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GRAPHS WITH TOTAL MUTUAL-VISIBILITY NUMBER ZERO AND TOTAL MUTUAL-VISIBILITY IN CARTESIAN PRODUCTS

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Abstract

If G is a graph and $X \subseteq V(G)$, then X is a total mutual-visibility set if every pair of vertices x and y of G admits a shortest x,y-path P with $V(P) \cap X \subseteq \{x,y\}$. The cardinality of a largest total mutual-visibility set of G is the total mutual-visibility number $\mu_{\rm t}(G)$ of G. Graphs with $\mu_{\rm t}(G) = 0$ are characterized as the graphs in which every vertex is the central vertex of a convex P_3. The total mutual-visibility number of Cartesian products is bounded and several exact results proved. For instance, $\mu_{\rm t}(K_n \,\square\, K_m) = \max\{n,m\}$ and $\mu_{\rm t}(T \,\square\, H) = \mu_{\rm t}(T)\mu_{\rm t}(H)$, where T is a tree and H an arbitrary graph. It is also demonstrated that $\mu_{\rm t}(G \,\square\, H)$ can be arbitrary larger than $\mu_{\rm t}(G)\mu_{\rm t}(H)$.

Keywords: mutual-visibility set, total mutual-visibility set, bypass vertex, Cartesian product of graphs, tree.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph and $X \subseteq V(G)$. Then two vertices x and y of X are *X-visible*, if there exists a shortest x, y-path P such that $V(P) \cap X = \{x, y\}$. The set X is a *mutual-visibility set* if its vertices are pairwise X-visible. The cardinality of a largest mutual-visibility set is the *mutual-visibility number* $\mu(G)$ of G and a largest mutual-visibility set is called a *μ-set* of G.

These concepts were introduced and studied for the first time by Di Stefano in [4]. The study was motivated in many ways, notably by the role that mutual-visibility plays in problems arising in the context of distributed computing by mobile entities, and by the fact that vertices in mutual-visibility may represent entities on some nodes of a network that want to efficiently communicate in such a way that the messages do not pass through other entities. We also mention related concepts in computer science that have been explored: distributed computing by mobile entities [5], mutual-visibility tasks [3], and fat entities modelled as disks in the Euclidean plane [12]. A related graph theory topic is the general position in graphs, introduced in [10, 15] and extensively studied by now, cf. [11, 17]. The general position problem was investigated in detail on Cartesian product graphs [6, 8, 9, 13, 14].

In [1], the mutual-visibility problem was studied on Cartesian products and on triangle-free graphs, while in [2] the focus was on strong products. In these studies, the following tools have proven to be extremely useful. We say that $X \subseteq V(G)$ is a *total mutual-visibility set* of G if every pair of vertices x and y of G is X-visible, that is, there exists a shortest x, y-path P with $V(P) \cap X \subseteq \{x, y\}$. Note that, by definition, the empty set is a total mutual-visibility set. The cardinality of a largest total mutual-visibility set of G is the *total mutual-visibility number* $\mu_{\rm t}(G)$ of G. Hence, if the empty set is the only total mutual-visibility set of G, then $\mu_{\rm t}(G) = 0$. Further, X is a *$\mu_{\rm t}$-set* if it is a total mutual-visibility set with $|X| = \mu_{\rm t}(G)$.

As observed in [1], there exist graphs G with $\mu_{\rm t}(G) = 0$. Partial results on such graphs were proved, in particular cactus graphs G with $\mu_{\rm t}(G) = 0$ were characterized. In Section 3 we characterize general graphs G for which $\mu_{\rm t}(G) = 0$ holds as the graphs that contain no bypass vertices. We introduce the latter concept in Section 2, where we also give further definitions needed, recall some know results, and add a few additional preparatory results. In Section 4 we prove bounds on the total mutual-visibility number of Cartesian product graphs and demonstrate their sharpness by several exact results. For instance, $\mu_{\rm t}(K_n \,\square\, K_m) = \max\{n, m\}$ and $\mu_{\rm t}(T \,\square\, H) = \mu_{\rm t}(T)\mu_{\rm t}(H)$, where T is a tree. In Section 5 we continue by the investigation of Cartesian products by considering the estimate $\mu_{\rm t}(G \,\square\, H) \le \mu_{\rm t}(G)\mu_{\rm t}(H)$. It holds in many cases, but on the other hand we show that $\mu_{\rm t}(G \,\square\, H)$ can be arbitrary larger than $\mu_{\rm t}(G)\mu_{\rm t}(H)$. We conclude by suggesting some open problems and directions for further investigation.

