i-1) \rfloor + 1, \lceil \log_2(n + k - i + \sigma(n)) \rceil\} \\
&\geq \lceil \log_2(n + k + \sigma(n)) \rceil - 1.
\end{aligned} \tag{5}
$$

If $m = \lfloor \log_2(i-1) \rfloor + 1$, Dominator's reply to $s_1 = u_i$ is $d_1 = u_{i+1}$. Then, the game continues on $\mathcal{H}' = (\mathcal{H}_{T(n,k)} \mid u_i) - u_{i+1}$. Adding a new winning set $\{u_{i+2}, u_{i+3}\}$ to $\mathcal{H}'$, we obtain $\mathcal{H}''$. Note that playing on $\mathcal{H}''$ is at least as advantageous for Staller as on $\mathcal{H}'$. One component of $\mathcal{H}''$ is the closed neighborhood hypergraph of a tadpole graph $T' = T(n, k-i-1)$. As $k-i-1$ is even, $\gamma'_{\mathrm{SMB}}(T') = \infty$. The other component of $\mathcal{H}''$ is the closed neighborhood hypergraph of a path of order $i - 1$. By Proposition 2.4, Staller has to play at least m further vertices to win. As (5) shows, the total number of Staller's moves is at least $m + 1 \geq \lceil \log_2(n + k + \sigma(n)) \rceil$.

The proof is similar if $m = \lceil \log_2(n + k - i + \sigma(n)) \rceil$. Then, Dominator plays $d_1 = u_{i-1}$ and then, assuming optimal strategies in the continuation, the game is played on the closed neighborhood hypergraph of a tadpole graph $T(n, k - i)$ and Staller has to play at least $m+1 = \lceil \log_2(n+k-i+\sigma(n)) \rceil + 1$ vertices to win. By (5), it is at least $\lceil \log_2(n+k+\sigma(n)) \rceil$. □

7. Concluding remarks

Concerning the largest possible finite values of the SMBD-numbers, we have the following:

Proposition 7.1. *Let G be a graph on n vertices.*

- *(i) If $\gamma'_{\mathrm{SMB}}(G) < \infty$, then $\gamma'_{\mathrm{SMB}}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*
- *(ii) If $\gamma_{\mathrm{SMB}}(G) < \infty$, then $\gamma_{\mathrm{SMB}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

Moreover, both upper bounds are sharp.

Proof. Staller and Dominator alternately select vertices in the MBD game. It readily follows that if Staller can win in an S-game or D-game, then she plays at most $\lceil n/2 \rceil$ or at most $\lfloor n/2 \rfloor$ vertices, respectively.

To show the sharpness of the upper bounds, first consider the S-game on a subdivided star S_k^1 which obtained from a star S_k by subdividing each of its k edges exactly once (see Fig. 3). Let Dominator play according the following strategy. If Staller plays a support vertex, Dominator replies by playing the adjacent leaf, and vice versa. If Staller plays the center of the star, Dominator plays one of its neighbors (if possible). This strategy of Dominator ensures that Staller cannot claim the closed neighborhood of any support vertices or leaves. As claiming the closed neighborhood of the center needs at

Fig. 3. Graphs S_4^1 and S_4^2.

least $k + 1$ moves, we have $\gamma'_{\text{SMB}}(S_k^1) \geq k + 1$. On the other hand, if Staller plays support vertices, it forces Dominator to respond with playing the adjacent leaves and Staller can win the game by playing the last leaf or the center of the star. Therefore $\gamma'_{\text{SMB}}(S_k^1) < \infty$, and by the upper bound $\lceil \frac{n}{2} \rceil = k + 1$ we conclude $\gamma'_{\text{SMB}}(S_k^1) = k + 1$. This proves that for every odd integer n, there exists a graph with $\gamma'_{\text{SMB}} = \lceil n/2 \rceil$. If n is even, let $k = (n-2)/2$. Supplementing S_k^1 with a new vertex that is adjacent to all support vertices, we obtain a graph S_k^2 on $n = 2k + 2$ vertices which satisfies $\gamma'_{\text{SMB}}(S_k^2) = k + 1$ (see Fig. 3). To obtain sharp examples for the D-game, we may add an isolated vertex to the graphs S_k^1 and S_k^2. □

Even if the upper bounds in Proposition 7.1 are tight, we do not know sharp examples with $\delta(G) \geq 2$. We conjecture that the inequalities can be improved as follows.

Conjecture 7.2. *Let G be a graph n vertices.*

(i) *If $\gamma'_{\text{SMB}}(G) < \infty$, then $\gamma'_{\text{SMB}}(G) \leq \left\lceil \frac{n}{2} \right\rceil - \delta(G) + 1$.*
(ii) *If $\gamma_{\text{SMB}}(G) < \infty$, then $\gamma_{\text{SMB}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - \delta(G) + 1$.*

Data availability

No data was used for the research described in the article.

Acknowledgments

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