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ON G-DRAZIN PARTIAL ORDER IN RINGS

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Abstract. We extend the concept of a G-Drazin inverse from the set M_n of all $n \times n$ complex matrices to the set $\mathcal{R}^D$ of all Drazin invertible elements in a ring $\mathcal{R}$ with identity. We also generalize a partial order induced by G-Drazin inverses from M_n to the set of all regular elements in $\mathcal{R}^D$, study its properties, compare it to known partial orders, and generalize some known results.

1. Introduction

Generalized inverses and induced partial orders were initially often studied on real and complex matrices. Their development was stimulated by, among other things, a wealth of applications, e.g., in statistics, the theory of differential equations, and numerical analysis, to name but a few (see, e.g., [4,15,17]). Recently, many of those concepts were extended from matrices to rings (satisfying suitable additional conditions) [7,13,24,25,29]. In the

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present paper we follow this trend and generalize a G-Drazin inverse and its induced partial order.

Throughout the paper, the term ring means an associative ring with identity 1. Let $\mathcal{R}$ be a $*$-ring, i.e., a ring equipped with involution $*$. One of the first among generalized inverses was introduced independently by Moore, Bjerhammar, and Penrose [2,19,23] on M_n, the $*$-ring of all $n \times n$ complex matrices. An element $a \in \mathcal{R}$ is said to be *Moore–Penrose invertible* if there exists $x \in \mathcal{R}$ that satisfies the following four equations:

$$(1.1) \qquad axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

If such x exists, it is unique and we write $x = a^\dagger$ and call it *the Moore–Penrose inverse* of a. There are many applications of the Moore–Penrose inverse and in some of them one may get by with a weaker type of a generalized inverse (see, e.g., [6,27]). For a ring $\mathcal{R}$, we say that $a \in \mathcal{R}$ has an inner generalized inverse or $\{1\}$-inverse $a^- \in \mathcal{R}$ if $x = a^-$ satisfies the first equation in (1.1), i.e., $aa^-a = a$. In this case we say that a is a regular element in $\mathcal{R}$. We denote the set of all $\{1\}$-inverses of a by $a\{1\}$ and by $\mathcal{R}^{(1)}$ the set of all regular elements in $\mathcal{R}$. If every element in $\mathcal{R}$ is regular, then we say that $\mathcal{R}$ is a von Neumann regular ring. The Moore–Penrose inverse has on the one hand many properties, that, in general, a $\{1\}$-inverse does not have, but on the other hand, it is often easier to find a $\{1\}$-inverse than the Moore–Penrose inverse.

Let $\mathcal{R}$ be a ring. If $x = a^\sharp$ satisfies the first two equations in (1.1) and commutes with $a \in \mathcal{R}$, then we call $a^\sharp$ *the group inverse* of a (see [11]). It turns out that the group inverse is a special case of an inverse known as *the Drazin inverse* that has many applications in the theories of control theory [3], finite Markov chains [4], singular differential and difference equations [4], cryptography [10], and iterative methods in numerical analysis [15]. We say that an element $a \in \mathcal{R}$ has a Drazin inverse $x = a^D \in \mathcal{R}$ if

$$(1.2) \qquad ax = xa, \quad x = ax^2, \quad a^k = a^{k+1}x$$

for some non-negative integer k. Note that for $k = 0$ we define $a^0 = 1$. If a has a Drazin inverse a^D, then we say that a is *Drazin invertible* and the smallest non-negative integer k in (1.2) is called *the Drazin index* $i(a)$ of a. It is well known that there is at most one $x = a^D$ such that (1.2) holds (see [8]). If $i(a) \leq 1$, the Drazin inverse x of a is the group inverse of a. We denote by $\mathcal{R}^D$ and $\mathcal{R}^\sharp$ the subsets of all Drazin invertible and group invertible elements in $\mathcal{R}$, respectively, and let $\mathcal{R}^{(1,D)} = \mathcal{R}^D \cap \mathcal{R}^{(1)}$.

Campbell and Meyer noted in [4] that it could be sometimes difficult to compute the Drazin invese of $A \in M_n$ and that one way to lessen this problem is to look for a generalized inverse that would play the same role for A^D as $\{1\}$-inverses play for $A^\dagger$. Therefore they introduced the following generalized inverse: For $A \in M_n$ with $i(A) = k$ we say that $X \in M_n$ is *a weak Drazin inverse* of A if $XA^{k+1} = A^k$. A special kind of a weak Drazin inverse called a G-Drazin (or GD) inverse was defined by Wang and Liu in [28]. We now extend [28, Definition 1.1] to rings.

Definition 1.1. Let $\mathcal{R}$ be a ring and let $a \in \mathcal{R}^D$ with $i(a) = k$. We say that $x = a^{GD}$ is a G-Drazin (or GD) inverse of a if

$$axa = a, \quad xa^{k+1} = a^k, \quad a^{k+1}x = a^k.$$

In general, the G-Drazin inverse of $a \in \mathcal{R}^D$, if it exists, is not unique [28]. We denote the set of all G-Drazin inverses of $a \in \mathcal{R}^D$ by $a\{GD\}$.

