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Singular p-biharmonic problem with the Hardy potential

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Abstract. The aim of this paper is to study existence results for a singular problem involving the p-biharmonic operator and the Hardy potential. More precisely, by combining monotonicity arguments with the variational method, the existence of solutions is established. By using the Nehari manifold method, the multiplicity of solutions is proved. An example is also given to illustrate the importance of these results.

Keywords: p-biharmonic equation, variational methods, existence of solutions, Hardy potential, Nehari manifold, fibering map.

1 Introduction

The aim of this work is to study the following p-biharmonic problem with singular nonlinearity and Hardy potential:

$$\Delta_p^2\varphi - \lambda\frac{|\varphi|^{p-2}\varphi}{|z|^{2p}} + \Delta_p\varphi = \frac{a(z)}{\varphi^\theta} + \mu g(z,\varphi) \quad \text{for all } \varphi \in W^{2,p}\bigl(\mathbb{R}^N\bigr), \tag{1}$$

where $1 < p < N/2$, $0 < \theta < 1$, and λ, μ are positive constants. The operators Δ_p and Δ_p^2 are the p-Laplacian operator and the p-biharmonic operator, respectively, defined by

$$\Delta_p\varphi = \operatorname{div}\bigl(|\nabla\varphi|^{p-2}\nabla\varphi\bigr) \quad \text{and} \quad \Delta_p^2\varphi = \Delta\bigl(|\Delta\varphi|^{p-2}\Delta\varphi\bigr).$$

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Nonlinear elliptic equations with singularities can model several phenomena like non-Newtonian fluids and chemical heterogeneity; for more details and other applications, see, for example, Alsaedi et al. [1], Callegari and Nachman [4], Candito et al. [5, 6], Molica Bisci and Rădulescu [14], Nachman and Callegari [16] Papageorgiou [17], Papageorgiou et al. [19], and Pimenta and Servadei [20]. In recent years, problems involving p-biharmonic operator have been extensively studied; see, for instance, Bhakta [2], Dhifli and Alsaedi [8], Huang and Liu [12], Molica Bisci and Repovš [15], Sun et al. [23], Wang and Zhao [26], and Yang et al. [27]. In particular, Dhifli and Alsaedi [8] considered the analysis of the fibering map on the Nehari manifold sets to prove the existence of multiple solutions for the following system:

$$\begin{aligned}&\Delta_p^2\varphi - \Delta_p\varphi + V(z)|\varphi|^{p-2}\varphi\\ &\quad = \lambda f(z)|\varphi|^{q-2}\varphi + a(z)|\varphi|^{m-2}\varphi \quad \text{for all } \varphi \in W^{2,p}(\mathbb{R}^N).\end{aligned}$$

Very recently, several researchers have concentrated on the study of singular p-biharmonic equations; see Sun et al. [23] and Sun and Wu [24, 25], whereas singular problem involving p-biharmonic operator and Hardy potential has not received that much attention – we refer the reader to Drissi et al. [10] and Huang and Liu [12] for related work.

Ferrara and Molica Bisci [11] used the variational principle of Ricceri [22] to prove the multiplicity of solutions for the following problem:

$$\begin{aligned}-\Delta_p\varphi &= \mu\frac{|\varphi|^{p-2}\varphi}{|z|^{2p}} + \lambda f(z,\varphi) \quad \text{in } \Omega,\\ \varphi &= \Delta\varphi = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

Motivated by [11], Huang and Liu [12] considered the following p-biharmonic problem:

$$\begin{aligned}&-\Delta_p^2\varphi - \mu\frac{|\varphi|^{p-2}\varphi}{|z|^{2p}} = \mu h(z,\varphi) \quad \text{in } \Omega,\\ &\varphi = \Delta\varphi = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

More precisely, they used the invariant sets of descending flows method and proved that under suitable conditions on the parameter μ and the nonlinearity h, such a problem admits a nontrivial solution that changes sign.

In the present paper, we shall combine variational methods with monotonicity arguments to prove the existence of a nontrivial solution for problem (1). Next, we shall use the Nehari manifold method to prove the multiplicity of solutions. We note that this problem is very important since it involves the p-biharmonic operator, the p-Laplacian operator, a singular nonlinearity, and the Hardy potential.

