Bivariate measure-inducing quasi-copulas

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ABSTRACT

It is well known that every bivariate copula induces a positive measure on the Borel σ-algebra on $[0,1]^2$, but there exist bivariate quasi-copulas that do not induce a signed measure on the same σ-algebra. In this paper we show that a signed measure induced by a bivariate quasi-copula can always be expressed as an infinite combination of measures induced by copulas. With this we are able to give the first characterization of measure-inducing quasi-copulas in the bivariate setting.

1. Introduction

Copulas, introduced in 1959 by Sklar, are one of the main tools for modeling dependence of random variables in statistical literature. These are multivariate functions that link the cumulative distribution function of a random vector and its one-dimensional marginal distributions. Copulas have found widespread use in various practical applications such as finance [13], biology [21], environmental sciences [4,10] and many others.

Quasi-copulas, a generalization of copulas, were introduced by Alsina, Nelsen, and Schweizer [1] in order to characterize operations on distribution functions that can be derived from operations on random variables. Their importance is explained by the following property: a point-wise infimum respectively supremum of a given set of copulas is always a quasi-copula. Thus, quasi-copulas are indispensable in the theory of imprecise probabilities, which model situations when the exact dependence structure between random variables, i.e. copula, is not known and is therefore replaced by a set of copulas.

The set of all quasi-copulas has been studied intensively in recent years and compared to the set of all copulas. The lattice theoretical properties of both sets were investigated in [17,11,2,19], while topological properties, particularly from the perspective of Baire categories, were studied in [8,9]. From a measure-theoretic point of view one of the major differences between copulas and quasi-copulas is that, while every n-variate copula C induces a positive measure μ_C on the Borel σ-algebra on $[0,1]^n$ (see [14]), there exist n-variate quasi-copulas (for all $n \geq 2$) that do not induce a signed measure on the same σ-algebra [16,12]. This has stimulated numerous investigations of the mass distribution of quasi-copulas [15,5,24,23,7] and related concepts [6,18,20], aimed at a better understanding of the behavior of quasi-copulas. As evidenced by several very recent papers, this is an active area of research, where there is still much to be done. In fact, a full characterization of quasi-copulas that do induce a signed measure on the Borel σ-algebra on $[0,1]^n$ is still an open problem, see [3, Problem 4].

In this paper we study bivariate quasi-copulas that induce a signed measure on the Borel σ-algebra on $[0,1]^2$. We show that the signed measure μ_Q induced by such a quasi-copula Q can always be expressed with measures induced by bivariate copulas. This is done by making use of a recent result in [7] giving a characterization of those quasi-copulas that can be expressed as linear combinations of two copulas. Note that any such quasi-copula automatically induces a signed measure, but not all measure-inducing quasi-copulas can be expressed as linear combinations of copulas, see [7, Example 13]. However, the same paper also initiated the study of quasi-copulas using infinite series of copulas and this idea is key to our result. In particular, our main theorem reads as follows.

Theorem 1. *For a bivariate quasi-copula Q the following two conditions are equivalent.*

(*i*) *There exists a signed measure μ_Q defined on the Borel σ-algebra on $[0,1]^2$ such that*

$$\mu_Q([0,x] \times [0,y]) = Q(x,y) \qquad \text{for all } x,y \in [0,1].$$

(*ii*) *There exists a sequence of bivariate copulas C_n and a sequence of real numbers γ_n such that*

    (*a*) *the series of functions $\sum_{n=1}^{\infty} \gamma_n C_n$ converges uniformly to Q and*
    (*b*) *the series of induced measures $\sum_{n=1}^{\infty} \gamma_n \mu_{C_n}$ converges in the total variation norm to some finite signed measure.*

This result can be seen as a characterization of bivariate quasi-copulas that induce a signed measure on $[0,1]^2$, so it gives an answer to the open problem [3, Problem 4] mentioned above in the bivariate case. However, we do not consider the bivariate case to be completely resolved, since it would still be beneficial to obtain a characterization of measure-inducing quasi-copulas in operationally simpler terms.

We note that in [7] it was shown that condition (*ii*)(*a*) of Theorem 1 is satisfied by any quasi-copula, i.e. any quasi-copula can be expressed as an infinite linear combination of copulas (converging in the supremum norm). So the crucial part of the above equivalence is condition (*ii*)(*b*). Any quasi-copula that does not induce a signed measure satisfies (*ii*)(*a*) for some choice of γ_n and C_n but fails to satisfy (*ii*)(*b*).

The paper is structured as follows. In Section 2 we recall some basic notions from measure theory and some results on bivariate quasi-copulas that we will need in our proofs. The main part of the paper is devoted to the proof of Theorem 1. Assuming a quasi-copula Q induces a signed measure μ_Q, we construct in Section 3 a sequence of measure-inducing quasi-copulas Q_N that converge to Q and whose induced measures converge to μ_Q. In addition, all these quasi-copulas are linear combinations of copulas. In Section 4 we convert the sequence Q_N into a series of multiples of copulas C_n, and finally prove Theorem 1. An example that demonstrates our result is given in Section 5.

2. Preliminaries

Throughout the paper we will denote the unit interval by $\mathbb{I} = [0,1]$ and the unit square by $\mathbb{I}^2$. A function $Q: \mathbb{I}^2 \to \mathbb{I}$ is a (bivariate) *quasi-copula* if it satisfies the following conditions:

(*i*) Q is *grounded*: $Q(x,0) = Q(0,y) = 0$ for all $x,y \in \mathbb{I}$,
(*ii*) Q has *uniform marginals*: $Q(x,1) = x$ and $Q(1,y) = y$ for all $x,y \in \mathbb{I}$,
(*iii*) Q is *increasing* in each variable,
(*iv*) Q is *1-Lipschitz*:

$$|Q(x_2, y_2) - Q(x_1, y_1)| \le |x_2 - x_1| + |y_2 - y_1|$$

    for all $x_1, y_1, x_2, y_2 \in \mathbb{I}$.

A function $Q: \mathbb{I}^2 \to \mathbb{I}$ that satisfies conditions (*i*), (*ii*), and

(*v*) Q is *2-increasing*:

$$Q(x_2, y_2) - Q(x_1, y_2) - Q(x_2, y_1) + Q(x_1, y_1) \ge 0$$

    for all $x_1, y_1, x_2, y_2 \in \mathbb{I}$ with $x_1 \le x_2$ and $y_1 \le y_2$,

is called a (bivariate) *copula*.

Let $(X, \mathcal{A})$ be a measurable space equipped with a finite signed measure μ. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of measure μ, i.e., μ^+ and μ^- are finite positive measures with disjoint supports. If S^+ and S^- denote the supports of μ^+ and μ^-, respectively, then $\mu^+(A) = \mu(A \cap S^+)$ and $\mu^-(A) = -\mu(A \cap S^-)$ for all $A \in \mathcal{A}$. The positive measure $|\mu| = \mu^+ + \mu^-$ is called the *total variation measure* of μ. It satisfies the inequality $|\mu(A)| \le |\mu|(A)$ for all $A \in \mathcal{A}$. The *total variation norm* of μ is given by

$$\|\mu\|_{TV} = |\mu|(X) = \mu^+(X) + \mu^-(X) = \mu(S^+) - \mu(S^-). \tag{1}$$

The vector space of all finite signed measures on $(X, \mathcal{A})$ equipped with the total variation norm is a Banach space. For two finite measures μ_1 and μ_2 on $\mathcal{A}$ we write $\mu_1 \le \mu_2$ if $\mu_1(A) \le \mu_2(A)$ for all $A \in \mathcal{A}$. In particular, we have $-\mu^- \le \mu \le \mu^+$ for any finite signed measure μ.

The Borel σ-algebras on $\mathbb{I}$ and $\mathbb{I}^2$ will be denoted by $\mathcal{B}(\mathbb{I})$ and $\mathcal{B}(\mathbb{I}^2)$, respectively. Note that $\mathcal{B}(\mathbb{I}^2) = \mathcal{B}(\mathbb{I}) \otimes \mathcal{B}(\mathbb{I})$ is the smallest σ-algebra on $\mathbb{I}^2$ that contains all rectangles of the form $R = [0, x] \times [0, y]$ for some $x, y \in \mathbb{I}$. It also contains all rectangles that are either closed of open on any of their four sides. In particular, it contains all half-open rectangles of the form $R = (x_1, x_2] \times (y_1, y_2]$ for some $x_1, x_2, y_1, y_2 \in \mathbb{I}$ with $x_1 < x_2$ and $y_1 < y_2$.

A bivariate quasi-copula Q is said to *induce a signed measure* on $\mathcal{B}(\mathbb{I}^2)$ if there exists a signed measure μ_Q on $\mathcal{B}(\mathbb{I}^2)$ such that $\mu_Q([0, x] \times [0, y]) = Q(x, y)$ for all $(x, y) \in \mathbb{I}^2$. Measure μ_Q is automatically finite and for any half-open rectangle $R = (x_1, x_2] \times (y_1, y_2] \subseteq \mathbb{I}^2$ we have $\mu_Q(R) = V_Q(R)$, where

$$V_Q(R) = Q(x_2, y_2) - Q(x_1, y_2) - Q(x_2, y_1) + Q(x_1, y_1)$$

is the *volume* of R with respect to Q. Since quasi-copulas are continuous functions, the equality $\mu_Q(R) = V_Q(R)$ holds also if the rectangle R is either open or closed on any of its sides, because the μ_Q measure of a vertical or horizontal segment is 0. For example,

$$\mu_Q\big((x_1, x_2] \times \{y_0\}\big) = \mu_Q\Big(\bigcap_{n=1}^{\infty}(x_1, x_2] \times (\tfrac{n-1}{n}y_0, y_0]\Big) = \lim_{n \to \infty} \mu_Q\big((x_1, x_2] \times (\tfrac{n-1}{n}y_0, y_0]\big)$$
$$= \lim_{n \to \infty} \big(Q(x_2, y_0) - Q(x_1, y_0) - Q(x_2, \tfrac{n-1}{n}y_0) + Q(x_1, \tfrac{n-1}{n}y_0)\big)$$
$$= Q(x_2, y_0) - Q(x_1, y_0) - Q(x_2, y_0) + Q(x_1, y_0) = 0.$$

It is well known that any copula C induces a (positive) measure μ_C on $\mathcal{B}(\mathbb{I}^2)$ and this measure is *stochastic* in the sense that $\mu_C(A \times \mathbb{I}) = \mu_C(\mathbb{I} \times A) = \lambda(A)$, where λ denotes the Lebesgue measure on $\mathbb{I}$.

We recall a result from [7] that will be crucial for our constructions. For a positive integer m we will denote $[m] = \{1, 2, \dots, m\}$.

Theorem 2 *([7, Theorem 10]). For a bivariate quasi-copula Q the following conditions are equivalent.*

(i) *There exist bivariate copulas A and B and real numbers α and β such that*

$$Q(x, y) = \alpha A(x, y) + \beta B(x, y) \qquad \text{for all } (x, y) \in \mathbb{I}^2.$$

(ii) *Quasi-copula Q satisfies the condition $\alpha_Q < \infty$, where*

$$\alpha_Q = \sup_{n \ge 1}\left\{\max_{k \in [2^n]} 2^n \sum_{l=1}^{2^n} V_Q(R_{kl}^n)^+, \max_{l \in [2^n]} 2^n \sum_{k=1}^{2^n} V_Q(R_{kl}^n)^+\right\},$$

$R_{kl}^n = [\tfrac{k-1}{2^n}, \tfrac{k}{2^n}] \times [\tfrac{l-1}{2^n}, \tfrac{l}{2^n}]$ *for all $k, l \in [2^n]$, and $x^+ = \max\{x, 0\}$ for all $x \in \mathbb{R}$.*

Informally speaking, what the coefficient α_Q does, is approximately detect at most how much positive mass quasi-copula Q accumulates on any vertical/horizontal strip relative to the width/height of the strip. With increasing n the detection is more and more accurate.