With all of the mentioned generalized inverses we may define relations on $\mathcal{R}$. For $a, b \in \mathcal{R}$ we say that a is below b with respect to the minus relation and write

$$a \leq^- b \quad \text{if} \quad a^-a = a^-b \ \text{ and } \ aa^- = ba^- \tag{1.3}$$

for some $a^- \in a\{1\}$. It turns out (see [9]) that this relation is a partial order on $\mathcal{R}^{(1)}$. There are many equivalent definitions of the minus partial order (see, e.g., [22]). Moreover, we may extend this partial order to a more general setting of rings or even semigroups [18]. Let $\mathcal{R}$ be a ring. For $a, b \in \mathcal{R}$ we write

$$a \leq^- b \quad \text{if} \quad a = xb = by \ \text{ and } \ xa = a \tag{1.4}$$

for some $x, y \in \mathcal{R}$. It turns out that this is indeed a partial order for any ring $\mathcal{R}$ (see [18]) and that definitions (1.3) and (1.4) are equivalent on $\mathcal{R}^{(1)}$.

The sharp partial order $\leq^\sharp$ was introduced in [16] on the set of all $n \times n$ matrices over a field $\mathbb{F}$ which have the group inverse. This order was generalized in [13] and independently in [25] to rings. Namely, for $a \in \mathcal{R}^\sharp$ and $b \in \mathcal{R}$, we write

$$a \leq^\sharp b \quad \text{if} \quad a^\sharp a = a^\sharp b \ \text{ and } \ aa^\sharp = ba^\sharp. \tag{1.5}$$

It was shown in [13] that $\leq^\sharp$ is indeed a partial order on $\mathcal{R}^\sharp$.

In [28] a new matrix partial order was introduced on M_n in terms of G-Drazin inverses. Namely, for $A, B \in M_n$ we say that A is below B under the G-Drazin order if there exist G-Drazin inverses A_1^{GD} and A_2^{GD} of A such that

$$A_1^{GD}A = A_1^{GD}B \quad \text{and} \quad AA_2^{GD} = BA_2^{GD}.$$

It was proved in [28] that this relation is indeed a partial order. It is the aim of this paper to generalize and study the G-Drazin partial order to the set $\mathcal{R}^D$ of all Drazin invertible elements in a unital ring $\mathcal{R}$. The paper is structured as follows. In the second section we recall the concept of the core-nilpotent decomposition and present some definitions and auxiliary results. In the third section we study G-Drazin invertibility in rings, characterize G-Drazin inverses, and also describe the set of all G-Drazin inverses of a given $a \in \mathcal{R}^{(1,D)}$. In the fourth section we introduce the G-Drazin relation on rings, present some characterizations of this relation, and then show that it is indeed a partial order on the set $\mathcal{R}^{(1,D)}$ of all Drazin invertible, regular elements in a ring $\mathcal{R}$. We then extend the concept of G-Drazin partial order to the set of all Drazin invertible elements in a ring and compare the G-Drazin partial order to some known partial orders.

2. Preliminaries

Let from now on $\mathcal{R}$ be a ring (i.e., an associative ring with identity 1). Denote by $\mathcal{N}(\mathcal{R})$ the set of all nilpotent elements in $\mathcal{R}$. Koliha gave in [12] an equivalent definition of the Drazin inverse. Namely, for $a, b \in \mathcal{R}$, (1.2) is equivalent to

$$ab = ba, \quad b = ab^2, \quad a - a^2 b \in \mathcal{N}(\mathcal{R}). \tag{2.1}$$

Moreover, the index $i(a)$ of a is equal to the nilpotency index of $a - a^2 b$. Suppose $a \in \mathcal{R}^D$. It is known (see [26]) that then

$$a = c + n \tag{2.2}$$

where $c \in \mathcal{R}^\sharp$, $n \in \mathcal{N}(\mathcal{R})$ with index of nilpotency equal to $i(a)$, and $cn = nc = 0$. Then c is called *the core part* of a and n *the nilpotent part* of a. Since $c^\sharp c c^\sharp = c^\sharp$ and $c^\sharp c = c c^\sharp$, it follows $c^\sharp n = 0 = n c^\sharp$, and therefore it is easy to see by (2.1) that $a^D = c^\#$. Since the Drazin inverse of every element in $\mathcal{R}$ is unique if it exists, we may conclude that c and n from (2.2) are unique. In fact,

$$c = a^2 a^D \quad \text{and} \quad n = a - a^2 a^D. \tag{2.3}$$

We refer to $c + n$ as *the core-nilpotent decomposition* of a.

For $a, b \in \mathcal{R}^D$, let $a = c_a + n_a$ and $b = c_b + n_b$ be the core-nilpotent decompositions of a and b respectively, where c_a is the core part of a, c_b is the core part of b, n_a is the nilpotent part of a, and n_b is the nilpotent part of b. The element a is said to be below the element b under *the Drazin order* if $c_a \leq^\sharp c_b$. When this happens, we write $a \leq^D b$ (see [14, Definition 4]). Note that the Drazin order is in fact a pre-order. Namely, since the sharp order (1.5) is a partial order on the set of all group invertible elements in a general ring with identity, it clearly follows that the Drazin order is reflexive and transitive however it is not anti-symmetric (see [17, Example 4.4.5]). The following result which was proved in [14] will be used in the continuation.