In the first main result of this paper, we shall assume that

$$g(z,\varphi) = f(z)h(\varphi) \quad \text{for all } (z,\varphi) \in \mathbb{R}^N \times \mathbb{R}$$

and that the functions f, h are measurable and satisfy the following hypotheses.

(H1) There exist $c_1 > 0$, $1 < r < p < N/2$, and $s \in (p^*/(p^* - r), p/(p - r))$ such that

$$f \in L^{p^*/(p^*-r)}\big(\mathbb{R}^N\big) \cap L^s_{\mathrm{loc}}\big(\mathbb{R}^N\big) \quad \text{and} \quad h(\varphi) \leqslant c_1|\varphi|^{r-1} \quad \text{for all } \varphi \in \mathbb{R}.$$

(H2) There exists $M > 0$ such that for all $(z, \varphi) \in \mathbb{R}^N \times \mathbb{R}$, we have

$$0 < rf(z)H(\varphi) \leqslant f(z)h(\varphi)\varphi \quad \text{for all } |\varphi| \geqslant M,$$

where

$$H(t) = \int_0^t h(s)\,\mathrm{d}s.$$

(H3) $a \in L^{p^*}(p^* + \theta - 1)(\mathbb{R}^N) \cap L^\beta_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\beta \in (p^*/(p^* + \theta - 1), p/(\theta + p - 1))$.

The first main result of this paper is the following theorem.

Theorem 1. *Suppose that hypotheses* (H1)–(H3) *hold. Then for all $\delta, \mu > 0$, problem* (1) *admits at least one nontrivial weak solution φ_μ, provided that $\lambda > 0$ is small enough.*

In the second main result of this paper, we shall assume the following hypotheses.

(H4) $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, defined by $G(z, \varphi) = \int_0^\varphi g(z, s)\,\mathrm{d}s$, is a C^1-function such that

$$G(z, t\varphi) = t^r G(z, \varphi) \quad \text{for all } (z, \varphi) \in \mathbb{R}^N \times \mathbb{R},\ t > 0.$$

Moreover, if $\varphi \neq 0$, then $G(z, \varphi) > 0$, where $0 < 1 - \theta < 1 < p < r$.

(H5) $a : \mathbb{R}^N \to (0, \infty)$ satisfies

$$a \in L^{p/(\theta+p-1)}\big(\mathbb{R}^N\big).$$

We note that by hypothesis (H4), we can find $M > 0$ such that

$$\varphi g(z, \varphi) = rG(z, \varphi) \quad \text{and} \quad \big|G(z, \varphi)\big| \leqslant M|\varphi|^r \quad \text{for all } (z, \varphi) \in \mathbb{R}^N \times \mathbb{R}. \tag{2}$$

The second main result of this paper is the following theorem.

Theorem 2. *Assume that hypotheses* (H4) *and* (H5) *hold. Then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem* (1) *admits two nontrivial solutions.*

The paper is organized as follows: In Section 2, we shall present some preliminary material needed in the paper. In Section 3, we shall prove the first main result of this paper, i.e., the existence of solutions (Theorem 1). In Section 4, we shall study fibering maps on Nehari manifold sets. In Section 5, we shall prove the second main result of this paper, i.e., the multiplicity of solutions (Theorem 2). In Section 6, we shall give an illustrative example.

2 Preliminaries

In this section, we shall present some preliminary material needed in the paper. For other necessary background facts, we recommend the comprehensive monograph Papageorgiou et al. [18].

The Hardy potential is related to the following Rellich inequality:

$$\int_{\mathbb{R}^N} \frac{|\varphi(z)|^p}{|z|^{2p}} \,\mathrm{d}z \leqslant \left(\frac{p^2}{N(p-1)(N-2p)}\right)^p \int_{\mathbb{R}^N} \left|\Delta\varphi(z)\right|^p \mathrm{d}z \quad \text{for all } \varphi \in E, \tag{3}$$

where $E := W^{2,p}(\mathbb{R}^N)$ is the Sobolev space, which is defined as follows:

$$W^{2,p}\left(\mathbb{R}^N\right) = \left\{\varphi \in L^p\left(\mathbb{R}^N\right):\ \Delta\varphi, |\nabla\varphi| \in L^p\left(\mathbb{R}^N\right)\right\}.$$

For the interested reader, properties of these spaces can be found in Davies and Hinz [7], Mitidieri [13], and Rellich [21]. According to the Rellich inequality (3), if λ satisfies

$$0 < \lambda < \left(\frac{N(p-1)(N-2p)}{p^2}\right)^p, \tag{4}$$

then $\|\cdot\| : E \to \mathbb{R}$, defined by

$$\|\varphi\| = \left(\int_{\mathbb{R}^N} \left|\Delta\varphi(z)\right|^p - \lambda\frac{|\varphi(z)|^p}{|z|^{2p}} + \left|\nabla\varphi(z)\right|^p \mathrm{d}z\right)^{1/p},$$

is a norm in E.