3. Measure induced by a quasi-copula

We assume throughout this section that Q is a bivariate quasi-copula that induces a signed measure μ_Q on $\mathcal{B}(\mathbb{I}^2)$. We will denote by $\mu_Q = \mu_Q^+ - \mu_Q^-$ the Jordan decomposition of measure μ_Q. Since $\mu_Q^+(\mathbb{I}^2) - \mu_Q^-(\mathbb{I}^2) = 1$, both μ_Q^+ and μ_Q^- are finite measures. Measure μ_Q is stochastic, but measures μ_Q^+ and μ_Q^- are not, unless Q is a copula, in which case $\mu_Q^+ = \mu_Q$ is stochastic and μ_Q^- is the zero measure.

The goal of this section is to construct a sequence of bivariate quasi-copulas Q_N with the following properties:

(a) sequence Q_N converges to Q uniformly,
(b) for every N, Q_N induces a signed measure μ_N,
(c) for every N, Q_N is a linear combination of two copulas,
(d) sequence μ_N converges to μ_Q in the total variation norm.

Let us give an outline of the construction, which is split into several steps to make it easier to follow. The main idea is to start with condition (c), using Theorem 2. We need to approximate quasi-copula Q, which need not satisfy $\alpha_Q < \infty$ (see condition (ii) of Theorem 2), with a quasi-copula Q_N, that does satisfy $\alpha_{Q_N} < \infty$. To this end we identify sets of the form $K_N \times \mathbb{I}$ and $\mathbb{I} \times L_N$ which cause α_Q to be greater than cN for some fixed normalising constant c. These sets are defined in Subsection 3.1. Quasi-copula Q_N (for every $N \ge 1$) is constructed in Subsection 3.3 by "smoothing out" the mass distribution of Q on the set $(K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)$ and leaving it unchanged elsewhere. For this to work, the sets K_N and L_N need to be constructed carefully, because, to ensure

property (a), they need to have small Lebesgue measure, i.e., in the limit when N goes to infinity the measure must tend to 0, and, to ensure property (d), the sets $(K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)$ need to have small $|\mu_Q|$ measure in the same sense. Both of these properties are verified in Subsection 3.2. We then prove property (b) by explicitly constructing signed measures μ_N and showing with a direct calculation that they are induced by quasi-copulas Q_N. This is done in Subsection 3.4, where property (d) is also verified. Finally, in Subsection 3.5 we prove that quasi-copulas Q_N satisfy condition (ii) of Theorem 2, which then give us property (c) and consequently also property (a).

3.1. Construction of sets K_N

We start with the construction of sets K_N, the sets L_N will be defined later. For an integer $n \geq 0$ we introduce a family of open intervals

$$\mathcal{P}_n = \left\{ \left(\tfrac{i-1}{2^n}, \tfrac{i}{2^n} \right) \mid i \in [2^n] \right\},$$

which essentially form a partition of $\mathbb{I}$. For integers $n \geq 0$ and $N \geq 1$ let

$$\mathcal{J}_{n,N} = \left\{ S \in \mathcal{P}_n \mid 2^n \mu_Q^+(S \times \mathbb{I}) > N \cdot \mu_Q^+(\mathbb{I}^2) \right\} \subseteq \mathcal{P}_n \qquad \text{and} \qquad J_{n,N} = \bigcup_{S \in \mathcal{J}_{n,N}} S. \tag{2}$$

Note that for every $S \in \mathcal{P}_n$ we have $\lambda(S) = \tfrac{1}{2^n}$, so that

$$S \in \mathcal{J}_{n,N} \qquad \text{if and only if} \qquad S \in \mathcal{P}_n \text{ and } \frac{\mu_Q^+(S \times \mathbb{I})}{\lambda(S)} > N \cdot \mu_Q^+(\mathbb{I}^2). \tag{3}$$

To make sense of what follows, we make the following remark. Intuitively, the sets $J_{n,N} \times \mathbb{I}$ are the "bad" strips, which make α_Q from condition (ii) of Theorem 2 large and possibly infinite. We will later "smooth out" the mass distribution of Q on the bad strips in order to make α_Q finite. For this to work, we actually need to slightly enlarge the sets $J_{n,N}$, so that the smoothing will also affect the boundary of the bad strips.

For every $n \geq 1$ and $S \in \mathcal{P}_n$ there exists a uniquely determined $\widehat{S} \in \mathcal{P}_{n-1}$ such that $S \subseteq \widehat{S}$. For all $N \geq 1$ let

$$\mathcal{K}_{0,N} = \left\{ \widehat{S} \mid S \in \mathcal{J}_{1,N} \right\} \subseteq \mathcal{P}_0 \qquad \text{and} \qquad K_{0,N} = \bigcup_{A \in \mathcal{K}_{0,N}} A, \tag{4}$$

and define inductively for all $n \geq 1$

$$\mathcal{K}_{n,N} = \left\{ \widehat{S} \mid S \in \mathcal{J}_{n+1,N}, \ S \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset \right\} \subseteq \mathcal{P}_n \qquad \text{and} \qquad K_{n,N} = \bigcup_{A \in \mathcal{K}_{n,N}} A. \tag{5}$$

We give an example to demonstrate the sets introduced above.

Example 3. Let Q be a quasi-copula with mass distributed as depicted in Fig. 1 left, where the unit square $\mathbb{I}^2$ is divided into 16×16 small squares of dimensions $\tfrac{1}{16} \times \tfrac{1}{16}$. The dark gray squares contain a mass of $\tfrac{1}{16}$ while the light gray squares contain a mass of $-\tfrac{1}{16}$ distributed uniformly over the square. All other squares contain no mass.

The corresponding (nonempty) sets $J_{n,N}$ and $K_{n,N}$, where $0 \leq n \leq 4$ and $1 \leq N \leq 2$, are depicted on Fig. 1 right, along with the values $\mu_Q^+(S \times \mathbb{I})$ for $S \in \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ on top (note that $\mu_Q^+(\mathbb{I}^2) = \tfrac{28}{16}$). In particular,

$$J_{0,1} = \emptyset, \quad J_{1,1} = \emptyset, \quad J_{2,1} = (\tfrac{2}{4}, \tfrac{3}{4}), \quad J_{3,1} = (\tfrac{1}{8}, \tfrac{2}{8}) \cup (\tfrac{3}{8}, \tfrac{4}{8}) \cup (\tfrac{4}{8}, \tfrac{5}{8}) \cup (\tfrac{6}{8}, \tfrac{7}{8}),$$

$$J_{4,1} = (\tfrac{1}{16}, \tfrac{2}{16}) \cup (\tfrac{2}{16}, \tfrac{3}{16}) \cup (\tfrac{7}{16}, \tfrac{8}{16}) \cup (\tfrac{8}{16}, \tfrac{9}{16}) \cup (\tfrac{9}{16}, \tfrac{10}{16}) \cup (\tfrac{11}{16}, \tfrac{12}{16}) \cup (\tfrac{12}{16}, \tfrac{13}{16}) \cup (\tfrac{13}{16}, \tfrac{14}{16}),$$

$$K_{0,1} = \emptyset, \quad K_{1,1} = (\tfrac{1}{2}, 1), \quad K_{2,1} = (0, \tfrac{1}{4}) \cup (\tfrac{1}{4}, \tfrac{2}{4}), \quad K_{3,1} = \emptyset, \quad K_{4,1} = \emptyset,$$

$$J_{0,2} = \emptyset, \quad J_{1,2} = \emptyset, \quad J_{2,2} = \emptyset, \quad J_{3,2} = \emptyset, \quad J_{4,2} = (\tfrac{7}{16}, \tfrac{8}{16}),$$

$$K_{0,2} = \emptyset, \quad K_{1,2} = \emptyset, \quad K_{2,2} = \emptyset, \quad K_{3,2} = (\tfrac{3}{8}, \tfrac{4}{8}), \quad K_{4,2} = \emptyset.$$

Next, we give some basic properties of the sets defined above.

Lemma 4. *For all $n \geq 1$ we have*

$$J_{n,N} \subseteq \bigcup_{k=0}^{n-1} K_{k,N}.$$

Proof. We prove this by induction on n. From equations (2) and (4) if follows that $J_{1,N} = \bigcup_{S \in \mathcal{J}_{1,N}} S \subseteq \bigcup_{S \in \mathcal{J}_{1,N}} \widehat{S} = K_{0,N}$, so the claim holds for $n = 1$. Suppose it holds for some $n \geq 1$ and let $S \in \mathcal{J}_{n+1,N}$. Then, by equation (5), either $\widehat{S} \in \mathcal{K}_{n,N}$ or $S \cap \bigcup_{k=0}^{n-1} K_{k,N} \neq \emptyset$.
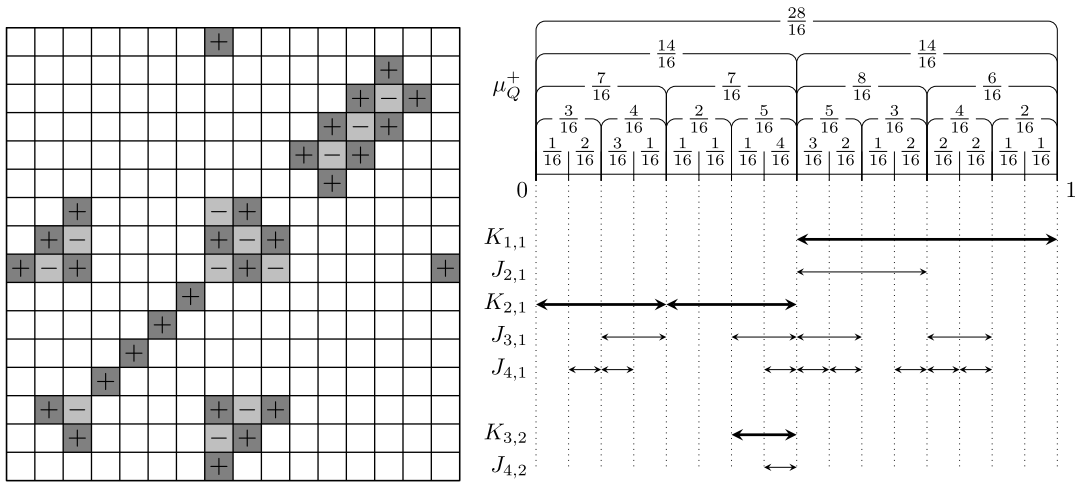
Fig. 1. Mass distribution of a quasi-copula Q from Example 3 (left) and the corresponding sets $J_{n,N}$ and $K_{n,N}$ (right). Each dark gray square contains a mass of $\frac{1}{16}$ and each light gray square contains a mass of $-\frac{1}{16}$.

In the first case $S \subseteq \widehat{S} \subseteq K_{n,N}$. In the second case $S \cap K_{k,N} \neq \emptyset$ for some $0 \leq k \leq n-1$. Since $S \in \mathcal{P}_{n+1}$ and $K_{k,N}$ is a union of some subfamily of $\mathcal{P}_k$, it follows that $S \subseteq K_{k,N}$, because $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_k$ for any $k \leq n$. By equation (2) we conclude that $J_{n+1,N} \subseteq \bigcup_{k=0}^{n} K_{k,N}$, which finishes the proof. $\square$

Lemma 5. *For any $N \geq 1$ the sets $K_{n,N}$ with $n \geq 0$ are disjoint.*

Proof. If $S \in \mathcal{J}_{n+1,N}$, then $\widehat{S} \in \mathcal{P}_n$, while $\bigcup_{k=0}^{n-1} K_{k,N}$ is a union of some subfamily of $\bigcup_{k=0}^{n-1} \mathcal{P}_k$. Hence, either $S \subseteq \widehat{S} \subseteq \bigcup_{k=0}^{n-1} K_{k,N}$ or $\widehat{S} \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset$. So if $S \in \mathcal{J}_{n+1,N}$ and $S \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset$, then $\widehat{S} \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset$. By equation (5) this implies that the sets $K_{n,N}$ and $\bigcup_{k=0}^{n-1} K_{k,N}$ are disjoint. The claim now follows. $\square$

Lemma 5 implies that for every $N \geq 1$ we have a disjoint (double) union

$$K_N = \bigcup_{n=0}^{\infty} K_{n,N} = \bigcup_{n=0}^{\infty} \bigcup_{A \in \mathcal{K}_{n,N}} A. \tag{6}$$

Next lemma shows that the sequence of sets K_N is decreasing in N with respect to inclusion.