PROPOSITION 2.1. *Let $a, b \in \mathcal{R}^D$. The following statements are then equivalent.*

(i) $a \leq^D b$.
(ii) $aa^D = ba^D = a^D b$.
(iii) $a^k b = ba^k = a^{k+1}$ *where* $k = i(a)$.
(iv) $a^k b = ba^k = a^{k+1}$ *for some integer* $k \geq 0$.

If for $p \in \mathcal{R}$, $p^2 = p$, then p is said to be an *idempotent*. The equality $1 = e_1 + e_2 + \cdots + e_n$, where e_1, e_2,..., e_n are idempotents in $\mathcal{R}$ and $e_i e_j = 0$ for $i \neq j$, is called a *decomposition of the identity of* $\mathcal{R}$. Let $1 = e_1 + e_2 + \cdots + e_n$ and $1 = f_1 + f_2 + \cdots + f_n$ be two decompositions of the identity of $\mathcal{R}$. We have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j.$$

Then any $x \in \mathcal{R}$ can be uniquely represented in the following matrix form:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f}$$

where $x_{ij} = e_i x f_j \in e_i \mathcal{R} f_j$. With $e \times f$ we emphasize the use of the decompositions of the identity $1 = e_1 + e_2 + \cdots + e_n$ on the left side and $1 = f_1 + f_2 + \cdots + f_n$ on the right side of $x = 1 \cdot x \cdot 1$. If $x = (x_{ij})_{e \times f}$ and $y = (y_{ij})_{e \times f}$, then $x + y = (x_{ij} + y_{ij})_{e \times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is another decomposition of the identity of $\mathcal{R}$ and $z = (z_{ij})_{f \times g}$, then, by the orthogonality of the idempotents involved, $xz = \big(\sum_{k=1}^{n} x_{ik} z_{kj} \big)_{e \times g}$. Thus, if we have decompositions of the identity of $\mathcal{R}$, then the usual algebraic operations in $\mathcal{R}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{R}$. When $n = 2$ and $p, q \in \mathcal{R}$ are idempotents, we may write

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p \times q}.$$

Here $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$, and $x_{22} = (1-p)x(1-q)$.

Let $a \in \mathcal{R}$ and let a° denote the right annihilator of a, i.e., the set $a^\circ = \{x \in \mathcal{R} : ax = 0\}$. Similarly we denote the left annihilator ${}^\circ a$ of a, i.e., the set ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$. Suppose that $p, q \in \mathcal{R}$ are such idempotents that ${}^\circ a = {}^\circ p$ and $a^\circ = q^\circ$. Observe (or see [7, Lemma 2.2]) that ${}^\circ p = \mathcal{R}(1-p)$ and $q^\circ = (1-q)\mathcal{R}$. It follows that then $a = paq$, i.e.,

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q}. \tag{2.4}$$

Let $a \in \mathcal{R}^D$, $p = aa^D$, and let $a = c_a + n_a$ be the core-nilpotent decomposition of a. Observe that then p is an idempotent and that (see [14, Section 2])

$$a = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p \times p}. \tag{2.5}$$

Here the index of nilpotency of n_a equals $i(a) = k$. Note that

$$p = aa^D = (c_a + n_a)c_a^{\#} = c_a c_a^{\#}. \tag{2.6}$$

We end this section with an auxiliary result which was proved in [14].

Proposition 2.2. *Let $a \in \mathcal{R}^D$, $p = aa^D$, and let $a = c_a + n_a$, be the core-nilpotent decomposition of a. For $b \in \mathcal{R}^D$, we have $a \leq^D b$ if and only if*

$$b = \begin{bmatrix} c_a & 0 \\ 0 & t \end{bmatrix}_{p \times p}$$

where $t \in (1-p)\mathcal{R}(1-p)$.

3. G-Drazin invertibility in rings

In this section we study and characterize G-Drazin inverses of elements in rings. First, let us use the core-nilpotent decomposition to derive the matrix form of a G-Drazin inverse of $a \in \mathcal{R}^{(1,D)}$.

Lemma 3.1. *Let $a \in \mathcal{R}^{(1,D)}$, $p = aa^D$, and let $a = c_a + n_a$ be the core-nilpotent decomposition of a. Then a^{GD} is a G-Drazin inverse of a if and only if*

$$a^{GD} = \begin{bmatrix} c_a^{\sharp} & 0 \\ 0 & n_a^- \end{bmatrix}_{p \times p}$$

where $n_a^- \in n_a\{1\}$.

Proof. Let $i(a) = k$ and let first

$$b = \begin{bmatrix} c_a^{\sharp} & 0 \\ 0 & n_a^- \end{bmatrix}_{p \times p}$$

with $n_a^- \in n_a\{1\}$. Then

$$aba = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p \times p} \begin{bmatrix} c_a^{\sharp} & 0 \\ 0 & n_a^- \end{bmatrix}_{p \times p} \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p \times p}$$

$$= \begin{bmatrix} c_a c_a^\sharp c_a & 0 \\ 0 & n_a n_a^- n_a \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p\times p} = a.$$

We have

$$ba^{k+1} = \begin{bmatrix} c_a^\sharp c_a^{k+1} & 0 \\ 0 & n_a^- n_a^{k+1} \end{bmatrix}_{p\times p} \quad \text{and} \quad a^k = \begin{bmatrix} c_a^k & 0 \\ 0 & n_a^k \end{bmatrix}_{p\times p}.$$

By (1.2) and since $n_a^k = 0$, we obtain $ba^{k+1} = a^k$. Similarly, we get that $a^{k+1}b = a^k$. Thus, $b \in a\{GD\}$.