2. Preliminaries and Bypass Vertices

We recall here some definitions, for other undefined graph theory concepts, see [7]. All the graphs in this paper are simple and connected, unless stated otherwise. The order of a graph $G = (V(G), E(G))$ is denoted by $n(G)$, the minimum degree of G is denoted by $\delta(G)$, and the subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. $S \subseteq V(G)$ is an *independent set* if $G[S]$ is an edgeless graph. The cardinality of a largest independent set is the *independence number* $\alpha(G)$ of G. A subgraph H of G is *isometric* if for each vertices $x, y \in V(H)$ the distance between them is the same in H and in G. Further, H is *convex* if for each vertices $x, y \in V(H)$, all shortest x, y-paths in G lie completely in H. The *girth* $g(G)$ of a graph G with a cycle is the length of a shortest cycle of G. If τ and τ' are two graph invariants, then we say that G is a (τ, τ')-*graph* if $\tau(G) = \tau'(G)$.

The Cartesian product $G \,\Box\, H$ of two graphs G and H has the vertex set $V(G \,\Box\, H) = V(G) \times V(H)$, vertices (g, h) and (g', h') are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. Given a vertex $h \in V(H)$, the subgraph of $G \,\Box\, H$ induced by the set $\{(g, h) : g \in V(G)\}$ is a G-*layer* and is denoted by G^h. H-layers ${}^gH$ are defined analogously. Each G-layer and each H-layer is isomorphic to G and H, respectively. Moreover, it is also well-known that each layer of a Cartesian product is its convex subgraph, see [7, Lemma 12.3]. We will use this fact later on many times, sometimes implicitly. More generally, a subgraph K of a Cartesian product $G \,\Box\, H$ is convex if and only if the projections of K on G and on H are convex [7, Proposition 13.3].

We next recall some known results on the (total) mutual-visibility number. (Recall that a block graph is a graph in which all blocks are complete.)

Proposition 1 [4, Corollary 4.3]**.** *Let T be a tree and L the set of its leaves. Then L is a mutual-visibility set and $\mu(T) = |L|$.*

Proposition 2 [2, Proposition 3.3]**.** *Block graphs* (*and hence trees and complete graphs*) *and graphs containing a universal vertex are all* $(\mu, \mu_{\rm t})$-*graphs.*

Proposition 3 [2, Proposition 3.1]**.** *Let G be a graph. If $V(G) = \bigcup_{i=1}^{k} V_i$, where $G[V_i]$ is a convex subgraph of G and $\mu_{\rm t}(G[V_i]) = 0$ for each $i \in [k]$, then $\mu_{\rm t}(G) = 0$.*

Here and later on, $[k]$ stands for $\{1, \ldots, k\}$. The following straightforward fact will be used several times later on.

Proposition 4. *If X is a total mutual-visibility set of a graph G and $Y \subseteq X$, then Y is also a total mutual-visibility set of G.*

To conclude the preliminaries we introduce the following concept which appears essential in the investigation of the total mutual-visibility concept. We say that a vertex u of a graph G is a *bypass vertex* if u is not the middle vertex of a convex P_3 in G. Otherwise, u is a *non-bypass vertex*. Let $BP(G)$ denote the set of all bypass vertices of G and let $bp(G) = |BP(G)|$. For instance, if $n \geq 1$, then $BP(K_n) = V(K_n)$ because there are no convex paths P_3 in a complete graph. Hence $bp(K_n) = n$. Similarly, $bp(K_{n,m}) = n + m$ for $n, m \geq 2$. Indeed, if u, v, w induce a P_3 in $K_{n,m}$, then since $n, m \geq 2$, there exists a common neighbor v' of u and w, where $v' \neq v$, hence no P_3 in $K_{n,m}$ is convex. On the other hand, if $n \geq 5$, then $bp(C_n) = 0$.

The basic fact on bypass vertices is the following.

Lemma 5. *If u is a non-bypass vertex of a graph G and X is a total mutual-visibility set of G, then $u \notin X$.*

Proof. Since u is a non-bypass vertex of G, it is the central vertex of a convex P_3. If x and y are the neighbors of u on this P_3 and u would lie in X, then x and y would not be X-visible. Hence $u \notin X$. ■

Lemma 5 implies that

$$\mu_{\mathrm{t}}(G) \leq bp(G). \tag{1}$$

This bound is sharp. If T is a tree with $n(T) \geq 3$, then using Proposition 1 and Proposition 2 we get $\mu_{\mathrm{t}}(T) = bp(T)$. Similarly, $\mu_{\mathrm{t}}(K_n) = bp(K_n) = n$. On the other hand, consider complete bipartite graphs $K_{n,m}$, $n, m \geq 3$. From [2, Corollary 3.6] and [4, Theorem 4.9] we know that $\mu_{\mathrm{t}}(K_{n,m}) = \mu(K_{n,m}) = n+m-2$, but $bp(K_{n,m}) = n+m$. The graph G from Figure 1 is another sporadic example for which the bound (1) is strict. We have $\mu_{\mathrm{t}}(G) = 1$ and $bp(G) = 2$, where $BP(G) = \{g_6, g_7\}$.