Lemma 6. $K_{N+1} \subseteq K_N$ *for all $N \geq 1$.*

Proof. Let $A \in \mathcal{K}_{n,N+1}$ for some $n \geq 0$. Then $A = \widehat{S}$ for some $S \in \mathcal{J}_{n+1,N+1}$. Clearly, $\mathcal{J}_{n+1,N+1} \subseteq \mathcal{J}_{n+1,N}$, so that $S \in \mathcal{J}_{n+1,N}$. From equation (2) and Lemma 4 it follows that $S \subseteq J_{n+1,N} \subseteq \bigcup_{k=0}^{n} K_{k,N} = \bigcup_{k=0}^{n} \bigcup_{B \in \mathcal{K}_{k,N}} B$, so there exists $0 \leq k \leq n$ and $B \in \mathcal{K}_{k,N}$ such that $S \cap B \neq \emptyset$. Since $S \in \mathcal{P}_{n+1}$ and $B \in \mathcal{P}_k$ with $k \leq n$, we infer $S \subseteq B$ and consequently even $A = \widehat{S} \subseteq B$, because $k < n+1$. We have thus shown that $K_{n,N+1} \subseteq \bigcup_{k=0}^{n} K_{k,N}$, which implies $K_{N+1} \subseteq K_N$. $\square$

3.2. Measure-theoretic properties of sets K_N and L_N

Note that all subsets of $\mathbb{I}$ considered above are open, so they are Borel measurable. We will now show that the Lebesgue measure λ of K_N is small.

Lemma 7. $\lambda(K_N) \leq \frac{2}{N}$ *for all $N \geq 1$.*

Proof. By Lemma 5 the double union in (6) is disjoint, so we have

$$\lambda(K_N) = \sum_{n=0}^{\infty} \sum_{A \in \mathcal{K}_{n,N}} \lambda(A).$$

From equations (4) and (5) it follows that

$$\lambda(K_N) \le \sum_{S \in \mathcal{J}_{1,N}} \lambda(\widehat{S}) + \sum_{n=1}^{\infty} \sum_{\substack{S \in \mathcal{J}_{n+1,N} \\ S \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset}} \lambda(\widehat{S}) = \sum_{S \in \mathcal{J}_{1,N}} 2\lambda(S) + \sum_{n=1}^{\infty} \sum_{\substack{S \in \mathcal{J}_{n+1,N} \\ S \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset}} 2\lambda(S).$$

Using property (3) we obtain

$$\lambda(K_N) \le \sum_{S \in \mathcal{J}_{1,N}} \frac{2\mu_Q^+(S \times \mathbb{I})}{N \cdot \mu_Q^+(\mathbb{I}^2)} + \sum_{n=1}^{\infty} \sum_{\substack{S \in \mathcal{J}_{n+1,N} \\ S \cap \bigcup_{k=0}^{n-1} K_{k,N} = \emptyset}} \frac{2\mu_Q^+(S \times \mathbb{I})}{N \cdot \mu_Q^+(\mathbb{I}^2)}.$$

By Lemma 4 we have $\bigcup_{k=1}^{n} J_{k,N} \subseteq \bigcup_{k=0}^{n-1} K_{k,N}$, so we can further estimate

$$\lambda(K_N) \le \sum_{S \in \mathcal{J}_{1,N}} \frac{2\mu_Q^+(S \times \mathbb{I})}{N \cdot \mu_Q^+(\mathbb{I}^2)} + \sum_{n=1}^{\infty} \sum_{\substack{S \in \mathcal{J}_{n+1,N} \\ S \cap \bigcup_{k=1}^{n} J_{k,N} = \emptyset}} \frac{2\mu_Q^+(S \times \mathbb{I})}{N \cdot \mu_Q^+(\mathbb{I}^2)}$$

$$= \frac{2}{N \cdot \mu_Q^+(\mathbb{I}^2)} \cdot \sum_{\substack{S \in \mathcal{P}_1 \\ S \subseteq J_{1,N}}} \mu_Q^+(S \times \mathbb{I}) + \frac{2}{N \cdot \mu_Q^+(\mathbb{I}^2)} \cdot \sum_{n=1}^{\infty} \sum_{\substack{S \in \mathcal{P}_{n+1} \\ S \subseteq J_{n+1,N} \setminus \bigcup_{k=1}^{n} J_{k,N}}} \mu_Q^+(S \times \mathbb{I})$$

$$\le \frac{2}{N \cdot \mu_Q^+(\mathbb{I}^2)} \cdot \mu_Q^+(J_{1,N} \times \mathbb{I}) + \frac{2}{N \cdot \mu_Q^+(\mathbb{I}^2)} \cdot \sum_{n=1}^{\infty} \mu_Q^+\left(\left(J_{n+1,N} \setminus \bigcup_{k=1}^{n} J_{k,N}\right) \times \mathbb{I}\right)$$

$$= \frac{2}{N \cdot \mu_Q^+(\mathbb{I}^2)} \cdot \mu_Q^+\left(\bigcup_{n=1}^{\infty} J_{n,N} \times \mathbb{I}\right) \le \frac{2}{N}. \quad \square$$

We can now give one of the key points of this construction. Let

$$K = \bigcap_{N=1}^{\infty} K_N. \tag{7}$$

The first crucial property of the set K is that its Lebesgue measure is 0.

Lemma 8. *We have $\lambda(K) = \lim_{N \to \infty} \lambda(K_N) = 0$.*

Proof. By Lemma 6 the sequence of sets K_N is decreasing in N. Hence, the conclusion follows directly from equation (7) and Lemma 7. $\quad \square$

Furthermore, the restriction of measure μ_Q to the set $K \times \mathbb{I}$ is the zero measure.

Lemma 9. *The measures μ' defined for all $B \in \mathcal{B}(\mathbb{I}^2)$ by*

$$\mu'(B) = \mu_Q\left(B \cap (K \times \mathbb{I})\right)$$

is the zero measure.

Proof. Let $(x, y) \in \mathbb{I}^2$ be arbitrary and let $R = [0, x] \times [0, y]$. Then

$$|\mu'(R)| = \left|\mu_Q\left(([0, x] \cap K) \times [0, y]\right)\right| = \left|\mu_Q\left(\bigcap_{N=1}^{\infty} ([0, x] \cap K_N) \times [0, y]\right)\right|.$$

Taking into account the monotonicity of K_N, see Lemma 6, and the fact that μ_Q is a finite signed measure, we obtain

$$|\mu'(R)| = \left|\lim_{N \to \infty} \mu_Q\left(([0, x] \cap K_N) \times [0, y]\right)\right|.$$

Using equation (6) and the fact that the union in (6) is disjoint by Lemma 5, we obtain

$$|\mu'(R)| = \left|\lim_{N \to \infty} \sum_{n=0}^{\infty} \sum_{A \in \mathcal{K}_{n,N}} \mu_Q\left(([0, x] \cap A) \times [0, y]\right)\right| \le \lim_{N \to \infty} \sum_{n=0}^{\infty} \sum_{A \in \mathcal{K}_{n,N}} \left|\mu_Q\left(([0, x] \cap A) \times [0, y]\right)\right|. \tag{8}$$

For any rectangle $(a, b\} \times [0, y]$, where $\}$ stands for either $)$ or $]$, the continuity, groundedness and 1-Lipschitz property of quasi-copulas imply

$$\left|\mu_Q\big((a,b\}\times[0,y]\big)\right|=|Q(b,y)-Q(a,y)|\le|b-a|=\lambda((a,b\}),$$

regardless of whether $(a,b\}$ is empty or not. Applying this inequality to the rectangle $([0,x]\cap A)\times[0,y]$ in (8) and using also Lemma 8 we obtain

$$|\mu'(R)|\le\lim_{N\to\infty}\sum_{n=0}^{\infty}\sum_{A\in\mathcal{K}_{n,N}}\lambda([0,x]\cap A)\le\lim_{N\to\infty}\sum_{n=0}^{\infty}\sum_{A\in\mathcal{K}_{n,N}}\lambda(A)=\lim_{N\to\infty}\lambda(K_N)=\lambda(K)=0$$

and consequently $\mu'(R)=0$. To prove that μ' is the zero measure on $\mathcal{B}(\mathbb{I}^2)$ we use a standard argument. By the above the collection $\mathcal{B}'=\{B\in\mathcal{B}(\mathbb{I}^2)\mid\mu'(B)=0\}$ is a σ-algebra that contains all rectangles of the form $[0,x]\times[0,y]$. Hence, $\mathcal{B}'=\mathcal{B}(\mathbb{I}^2)$ and μ' is the zero measure. $\square$

As a consequence to Lemma 9, we obtain the second crucial property of the set K, which will be essential for the convergence of measures constructed later.

Lemma 10. *We have*

$$|\mu_Q|\big(K\times\mathbb{I}\big)=\lim_{N\to\infty}|\mu_Q|\big(K_N\times\mathbb{I}\big)=0.$$

Proof. The sequence $K_N\times\mathbb{I}$ is decreasing in N by Lemma 6, and we have $\bigcap_{N=1}^{\infty}(K_N\times\mathbb{I})=K\times\mathbb{I}$ by equation (7). This implies

$$\lim_{N\to\infty}|\mu_Q|\big(K_N\times\mathbb{I}\big)=|\mu_Q|\big(K\times\mathbb{I}\big).$$

By Lemma 9, the measure μ' defined in that lemma is a zero measure. Denote the support of μ_Q^+ by P^+. Then

$$\mu_Q^+\big(K\times\mathbb{I}\big)=\mu_Q\big(P^+\cap(K\times\mathbb{I})\big)=\mu'(P^+)=0,$$

and similarly $\mu_Q^-\big(K\times\mathbb{I}\big)=0$. The claim now follows since $|\mu_Q|=\mu_Q^++\mu_Q^-$. $\square$

Switching the first and second coordinate in the above constructions, we can analogously define for all $n\ge 0$ and $N\ge 1$ the sets

$$\mathcal{I}_{n,N}=\left\{S\in\mathcal{P}_n\mid 2^n\mu_Q^+(\mathbb{I}\times S)>N\cdot\mu_Q^+(\mathbb{I}^2)\right\}\subseteq\mathcal{P}_n,\qquad I_{n,N}=\bigcup_{S\in\mathcal{I}_{n,N}}S,$$

$$\mathcal{L}_{0,N}=\left\{\hat{S}\mid S\in\mathcal{I}_{1,N}\right\}\subseteq\mathcal{P}_0,\qquad L_{0,N}=\bigcup_{A\in\mathcal{L}_{0,N}}A,$$

$$\mathcal{L}_{n,N}=\left\{\hat{S}\mid S\in\mathcal{I}_{n+1,N},\ S\cap\bigcup_{k=0}^{n-1}L_{k,N}=\emptyset\right\}\subseteq\mathcal{P}_n,\qquad L_{n,N}=\bigcup_{A\in\mathcal{L}_{n,N}}A,$$

$$L_N=\bigcup_{n=0}^{\infty}L_{n,N}=\bigcup_{n=0}^{\infty}\bigcup_{A\in\mathcal{L}_{n,N}}A,\qquad L=\bigcap_{N=1}^{\infty}L_N.\tag{9}$$

By symmetry the versions of Lemmas 7–10 for the sets L_N and L also hold. In particular, $\lambda(L_N)\le\frac{2}{N}$ for all $N\ge 1$, $\lambda(L)=0$, the measure μ'' defined for all $B\in\mathcal{B}(\mathbb{I}^2)$ by

$$\mu''(B)=\mu_Q\big(B\cap(\mathbb{I}\times L)\big)$$

is the zero measure, and the following lemma holds.

Lemma 11. *We have*

$$|\mu_Q|\big(\mathbb{I}\times L\big)=\lim_{N\to\infty}|\mu_Q|\big(\mathbb{I}\times L_N\big)=0.$$

3.3. Construction of quasi-copulas Q_N

We now turn our attention to the construction of a sequence of quasi-copulas Q_N. For every positive integer N denote the complements

$$K_N'=\mathbb{I}\setminus K_N\qquad\text{and}\qquad L_N'=\mathbb{I}\setminus L_N.$$
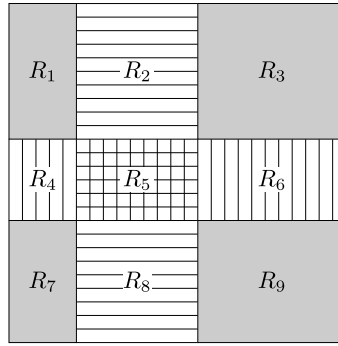
Fig. 2. The extension Q_N of the sub-quasi-copula $Q|_{K'_N \times L'_N}$ defined on the shaded region $K'_N \times L'_N$. The straight sections indicate in which direction the extension Q_N is linear on $(K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)$.