Conversely, let $x \in a\{GD\}$ and write

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p\times p}.$$

From $xa^{k+1} = a^k$ we obtain

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p\times p} \begin{bmatrix} c_a^{k+1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} x_1 c_a^{k+1} & 0 \\ x_3 c_a^{k+1} & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a^k & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and so $x_1 c_a^{k+1} = c_a^k$ and $x_3 c_a^{k+1} = 0$. By the first equation, we have

$$x_1 c_a^{k+1} \big(c_a^\sharp\big)^{k+1} = c_a^k \big(c_a^\sharp\big)^{k+1},$$

hence $x_1 c_a c_a^\sharp = c_a^\sharp$, and so by (2.6) $x_1 p = c_a^\sharp$. Since $x_1 \in \mathcal{R}p$, $x_1 = c_a^\sharp$. The second equation yields that $x_3 c_a^{k+1} (c_a^\sharp)^{k+1} = 0$ and hence $x_3 c_a c_a^\sharp = 0$. By (2.6), $x_3 p = 0$ and since $x_3 \in \mathcal{R}p$, we have $x_3 = 0$. From $a^{k+1}x = a^k$ we similarly get $c_a^{k+1} x_2 = 0$ and thus $(c_a^\sharp)^{k+1} c_a^{k+1} x_2 = 0$. So, $0 = c_a^\sharp c_a x_2 = p x_2$ and since $x_2 \in p\mathcal{R}$, $x_2 = 0$. From $axa = a$ we now obtain

$$\begin{bmatrix} c_a x_1 c_a & 0 \\ 0 & n_a x_4 n_a \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p\times p}$$

and therefore $n_a x_4 n_a = n_a$. Thus,

$$x = \begin{bmatrix} c_a^\sharp & 0 \\ 0 & x_4 \end{bmatrix}_{p\times p}$$

where $x_4 \in n_a\{1\}$. $\square$

The equivalence between statements (i) and (ii) of the next result recovers [5, Corollary 2.1] which was proved using corresponding matrix decompositions.

THEOREM 3.2. *Let $a \in \mathcal{R}^D$ with $i(a) = k$ and $x \in \mathcal{R}$. The following statements are equivalent*:

(i) $x = a^{GD}$.

(ii) $axa = a$ *and* $a^k x = xa^k$.

(iii) $axa = a$ *and* $a^D x = xa^D$.

(iv) $axa = a$ *and* $a^D ax = xa^D a$.

(v) $axa = a$ *and* $a^2 a^D x = xa^2 a^D$.

PROOF. For $x = a^{GD}$, we know that $axa = a$ and $a^{k+1}x = a^k = xa^{k+1}$. So,

$$a^k x = a^D(a^{k+1}x) = a^D a^k = a^k a^D = xa^{k+1}a^D = xa^k,$$

i.e., (ii) holds.

If $axa = a$ and $a^k x = xa^k$, then $a^{k+1}x = a(a^k x) = axa^k = a^k$ and similarly $xa^{k+1} = a^k$. Hence, $x = a^{GD}$ and (i) is satisfied.

The rest of the equivalences follow as in [20, Corollary 4.3]. □

We can also characterize G-Drazin invertibility by idempotents.

THEOREM 3.3. *Let $a \in \mathcal{R}^D$ with $i(a) = k$. The following statements are equivalent*:

(i) $a\{GD\} \neq \emptyset$.

(ii) *There exist idempotents $p, q \in \mathcal{R}$ such that $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$ and $a^k p = a^k = qa^k$.*

(iii) *There exist idempotents $p, q \in \mathcal{R}$ such that $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$ and $a^D p = qa^D$.*

Moreover, for arbitrary $a^- \in a\{1\}$, $qa^- p \in a\{GD\}$, that is,

$$q \cdot a\{1\} \cdot p \subseteq a\{GD\}.$$

PROOF. Assume that $x = a^{GD}$. Set $p = ax$ and $q = xa$. Then $p = p^2$, $q = q^2$, $p\mathcal{R} = ax\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}xa = \mathcal{R}a$ and $a^k p = a^{k+1}x = a^k = xa^{k+1} = qa^k$.

Let (ii) hold. Since $p\mathcal{R} = a\mathcal{R}$, then $a = pa$ and $p = av$, for some $v \in \mathcal{R}$, which gives $ava = a$ and so a is regular. Denote by $x = qa^- p$, for $a^- \in a\{1\}$. Further, $p = av = aa^-(av) = aa^- p$, and, by $\mathcal{R}q = \mathcal{R}a$, $a = aq$ and $q = qa^- a$. Thus, $axa = (aq)a^- pa = (aa^- p)a = pa = a$,

$$\begin{aligned} xa^{k+1} &= qa^- pa^{k+1} = qa^- a^{k+1} = qa^k = a^k \\ &= a^k p = a^k aa^- p = a^{k+1} qa^- p = a^{k+1} x \end{aligned}$$

which yields that $x \in a\{GD\}$.