For every $r \in [p, p^*]$, there exists a continuous embedding from E into $L^r(\mathbb{R}^N)$. On the other hand, if $r \in (p, p^*)$, then there exists a compact embedding from E into $L^r_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, we have

$$S_r|\varphi|_r^p \leqslant \|\varphi\|^p \quad \text{for all } \varphi \in E \text{ and } r \in [p, p^*], \tag{5}$$

where $p^* = Np/(N-2p)$, $|\varphi|_r$ denotes the usual $L^r(\mathbb{R}^N)$-norm, and S_r is the best Sobolev constant given by

$$S_r = \inf_{\varphi \in W^{2,p}\left(\mathbb{R}^N\right)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\Delta\varphi(z)|^p - \lambda\frac{|\varphi(z)|^p}{|z|^{2p}} + |\nabla\varphi(z)|^p \,\mathrm{d}z}{(\int_{\mathbb{R}^N} |\varphi(z)|^r \,\mathrm{d}z)^{p/r}}.$$

If ψ is a positive function on $\mathbb{R}^N$ and $1 \leqslant \sigma < \infty$, then we can define the weighted Lebesgue space $L^\sigma(\mathbb{R}^{\mathbb{N}}, \psi)$ by

$$L^\sigma\left(\mathbb{R}^{\mathbb{N}}, \psi\right) = \left\{\varphi : \mathbb{R}^N \to \mathbb{R} \text{ measurable: } \int_{\mathbb{R}^N} \psi(z)\left|\varphi(z)\right|^\sigma \mathrm{d}z < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_{\sigma,\psi} = \left(\int\limits_{\mathbb{R}^N} \psi(z) \bigl|\varphi(z)\bigr|^{\sigma} \,\mathrm{d}z \right)^{1/\sigma}.$$

Then $L^{\sigma}(\mathbb{R}^{\mathbb{N}}, \psi)$ is a uniformly convex Banach space. Dhifli and Alsaedi [8] have proved that if $\psi \in L^{p^*/(p^*-r)}(\mathbb{R}^N) \cap L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for some $s \in (p^*/(p^*-r), p/(p-r))$, then the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N, \psi)$ is continuous and compact. Moreover, we have the following estimate:

$$\|\varphi\|^r_{r,\psi} \leqslant S^{-r/p}_{p^*} |f|_{p^*/(p^*-r)} \|\varphi\|^r \quad \text{for all } \varphi \in E. \tag{6}$$

Remark 1. We get an inequality similar to (6) if we replace r by $1-\theta$ and f by a. More precisely, we have

$$\int\limits_{\mathbb{R}^N} a(z) \bigl|\varphi(z)\bigr|^{1-\theta} \,\mathrm{d}z \leqslant S^{-(1-\theta)/p}_{p^*} |f|_{p^*/(p^*+\theta-1)} \|\varphi\|^{1-\theta}.$$

Indeed, from Eq. (5) and the Hölder inequality we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^N} a(z) \bigl|\varphi(z)\bigr|^{1-\theta} \mathrm{d}z &\leqslant \left(\int\limits_{\mathbb{R}^N} \bigl|a(z)\bigr|^{p^*/(p^*+\theta-1)} \mathrm{d}z \right)^{(p^*+\theta-1)/p^*} \left(\int\limits_{\mathbb{R}^N} \bigl|u(z)\bigr|^{p^*} \mathrm{d}z \right)^{(1-\theta)/p^*} \\ &\leqslant S^{-(1-\theta)/p}_{p^*} |f|_{p^*/(p^*+\theta-1)} \|\varphi\|^{1-\theta}. \end{aligned}$$

3 The proof of Theorem 1

We recall that a function $\varphi \in E$ is called a weak solution for problem (1) if, for all $v \in E$, one has