By equations (6) and (9) the sets K_N and L_N are open, so the sets K'_N and L'_N are closed and contain 0 and 1. The restriction $Q|_{K'_N \times L'_N}$ of quasi-copula Q to the set $K'_N \times L'_N$ is a sub-quasi-copula according to [22, Definition 2.3]. We can extend sub-quasi-copula $Q|_{K'_N \times L'_N}$ to a quasi-copula Q_N using the formula from the proof of [22, Theorem 2.3], which we recall now. Let (x, y) be an arbitrary point in $\mathbb{I}^2$ and denote

$$x_1 = \max\left([0, x] \cap K'_N\right), \qquad x_2 = \min\left([x, 1] \cap K'_N\right),$$

$$y_1 = \max\left([0, y] \cap L'_N\right), \qquad y_2 = \min\left([y, 1] \cap L'_N\right).$$

Then $x_1 \le x \le x_2$ and $y_1 \le y \le y_2$. If $x \in K'_N$, then $x_1 = x_2$, and if $y \in L'_N$, then $y_1 = y_2$. Now let

$$\alpha = \begin{cases} \frac{x - x_1}{x_2 - x_1}, & x_1 < x_2, \\ 1, & x_1 = x_2, \end{cases} \quad \text{and} \quad \beta = \begin{cases} \frac{y - y_1}{y_2 - y_1}, & y_1 < y_2, \\ 1, & y_1 = y_2. \end{cases}$$

Then the value of the extension at (x, y) is given by

$$Q_N(x, y) = (1 - \alpha)(1 - \beta)Q(x_1, y_1) + (1 - \alpha)\beta Q(x_1, y_2) + \alpha(1 - \beta)Q(x_2, y_1) + \alpha\beta Q(x_2, y_2). \tag{10}$$

Note that Q_N is bilinear on $[x_1, x_2] \times [y_1, y_2]$. In particular, if R is a closed rectangle such that only the vertices of R lie in $K'_N \times L'_N$, then Q_N is bilinear on R (e.g. R_5 on Fig. 2). If only the left and right side of R lie in $K'_N \times L'_N$, then Q_N is linear in x on R but not necessary in y (e.g. R_2 and R_8 on Fig. 2), if only the bottom and top side of R lie in $K'_N \times L'_N$, then Q_N is linear in y on R but not necessary in x (e.g. R_4 and R_6 on Fig. 2), and if the whole R lies in $K'_N \times L'_N$, then Q_N on R is not linear in any coordinate in general (e.g. R_1, R_3, R_7, and R_9 on Fig. 2). So the extension Q_N is as linear as possible. This means that it spreads mass on $\mathbb{I} \setminus (K'_N \times L'_N) = (K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)$ uniformly in certain directions, which will be important later on.

3.4. Construction of measures μ_N

We aim to prove that quasi-copulas Q_N induce signed measures on $\mathcal{B}(\mathbb{I}^2)$. We now construct these measures. From equations (6) and (9) it follows that the sets K_N and L_N are countable unions of disjoint open intervals, say

$$K_N = \bigcup_{i=0}^{\infty}(a_i^N, b_i^N) \quad \text{and} \quad L_N = \bigcup_{j=0}^{\infty}(c_i^N, d_i^N), \tag{11}$$

where each (a_i^N, b_i^N) and (c_i^N, d_i^N) is a member of $\bigcup_{n=0}^{\infty} \mathcal{P}_n$. We are assuming here without loss of generality that the unions are infinite. If they are finite the arguments are essentially the same.

For a set $E \in \mathcal{B}(\mathbb{I})$ with $\lambda(E) > 0$, define a probability measure λ_E on $\mathcal{B}(\mathbb{I})$ by

$$\lambda_E(A) = \lambda(A \cap E) / \lambda(E)$$

for all $A \in \mathcal{B}(\mathbb{I})$, and finite signed measures φ_E and ψ_E on $\mathcal{B}(\mathbb{I})$ by

$$\varphi_E(A) = \mu_Q\left(E \times (A \cap L'_N)\right) \qquad \text{and} \qquad \psi_E(A) = \mu_Q\left((A \cap K'_N) \times E\right) \tag{12}$$

for all $A \in \mathcal{B}(\mathbb{I})$. Furthermore, introduce finite signed measures $\hat{\mu}_N$ and $\overline{\mu}_N$ on $\mathcal{B}(\mathbb{I}^2)$ by

$$\hat{\mu}_N = \sum_{i=0}^{\infty} \lambda_{(a_i^N, b_i^N)} \times \varphi_{(a_i^N, b_i^N)} + \sum_{j=0}^{\infty} \psi_{(c_j^N, d_j^N)} \times \lambda_{(c_j^N, d_j^N)} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu_Q\left((a_i^N, b_i^N) \times (c_j^N, d_j^N)\right) \cdot \lambda_{(a_i^N, b_i^N)} \times \lambda_{(c_j^N, d_j^N)} \tag{13}$$

and

$$\overline{\mu}_N(B) = \mu_Q\big(B \cap (K'_N \times L'_N)\big) \tag{14}$$

for all $B \in \mathcal{B}(\mathbb{I}^2)$, where $\times$ denotes both the product of measures and the Cartesian product of sets. The idea behind the definition of $\widehat{\mu}_N$ is that it should correspond to the measure induced by the extension Q_N on the set $(K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)$. In particular, the first sum corresponds to the regions where Q_N is linear in x coordinate (regions R_2 and R_8 on Fig. 2), the second sum corresponds to the regions where Q_N is linear in y coordinate (regions R_4 and R_6 on Fig. 2) and the third sum corresponds to the regions where Q_N is bilinear (region R_5 on Fig. 2).

We first need to show that $\widehat{\mu}_N$ is well defined.

Lemma 12. *For every $N \geq 1$ the sum in equation* (13) *converges in the total variation norm, so $\widehat{\mu}_N$ is a finite signed measure on $\mathcal{B}(\mathbb{I}^2)$.*

Proof. For every $E \in \mathcal{B}(\mathbb{I})$ with $\lambda(E) > 0$ let $\varphi_E = \varphi_E^+ - \varphi_E^-$ be the Jordan decomposition of φ_E and denote the supports of φ_E^+ and φ_E^- by P_E^+ and P_E^-, respectively. Since λ_E is a probability measure, and $\lambda_E \times \varphi_E^+$ and $\lambda_E \times \varphi_E^-$ are positive measures, we can estimate using equation (1)

$$\begin{aligned} \|\lambda_E \times \varphi_E\|_{TV} &= \|\lambda_E \times \varphi_E^+ - \lambda_E \times \varphi_E^-\|_{TV} \\ &\leq \|\lambda_E \times \varphi_E^+\|_{TV} + \|\lambda_E \times \varphi_E^-\|_{TV} = \varphi_E^+(\mathbb{I}) + \varphi_E^-(\mathbb{I}) = \|\varphi_E\|_{TV}. \end{aligned} \tag{15}$$

Using equations (1) and (12) we estimate

$$\begin{aligned} \|\varphi_E\|_{TV} = \varphi_E(P_E^+) - \varphi_E(P_E^-) &= \mu_Q\big(E \times (P_E^+ \cap L'_N)\big) - \mu_Q\big(E \times (P_E^- \cap L'_N)\big) \\ &\leq \mu_Q^+\big(E \times (P_E^+ \cap L'_N)\big) + \mu_Q^-\big(E \times (P_E^- \cap L'_N)\big) \\ &\leq \mu_Q^+(E \times L'_N) + \mu_Q^-(E \times L'_N) \\ &= |\mu_Q|(E \times L'_N). \end{aligned} \tag{16}$$

Combining inequalities (15) and (16) gives

$$\|\lambda_E \times \varphi_E\|_{TV} \leq |\mu_Q|(E \times L'_N),$$

and similarly we obtain

$$\|\psi_E \times \lambda_E\|_{TV} \leq |\mu_Q|(K'_N \times E).$$

These two inequalities imply

$$\begin{aligned} &\sum_{i=0}^{\infty} \|\lambda_{(a_i^N, b_i^N)} \times \varphi_{(a_i^N, b_i^N)}\|_{TV} + \sum_{j=0}^{\infty} \|\psi_{(c_j^N, d_j^N)} \times \lambda_{(c_j^N, d_j^N)}\|_{TV} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left\| \mu_Q\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \cdot \lambda_{(a_i^N, b_i^N)} \times \lambda_{(c_j^N, d_j^N)} \right\|_{TV} \\ &\leq \sum_{i=0}^{\infty} |\mu_Q|\big((a_i^N, b_i^N) \times L'_N\big) + \sum_{j=0}^{\infty} |\mu_Q|\big(K'_N \times (c_j^N, d_j^N)\big) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left| \mu_Q\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \right| \\ &\leq \sum_{i=0}^{\infty} |\mu_Q|\big((a_i^N, b_i^N) \times L'_N\big) + \sum_{j=0}^{\infty} |\mu_Q|\big(K'_N \times (c_j^N, d_j^N)\big) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\mu_Q|\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \\ &= |\mu_Q|(K_N \times L'_N) + |\mu_Q|(K'_N \times L_N) + |\mu_Q|(K_N \times L_N) \\ &= |\mu_Q|\big((K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)\big) < \infty. \end{aligned} \tag{17}$$

Since the vector space of all finite measures equipped with the total variation norm is a Banach space, this implies that the sum in equation (13) converges in the total variation norm and $\widehat{\mu}_N$ is a finite signed measure. $\quad\square$

We can now prove that quasi-copulas Q_N induce signed measures. For every $N \geq 1$ define a finite signed measure

$$\mu_N = \widehat{\mu}_N + \overline{\mu}_N,$$

where $\widehat{\mu}_N$ and $\overline{\mu}_N$ are given in equations (13) and (14). We show with a direct calculation that measure μ_N is induced by Q_N.

Lemma 13. *For every $N \geq 1$ quasi-copula Q_N induces signed measure μ_N.*

Proof. For any $(x, y) \in \mathbb{I}^2$ we use equations (13) and (14) to get

$$
\begin{aligned}
\mu_N\big([0, x] \times [0, y]\big) &= \sum_{i=0}^{\infty} \frac{\lambda\big([0, x] \cap (a_i^N, b_i^N)\big)}{\lambda((a_i^N, b_i^N))} \cdot \mu_Q\big((a_i^N, b_i^N) \times ([0, y] \cap L_N')\big) \\
&\quad + \sum_{j=0}^{\infty} \mu_Q\big(([0, x] \cap K_N') \times (c_j^N, d_j^N)\big) \cdot \frac{\lambda\big([0, y] \cap (c_j^N, d_j^N)\big)}{\lambda((c_j^N, d_j^N))} \\
&\quad + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mu_Q\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \cdot \frac{\lambda\big([0, x] \cap (a_i^N, b_i^N)\big)}{\lambda((a_i^N, b_i^N))} \cdot \frac{\lambda\big([0, y] \cap (c_j^N, d_j^N)\big)}{\lambda((c_j^N, d_j^N))} \\
&\quad + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N' \times L_N')\big).
\end{aligned}
\tag{18}
$$

We consider four cases, depending on where the point (x, y) lies.