The equivalence (i) ⇔ (iii) follows similarly (see also [20]). □

Let $a \in \mathcal{R}^D$, with $i(a) = k$, and let $a = c_a + n_a$ be the core-nilpotent decomposition of a. We have $n_a^k = 0$ and thus $a^l = c_a^l$ for every $l \geq k$. So, for every $l \geq k$, a^l has the group inverse $(c_a^\sharp)^l$ and hence $a^l \in \mathcal{R}^\sharp \subseteq \mathcal{R}^{(1)}$. We now describe the set of all G-Drazin inverses of $a \in \mathcal{R}^{(1,D)}$ and extend [21, Corollary 3.11(ii)].

THEOREM 3.4. *Let $a \in \mathcal{R}^{(1,D)}$ with $i(a) = k$. Then*

$$a\{GD\} = \big\{a^-aa^- + (1-a^-a)a^D + a^D(1-aa^-) + u - a^Dau(1-aa^-) - (1-a^-a)uaa^D - a^-auaa^- : u \in \mathcal{R} \text{ is arbitrary}\big\}$$

for some $a^- \in a\{1\}$ and $(a^{k+1})^- \in (a^{k+1})\{1\}$.

PROOF. For

$$x = a^-aa^- + (1-a^-a)a^D + a^D(1-aa^-) + u - a^Dau(1-aa^-) - (1-a^-a)uaa^D - a^-auaa^-,$$

where $u \in \mathcal{R}$, we can verify that $axa = a$, $xa^{k+1} = a^k$ and $a^{k+1}x = a^k$. So, $x \in a\{GD\}$.

Now, we derive the general solution for the system of equations $axa = a$, $xa^{k+1} = a^k$, and $a^{k+1}x = a^k$. According to [1, p. 52], the equation $axa = a$ has the general solution

$$x = a^-aa^- + y - a^-ayaa^-, \tag{3.1}$$

for an arbitrary $y \in \mathcal{R}$. Substituting (3.1) in $xa^{k+1} = a^k$, we obtain

$$(1-a^-a)ya^{k+1} = (1-a^-a)a^k. \tag{3.2}$$

Since $1-a^-a \in (1-a^-a)\{1\}$, by [1, p. 52], (3.2) yields

$$y = (1-a^-a)a^k(a^{k+1})^- + z - (1-a^-a)za^{k+1}(a^{k+1})^- \tag{3.3}$$

for an arbitrary $z \in \mathcal{R}$. Using $a^{k+1}x = a^k$, (3.1), and (3.3), it follows

$$a^{k+1}z(1-aa^-) = a^k(1-aa^-). \tag{3.4}$$

Applying [1, p. 52] and $1-aa^- \in (1-aa^-)\{1\}$, the general solution to (3.4) is

$$z = (a^{k+1})^-a^k(1-aa^-) + u - (a^{k+1})^-a^{k+1}u(1-aa^-), \tag{3.5}$$

for an arbitrary $u \in \mathcal{R}$. From (3.1), (3.3), and (3.5), one can see that

$$x = a^-aa^- + (1-a^-a)a^D + a^D(1-aa^-) + u - a^Dau(1-aa^-) - (1-a^-a)uaa^D - a^-auaa^-. \qquad \square$$

4. G-Drazin partial order in rings

Let us extend the G-Drazin order to rings.

DEFINITION 4.1. Let $\mathcal{R}$ be a ring and let $a \in \mathcal{R}^D$ and $b \in \mathcal{R}$. We say that a is below b under the G-Drazin order and write $a \leq^{GD} b$ if there exist G-Drazin inverses a_1^{GD} and a_2^{GD} of a such that

$$a_1^{GD}a = a_1^{GD}b \quad \text{and} \quad aa_2^{GD} = ba_2^{GD}. \tag{4.1}$$

REMARK 4.2. For G-Drazin inverses a_1^{GD}, a_2^{GD} of $a \in \mathcal{R}^D$, observe that by Definition 1.1, $a_2^{GD}aa_1^{GD} \in a\{GD\}$. If (4.1) holds, then

$$a_2^{GD}aa_1^{GD}a = a_2^{GD}aa_1^{GD}b \quad \text{and} \quad aa_2^{GD}aa_1^{GD} = ba_2^{GD}aa_1^{GD}$$

and therefore we may equivalently reformulate Definition 4.1 as follows: $a \leq^{GD} b$ if there exists a G-Drazin inverse a^{GD} of a such that

$$a^{GD}a = a^{GD}b \quad \text{and} \quad aa^{GD} = ba^{GD}.$$

COROLLARY 4.3. *Let $a \in \mathcal{R}^D$ and $b \in \mathcal{R}$. The following statements are equivalent.*

(i) $a \leq^{GD} b$.
(ii) $a = ba^{GD}a = aa^{GD}b$ *for some* $a^{GD} \in a\{GD\}$.

PROOF. By Remark 4.2, if $a \leq^{GD} b$, there exists $a^{GD} \in a\{GD\}$ such that $a^{GD}a = a^{GD}b$ and $aa^{GD} = ba^{GD}$. Hence, $a = aa^{GD}a = aa^{GD}b$ and analogously $a = ba^{GD}a$.