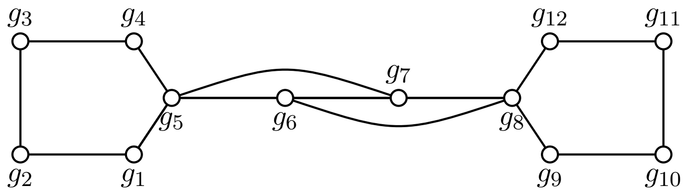

Figure 1. Graph G.

3. GRAPHS WITH $\mu_{\mathrm{t}} = 0$

In this section we characterize graphs with $\mu_{\mathrm{t}} = 0$ and give several applications of the characterization. We begin by two lemmas, where the first one follows immediately from Proposition 4, and the second one being of independent interest.

Lemma 6. *Let G be a graph. Then $\mu_t(G) = 0$ if and only if for every $x \in V(G)$, the set $\{x\}$ is not a total mutual-visibility set of G.*

Lemma 7. *Let G be a graph with $n(G) \geq 2$ and $u \in V(G)$. Then $\{u\}$ is a total mutual-visibility set of G if and only if u is a bypass vertex.*

Proof. First assume that u is a bypass vertex of G. Then by definition, $G - u$ is an isometric subgraph of G. It follows that in G each pair of vertices is connected by shortest path avoiding u, hence $\{u\}$ is a total mutual-visibility set.

The other direction follows by Lemma 5. ■

Note that Lemma 7 does not extend to two vertices. For instance, two opposite vertices of C_4 are bypass vertices, but they do not form a total mutual-visibility set.

The announced characterization now reads as follows.

Theorem 8. *Let G be a graph with $n(G) \geq 2$. Then $\mu_t(G) = 0$ if and only if $bp(G) = 0$.*

Proof. If $\mu_t(G) = 0$, then $bp(G) = 0$ by Lemmas 6 and 7. Conversely, if $bp(G) = 0$, then Lemma 7 says that G has no singleton total mutual-visibility set, and so by Lemma 6, $\mu_t(G) = 0$. ■

Clearly, to check whether a vertex is a bypass vertex is algorithmically simple. Hence the characterization of graphs G with $\mu_t(G) = 0$ from Theorem 8 is efficient.

Note that Theorem 8 implies that if $\mu_t(G) = 0$, then $\delta(G) \geq 2$. Another consequence of the theorem is the following.

Corollary 9. *Let G be a graph with $g(G) \geq 5$. Then $\mu_t(G) = 0$ if and only if $\delta(G) \geq 2$.*

The Petersen graph applies to Corollary 9. In addition, the corollary implies the characterization of cactus graphs G with $\mu_t(G) = 0$ as given in [2, Proposition 3.2]. For a sporadic example of a graph G with $\mu_t(G) = 0$ see Figure 2.

As another application of Theorem 8 we next determine the theta graphs with $\mu_t = 0$. For any positive integer $k \geq 2$ and $1 \leq p_1 \leq \cdots \leq p_k$, the *theta graph* $\Theta(p_1, \ldots, p_k)$ is the graph consisting of two vertices a and b which are joined by k internally disjoint paths of respective lengths $p_1, \ldots, p_k$, where $p_2 \geq 2$. (We add that several authors use the name theta graph restricted to the case $k = 3$ in our definition, cf. [19].)

Corollary 10. *If $1 \leq p_1 \leq \cdots \leq p_k$, $k \geq 2$, $p_2 \geq 2$, then $\mu_t(\Theta(p_1, \ldots, p_k)) = 0$ if and only if the following cases hold:*

(i) *$p_1 = 1$ and $p_2 \geq 4$;*

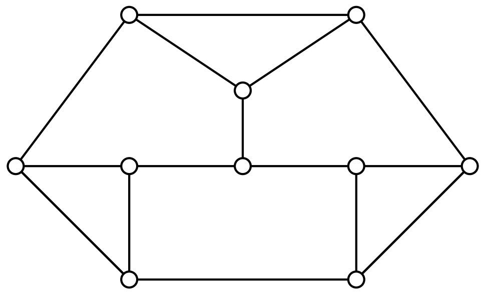

Figure 2. A graph with the total mutual-visibility number 0.

(ii) $p_1 = 2$ *and* $p_2 \geq 3$*;*

(iii) $p_1 \geq 3$.

Proof. Set $\Theta = \Theta(p_1, \ldots, p_k)$ and let $P_1, \ldots, P_k$ be the respective paths of Θ connecting a and b. Then $C^i = P_1 \cup P_i$ is an isometric cycle of Θ for each $i \in \{2, \ldots, k\}$. From this fact we infer that $\mu_{\rm t}(\Theta) = 0$ if and only if each of the cycles C^i is of length at least 5. This condition then yields the cases (i)–(iii). ∎

We conclude the section with a description of Cartesian products with $\mu_{\rm t} = 0$.