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} \left(|\Delta\varphi|^{p-2} \Delta\varphi \Delta v - \lambda \frac{|\varphi|^{p-2}\varphi v}{|z|^{2p}} + |\nabla\varphi|^{p-2} \nabla\varphi \nabla v \right) \mathrm{d}z \\ &\quad = \int\limits_{\mathbb{R}^N} a(z) \varphi^{-\theta} v \,\mathrm{d}z + \mu \int\limits_{\mathbb{R}^N} g(z, \varphi) v \,\mathrm{d}z. \end{aligned}$$

Associated to problem (1), we define the energy functional $J_\mu : E \to \mathbb{R}$ by

$$J_\mu(\varphi) = \frac{1}{p} \|\varphi\|^p - \frac{1}{1-\theta} \int\limits_{\mathbb{R}^N} a(z) \varphi^{1-\theta} \,\mathrm{d}z - \mu \int\limits_{\mathbb{R}^N} G\bigl(z, \varphi(z)\bigr) \,\mathrm{d}z. \tag{7}$$

Several lemmas will be needed for the proof of Theorem 1.

Lemma 1. *Under hypotheses* (H1)–(H3)*, the functional J_μ is coercive and bounded from below on E.*

Proof. Let $\varphi \in E$. Assume that hypotheses (H1)–(H3) hold. Then it follows by (6) and Remark 1 that

$$\begin{aligned}
J_\mu(\varphi) &= \frac{1}{p}\|\varphi\|^p - \frac{1}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z - \mu\int\limits_{\mathbb{R}^N} f(z)H(\varphi)\,\mathrm{d}z \\
&\geqslant \frac{1}{p}\|\varphi\|^p - \frac{S_{p^*}^{-(1-\theta)/p}}{1-\theta}|a|_{p^*/(p^*+\theta-1)}\|\varphi\|^{1-\theta} - \frac{\mu}{r}\|\varphi\|_{r,h}^r \\
&\geqslant \frac{1}{p}\|\varphi\|^p - \frac{S_{p^*}^{-(1-\theta)/p}}{1-\theta}|a|_{p^*/(p^*+\theta-1)}\|\varphi\|^{1-\theta} - \frac{\mu S_{p^*}^{-r/p}}{r}|f|_{p^*/(p^*-r)}\|\varphi\|^r.
\end{aligned}$$

Since $0 < 1-\theta < r < p$, we can infer that

$$\lim_{\|\varphi\|\to\infty} J_\mu(\varphi) = \infty.$$

In other words, J_μ is indeed coercive and bounded from below on E. This completes the proof of Lemma 1. □

Lemma 2. *Assume that hypotheses* (H1)–(H3) *hold. Then there exists a nonnegative nontrivial function $\phi \in E$ such that $J_\mu(t\phi) < 0$, provided that $t > 0$ is small enough.*

Proof. Let $t > 0$ and $\phi \in C^\infty(\mathbb{R}^N)$. Assume that for some bounded subsets Ω_0 and Ω_1, we have $\Omega_0 \subset \operatorname{supp}(\phi) \subset \Omega_1 \subset \mathbb{R}^N$, $0 \leqslant \phi \leqslant 1$, on Ω_1 and $\phi = 1$ on Ω_0. Then by (H2), we can find $K > 0$ such that for all $(z,t) \in \mathbb{R}^N \times \mathbb{R}$, we have

$$f(z)H(t) \geqslant Kf(z)|t|^r.$$

Invoking (H1)–(H3) and Eq. (6), we get

$$\begin{aligned}
J_\mu(t\phi) &= \frac{t^p}{p}\|\phi\|^p - \frac{t^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\phi^{1-\theta}\,\mathrm{d}z - \mu\int\limits_{\mathbb{R}^N} f(z)H(t\phi)\,\mathrm{d}z \\
&\leqslant \frac{t^p}{p}\|\phi\|^p - \frac{t^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\phi^{1-\theta}\,\mathrm{d}z - \mu K t^r\|\phi\|_{r,f}^r \\
&\leqslant t^r\left(\frac{1}{p}\|\phi\|^p + \mu K\|\phi\|_{r,f}^r\right) - \frac{t^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\phi^{1-\theta}\,\mathrm{d}z \\
&\leqslant t^{1-\theta}\left[t^{r+\theta-1}\left(\frac{1}{p}\|\phi\|^p + \mu K\|\phi\|_{r,f}^r\right) - \frac{1}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\phi^{1-\theta}\,\mathrm{d}z\right] \\
&< 0 \quad \text{for all } t \in \bigl(0, \xi^{1/(r+\theta-1)}\bigr),
\end{aligned}$$