Suppose that $a = ba^{GD}a = aa^{GD}b$ for some $a^{GD} \in a\{GD\}$. Then, for $x = a^{GD}aa^{GD}$, we have $x \in a\{GD\}$, $ax = aa^{GD}aa^{GD} = aa^{GD} = ba^{GD}aa^{GD} = bx$ and similarly $xa = xb$. $\square$

With the next theorem we characterize the G-Drazin order in terms of core-nilpotent decomposition.

THEOREM 4.4. *Let $a \in \mathcal{R}^{(1,D)}$, $p = aa^D$, $b \in \mathcal{R}$, and let $a = c_a + n_a$ be the core-nilpotent decomposition of a. Then $a \leq^{GD} b$ if and only if*

$$b = \begin{bmatrix} c_a & 0 \\ 0 & b_4 \end{bmatrix}_{p \times p}$$

where $n_a \leq^- b_4$.

PROOF. Let

$$b = \begin{bmatrix} c_a & 0 \\ 0 & b_4 \end{bmatrix}_{p \times p}$$

with $n_a \leq^- b_4$. There exists $n_a^- \in n_a\{1\}$ such that $n_a^- n_a = n_a^- b_4$ and $n_a n_a^- = b_4 n_a^-$. Let

$$a^{GD} = \begin{bmatrix} c_a^\sharp & 0 \\ 0 & n_a^- \end{bmatrix}_{p\times p}.$$

By Lemma 3.1, $a^{GD} \in a\{GD\}$. Also,

$$a^{GD}a = \begin{bmatrix} c_a^\sharp c_a & 0 \\ 0 & n_a^- n_a \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a^\sharp c_a & 0 \\ 0 & n_a^- b_4 \end{bmatrix}_{p\times p} = a^{GD}b$$

and similarly $aa^{GD} = a^{GD}b$. Thus, $a \leq^{GD} b$.

Conversely, let $a \leq^{GD} b$ and

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p}.$$

There exists $a^{GD} \in a\{GD\}$ such that $a^{GD}a = a^{GD}b$ and $aa^{GD} = ba^{GD}$. By Lemma 3.1 there exists $n_a^- \in n_a\{1\}$ such that

$$a^{GD} = \begin{bmatrix} c_a^\sharp & 0 \\ 0 & n_a^- \end{bmatrix}_{p\times p}.$$

Then $a^{GD}a = a^{GD}b$ yields

$$\begin{bmatrix} c_a^\sharp c_a & 0 \\ 0 & n_a^- n_a \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a^\sharp b_1 & c_a^\sharp b_2 \\ n_a^- b_3 & n_a^- b_4 \end{bmatrix}_{p\times p}$$

and so $c_a^\sharp c_a = c_a^\sharp b_1$, $c_a^\sharp b_2 = 0$, and $n_a^- n_a = n_a^- b_4$. Since $b_1, b_2 \in p\mathcal{R}$, we have $c_a = c_a c_a^\sharp b_1 = pb_1 = b_1$ and $0 = c_a c_a^\sharp b_2 = pb_2 = b_2$. From $aa^{GD} = ba^{GD}$ we get

$$\begin{bmatrix} c_a c_a^\sharp & 0 \\ 0 & n_a n_a^- \end{bmatrix}_{p\times p} = \begin{bmatrix} b_1 c_a^\sharp & b_2 n_a^- \\ b_3 c_a^\sharp & b_4 n_a^- \end{bmatrix}_{p\times p}.$$

Thus, $b_3 c_a^\sharp = 0$ and since $b_3 \in \mathcal{R}p$, $0 = b_3 c_a^\sharp c_a = b_3 p = b_3$. Also, $n_a n_a^- = b_4 n_a^-$. Therefore,

$$b = \begin{bmatrix} c_a & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $n_a \leq^- b_4$. $\square$

The G-Drazin order $\leq^{GD}$ can also be characterized by idempotents.

THEOREM 4.5. *Let* $a \in \mathcal{R}^D$ *with* $i(a) = k$ *and let* $b \in \mathcal{R}$. *The following statements are equivalent*:

(i) $a \leq^{GD} b$.

(ii) *There exist idempotents* $p, q \in \mathcal{R}$ *such that* $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$, $a^k p = a^k = qa^k$ *and* $pb = a = bq$.

(iii) *There exist idempotents* $p, q \in \mathcal{R}$ *such that* $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$, $a^D p = qa^D$ *and* $pb = a = bq$.

PROOF. Using Remark 4.2, the assumption $a \leq^{GD} b$ implies that there exists $x \in a\{GD\}$ such that $xa = xb$ and $ax = bx$. As in the proof of Theorem 3.3, we observe that $p = ax$ and $q = xa$ satisfy $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$, $a^k p = a^k = qa^k$. Also, $pb = a(xb) = axa = a$ and $bq = (bx)a = axa = a$, and hence (i) implies (ii).

Assume that (ii) holds. According to Theorem 3.3, $x = qa^- p \in a\{GD\}$, for $a^- \in a\{1\}$. Since q is an idempotent, $a = bq$ implies $a = aq$, and thus $ax = (aq)a^- p = aa^- p = bqa^- p = bx$. Similarly, $xa = ab$, that is, $a \leq^{GD} b$.