Theorem 11. *If G and H are graphs, then $\mu_{\rm t}(G \,\Box\, H) = 0$ if and only if $\mu_{\rm t}(G) = 0$ or $\mu_{\rm t}(H) = 0$.*

Proof. Assume first that $\mu_{\rm t}(G \,\Box\, H) = 0$. Suppose on the contrary that $\mu_{\rm t}(G) \geq 1$ with a total mutual-visibility set X and $\mu_{\rm t}(H) \geq 1$ with a total mutual-visibility set Y. By Proposition 4, there exist two vertices $x \in X$ and $y \in Y$ such that the sets $\{x\}$ and $\{y\}$ are total mutual-visibility set of G and H. Then we claim that $U = \{u\}$ with $u = (x, y)$ is a total mutual-visibility set of $G \,\Box\, H$.

Let v, w be arbitrary vertices of $G \,\Box\, H$. We need to show that they are U-visible. If $v = u$ or $w = u$, there is nothing to be proved. If v and w lie in the same G-layer, then v and w are U-visible because their projections onto G are $\{x\}$-visible and since layers in Cartesian product graphs are convex. By the same argument we see that v and w are U-visible when v and w lie in the same H-layer. The last case to consider is when v and w neither lie in a common G-layer or a common H-layer. Then $w = (g_1, h_1)$ and $v = (g_2, h_2)$, where $g_1 \neq g_2$ and $h_1 \neq h_2$. Then it is well known that there exist two internally disjoint v, w-shortest paths. Since at least one of these two paths does not contain u, the vertices v and w are U-visible in $G \,\Box\, H$ also in this case. Hence we conclude that $\mu_{\rm t}(G \,\Box\, H) \geq 1$.

To prove the converse, we may assume, without loss of generality, that $\mu_{\rm t}(G) = 0$. Since G^h is a convex subgraph of $G \,\Box\, H$ for any $h \in V(H)$, by Proposition 3, we have $\mu_{\rm t}(G \,\Box\, H) = 0$. By symmetry, the same result also holds if $\mu_{\rm t}(H) = 0$, as desired. ∎

Theorem 11 extends to an arbitrary number of factors as follows.

Corollary 12. *If* $G = G_1 \,\square\, \cdots \,\square\, G_k$*, where* $k \geq 2$*, then* $\mu_{\rm t}(G) = 0$ *if and only if* $\mu_{\rm t}(G_i) = 0$ *for at least one* $i \in [k]$*.*

4. TOTAL MUTUAL-VISIBILITY IN CARTESIAN PRODUCTS

In this section we consider the total mutual-visibility number of Cartesian product graphs. In the previous section we have seen that if $\mu_{\rm t}(G) = 0$ or $\mu_{\rm t}(H) = 0$, then $\mu_{\rm t}(G \,\square\, H) = 0$. Hence we may restrict our attention here to factor graphs with the total mutual-visibility number at least 1.

To give general bounds we need the following concept. The *independent total mutual-visibility number* $\mu_{\rm it}(G)$ of G is the cardinality of a largest independent total mutual-visibility set. Setting $\ell(G)$ to be the number of leaves of G, it follows from definitions that for any graph G, $\ell(G) \leq \mu_{\rm it}(G) \leq \min\{\mu_{\rm t}(G), \alpha(G)\}$. From Propositions 1 and 2 we know that the leaves set L is a total mutual-visibility set of a tree T and $\mu_{\rm t}(G) = |L|$. Hence, if $n(T) \geq 3$, then $\mu_{\rm t}(T) = |L| = \mu_{\rm it}(T)$. Note in addition that $\mu_{\rm it}(K_n) = 1$ while $\mu_{\rm t}(K_n) = n$.

Theorem 13. *If* G *and* H *are graphs of order at least* 2*,* $\mu_{\rm t}(G) \geq 1$*, and* $\mu_{\rm t}(H) \geq 1$*, then*

$$\max\{\mu_{\rm it}(H)\mu_{\rm t}(G), \mu_{\rm it}(G)\mu_{\rm t}(H)\} \leq \mu_{\rm t}(G \,\square\, H) \leq \min\{\mu_{\rm t}(G)n(H), \mu_{\rm t}(H)n(G)\}.$$

Proof. Let I_G be an independent total mutual-visibility set of G with $|I_G| = \mu_{\rm it}(G)$, and let X_H be a $\mu_{\rm t}$-set of H. Set $U = I_G \times X_H$. We claim that U is a total mutual-visibility set of $G \,\square\, H$.

Let (g, h) and (g', h') be arbitrary, different vertices of $G \,\square\, H$. If (g, h) and (g', h') lie in the same G-layer or the same H-layer, then x and y are U-visible as layers are convex subgraphs of the product. Hence assume in the rest that $g \neq g'$ and $h \neq h'$.