where

$$\xi = \min\left(1, \frac{\frac{t^{1-\theta}}{1-\theta}\int_{\mathbb{R}^N} a(z)\phi^{1-\theta}\,\mathrm{d}z}{\frac{1}{p}\|\phi\|^p + \mu K\|\phi\|_{r,f}^r}\right).$$

This completes the proof of Lemma 2. □

We note that according to Lemma 1, we can define the following:

$$m_\mu = \inf_{\varphi \in E} J_\mu(\varphi),$$

and by Lemma 2, we have $m_\mu < 0$.

Lemma 3. *The functional J_μ attains its global minimizer on E. That is, there exists $\varphi_\mu \in E$ such that*

$$J_\mu(\varphi_\mu) = m_\mu < 0.$$

Proof. Let $\{\varphi_n\}$ be a minimizing sequence for J_μ, which means that $J_\mu(\varphi_n) \to m_\mu$ as $n \to \infty$. Since J_μ is coercive, it follows that $\{\varphi_n\}$ is bounded on E. Indeed, if not, then up to a subsequence, we can assume that $\|\varphi_n\| \to \infty$. Therefore, the coercivity of J_μ implies that $J_\mu(\varphi_n) \to \infty$, which is a contradiction. Hence, $\{\varphi_n\}$ is bounded. Therefore, there exist $\varphi_\mu \in E$ and a subsequence still denoted by $\{\varphi_n\}$ such that, as n tends to infinity, we have

$$\begin{aligned} &\varphi_n \hookrightarrow \varphi_\mu \quad \text{weakly in } E,\\ &\varphi_n \to \varphi_\mu \quad \text{strongly in } L^r\big(\mathbb{R}^N, f\big),\\ &\varphi_n \to \varphi_\mu \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{8}$$

Since $\{\varphi_n\}$ is bounded on E, it follows by the Sobolev embedding theorem that $\{\varphi_n\}$ is bounded on $L^{p^*}(\mathbb{R}^N)$. On the other hand, by Remark 1, we have

$$\int_{\mathbb{R}^N} a(z)|\varphi_n|^{1-\theta}\,\mathrm{d}z \leqslant S_{p^*}^{-(1-\theta)/p}|a|_{p^*/(p^*+\theta-1)}\|\varphi_n\|^{1-\theta}.$$

So, by absolute continuity of $|a|_{p^*/(p^*+\theta-1)}$, we can deduce that

$$\left\{\int_{\mathbb{R}^N} a(z)|\varphi_n|^{1-\theta}\,\mathrm{d}z,\ n \in \mathbb{N}\right\}$$

is equi-absolutely continuous. Therefore, by the Vitali theorem (see Brooks [3]), one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(z)|\varphi_n|^{1-\theta}\,\mathrm{d}z = \int_{\mathbb{R}^N} a(z)|\varphi_\mu|^{1-\theta}\,\mathrm{d}z. \tag{9}$$

Finally, by (8) and weak lower semi-continuity of the norm, we obtain

$$m_\mu \leqslant J_\mu(\varphi_\mu) \leqslant \lim_{n\to\infty} J_\mu(\varphi_n) = m_\mu,$$

hence,

$$J_\mu(\varphi_\mu) = m_\mu < 0. \tag{10}$$

This completes the proof of Lemma 3. □

Now we are ready to present the proof of Theorem 1.