Case 1: $x \notin K_N$ and $y \notin L_N$. On Fig. 2 this corresponds to $(x, y) \in R_1 \cup R_3 \cup R_7 \cup R_9$. In this case x is not contained in any (a_i^N, b_i^N) and y is not contained in any (c_j^N, d_j^N) so equation (18) becomes

$$
\begin{aligned}
\mu_N\big([0, x] \times [0, y]\big) &= \sum_{(a_i^N, b_i^N) \subseteq [0, x]} \mu_Q\big((a_i^N, b_i^N) \times ([0, y] \cap L_N')\big) + \sum_{(c_j^N, d_j^N) \subseteq [0, y]} \mu_Q\big(([0, x] \cap K_N') \times (c_j^N, d_j^N)\big) \\
&\quad + \sum_{(a_i^N, b_i^N) \subseteq [0, x]} \sum_{(c_j^N, d_j^N) \subseteq [0, y]} \mu_Q\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N' \times L_N')\big) \\
&= \mu_Q\big(([0, x] \cap K_N) \times ([0, y] \cap L_N')\big) + \mu_Q\big(([0, x] \cap K_N') \times ([0, y] \cap L_N)\big) \\
&\quad + \mu_Q\big(([0, x] \cap K_N) \times ([0, y] \cap L_N)\big) + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N' \times L_N')\big) \\
&= \mu_Q\big(([0, x] \times [0, y]) \cap (K_N \times L_N')\big) + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N' \times L_N)\big) \\
&\quad + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N \times L_N)\big) + \mu_Q\big(([0, x] \times [0, y]) \cap (K_N' \times L_N')\big).
\end{aligned}
$$

Since

$$
(K_N \times L_N') \cup (K_N' \times L_N) \cup (K_N \times L_N) \cup (K_N' \times L_N') = \mathbb{I}^2
$$

and this union is disjoint, we conclude

$$
\mu_N\big([0, x] \times [0, y]\big) = \mu_Q\big([0, x] \times [0, y]\big) = Q(x, y).
\tag{19}
$$

By equation (10) this is equal to $Q_N(x, y)$ because $x \in K_N'$ and $y \in L_N'$.

Case 2: $x \in K_N$ and $y \notin L_N$. On Fig. 2 this corresponds to $(x, y) \in R_2 \cup R_8$. We may assume without loss of generality that $x \in (a_0^N, b_0^N)$ since the order of the intervals in equation (11) is arbitrary. On the other hand, y is not contained in any (c_j^N, d_j^N). By splitting $[0, x] \times [0, y] = \big([0, a_0^N] \times [0, y]\big) \cup \big((a_0^N, x] \times [0, y]\big)$, we infer that equation (18) can be written as

$$
\begin{aligned}
\mu_N\big([0, x] \times [0, y]\big) &= \mu_N\big([0, a_0^N] \times [0, y]\big) \\
&\quad + \frac{\lambda((a_0^N, x])}{\lambda((a_0^N, b_0^N))} \cdot \mu_Q\big((a_0^N, b_0^N) \times ([0, y] \cap L_N')\big) \\
&\quad + \sum_{(c_j^N, d_j^N) \subseteq [0, y]} \mu_Q\big(((a_0^N, x] \cap K_N') \times (c_j^N, d_j^N)\big) \\
&\quad + \sum_{(c_j^N, d_j^N) \subseteq [0, y]} \mu_Q\big((a_0^N, b_0^N) \times (c_j^N, d_j^N)\big) \cdot \frac{\lambda((a_0^N, x])}{\lambda((a_0^N, b_0^N))} \\
&\quad + \mu_Q\big(((a_0^N, x] \times [0, y]) \cap (K_N' \times L_N')\big)
\end{aligned}
$$

Note that $a_0^N \notin K_N$, so we may use Case 1 to evaluate the first term in the above expression. Using equation (19), the first term is equal to $\mu_N\big([0, a_0^N] \times [0, y]\big) = Q(a_0^N, y)$. Furthermore, the last term and the first of the two sums $\sum_{(c_j^N, d_j^N) \subseteq [0, y]}$ are equal to 0 because $(a_0^N, x] \subseteq K_N$. Hence,

$$
\begin{aligned}
\mu_N\big([0,x]\times[0,y]\big) &= Q(a_0^N,y) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times([0,y]\cap L_N')\big)\\
&\quad+\frac{x-a_0^N}{b_0^N-a_0^N}\cdot\sum_{(c_j^N,d_j^N)\subseteq[0,y]}\mu_Q\big((a_0^N,b_0^N)\times(c_j^N,d_j^N)\big)\\
&= Q(a_0^N,y) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times([0,y]\cap L_N')\big)\\
&\quad+\frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times([0,y]\cap L_N)\big)\\
&= Q(a_0^N,y) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times[0,y]\big).
\end{aligned}
\tag{20}
$$

Finally, by continuity of Q we can express

$$
\mu_N\big([0,x]\times[0,y]\big)= Q(a_0^N,y) + \frac{x-a_0^N}{b_0^N-a_0^N}\big(Q(b_0^N,y)-Q(a_0^N,y)\big)=\frac{b_0^N-x}{b_0^N-a_0^N}Q(a_0^N,y)+\frac{x-a_0^N}{b_0^N-a_0^N}Q(b_0^N,y).
$$

By equation (10) this is equal to $Q_N(x,y)$ because $x\in K_N$ and $y\in L_N'$.

Case 3: $x\notin K_N$ and $y\in L_N$. On Fig. 2 this corresponds to $(x,y)\in R_4\cup R_6$. This case is treated similarly as Case 2. Assuming $y\in(c_0^N,d_0^N)$, we obtain

$$
\mu_N\big([0,x]\times[0,y]\big)=\frac{d_0^N-y}{d_0^N-c_0^N}Q(x,c_0^N)+\frac{y-c_0^N}{d_0^N-c_0^N}Q(x,d_0^N),
$$

which is again equal to $Q_N(x,y)$ by equation (10).

Case 4: $x\in K_N$ and $y\in L_N$. On Fig. 2 this corresponds to $(x,y)\in R_5$. We may assume without loss of generality that $x\in(a_0^N,b_0^N)$ and $y\in(c_0^N,d_0^N)$. By splitting $[0,x]\times[0,y]=([0,x]\times[0,c_0^N])\cup([0,x]\times(c_0^N,y])$, we infer that equation (18) can be written as

$$
\begin{aligned}
\mu_N\big([0,x]\times[0,y]\big) &= \mu_N\big([0,x]\times[0,c_0^N]\big)\\
&\quad+\sum_{i=0}^{\infty}\frac{\lambda\big([0,x]\cap(a_i^N,b_i^N)\big)}{\lambda((a_i^N,b_i^N))}\cdot\mu_Q\big((a_i^N,b_i^N)\times((c_0^N,y]\cap L_N')\big)\\
&\quad+\mu_Q\big(([0,x]\cap K_N')\times(c_0^N,d_0^N)\big)\cdot\frac{\lambda((c_0^N,y])}{\lambda((c_0^N,d_0^N))}\\
&\quad+\sum_{i=0}^{\infty}\mu_Q\big((a_i^N,b_i^N)\times(c_0^N,d_0^N)\big)\cdot\frac{\lambda\big([0,x]\cap(a_i^N,b_i^N)\big)}{\lambda((a_i^N,b_i^N))}\cdot\frac{\lambda((c_0^N,y])}{\lambda((c_0^N,d_0^N))}\\
&\quad+\mu_Q\big(([0,x]\times(c_0^N,y])\cap(K_N'\times L_N')\big).
\end{aligned}
$$

Note that $c_0^N\notin L_N$, so by Case 2, using equation (20), the first term in the above expression is equal to

$$
\mu_N\big([0,x]\times[0,c_0^N]\big)=Q(a_0^N,c_0^N)+\frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times[0,c_0^N]\big).
$$

Furthermore, the last term and the first of the two sums $\sum_{i=0}^{\infty}$ are equal to 0 because $(c_0^N,y]\subseteq L_N$. Hence,

$$
\begin{aligned}
\mu_N\big([0,x]\times[0,y]\big) &= Q(a_0^N,c_0^N)+\frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times[0,c_0^N]\big)\\
&\quad+\frac{y-c_0^N}{d_0^N-c_0^N}\cdot\mu_Q\big(([0,x]\cap K_N')\times(c_0^N,d_0^N)\big)\\
&\quad+\frac{y-c_0^N}{d_0^N-c_0^N}\cdot\sum_{i=0}^{\infty}\mu_Q\big((a_i^N,b_i^N)\times(c_0^N,d_0^N)\big)\cdot\frac{\lambda\big([0,x]\cap(a_i^N,b_i^N)\big)}{\lambda((a_i^N,b_i^N))}.
\end{aligned}
$$

We can simplify the last sum, also using $x\in(a_0^N,b_0^N)$, to obtain

$$\mu_N\big([0,x]\times[0,y]\big) = Q(a_0^N,c_0^N) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times[0,c_0^N]\big)$$

$$+ \frac{y-c_0^N}{d_0^N-c_0^N}\cdot\mu_Q\big(([0,x]\cap K_N')\times(c_0^N,d_0^N)\big)$$

$$+ \frac{y-c_0^N}{d_0^N-c_0^N}\cdot\sum_{(a_i^N,b_i^N)\subseteq[0,x]}\mu_Q\big((a_i^N,b_i^N)\times(c_0^N,d_0^N)\big) \qquad (21)$$

$$+ \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\frac{y-c_0^N}{d_0^N-c_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times(c_0^N,d_0^N)\big).$$

Note that

$$\bigcup_{(a_i^N,b_i^N)\subseteq[0,x]}(a_i^N,b_i^N) = [0,x]\cap\big(K_N\setminus(a_0^N,x]\big) \qquad\text{and}\qquad K_N'\cup\big(K_N\setminus(a_0^N,x]\big) = \mathbb{I}\setminus(a_0^N,x],$$

so we may combine the second and third row of equation (21) to get

$$\mu_N\big([0,x]\times[0,y]\big) = Q(a_0^N,c_0^N) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times[0,c_0^N]\big)$$

$$+ \frac{y-c_0^N}{d_0^N-c_0^N}\cdot\mu_Q\big([0,a_0^N]\times(c_0^N,d_0^N)\big)$$

$$+ \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\frac{y-c_0^N}{d_0^N-c_0^N}\cdot\mu_Q\big((a_0^N,b_0^N)\times(c_0^N,d_0^N)\big).$$

The continuity of Q implies

$$\mu_Q\big((a_0^N,b_0^N)\times[0,c_0^N]\big) = Q(b_0^N,c_0^N) - Q(a_0^N,c_0^N),$$

$$\mu_Q\big([0,a_0^N]\times(c_0^N,d_0^N)\big) = Q(a_0^N,d_0^N) - Q(a_0^N,c_0^N),$$

$$\mu_Q\big((a_0^N,b_0^N)\times(c_0^N,d_0^N)\big) = Q(b_0^N,d_0^N) - Q(a_0^N,d_0^N) - Q(b_0^N,c_0^N) + Q(a_0^N,c_0^N),$$

so that

$$\mu_N\big([0,x]\times[0,y]\big) = Q(a_0^N,c_0^N) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\big(Q(b_0^N,c_0^N) - Q(a_0^N,c_0^N)\big)$$

$$+ \frac{y-c_0^N}{d_0^N-c_0^N}\cdot\big(Q(a_0^N,d_0^N) - Q(a_0^N,c_0^N)\big)$$

$$+ \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\frac{y-c_0^N}{d_0^N-c_0^N}\cdot\big(Q(b_0^N,d_0^N) - Q(a_0^N,d_0^N) - Q(b_0^N,c_0^N) + Q(a_0^N,c_0^N)\big)$$

$$= \frac{b_0^N-x}{b_0^N-a_0^N}\cdot\frac{d_0^N-y}{d_0^N-c_0^N}\cdot Q(a_0^N,c_0^N) + \frac{b_0^N-x}{b_0^N-a_0^N}\cdot\frac{y-c_0^N}{d_0^N-c_0^N}\cdot Q(a_0^N,d_0^N)$$

$$+ \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\frac{d_0^N-y}{d_0^N-c_0^N}\cdot Q(b_0^N,c_0^N) + \frac{x-a_0^N}{b_0^N-a_0^N}\cdot\frac{y-c_0^N}{d_0^N-c_0^N}\cdot Q(b_0^N,d_0^N).$$

By equation (10) this is again equal to $Q_N(x,y)$ because $x\in K_N$ and $y\in L_N$. $\quad\square$

Next step is to establish the convergence of finite signed measures μ_N.