In an analogous manner, we prove that (i) $\Leftrightarrow$ (iii). $\square$

Since every G-Drazin inverse of $a \in \mathcal{R}^D$ is also its inner generalized inverse, it follows by (1.3) and Definition 4.1 that the G-Drazin order implies the minus order, i.e., if $a \leq^{GD} b$ for $a \in \mathcal{R}^D$ and $b \in \mathcal{R}$, then $a \leq^- b$. We will present some constraints under which the converse is true. Let us first prove an auxiliary result.

LEMMA 4.6. *Let* $a \in \mathcal{R}^{(1,D)}$, $p = aa^D$, $b \in \mathcal{R}$, *and let* $a = c_a + n_a$ *be the core-nilpotent decomposition of* a. *Let* $b = b_1 + b_4$ *for some* $b_1 \in p\mathcal{R}p$ *and* $b_4 \in (1-p)\mathcal{R}(1-p)$. *If* $a \leq^- b$, *then* $b_1 = c_a$ *and* $n_a \leq^- b_4$.

PROOF. Since $a \leq^- b$, there exists $x \in \mathcal{R}$ such that $axa = a$, $xa = xb$, and $ax = bx$. Let

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p \times p}.$$

By $axa = a$, we get

$$\begin{bmatrix} c_a x_1 c_a & c_a x_2 n_a \\ n_a x_3 c_a & n_a x_4 n_a \end{bmatrix}_{p \times p} = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p \times p}$$

and therefore $c_a x_1 c_a = c_a$ and $n_a x_4 n_a = n_a$. Multiplying the former equation first from the left and next from the right by $c_a^\sharp$ we obtain

$$c_a x_1 p = p = p x_1 c_a,$$

and since $x_1 \in p\mathcal{R}p$, we get $c_a x_1 = x_1 c_a$ and $x_1 = x_1 c_a x_1$. Thus, $x_1 = c_a^\sharp$. Since $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$, we have by $xa = xb$,

$$\begin{bmatrix} c_a^\sharp c_a & x_2 n_a \\ x_3 c_a & x_4 n_a \end{bmatrix}_{p\times p} = \begin{bmatrix} c_a^\sharp b_1 & x_2 b_4 \\ x_3 b_1 & x_4 b_4 \end{bmatrix}_{p\times p} \tag{4.2}$$

and by $ax = bx$,

$$\begin{bmatrix} c_a c_a^\sharp & c_a x_2 \\ n_a x_3 & n_a x_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} b_1 c_a^\sharp & b_1 x_2 \\ b_4 x_3 & b_4 x_4 \end{bmatrix}_{p\times p}. \tag{4.3}$$

It follows that $p = c_a^\sharp c_a = c_a^\sharp b_1$ and $p = c_a c_a^\sharp = b_1 c_a^\sharp$. So, $c_a^\sharp b_1 = b_1 c_a^\sharp$. Since $b_1, c_a^\sharp \in p\mathcal{R}p$, we also have $b_1 = b_1 c_a^\sharp b_1$ and $c_a^\sharp = c_a^\sharp b_1 c_a^\sharp$. Thus, $b_1 = \left(c_a^\sharp\right)^\sharp = c_a$. By (4.2) and (4.3) we also get $x_4 n_a = x_4 b_4$ and $n_a x_4 = b_4 x_4$, and since $n_a x_4 n_a = n_a$, we establish that $n_a \leq^- b_4$. $\square$

THEOREM 4.7. *Let $a \in \mathcal{R}^{(1,D)}$, $i(a) = k$, and $b \in \mathcal{R}^D$. The following statements are equivalent.*

(i) $a \leq^{GD} b$.
(ii) $a \leq^- b$ *and* $a \leq^D b$.
(iii) $a \leq^- b$ *and* $a^k b = b a^k$.
(iv) $a \leq^- b$, ${}^\circ(a^k) \subseteq {}^\circ(ba^k)$, *and* $(a^k)^\circ \subseteq (a^k b)^\circ$.

PROOF. Let us first show that (i) implies (ii)–(iv). Let $a \leq^{GD} b$. Then $a \leq^- b$. Let $a = c_a + n_a$ be the core-niplotent decomposition of a and let $p = aa^D$. By Theorem 4.4, we may write

$$a = \begin{bmatrix} c_a & 0 \\ 0 & n_a \end{bmatrix}_{p\times p} \quad \text{and} \quad b = \begin{bmatrix} c_a & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $n_a \leq^- b_4$. Since $n_a^k = 0$, we have

$$a^{k+1} = a^k b = b a^k \tag{4.4}$$

and therefore by Proposition 2.1, $a \leq^D b$. It follows that (i) implies (ii) and (iii). Let $sa^k = 0 = a^k z$ for some $s, z \in \mathcal{R}$. By (4.4), $0 = sa^{k+1} = sba^k$ and similarly $0 = a^k bz$. Therefore, (i) also implies (iv).

As above, let from now on a have the form (2.5) and let us first prove that (ii) implies (i). So, suppose $a \leq^- b$ and $a \leq^D b$. By Proposition 2.2,

$$b = \begin{bmatrix} c_a & 0 \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $b_4 \in (1-p)\mathcal{R}(1-p)$. Lemma 4.6 implies then that $n_a \leq^- b_4$. Therefore, by Theorem 4.4, $a \leq^{GD} b$.