Let $g = g_0, g_1, \ldots, g_k = g'$ be the consecutive vertices of a shortest g, g'-path in G whose internal vertices are not in I_G. Similarly, let $h = h_0, h_1, \ldots, h_\ell = h'$ be the consecutive vertices of a shortest h, h'-path in H whose internal vertices are not in X_H. Assume first that $k = 1$, that is, $gg' \in E(G)$. If $g \notin I_G$, then the path $(g, h) = (g_0, h_0), (g_0, h_1), \ldots, (g_0, h_\ell), (g_1, h_\ell) = (g', h')$ demonstrates that (g, h) and (g', h') are U-visible. If $g \in I_G$, then $g' \notin I_G$ and then the path $(g, h) = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell) = (g', h')$ demonstrates that (g, h) and (g', h') are U-visible. Assume in the following that $k \geq 2$. Consider now the path P with the consecutive vertices

$$(g, h) = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell), (g_2, h_\ell), \ldots, (g_k, h_\ell) = (g', h').$$

The path P is a shortest $(g,h),(g',h')$-path with no internal vertex in U. Note that in all the above cases it is possible that $\ell = 1$ which happens if $hh' \in E(H)$.

This proves the claim which implies that $\mu_{\rm t}(G \,\square\, H) \geq |U| = \mu_{\rm it}(G)\mu_{\rm t}(H)$. By the commutativity of the Cartesian product, $\mu_{\rm t}(G \,\square\, H) \geq \mu_{\rm it}(H)\mu_{\rm t}(G)$ and the lower bound follows.

Let X be a total mutual-visibility set of $G \,\square\, H$. Since each G-layer G^h is convex in $G \,\square\, H$ we have $|X \cap V(G^h)| \leq \mu_{\rm t}(G)$, hence $\mu_{\rm t}(G \,\square\, H) \leq \mu_{\rm t}(G)n(H)$. Analogously $\mu_{\rm t}(G \,\square\, H) \leq \mu_{\rm t}(H)n(G)$. ■

In the rest of the section we give several exact results on $\mu_{\rm t}(G \,\square\, H)$ which also demonstrate that the bounds of Theorem 13 can be attained. We begin with the following sharpness result for the lower bound.

Corollary 14. *If $\mu_{\rm t}(G \,\square\, H) = 1$, then $\mu_{\rm t}(G) = 1$ and $\mu_{\rm t}(H) = 1$.*

Proof. By Theorem 11 we have $\mu_{\rm t}(G) \geq 1$ and $\mu_{\rm t}(H) \geq 1$. Hence by the lower bound of Theorem 13 we conclude that $\mu_{\rm t}(G) = 1$ and $\mu_{\rm t}(H) = 1$. ■

The converse of Corollary 14 does not hold. For instance, consider the theta graph $\Theta(2,2,4)$ as presented in Figure 3.

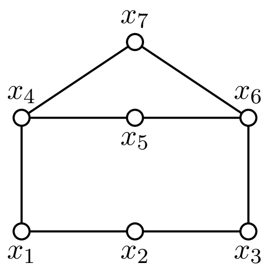

Figure 3. The theta graph $\Theta(2,2,4)$.

It is straightforward to see that $\mu_{\rm t}(\Theta(2,2,4)) = 1$ and that $\{x_5\}$ and $\{x_7\}$ are $\mu_{\rm t}$-sets. In the product $\Theta(2,2,4) \,\square\, \Theta(2,2,4)$ one can see that $\{(x_5,x_5),(x_7,x_7)\}$ is a total mutual-visibility set, hence $\mu_{\rm t}(\Theta(2,2,4) \,\square\, \Theta(2,2,4)) \geq 2$.

[1, Corollary 3.7] asserts that $\mu(K_n \,\square\, K_m) = z(n,m;2,2)$, where $z(n,m;2,2)$ is the Zarankiewitz's number. To determine $z(n,m;2,2)$ is a notorious open problem [16, 18]. Interestingly, the total mutual-visibility number of Cartesian products of complete graphs can be determined as follows, which further demonstrates that the lower bound of Theorem 13 is sharp.

Proposition 15. *If $n,m \geq 2$, then $\mu_{\rm t}(K_n \,\square\, K_m) = \max\{n,m\}$.*

Proof. Note first that the vertices of a single K_n-layer (or a single K_m-layer) form a total mutual-visibility set. Hence $\mu_t(K_n \,\square\, K_m) \geq \max\{n, m\}$. To prove the other inequality, set $V(K_n) = [n]$ and $V(K_m) = [m]$, so that $V(K_n \,\square\, K_m) = [n] \times [m]$. Suppose without loss of generality that $n \geq m$. Let U be an arbitrary total mutual-visibility set of $G \,\square\, H$. If each K_n-layer contains at most one vertex of U there is nothing to be proved. Assume now that some K_n-layer contains (at least) two vertices of U. By the symmetry we may assume that $(1,1) \in U$ and $(2,1) \in U$. We claim first that $(i,j) \notin U$, where $i,j \geq 2$. Indeed, if $(i,j) \in U$, then the vertices $(1,j)$ and $(i,1)$ are not U-visible. We claim second that $(1,j) \notin U$ for $2 \leq j \leq m$. Indeed, if $(1,j) \in U$ for some $j \geq 2$, then the vertices $(1,1)$ and $(2,j)$ are not visible. We conclude that if $(1,1),(2,1) \in U$, then $U \subseteq V(K_n^1)$ and consequently $|U| \leq n$. Analogously we see that if some K_m-layer contains (at least) two vertices of U, then $|U| \leq m$. In any case, $\mu_t(K_n \,\square\, K_m) \leq \max\{n, m\}$. ■

The next result (when $s \in \{3,4\}$) also demonstrates sharpness of the lower bound of Theorem 13.