Lemma 14. *The sequence of measures μ_N converges to μ_Q in the total variation norm.*

Proof. From the proof of Lemma 12, see in particular inequality (17), it follows that

$$\|\widehat{\mu}_N\|_{TV} \le |\mu_Q|\big((K_N\times\mathbb{I})\cup(\mathbb{I}\times L_N)\big) \le |\mu_Q|\big(K_N\times\mathbb{I}\big) + |\mu_Q|\big(\mathbb{I}\times L_N\big).$$

When N tends to infinity, the right-hand side converges to 0 by Lemmas 10 and 11, so the sequence of measures $\widehat{\mu}_N$ converges to the zero measure in the total variation norm. Furthermore, note that

$$(\mu_Q - \overline{\mu}_N)(B) = \mu_Q(B) - \mu_Q\big(B \cap (K'_N \times L'_N)\big) = \mu_Q\big(B \cap ((K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N))\big)$$

for all $B \in \mathcal{B}(\mathbb{I}^2)$. So we can use equation (1) and an analogous calculation as in inequality (16) to obtain

$$\|\mu_Q - \overline{\mu}_N\|_{TV} \leq |\mu_Q|\big((K_N \times \mathbb{I}) \cup (\mathbb{I} \times L_N)\big).$$

Using a similar argument as for $\widehat{\mu}_N$, we infer that the sequence $\mu_Q - \overline{\mu}_N$ converges to the zero measure, so that $\overline{\mu}_N$ converges to μ_Q in the total variation norm. We conclude that the sequence of finite measures $\mu_N = \widehat{\mu}_N + \overline{\mu}_N$ converges to μ_Q. $\quad\square$

3.5. Decomposition of quasi-copulas Q_N

We have so far shown that quasi-copulas Q_N induce signed measures μ_N and the measures μ_N converge in the total variation norm to measure μ_Q, which is induced by Q.

The final property of quasi-copulas Q_N that we need is that every Q_N is a linear combination of two copulas. To prove this we will show that each Q_N satisfies condition (ii) of Theorem 2.

Lemma 15. *For every $N \geq 1$ there exist bivariate copulas A_N and B_N and real numbers α_N and β_N such that $Q_N = \alpha_N A_N + \beta_N B_N$. Consequently, $\mu_N = \alpha_N \mu_{A_N} + \beta_N \mu_{B_N}$.*

Proof. Fix $N \geq 1$. By Theorem 2 it suffices to prove that

$$\alpha_{Q_N} = \sup_{n \geq 1} \left\{ \max_{k \in [2^n]} 2^n \sum_{l=1}^{2^n} V_{Q_N}(R^n_{kl})^+ , \max_{l \in [2^n]} 2^n \sum_{k=1}^{2^n} V_{Q_N}(R^n_{kl})^+ \right\} < \infty,$$

where $R^n_{kl} = [\frac{k-1}{2^n}, \frac{k}{2^n}] \times [\frac{l-1}{2^n}, \frac{l}{2^n}]$ for all $k,l \in [2^n]$ and $x^+ = \max\{x, 0\}$ for all $x \in \mathbb{R}$. Let $n \geq 1$ and $k \in [2^n]$. The continuity of Q_N implies

$$V_{Q_N}(R^n_{kl})^+ = \mu_N(R^n_{kl})^+ = \mu_N\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times (\tfrac{l-1}{2^n}, \tfrac{l}{2^n})\big)^+ \leq \mu_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times (\tfrac{l-1}{2^n}, \tfrac{l}{2^n})\big),$$

therefore

$$2^n \sum_{l=1}^{2^n} V_{Q_N}(R^n_{kl})^+ \leq 2^n \sum_{l=1}^{2^n} \mu_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times (\tfrac{l-1}{2^n}, \tfrac{l}{2^n})\big) \leq 2^n \mu_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times \mathbb{I}\big). \tag{22}$$

Equation $\lambda_E \times \varphi_E = \lambda_E \times \varphi_E^+ - \lambda_E \times \varphi_E^-$ implies $(\lambda_E \times \varphi_E)^+ \leq \lambda_E \times \varphi_E^+$ for all $E \in \mathcal{B}(\mathbb{I})$. Similarly, $(\psi_E \times \lambda_E)^+ \leq \psi_E^+ \times \lambda_E$ for all $E \in \mathcal{B}(\mathbb{I})$. Hence, by equations (13) and (14),

$$\mu_N^+ \leq \widehat{\mu}_N^+ + \overline{\mu}_N^+ \leq \sum_{i=0}^{\infty} \lambda_{(a_i^N, b_i^N)} \times \varphi_{(a_i^N, b_i^N)}^+ + \sum_{j=0}^{\infty} \psi_{(c_j^N, d_j^N)}^+ \times \lambda_{(c_j^N, d_j^N)}$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mu_Q^+\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \cdot \lambda_{(a_i^N, b_i^N)} \times \lambda_{(c_j^N, d_j^N)}$$

$$+ \overline{\mu}_N^+,$$

where we also used $\mu_Q\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big)^+ \leq \mu_Q^+\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big)$. This implies

$$\mu_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times \mathbb{I}\big) \leq \sum_{i=0}^{\infty} \lambda_{(a_i^N, b_i^N)}((\tfrac{k-1}{2^n}, \tfrac{k}{2^n})) \cdot \varphi_{(a_i^N, b_i^N)}^+(\mathbb{I}) + \sum_{j=0}^{\infty} \psi_{(c_j^N, d_j^N)}^+((\tfrac{k-1}{2^n}, \tfrac{k}{2^n})) \cdot \lambda_{(c_j^N, d_j^N)}(\mathbb{I})$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mu_Q^+\big((a_i^N, b_i^N) \times (c_j^N, d_j^N)\big) \cdot \lambda_{(a_i^N, b_i^N)}((\tfrac{k-1}{2^n}, \tfrac{k}{2^n})) \cdot \lambda_{(c_j^N, d_j^N)}(\mathbb{I})$$

$$+ \overline{\mu}_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times \mathbb{I}\big).$$

From equations (12) and (14) we infer

$$\varphi_{(a_i^N, b_i^N)}^+(\mathbb{I}) \leq \mu_Q^+\big((a_i^N, b_i^N) \times L'_N\big),$$

$$\psi_{(c_j^N, d_j^N)}^+((\tfrac{k-1}{2^n}, \tfrac{k}{2^n})) \leq \mu_Q^+\big(((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \cap K'_N) \times (c_j^N, d_j^N)\big),$$

$$\overline{\mu}_N^+\big((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times \mathbb{I}\big) \leq \mu_Q^+\big(((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \times \mathbb{I}) \cap (K'_N \times L'_N)\big) = \mu_Q^+\big(((\tfrac{k-1}{2^n}, \tfrac{k}{2^n}) \cap K'_N) \times L'_N\big),$$

therefore

$$\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big) \le \sum_{i=0}^{\infty}\lambda_{(a_i^N,b_i^N)}\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)\cdot\mu_Q^+\big((a_i^N,b_i^N)\times L_N'\big) + \sum_{j=0}^{\infty}\mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times(c_j^N,d_j^N)\big)$$

$$+\sum_{i=0}^{\infty}\lambda_{(a_i^N,b_i^N)}\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)\sum_{j=0}^{\infty}\mu_Q^+\big((a_i^N,b_i^N)\times(c_j^N,d_j^N)\big) + \mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times L_N'\big)$$

$$=\sum_{i=0}^{\infty}\lambda_{(a_i^N,b_i^N)}\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)\cdot\mu_Q^+\big((a_i^N,b_i^N)\times L_N'\big) + \mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times L_N\big) \tag{23}$$

$$+\sum_{i=0}^{\infty}\lambda_{(a_i^N,b_i^N)}\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)\cdot\mu_Q^+\big((a_i^N,b_i^N)\times L_N\big) + \mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times L_N'\big)$$

$$=\sum_{i=0}^{\infty}\lambda_{(a_i^N,b_i^N)}\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)\cdot\mu_Q^+\big((a_i^N,b_i^N)\times\mathbb{I}\big) + \mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times\mathbb{I}\big)$$

$$=\sum_{i=0}^{\infty}\frac{\lambda\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap(a_i^N,b_i^N)\big)}{\lambda\big((a_i^N,b_i^N)\big)}\cdot\mu_Q^+\big((a_i^N,b_i^N)\times\mathbb{I}\big) + \mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times\mathbb{I}\big).$$

We now consider two cases.

Case 1: $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\subseteq K_N$. Note that $\tfrac{2k-1}{2^{n+1}}\in\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ but $\tfrac{2k-1}{2^{n+1}}\notin\bigcup_{m=n+1}^{\infty}K_{m,N}$ because $\mathcal{K}_{m,N}\subseteq\mathcal{P}_m$. Hence, $\tfrac{2k-1}{2^{n+1}}\in\bigcup_{m=0}^{n}K_{m,N}$ by equation (6). This implies that there exists m with $0\le m\le n$ and $A\in\mathcal{K}_{m,N}$ such that $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ intersects A. But then $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\subseteq A$ because $A\in\mathcal{P}_m$ and $m\le n$. By equation (11) we may assume without loss of generality that $A=(a_0^N,b_0^N)$. Hence, $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ does not intersect any (a_i^N,b_i^N) with $i\ge 1$. It now follows from inequality (23) and from $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'=\emptyset$ that

$$\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)\le\frac{\lambda\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)}{\lambda\big((a_0^N,b_0^N)\big)}\cdot\mu_Q^+\big((a_0^N,b_0^N)\times\mathbb{I}\big),$$

or equivalently

$$2^n\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)\le\frac{\mu_Q^+(A\times\mathbb{I})}{\lambda(A)}.$$

Since $A\in\mathcal{K}_{m,N}$ with $m\le n$ and the set $K_{m,N}$ is disjoint from $\bigcup_{l=0}^{m-1}K_{l,N}$, Lemma 4 and equation (2) imply that $A\notin\mathcal{J}_{m,N}$. By condition (3) we have $\frac{\mu_Q^+(A\times\mathbb{I})}{\lambda(A)}\le N\cdot\mu_Q^+(\mathbb{I}^2)$, hence

$$2^n\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)\le N\cdot\mu_Q^+(\mathbb{I}^2). \tag{24}$$

Case 2: $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\nsubseteq K_N$. Note that each (a_i^N,b_i^N) is a member of $\bigcup_{n=0}^{\infty}\mathcal{P}_n$. Hence, we have three options for each (a_i^N,b_i^N), namely, (i) the interval $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ does not intersect (a_i^N,b_i^N), (ii) the interval $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ is contained in (a_i^N,b_i^N), or (iii) the interval $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ contains (a_i^N,b_i^N). By the case assumption and equation (11), option (ii) cannot happen, so the interval $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)$ contains all (a_i^N,b_i^N) that it intersects. In particular,

$$\bigcup_{(a_i^N,b_i^N)\subseteq(\tfrac{k-1}{2^n},\tfrac{k}{2^n})}(a_i^N,b_i^N)=\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N.$$

Thus, inequality (23) implies

$$\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)\le\sum_{(a_i^N,b_i^N)\subseteq(\tfrac{k-1}{2^n},\tfrac{k}{2^n})}\mu_Q^+\big((a_i^N,b_i^N)\times\mathbb{I}\big)+\mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times\mathbb{I}\big)$$

$$=\mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N\big)\times\mathbb{I}\big)+\mu_Q^+\big(\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\cap K_N'\big)\times\mathbb{I}\big) \tag{25}$$

$$=\mu_Q^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big).$$

Furthermore, $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\notin\mathcal{J}_{n,N}$, otherwise Lemma 4 and equation (6) would imply that $\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\subseteq J_{n,N}\subseteq K_N$. By condition (3) this implies

$$2^n\mu_Q^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)=\frac{\mu_Q^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)}{\lambda\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\big)}\le N\cdot\mu_Q^+(\mathbb{I}^2).$$

This, together with inequality (25), gives

$$2^n\mu_N^+\big(\big(\tfrac{k-1}{2^n},\tfrac{k}{2^n}\big)\times\mathbb{I}\big)\le N\cdot\mu_Q^+(\mathbb{I}^2). \tag{26}$$

Combining inequalities (24) and (26) with inequality (22) gives

$$2^n \sum_{l=1}^{2^n} V_{Q_N}(R_{kl}^n)^+ \le N \cdot \mu_Q^+(\mathbb{I}^2).$$

Since n and k were arbitrary, we infer

$$\sup_{n \ge 1} \left\{ \max_{k \in [2^n]} 2^n \sum_{l=1}^{2^n} V_{Q_N}(R_{kl}^n)^+ \right\} \le N \cdot \mu_Q^+(\mathbb{I}^2).$$

By symmetry we also obtain

$$\sup_{n \ge 1} \left\{ \max_{l \in [2^n]} 2^n \sum_{k=1}^{2^n} V_{Q_N}(R_{kl}^n)^+ \right\} \le N \cdot \mu_Q^+(\mathbb{I}^2).$$

Consequently, $\alpha_{Q_N} \le N \cdot \mu_Q^+(\mathbb{I}^2) < \infty$, as required. $\quad\square$

Let us summarize the results of this section that we will need for the proof of Theorem 1.