Proof of Theorem 1. From Lemma 3 we see that φ_μ is a global minimizer for J_μ, hence, φ_μ satisfies

$$0 \leqslant J_\mu(\varphi_\mu + t\varphi) - J_\mu(\varphi_\mu) \quad \text{for all } (t, \varphi) \in (0, \infty) \times E.$$

Dividing the above inequality by $t > 0$ and letting t tend to zero, we obtain

$$\begin{aligned} 0 \leqslant & \int_{\mathbb{R}^N} \left(|\Delta\varphi_\mu|^{p-2} \Delta\varphi_\mu \Delta\varphi - \lambda \frac{|\varphi_\mu|^{p-2}\varphi_\mu\varphi}{|z|^{2p}} + |\nabla\varphi_\mu|^{p-2} \nabla\varphi_\mu \nabla\varphi \right) \mathrm{d}z \\ & - \int_{\mathbb{R}^N} a(z)\varphi_\mu^{-\theta}\varphi \,\mathrm{d}z - \mu \int_{\mathbb{R}^N} f(z)h(\varphi_\mu)\varphi \,\mathrm{d}z. \end{aligned}$$

The fact that φ is arbitrary in E implies that in the last inequality, we can replace φ by $-\varphi$, so, for any $\varphi \in E$, we get

$$\begin{aligned} 0 = & \int_{\mathbb{R}^N} \left(|\Delta\varphi_\mu|^{p-2} \Delta\varphi_\mu \Delta\varphi - \lambda \frac{|\varphi_\mu|^{p-2}\varphi_\mu\varphi}{|z|^{2p}} + |\nabla\varphi_\mu|^{p-2} \nabla\varphi_\mu \nabla\varphi \right) \mathrm{d}z \\ & - \int_{\mathbb{R}^N} a(z)\varphi_\mu^{-\theta}\varphi \,\mathrm{d}z - \mu \int_{\mathbb{R}^N} f(z)h(\varphi_\mu)\varphi \,\mathrm{d}z. \end{aligned}$$

That is, φ_μ is a weak solution for problem (1). Moreover, from Eq. (10) we see that φ_μ is nontrivial. This completes the proof of Theorem 1. □

4 Fibering maps on Nehari manifold sets

In order to prove Theorem 2, we first need to study the fibering maps on Nehari manifold sets. First, let us mention that the functional J_μ defined in Eq. (7) is Fréchet differentiable. Moreover, for all $(\varphi, \psi) \in E \times E$, we have

$$\begin{aligned} J'_\mu(\varphi)\psi = & \int_{\mathbb{R}^N} \left(|\Delta\varphi|^{p-2} \Delta\varphi \Delta\psi - \lambda \frac{|\varphi|^{p-2}\varphi\psi}{|z|^{2p}} + |\nabla\varphi|^{p-2} \nabla\varphi \nabla\psi \right) \mathrm{d}z \\ & - \int_{\mathbb{R}^N} a(z)\varphi^{-\theta}\psi \,\mathrm{d}z - \frac{\mu}{r} \int_{\mathbb{R}^N} g(z,\varphi)\psi \,\mathrm{d}z. \end{aligned}$$

It is obvious that J_μ is not bounded from below on E. We introduce the following set:

$$N_\mu = \bigl\{ \varphi \in E\colon\ J'_\mu(\varphi)\varphi = 0 \bigr\}.$$

Note that a function $\varphi \in E$ is a weak solution for problem (1) if it satisfies $J'_\mu(\varphi) = 0$, that is, φ is a critical value for J_μ. Clearly, $\varphi \in N_\mu$ if and only if

$$\|\varphi\|^p - \int_{\mathbb{R}^N} a(z)\varphi^{1-\theta} \,\mathrm{d}z - \mu \int_{\mathbb{R}^N} G\bigl(z, \varphi(z)\bigr) \,\mathrm{d}z = 0. \tag{11}$$

Lemma 4. *The functional J_μ is coercive and bounded from below on N_μ.*

Proof. Let $\varphi \in N_\mu$. Then, by Eqs. (5), (11) and the Hölder inequality, we obtain

$$\begin{aligned} J_\mu(\varphi) &= \frac{1}{p}\|\varphi\|^p - \frac{1}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z - \frac{\mu}{r}\int\limits_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z \\ &\geqslant \frac{r-p}{pr}\|\varphi\|^p - \frac{\theta+r-1}{r(1-\theta)}\int\limits_{\mathbb{R}^N} a(z)|\varphi|^{1-\theta}\,\mathrm{d}z \\ &\geqslant \frac{r-p}{pr}\|\varphi\|^p - \frac{\theta+r-1}{r(1-\theta)} S_p^{(\theta-1)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta}. \end{aligned} \tag{12}$$

Since $0 < 1-\theta < 1 < p < r$, it follows that J_μ is coercive and bounded from below on N_μ. This completes the proof of Lemma 4. $\square$