To show that (iii) implies (i), assume $a \leq^- b$ and $a^k b = ba^k$. Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p}.$$

Since $n_a^k = 0$, we get

$$\begin{bmatrix} c_a^k b_1 & c_a^k b_2 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} b_1 c_a^k & 0 \\ b_3 c_a^k & 0 \end{bmatrix}_{p \times p}$$

and thus $c_a^k b_2 = 0$ and $b_3 c_a^k = 0$. Multiplying the first equation from the left and the second equation from the right by $(c_a^\sharp)^k$ we get $pb_2 = 0 = b_3 p$. Since $b_2 \in p\mathcal{R}$ and $b_3 \in \mathcal{R}p$, $b_2 = b_3 = 0$. Lemma 4.6 then yields that $b_1 = c_a$ and $n_a \leq^- b_4$, and so by Theorem 4.4, $a \leq^{GD} b$.

Assume now that (iv) holds. Again, let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p}.$$

Since $n_a^k = 0$ and thus $a^k = c_a^k \in p\mathcal{R}p$, we have $(1-p)a^k = 0$ and therefore, by assumption, $(1-p)ba^k = 0$. So,

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{p \times p} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p} \begin{bmatrix} c_a^k & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} 0 & 0 \\ (1-p)b_3 c_a^k & 0 \end{bmatrix}_{p \times p}$$

and hence

$$0 = (1-p)b_3 c_a^k (c_a^\sharp)^k = (1-p)b_3 p^k = (1-p)b_3 p.$$

Since $b_3 \in (1-p)\mathcal{R}p$, $b_3 = 0$. Similarly, $a^k(1-p) = 0$ and thus, by assumption, $a^k b(1-p) = 0$ which yields $0 = pb_2(1-p) = b_2$. Therefore,

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_{p \times p}$$

and hence again by Lemma 4.6 and Theorem 4.4, $a \leq^{GD} b$. □

With Theorems 4.4 and 4.7 we generalized [28, Theorem 3.4]. The G-Drazin order $\leq^{GD}$ is clearly a reflexive relation on $\mathcal{R}^{(1,D)}$. Suppose $a \leq^{GD} b$ and $b \leq^{GD} a$ for some $a, b \in \mathcal{R}^{(1,D)}$. Since then statement (ii) of Theorem 4.7 implies $a \leq^- b$ and $b \leq^- a$, it follows that $a = b$. So, the G-Drazin order is an antisymmetric relation on $\mathcal{R}^{(1,D)}$. Both, the minus order $\le^-$ and the Drazin order $\le^D$ are transitive relations and therefore, again by Theorem 4.7, the G-Drazin order is also a transitive relation. We thus have the following result.

THEOREM 4.8. *The G-Drazin order* $\le^{GD}$ *is a partial order on* $\mathcal{R}^{(1,D)}$.

REMARK 4.9. Note that each of the statements (ii), (iii), (iv) of Theorem 4.7 together with the definition (1.4) of the minus partial order, which holds in a general ring $\mathcal{R}$ with identity, allows us to extend the G-Drazin partial order from the set $\mathcal{R}^{(1,D)}$ to the set $\mathcal{R}^D$. For example, we may define in this more general setting the G-Drazin partial order as follows. Let $a \in \mathcal{R}^D$ and $b \in \mathcal{R}$. Then $a \le^{GD} b$ when $a \le^- b$ and $a \le^D b$. It turns out that such a partial order has already been defined (but not much studied) in [14, Section 5]. It is called the S-minus partial order (see also [17]).

Recall that the Drazin order $\le^D$ is a pre-order on $\mathcal{R}^D$. Namely, the failure of anti-symmetry is due to the fact that the Drazin order ignores the nilpotent parts in the core-nilpotent decomposition. As a modification of the Drazin order so that the nilpotent parts are also involved another partial order was introduced on the set of all all $n \times n$ matrices over a field $\mathbb{F}$ in [17] and later extended in [14] to $\mathcal{R}^D$. In what follows, we use (1.4) as the definition of the minus partial order.

DEFINITION 4.10. Let $a, b \in \mathcal{R}^D$ and let $a = c_a + n_a$ and $b = c_b + n_b$ be the core-nilpotent decompositions of a and b respectively, where c_a is the core part of a, c_b is the core part of b, n_a is the nilpotent part of a, and n_b is the nilpotent part of b. The element a is said to be below the element b under *the C-N partial order* if $c_a \le^\sharp c_b$ and $n_a \le^- n_b$. When this happens, we write $a \le^{\sharp,-} b$.

With [14, Theorem 6] it was proved that the C-N partial order implies the minus partial order, i.e., if for $a, b \in \mathcal{R}^D$, $a \le^{\sharp,-} b$, then $a \le^- b$. It follows by statement (ii) of Theorem 4.7 that the C-N partial order implies the above extension of the G-Drazin partial order (i.e., the S-minus partial order) to the set of all Drazin invertible elements in a ring. The converse implication is in general not true and some constraints under which the S-minus partial implies the C-N partial order were presented in [14].

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