Theorem 16. *Let Q be a bivariate quasi-copula that induces a signed measure μ_Q on $\mathcal{B}(\mathbb{I}^2)$. Then there exists a sequence of bivariate quasi-copulas Q_N, $N \ge 1$, such that*

  (i) $Q_N = \alpha_N A_N + \beta_N B_N$ *for some bivariate copulas A_N and B_N and some real numbers α_N and β_N, and*
  (ii) the sequence of induced measures $\mu_N = \alpha_N \mu_{A_N} + \beta_N \mu_{B_N}$ converges to μ_Q in the total variation norm.

It was shown in [7, Corollary 9 and Theorem 10] that the set of quasi-copulas which are linear combinations of two copulas is dense in the set of all quasi-copulas equipped with the topology induced by the supremum norm. As a direct consequence of Theorem 16 we obtain the following variant for the topology induced by the total variation norm of signed measures.

Corollary 17. *The set of quasi-copulas which are linear combinations of two copulas is dense in the set of all measure-inducing quasi-copulas equipped with the total variation norm of the corresponding signed measures.*

4. Proof of the main theorem

In this section we continue assuming that Q is a bivariate quasi-copula that induces a signed measure μ_Q. By Theorem 16 we obtain a sequence of measures μ_N, induced by quasi-copulas Q_N, that converges to μ_Q in the total variation norm. We will now convert the sequence of measures μ_N into a series, which will be manipulated to produce a converging series of multiples of measures induced by copulas. We will employ a method that was used in [7] for converting a sequence of quasi-copulas into a function series converging in the supremum norm. We just need to apply the method to a sequence of measures converging in the total variation norm.

First we construct the series

$$\mu_Q = \mu_1 + \sum_{N=2}^{\infty} (\mu_N - \mu_{N-1}). \tag{27}$$

Partial sums of this series are measures μ_N, so by Theorem 16 the series converges in the total variation norm and its sum is μ_Q. Theorem 16 also implies that $Q_N = \alpha_N A_N + \beta_N B_N$ for some bivariate copulas A_N and B_N and some real numbers α_N and β_N. Note that $\alpha_N + \beta_N = Q_N(1,1) = 1$. If both α_N and β_N are positive, Q_N is a convex combination of copulas, so it is a copula itself and we may assume $\alpha_N = 1$, $\beta_N = 0$, and $A_N = Q_N$. If at least one of α_N and β_N is negative, we may assume $\beta_N < 0$, and consequently $\alpha_N > 1$. So we have $\alpha_N \ge 1$ and $\beta_N \le 0$ for all $N \ge 1$. Furthermore,

$$\mu_N - \mu_{N-1} = \alpha_N \mu_{A_N} + \beta_N \mu_{B_N} - \alpha_{N-1} \mu_{A_{N-1}} - \beta_{N-1} \mu_{B_{N-1}}$$

$$= (\alpha_N - \beta_{N-1}) \cdot \frac{\alpha_N \mu_{A_N} - \beta_{N-1} \mu_{B_{N-1}}}{\alpha_N - \beta_{N-1}} - (\alpha_{N-1} - \beta_N) \cdot \frac{\alpha_{N-1} \mu_{A_{N-1}} - \beta_N \mu_{B_N}}{\alpha_{N-1} - \beta_N}$$

for all $N \ge 2$, where $\alpha_N - \beta_{N-1} \ge 1$ and $\alpha_{N-1} - \beta_N \ge 1$. Since $\alpha_N \ge 1$ and $\beta_N \le 0$ for all $N \ge 1$, the functions

$$D_N = \frac{\alpha_N A_N - \beta_{N-1} B_{N-1}}{\alpha_N - \beta_{N-1}} \quad \text{and} \quad E_N = \frac{\alpha_{N-1} A_{N-1} - \beta_N B_N}{\alpha_{N-1} - \beta_N} \tag{28}$$

for all $N \ge 2$ are convex combinations of copulas, so they are copulas themselves. For all $N \ge 2$ denote also

$$\zeta_N = \alpha_N - \beta_{N-1} \ge 1 \quad \text{and} \quad \xi_N = -(\alpha_{N-1} - \beta_N) \le -1, \tag{29}$$

so that

$$\mu_{N+1} - \mu_N = \zeta_N \mu_{D_N} + \xi_N \mu_{E_N}.$$

In addition, let

$$D_1 = A_1, \quad E_1 = B_1, \quad \zeta_1 = \alpha_1, \quad \text{and} \quad \xi_1 = \beta_1. \tag{30}$$

Then the series in equation (27) is expressed as

$$\mu_Q = \sum_{N=1}^{\infty} (\zeta_N \mu_{D_N} + \xi_N \mu_{E_N}). \tag{31}$$

If we omit the parenthesis in series (31), the resulting series

$$\zeta_1 \mu_{D_1} + \xi_1 \mu_{E_1} + \zeta_2 \mu_{D_2} + \xi_2 \mu_{E_2} + \zeta_3 \mu_{D_3} + \xi_3 \mu_{E_3} + \dots$$

is not convergent in the total variation norm. For example, evaluating it on the set $\mathbb{I}^2$ we obtain the series $\zeta_1 + \xi_1 + \zeta_2 + \xi_2 + \zeta_3 + \xi_3 + \dots$, which is divergent, in fact, oscillating, since $\zeta_N \geq 1$ and $\xi_N \leq -1$. Before we omit the parenthesis in series (31), we need to split its terms into sums of terms with small enough norm, so that after omitting the parenthesis the "oscillation" will tend to 0. For every $N \geq 1$ we choose a positive integer $M_N > |\xi_N|$. We rewrite the series in equation (31) as

$$\mu_Q = \sum_{N=1}^{\infty} \sum_{i=1}^{NM_N} \left(\frac{\zeta_N}{NM_N} \mu_{D_N} + \frac{\xi_N}{NM_N} \mu_{E_N} \right) \tag{32}$$

and remove the parenthesis to obtain the series

$$\underbrace{\frac{\zeta_1}{1M_1} \mu_{D_1} + \frac{\xi_1}{1M_1} \mu_{E_1} + \frac{\zeta_1}{1M_1} \mu_{D_1} + \frac{\xi_1}{1M_1} \mu_{E_1} + \dots + \frac{\zeta_1}{1M_1} \mu_{D_1} + \frac{\xi_1}{1M_1} \mu_{E_1}}_{2M_1 \text{ terms}}$$

$$+ \underbrace{\frac{\zeta_2}{2M_2} \mu_{D_2} + \frac{\xi_2}{2M_2} \mu_{E_2} + \frac{\zeta_2}{2M_2} \mu_{D_2} + \frac{\xi_2}{2M_2} \mu_{E_2} + \dots + \frac{\zeta_2}{2M_2} \mu_{D_2} + \frac{\xi_2}{2M_2} \mu_{E_2}}_{4M_2 \text{ terms}} \tag{33}$$

$$+ \underbrace{\frac{\zeta_3}{3M_3} \mu_{D_3} + \frac{\xi_3}{3M_3} \mu_{E_3} + \frac{\zeta_3}{3M_3} \mu_{D_3} + \frac{\xi_3}{3M_3} \mu_{E_3} + \dots + \frac{\zeta_3}{3M_3} \mu_{D_3} + \frac{\xi_3}{3M_3} \mu_{E_3}}_{6M_3 \text{ terms}}$$

$$+ \dots$$

We now prove the following.

Lemma 18. *The series of finite signed measures* (33) *converges to* μ_Q *in the total variation norm.*

Proof. The difference between μ_Q and any partial sum of series (33), is of the form

$$\Delta = \delta \frac{\xi_m}{mM_m} \mu_{E_m} + \sum_{i=k+1}^{mM_m} \left(\frac{\zeta_m}{mM_m} \mu_{D_m} + \frac{\xi_m}{mM_m} \mu_{E_m} \right) + \sum_{N=m+1}^{\infty} \sum_{i=1}^{NM_N} \left(\frac{\zeta_N}{NM_N} \mu_{D_N} + \frac{\xi_N}{NM_N} \mu_{E_N} \right)$$

$$= \delta \frac{\xi_m}{mM_m} \mu_{E_m} + \frac{mM_m - k}{mM_m} (\zeta_m \mu_{D_m} + \xi_m \mu_{E_m}) + \sum_{N=m+1}^{\infty} (\zeta_N \mu_{D_N} + \xi_N \mu_{E_N})$$

for some $m \geq 1$, $k \in [mM_m]$, and $\delta \in \{0, 1\}$. We can estimate its total variation norm as follows

$$\|\Delta\|_{TV} \leq \delta \frac{|\xi_m|}{mM_m} \cdot \left\| \mu_{E_m} \right\|_{TV} + \frac{mM_m - k}{mM_m} \cdot \left\| \zeta_m \mu_{D_m} + \xi_m \mu_{E_m} \right\|_{TV} + \left\| \sum_{N=m+1}^{\infty} (\zeta_N \mu_{D_N} + \xi_N \mu_{E_N}) \right\|_{TV}.$$

Using $M_m > |\xi_m|$ and the fact that μ_{E_m} is a probability measure, we obtain

$$\|\Delta\|_{TV} \leq \frac{\delta}{m} + \left\| \zeta_m \mu_{D_m} + \xi_m \mu_{E_m} \right\|_{TV} + \left\| \sum_{N=m+1}^{\infty} (\zeta_N \mu_{D_N} + \xi_N \mu_{E_N}) \right\|_{TV}.$$

When m tends to infinity, the first term converges to 0, the second term converges to 0 because it is the norm of a single term of a converging series (31), and the last term converges to 0 because it is the norm of the tail of a converging series (31). This proves that the partial sums of series (33) converge to μ_Q in the total variation norm. $\quad\square$

We are now finally ready to prove Theorem 1.

Proof of Theorem 1.

$(i) \Longrightarrow (ii)$: Assume that a quasi-copula Q induces a signed measure μ_Q on $\mathcal{B}(\mathbb{I}^2)$. Let $C_1, C_2, C_3, \dots$ be the sequence of copulas

$$\underbrace{D_1, E_1, D_1, \dots, D_1, E_1}_{2M_1 \text{ terms}}, \underbrace{D_2, E_2, D_2, \dots, D_2, E_2}_{4M_2 \text{ terms}}, \underbrace{D_3, E_3, D_3, \dots, D_3, E_3}_{6M_3 \text{ terms}}, \dots$$

defined in equations (28) and (30), and let $\gamma_1, \gamma_2, \gamma_3, \dots$ be the sequence of real numbers

$$\underbrace{\frac{\zeta_1}{1M_1}, \frac{\xi_1}{1M_1}, \frac{\zeta_1}{1M_1}, \dots, \frac{\zeta_1}{1M_1}, \frac{\xi_1}{1M_1}}_{2M_1 \text{ terms}}, \underbrace{\frac{\zeta_2}{2M_2}, \frac{\xi_2}{2M_2}, \frac{\zeta_2}{2M_2}, \dots, \frac{\zeta_2}{2M_2}, \frac{\xi_2}{2M_2}}_{4M_2 \text{ terms}}, \underbrace{\frac{\zeta_3}{3M_3}, \frac{\xi_3}{3M_3}, \frac{\zeta_3}{3M_3}, \dots, \frac{\zeta_3}{3M_3}, \frac{\xi_3}{3M_3}}_{6M_3 \text{ terms}}, \dots$$

defined in equations (29) and (30). By Lemma 18 the series $\sum_{n=1}^{\infty} \gamma_n \mu_{C_n}$ converges in the total variation norm to μ_Q, which proves condition (b). For all $k \geq 1$ and $(x, y) \in \mathbb{I}^2$ we have

$$\left| Q(x, y) - \sum_{n=1}^{k} \gamma_n C_n(x, y) \right| = \left| \left(\mu_Q - \sum_{n=1}^{k} \gamma_n \mu_{C_n} \right) \left([0, x] \times [0, y] \right) \right| \leq \left| \mu_Q - \sum_{n=1}^{k} \gamma_n \mu_{C_n} \right| \left([0, x] \times [0, y] \right)$$

$$\leq \left| \mu_Q - \sum_{n=1}^{k} \gamma_n \mu_{C_n} \right| (\mathbb{I}^2) = \left\| \mu_Q - \sum_{n=1}^{k} \gamma_n \mu_{C_n} \right\|_{TV},$$

Hence, the convergence of the series $\sum_{n=1}^{\infty} \gamma_n \mu_{C_n}$ in the total variation norm implies that the series $\sum_{n=1}^{\infty} \gamma_n C_n$ converges uniformly to Q. This proves condition (a).