Next, we define a function $\phi_{\mu,\varphi}$ on $[0,+\infty)$, introduced in Drăbek and Pohožaev [9], as follows:

$$\phi_{\mu,\varphi}(t) := J_\mu(t\varphi) = \frac{t^p}{p}\|\varphi\|^p - \frac{t^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z - \frac{\mu t^r}{r}\int\limits_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z.$$

A simple calculation shows that

$$\phi'_{\mu,\varphi}(t) = t^{p-1}\|\varphi\|^p - t^{-\theta}\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z - \mu t^{r-1}\int\limits_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z$$

and

$$\begin{aligned} \phi''_{\mu,\varphi}(t) &= (p-1)t^{p-2}\|\varphi\|^p \\ &\quad + \theta t^{-\theta-1}\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z - \mu(r-1)t^{r-2}\int\limits_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z. \end{aligned}$$

Since $t\phi'_{\mu,\varphi}(t) = \langle J'_\mu(t\varphi), t\varphi\rangle$, it follows that for $t > 0$ and $\varphi \in E \setminus \{0\}$, we have

$$\phi'_{\mu,\varphi}(t) = 0 \quad \text{if and only if } t\varphi \in N_\mu.$$

In particular, $\varphi \in N_\mu$ if and only if $\phi'_{\mu,\varphi}(1) = 0$. On the other hand, it follows by Eq. (11) that for all $\varphi \in N_\mu$, one has

$$\phi''_{\mu,\varphi}(1) = (p-r)\|\varphi\|^p + (\theta+r-1)\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z \tag{13}$$

$$= (\theta+p-1)\|\varphi\|^p - \mu(\theta+r-1)\int\limits_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z. \tag{14}$$

Now, in order to obtain the multiplicity of solutions, we split N_μ into three parts:

$$N_\mu^+ = \{\varphi \in N_\mu \setminus \{0\}\colon \phi''_{\mu,\varphi}(1) > 0\},$$
$$N_\mu^- = \{\varphi \in N_\mu \setminus \{0\}\colon \phi''_{\mu,\varphi}(1) < 0\},$$

and

$$N_\mu^0 = \{\varphi \in N_\mu \setminus \{0\}\colon \phi''_{\mu,\varphi}(1) = 0\}.$$

In the following lemmas, we shall present some important properties related to the subsets introduced above.

Lemma 5. *If $u \notin N_\mu^0$ is a local mimimizer for J_μ on N_μ, then $J'_\mu(\varphi) = 0$.*

Proof. Since φ is a minimizer for J_μ under the following constraint

$$I_\mu(\varphi) := J'_\mu(\varphi)\varphi = 0,$$

the Lagrange multipliers theory implies the existence of $\xi \in \mathbb{R}$ such that $J'_\mu(\varphi) = I'_\mu(\varphi)\xi$. Thus

$$J'_\mu(\varphi)\varphi = (I'_\mu(\varphi)\varphi)\xi = \phi''_{\mu,\varphi}(1)\xi = 0.$$

The fact that $\varphi \notin N_\mu^0$ implies that $\phi''_{\mu,\varphi}(1) \neq 0$. So, $\xi = 0$, which completes the proof of Lemma 5. □

Lemma 6. *There exists μ_0 such that if $\mu \in (0, \mu_0)$, then the set N_μ^0 is empty.*

Proof. Put

$$\mu_0 = \frac{(\theta + p - 1)S_r^{r/p}}{(\theta + r - 1)M}\left(\frac{r - p}{(\theta + r - 1)\|a\|_{p/(\theta+p-1)}S_p^{(1-\theta)/p}}\right)^{(r-p)/(\theta+p-1)},$$

where M is defined as in Eq. (2), and let $\mu \in (0, \mu_0)$. We shall prove that $N_\mu^0 = \emptyset$. Suppose to the contrary and let $\varphi \in N_\mu^0$. Then we have

$$\begin{aligned} 0 &= \phi''_{\mu,\varphi}(1) \\ &= (p-1)\|\varphi\|^p + \theta \int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}(z)\,\mathrm{d}z - \mu(r-1)\int\limits_{\mathbb{R}^N} G(z,\varphi(z))\,\mathrm{d}z. \end{aligned}$$