Proposition 16. *If $s \geq 3$ and $n \geq 3$, then*

$$\mu_t(C_s \,\square\, K_n) = \begin{cases} 0; & s \geq 5, \\ n; & \textit{otherwise.} \end{cases}$$

Proof. If $s \geq 5$, then Corollary 9 and Theorem 11 yield $\mu_t(C_s \,\square\, K_n) = 0$. In addition, $\mu_t(C_3 \,\square\, K_n) = n$ by Proposition 15. Hence assume that $s = 4$ in the remaining proof.

By Theorem 13 we have $\mu_t(C_4 \,\square\, K_n) \geq n$. It remains to demonstrate that $\mu_t(C_4 \,\square\, K_n) \leq n$. Let R be a μ_t-set of $C_4 \,\square\, K_n$. If each C_4-layer contains at most one vertex of R, then $\mu_t(C_4 \,\square\, K_n) \leq n$ holds clearly. Suppose next that R contains at least two vertices from the some C_4-layer. We may without loss of generality assume that $(1,n),(2,n) \in R$. Suppose that there exists another vertex $(i,j) \in R$, where $i \in [4] \backslash [2]$ and $j \in [n-1]$. If $i = 3$, then the two vertices $(2,j)$ and $(3,n)$ are not R-visible. Similarly, the vertices $(1,j)$ and $(4,n)$ are not R-visible. This would thus mean that $|R| = 2$. We conclude that $\mu_t(C_4 \,\square\, K_n) = n$. ■

Theorem 17. *If T is tree with $n(T) \geq 3$ and H is a graph with $n(H) \geq 2$, then $\mu_t(T \,\square\, H) = \mu_t(T)\mu_t(H)$.*

Proof. The lower bound $\mu_t(T \,\square\, H) \geq \mu_t(T)\mu_t(H)$ follows by Theorem 13 and the fact that $\mu_t(T) = \mu_{it}(T)$.

To prove that $\mu_t(T \,\square\, H) \leq \mu_t(T)\mu_t(H)$, consider an arbitrary μ_t-set R of $T \,\square\, H$. Let $t \in V(T)$ be a vertex with $\deg_T(t) \geq 2$. Then t is a non-bypass vertex of T. Hence, if $h \in V(H)$, then (t,h) is a non-bypass vertex of $T \,\square\, H$, thus

$(t,h) \notin R$ by Lemma 5. Therefore, $R \cap V({}^tH) = \emptyset$. So R contains only vertices in H-layers corresponding to the leaves of T. Since each such layer can contain at most $\mu_t(H)$ vertices of R we conclude that $\mu_t(T \,\square\, H) \leq |R| \leq \mu_t(T)\mu_t(H)$. ∎

As a consequence of Theorem 17 we obtain the following result which demonstrates sharpness of the upper bound of Theorem 13.

Corollary 18. *If T is tree with $n(T) \geq 3$, then $\mu_t(T \,\square\, K_n) = n \cdot \mu_t(T)$.*

5. On the Inequality $\mu_t(G \,\square\, H) \leq \mu_t(G)\mu_t(H)$

All the exact results obtained in Section 4 fulfil the bound

$$\mu_t(G \,\square\, H) \leq \mu_t(G)\mu_t(H)\,. \tag{2}$$

Hence one may wonder whether the upper bound of Theorem 13 can be improved/replaced by (2). Before we answer the question, we prove another result where (2) holds.

A graph G is a *generalized complete graph* if it is obtained by the join of an isolated vertex with a disjoint union of $k \geq 1$ complete graphs [13]. We further say that G is a *non-trivial generalized complete graph* if $k \geq 2$. Note that if G is a non-trivial generalized complete graph, then $\mu_t(G) = n(G) - 1$.

Theorem 19. *If G and H are two non-trivial generalized complete graphs, then*

$$\mu_t(G \,\square\, H) \leq \mu_t(G)\mu_t(H) = (n(G) - 1)(n(H) - 1).$$

Moreover, the equality holds if and only if G or H is isomorphic to a star.

Proof. Let $V(G) = \{g_1, \ldots, g_{n(G)}\}$ and $V(H) = \{h_1, \ldots, h_{n(H)}\}$. Let g_1 and h_1 be the universal vertices of G and H, respectively.