$(ii) \Longrightarrow (i)$: By condition (b) the sum of the series $\sum_{n=1}^{\infty} \gamma_n \mu_{C_n}$ is a finite signed measure on $\mathcal{B}(\mathbb{I}^2)$. Denote this measure by μ. As above, this implies that the series $\sum_{n=1}^{\infty} \gamma_n C_n$ converges uniformly to the function $(x, y) \mapsto \mu\left([0, x] \times [0, y] \right)$. On the other hand, this series converges to Q by condition (a). Hence, $Q(x, y) = \mu\left([0, x] \times [0, y] \right)$ for all $(x, y) \in \mathbb{I}^2$. $\square$

We note that if condition (i) of Theorem 1 holds, then the series in condition $(ii)(b)$ converges to μ_Q. This implies that a measure induced by a bivariate quasi-copula can always be expressed as a converging sum of multiples of measures induced by bivariate copulas, i.e., as what we can call an *infinite linear combination* of measures induced by copulas.

Corollary 19. *Any measure μ induced by some measure-inducing bivariate quasi-copula can be expressed as $\mu = \sum_{n=1}^{\infty} \gamma_n \mu_n$, where each μ_n is a measure induced by some bivariate copula, each γ_n is a real number, and the series converges in the total variation norm.*

Clearly, the sequences γ_n and μ_n in the above representation are not unique, since we can easily interchange or even combine a few neighboring terms of the series to obtain a different series. If $\sum_{n=1}^{\infty} |\gamma_n| < \infty$ (note that $\|\mu_n\|_{TV} = 1$ for all n), then we can actually collect together all the positive and negative terms of the series and express μ as $\mu = \alpha \mu_A - \beta \mu_B$, where

$$\alpha = \sum_{\gamma_n > 0} |\gamma_n|, \qquad \mu_A = \frac{1}{\alpha} \sum_{\gamma_n > 0} |\gamma_n| \mu_n, \qquad \beta = \sum_{\gamma_n < 0} |\gamma_n|, \qquad \mu_B = \frac{1}{\beta} \sum_{\gamma_n < 0} |\gamma_n| \mu_n.$$

The measures μ_A and μ_B are induced by copulas since they are (infinite) convex combinations of measures induced by copulas. So if the series $\sum_{n=1}^{\infty} \gamma_n$ is absolutely convergent (which includes the case when the series is finite, i.e. $\gamma_n = 0$ for all n large enough), then the quasi-copula that induces μ is a linear combination of two copulas. If μ is not induced by a linear combination of two copulas, then $\sum_{n=1}^{\infty} \gamma_n$ must be conditionally convergent.

As another corollary to Theorem 1 we obtain the following interesting property.

Theorem 20. *Let Q be a bivariate quasi-copula that induces a signed measure μ_Q on $\mathcal{B}(\mathbb{I}^2)$. Then*

$$H(x, y) = \frac{|\mu_Q|\left([0, x] \times [0, y] \right)}{|\mu_Q|(\mathbb{I}^2)} \tag{34}$$

is a joint distribution function of two absolutely continuous random variables X and Y with ranges in $\mathbb{I}^2$.

Proof. Let C_n and γ_n be the sequences from Theorem 1, so that $\mu_Q = \sum_{n=1}^{\infty} \gamma_n \mu_{C_n}$. Function H is clearly a distribution function of two random variables X and Y with support in $\mathbb{I}$. The cumulative distribution of X is given by

$$F_X(x) = \frac{|\mu_Q|\left([0, x] \times \mathbb{I} \right)}{|\mu_Q|(\mathbb{I}^2)}.$$

Let μ_{F_X} be the positive measure induced by F_X, i.e., $\mu_{F_X}(A) = \frac{|\mu_Q|(A \times \mathbb{I})}{|\mu_Q|(\mathbb{I}^2)}$ for all $A \in \mathcal{B}(\mathbb{I})$. Suppose $A \in \mathcal{B}(\mathbb{I})$ satisfies $\lambda(A) = 0$. Let P^+ and P^- be the supports of measures μ_Q^+ and μ_Q^-, respectively. Then

$$
\begin{aligned}
\mu_{F_X}(A) &= \frac{|\mu_Q|(A \times \mathbb{I})}{|\mu_Q|(\mathbb{I}^2)} = \frac{\mu_Q\big((A \times \mathbb{I}) \cap P^+\big) - \mu_Q\big((A \times \mathbb{I}) \cap P^-\big)}{|\mu_Q|(\mathbb{I}^2)} \\
&= \frac{1}{|\mu_Q|(\mathbb{I}^2)} \Big(\sum_{n=1}^{\infty} \gamma_n \mu_{C_n}\big((A \times \mathbb{I}) \cap P^+\big) - \sum_{n=1}^{\infty} \gamma_n \mu_{C_n}\big((A \times \mathbb{I}) \cap P^-\big) \Big).
\end{aligned}
$$

Since μ_{C_n} is a positive measure, $\mu_{C_n}\big((A \times \mathbb{I}) \cap P^+\big) \le \mu_{C_n}(A \times \mathbb{I}) = \lambda(A) = 0$ and similarly $\mu_{C_n}\big((A \times \mathbb{I}) \cap P^-\big) = 0$ for all $n \ge 1$. Hence, $\mu_{F_X}(A) = 0$. This shows that measure μ_{F_X} is absolutely continuous with respect to Lebesgue measure. By Radon-Nikodym theorem there exists a $\mathcal{B}(\mathbb{I})$-measurable function f_X such that $F_X(x) = \mu_{F_X}([0,x]) = \int_{[0,x]} f_X \, d\lambda$. Therefore, X is absolutely continuous random variable with density f_X. Similarly, Y is also absolutely continuous. $\square$

Remark 21. Theorem 20 essentially states that the signed measures $\mu_1(A) = |\mu_Q|(A \times \mathbb{I})$ and $\mu_2(A) = |\mu_Q|(\mathbb{I} \times A)$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{I}$. The conclusion of Theorem 20 holds also if we replace measure $|\mu_Q|$ in formula (34) by either μ_Q^+ or μ_Q^-.

5. Example

We conclude this paper with an example illustrating Theorem 1 and Corollary 19. In [7, Example 13] the authors construct an example of a quasi-copula Q, that induces a finite signed measure, but cannot be written as a linear combination of two copulas. This implies that its induced measure μ_Q cannot be written as a linear combination of two measures induced by copulas. On the other hand, by Corollary 19, measure μ_Q can be expressed as an infinite linear combination of measures induced by copulas. We now find such a representation of μ_Q.

First, we briefly recall the definition of quasi-copula Q, for some additional details see [7, Example 13]. For a positive integer n let Q_n be a discrete quasi-copula defined on an equidistant mesh $\{0, \frac{1}{2n+1}, \frac{2}{2n+1}, \ldots, 1\}^2 \subseteq \mathbb{I}^2$, which has mass distributed in a checkerboard pattern (of positive and negative values) within the central diamond-shaped area (with no mass outside this area), so that there are exactly $(n+1)^2$ squares with positive mass $\frac{1}{2n+1}$ and n^2 squares with negative mass $-\frac{1}{2n+1}$. For example, the spread of mass of Q_3 is given by the matrix

$$
\begin{bmatrix}
0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & 0 & 0 \\
0 & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & 0 \\
\frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \\
0 & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & 0 \\
0 & 0 & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{7} & 0 & 0 & 0
\end{bmatrix}.
$$

If we denote by $\hat{Q}_n$ the bilinear extension of Q_n to $\mathbb{I}^2$, then quasi-copula Q is defined as an ordinal sum of quasi-copulas $\{\hat{Q}_n\}_{n=1}^{\infty}$, with respect to the partition $\mathcal{J} = \{J_n\}_{n=1}^{\infty}$, where $J_n = [a_{n-1}, a_n]$ and $a_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$. Since the length of J_n is $\frac{1}{2^n}$, the summand $\hat{Q}_n$ contributes a total mass of $\frac{1}{2^n}$ to the mass of Q.

Denote the product copula by Π. Note that for every positive integer n the function $\hat{C}_n = \frac{1}{2n}((2n+1)\Pi - \hat{Q}_n)$ is clearly grounded and has uniform marginals. In fact, it is a copula since its mass is nowhere negative and its total mass is equal to 1. We can express

$$
\hat{Q}_n = (2n+1)\Pi - 2n C_n = \Pi + 2n(\Pi - \hat{C}_n). \tag{35}
$$

For a positive integer n denote by C_n the ordinal sum with respect to partition $\mathcal{J}$, where the n-th summand is $\hat{C}_n$ and all other summands are Π. In addition, denote by P the ordinal sum with respect to partition $\mathcal{J}$, where all the summands are Π.

We claim that

$$
\mu_Q = \mu_P + 2(\mu_P - \mu_{C_1}) + 4(\mu_P - \mu_{C_2}) + 6(\mu_P - \mu_{C_3}) + \ldots + 2n(\mu_P - \mu_{C_n}) + \ldots \tag{36}
$$

Indeed, the series on the right converges in the total variation norm because the support of measure $2n(\mu_P - \mu_{C_n})$ is contained in $J_n \times J_n$, so by equation (35)

$$
\|2n(\mu_P - \mu_{C_n})\|_{TV} = \frac{1}{2^n} \|\mu_{\hat{Q}_n} - \mu_{\Pi}\|_{TV} = \frac{1}{2^n} \cdot 2 \cdot (\mu_{\hat{Q}_n} - \mu_{\Pi})^+(\mathbb{I}^2) = \frac{1}{2^n} \cdot 2 \cdot (n+1)^2 \cdot \Big(\frac{1}{2n+1} - \frac{1}{(2n+2)^2} \Big) = \frac{4n(n+1)^2}{2^n(2n+1)^2},
$$

and the series $\sum_{n=1}^{\infty} \frac{4n(n+1)^2}{2^n(2n+1)^2}$ is convergent with sum ≈ 2.842. In addition, on each $J_n \times J_n$ the sum of the series in (36) coincides with μ_Q by equation (35), because only the two terms μ_P and $2n(\mu_P - \mu_{C_n})$ are nonzero there. This implies that the sum of the series in (36) is indeed equal to μ_Q.

Using the trick from the proof of Lemma 18, we obtain from equation (36) the converging sum expression for μ_Q, namely

$$
\begin{aligned}
\mu_Q = \mu_P &+ \underbrace{\mu_P - \mu_{C_1} + \mu_P - \mu_{C_1}}_{4 \text{ terms}} \\
&+ \underbrace{\tfrac{1}{2}\mu_P - \tfrac{1}{2}\mu_{C_2} + \tfrac{1}{2}\mu_P - \tfrac{1}{2}\mu_{C_2} + \ldots + \tfrac{1}{2}\mu_P - \tfrac{1}{2}\mu_{C_2}}_{16 \text{ terms}} \\
&+ \underbrace{\tfrac{1}{3}\mu_P - \tfrac{1}{3}\mu_{C_3} + \tfrac{1}{3}\mu_P - \tfrac{1}{3}\mu_{C_3} + \ldots + \tfrac{1}{3}\mu_P - \tfrac{1}{3}\mu_{C_3}}_{36 \text{ terms}} \\
&+ \ldots \\
&+ \underbrace{\tfrac{1}{n}\mu_P - \tfrac{1}{n}\mu_{C_n} + \tfrac{1}{n}\mu_P - \tfrac{1}{n}\mu_{C_n} + \ldots + \tfrac{1}{n}\mu_P - \tfrac{1}{n}\mu_{C_n}}_{4n^2 \text{ terms}} \\
&+ \ldots
\end{aligned}
$$

Of course, this representation is not unique.

CRediT authorship contribution statement

Nik Stopar: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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