Note that g_1 is a non-bypass vertex of G and h_1 is a non-bypass vertex of H. It follows that each vertex from the layer ${}^{g_1}H$ is a non-bypass vertex of $G \,\square\, H$ as well as is each vertex from the layer G^{h_1}. Hence $G \,\square\, H$ contains at least $n(G) + n(H) - 1$ non-bypass vertices and so by (1),

$$\begin{aligned} \mu_t(G \,\square\, H) &\leq bp(G \,\square\, H) \\ &\leq n(G)n(H) - (n(G) + n(H) - 1) \\ &= (n(G) - 1)(n(H) - 1). \end{aligned}$$

To prove the equality case, by Theorem 17 we know that $\mu_t(G \,\square\, H) = (n(G) - 1)(n(H) - 1)$ if G or H is a star. Suppose in the rest that neither G nor H is a star. Then each of them contains an induced subgraph K_3. Without loss of generality, assume that g_1, g_2, g_3 induce a K_3 of G, and that h_1, h_2, h_3

induce a K_3 of H. Then the vertices $(g_2, h_2), (g_2, h_3), (g_3, h_3)$, and (g_3, h_2) induce a C_4 of $G \,\square\, H$. This C_4 is convex because it is the Cartesian product of K_2 (as a subgraph of G) and a K_2 (as a subgraph of H) and these two K_2 are clearly convex in respective factors, cf. [7, Proposition 13.3]. Since at most two vertices of this C_4 can lie in a total mutual-visibility set of $G \,\square\, H$, we conclude that $\mu_{\rm t}(G \,\square\, H) < bp(G \,\square\, H)$. ■

In the rest we demonstrate that (2) does not hold in general. For this sake we say that a graph G is *bypass over-visible* if it contains an independent bypass set of vertices U which contains a $\mu_{\rm t}$-set U' as a **proper** subset. Note that since U' is an independent set, a bypass over-visible graph is a $(\mu_{\rm it}, \mu_{\rm t})$-graph.

Before we state the last result of this paper, we construct two families of bypass over-visible graphs. (Another such family will be presented after the proof of the last result.)

If $k \geq 3$, then let H_k be the graph obtained by attaching two pendant vertices to each of the degree k vertices of $K_{2,k}$. See Figure 4 for H_5. Let U_k be the set of all the vertices of H_k of degree 1 or 2. Then U_k is an independent bypass set with $|U_k| = k + 4$, while every $\mu_{\rm t}$-set of H_k is obtained from U_k by removing one degree 2 vertex. Hence each H_k is a bypass over-visible graphs.

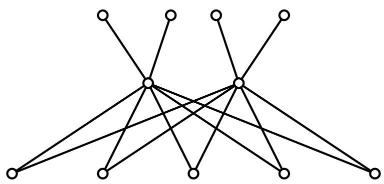

Figure 4. The graph H_5.

For another family of bypass over-visible graphs let Θ_i denote any theta graph $\Theta(p_1, \ldots, p_k)$, where $i \geq 2$, $k \geq 3$, and

$$2 = p_1 = \cdots = p_i < p_{i+1} \leq \cdots \leq p_k.$$

Let a and b be the vertices of Θ_i of degree k. Then $BP(\Theta_i)$ consists of the degree 2 vertices which are adjacent to a and to b, so that $bp(\Theta_i) = i$. Note in addition that $BP(\Theta_i)$ is an independent set. On the other hand, $BP(\Theta_i)$ is not a total mutual-visibility set, but becomes such a set if an arbitrary vertex is removed from it. Hence $\mu_{\rm t}(\Theta_i) = i - 1$. We conclude that Θ_i is a bypass over-visible graph.

Theorem 20. *If G and H are bypass over-visible graphs, then*

$$\mu_{\rm t}(G \,\square\, H) > \mu_{\rm t}(G)\mu_{\rm t}(H).$$

Proof. Since G and H are bypass over-visible graphs, there exist independent bypass vertex sets I_G and I_H of G and H, respectively, which contain μ_t-sets S_G and S_H as proper subsets. Hence there exist vertices $u \in I_G \setminus S_G$ and $v \in I_H \setminus S_H$. We set $U = (S_G \times S_H) \cup \{(u, v)\}$ and claim that U is a total mutual-visibility set of $G \,\square\, H$.

Consider two arbitrary vertices $x = (g, h)$ and $y = (g', h')$ from $G \,\square\, H$. Suppose first that $g = g'$. If $g \in S_G$, then x and y are U-visible because S_H is a total mutual-visibility set of H. If $g = u$, then x and y are U-visible by Lemma 7 applied to (u, v) and the layer ${}^gH$. In all the other cases $V({}^gH) \cap U = \emptyset$, hence there is nothing to prove. If $h = h'$, then x and y are U-visible by the same argument.

Assume in the rest that $g \neq g'$ and $h \neq h'$. Let $P_G : g = g_0, g_1, \ldots, g_k = g'$ be a shortest g, g'-path in G whose internal vertices are not in S_G. Similar, let $P_H : h = h_0, h_1, \ldots, h_\ell = h'$ be a shortest h, h'-path in H whose internal vertices are not in S_H. The copy of P_G in the layer G^w will be denoted by P_G^w and the copy of P_H in the layer ${}^zH$ will be denoted by ${}^zP_H$.

Consider first the case $k = 1$, that is, when $gg' \in E(G)$. Assume first that $(g_1, h_0) \notin U$. If $(u, v) \notin V({}^{g_1}P_H)$, then the concatenation of the edge $(g_0, h_0)(g_1, h_0)$ and the path ${}^{g_1}P_H$ is a required x, y-path. Suppose next that $(u, v) \in V({}^{g_1}P_H)$. Then, because $g_1 = u \in I_G$ and since I_G is independent, we infer that $g_0 \notin I_G$ and thus also $g_0 \notin S_G$. Consequently, $(g_0, h_\ell) \notin U$ and then the concatenation of the path ${}^{g_0}P_H$ with the edge $(g_0, h_\ell)(g_1, h_\ell)$ is a required x, y-path. Assume second that $(g_1, h_0) \in U$. Then by the same argument we see that the path ${}^{g_0}P_H$ followed by the edge $(g_0, h_\ell)(g_1, h_\ell)$ is again a path which ensures that x and y are U-visible. Similarly we see that x and y are U-visible if $\ell = 1$. Note that the argument also applies when $k = \ell = 1$.

We are left with the case when $k \geq 2$ and $\ell \geq 2$. Assume first that $u \neq g_i$ for $i \in [k-1]$. Then the vertices

$$x = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell), (g_2, h_\ell), \ldots, (g_k, h_\ell) = y$$

induce a shortest x, y-path and the internal vertices of it are not in U. Hence x and y are U-visible. Similarly we see that x and y are U-visible if $v \neq h_j$ for $j \in [\ell - 1]$. The remaining case is that $u = g_i$ for some $i \in [k-1]$ and $v = h_j$ for some $j \in [\ell - 1]$. Then the vertices

$$x = (g_0, h_0), \ldots, (g_{i-1}, h_0), (g_{i-1}, h_1), \ldots, (g_{i-1}, h_\ell), (g_i, h_\ell), \ldots, (g_k, h_\ell) = y$$

induce a shortest x, y-path in $G \,\square\, H$ with no internal vertices in U. We conclude that in any case x and y are U-visible. ∎

We next present another family of bypass over-visible graphs. Let $m \geq 1$. Then we define the graph G_m as follows. The vertex set is

$$V(G_m) = \{x_0, x_1, \ldots, x_{m+2}\} \cup \{y_1, \ldots, y_m\} \cup \{z_1, \ldots, z_m\}.$$

For any $i \in [m]$ we connect y_i and z_i with x_i and x_{i+1}. Finally add the edges x_0x_1 and $x_{m+1}x_{m+2}$. See Figure 5 for G_5.

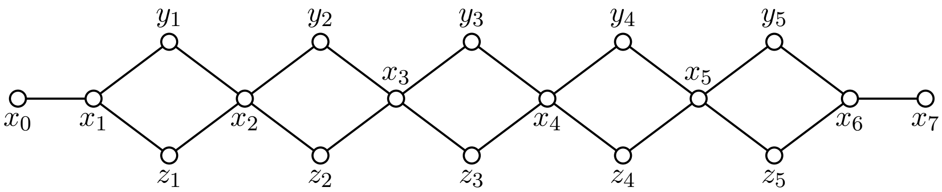

Figure 5. The graph G_5.

It is straightforward to see that $BP(G_m) = \{x_0, x_{m+2}\} \cup \{y_1, \ldots, y_m\} \cup \{z_1, \ldots, z_m\}$, hence $bp(G_m) = 2m + 2$ and $BP(G_m)$ is an independent set. In addition, since for any i, the vertices y_i and z_i cannot both lie in a total mutual-visibility set, the set $\{x_0, x_{m+2}, y_1, \ldots, y_m\}$ is an independent $\mu_{\rm t}$-set of G. Hence G_m is a bypass over-visible graph. Moreover, $bp(G_m) - \mu_{\rm t}(G_m) = m$. Now, by a parallel construction as in the proof of Theorem 20 we find out that $\mu_{\rm t}(G_m \,\square\, G_m) \geq (m + 2)^2 + m$, hence $\mu_{\rm t}(G \,\square\, H)$ can be arbitrary larger than $\mu_{\rm t}(G)\mu_{\rm t}(H)$.

6. Concluding Remarks

There are several possibilities how to continue the investigation of this paper, here we emphasize some of them.

We have characterized the graphs G with $\mu_{\rm t}(G) = 0$. The next step would be to characterize the graphs G with $\mu_{\rm t}(G) = 1$ (and maybe also with $\mu_{\rm t}(G) = 2$).

In view of (1) it would be interesting to consider the graphs G with $\mu_{\rm t}(G) = bp(G)$.

In this paper we had a closer look to the total mutual-visibility number of Cartesian product graphs. Some other graph operations also appear interesting for such investigations, in particular the strong product and the lexicographic product.

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