So, it follows from (13) and (14) that

$$(\theta + p - 1)\|\varphi\|^p = \mu(\theta + r - 1)\int\limits_{\mathbb{R}^N} G(z,\varphi(z))\,\mathrm{d}z \tag{15}$$

and

$$(r - p)\|\varphi\|^p = (\theta + r - 1)\int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}(z)\,\mathrm{d}z. \tag{16}$$

On the other hand, from (5) and the Hölder inequality we get

$$\int_{\mathbb{R}^N} a(z)\varphi^{1-\theta}(z)\,\mathrm{d}z \leqslant \left(\int_{\mathbb{R}^N} \left|\varphi(z)\right|^p \mathrm{d}z\right)^{(1-\theta)/p} \left(\int_{\mathbb{R}^N} \left|a(z)\right|^{p/(\theta+p-1)} \mathrm{d}z\right)^{(\theta+p-1)/p}$$
$$\leqslant |\varphi|_p^{1-\theta}\|a\|_{p/(\theta+p-1)} \leqslant S_p^{(1-\theta)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta}.$$

So, it follows from (16) that

$$\|\varphi\|^p = \frac{\theta+r-1}{r-p}\int_{\mathbb{R}^N} a(z)u^{1-\theta}(z)\,\mathrm{d}z \leqslant \frac{\theta+r-1}{r-p}S_p^{(1-\theta)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta},$$

that is,

$$\|\varphi\| \leqslant \left(\frac{\theta+r-1}{r-p}S_p^{(\theta-1)/p}\|a\|_{p/(\theta+p-1)}\right)^{1/(\theta+p-1)}. \tag{17}$$

From (5), (2), and (15) we have

$$\|\varphi\|^p = \mu\frac{(\theta+r-1)}{\theta+p-1}\int_{\mathbb{R}^N} G\big(z,\varphi(z)\big)\,\mathrm{d}z \leqslant \mu M\frac{(\theta+r-1)}{\theta+p-1}\int_{\mathbb{R}^N} \left|\varphi(z)\right|^r \mathrm{d}z$$
$$\leqslant \mu M\frac{(\theta+r-1)}{\theta+p-1}S_r^{-r/p}\|\varphi\|^r,$$

hence,

$$\|\varphi\| \geqslant \left(\frac{(\theta+p-1)S_p^{r/p}}{(\theta+r-1)M\mu}\right)^{1/(r-p)}. \tag{18}$$

By combining (17) with (18), we obtain $\mu \geqslant \mu_0$, which gives us the desired contradiction. This completes the proof of Lemma 6. $\square$

Lemma 7. *Let $\varphi \in E \setminus \{0\}$. Then there exists $\mu_1 > 0$ such that for all $0 < \mu < \mu_1$, ϕ_φ has exactly a local minimum at t_1 and a local maximum at t_2. That is, $t_1 u \in N_\mu^+$ and $t_2 u \in N_\mu^-$.*

Proof. Let $\varphi \in E$ be such that

$$\int_{\mathbb{R}^N} g(z,\varphi)\,\mathrm{d}z > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z > 0.$$

It is easy to see that for all $t > 0$, we have

$$\phi'_{\mu,\varphi}(t) = t^{-\theta}\left(m_\varphi(t) - \int_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z\right), \tag{19}$$

where $m_\varphi : [0,\infty) \to \mathbb{R}$ is defined by

$$m_\varphi(t) = t^{\theta+p-1}\|\varphi\|^p - t^{\theta+r-1}\int\limits_{\mathbb{R}^N} g(z,\varphi)\,\mathrm{d}z.$$

It is not difficult to show that $m'_\varphi(t) = 0$ if and only if $t = 0$ or $t = t_0$, where

$$t_0 = \left(\frac{(\theta+p-1)\|\varphi\|^p}{(\theta+r-1)\mu\int_{\mathbb{R}^N} g(z,\varphi)\,\mathrm{d}z}\right)^{1/(r-p)}. \tag{20}$$

Moreover,

$$\begin{aligned} m_\varphi(t_0) = {} & \left(\mu\int\limits_{\mathbb{R}^N} g(z,\varphi)\,\mathrm{d}z\right)^{-(\theta+p-1)/(r-p)} \\ & \times\left(\left(\frac{\theta+p-1}{\theta+r-1}\right)^{(\theta+p-1)/(r-p)} - \left(\frac{\theta+p-1}{\theta+r-1}\right)^{(\theta+r-1)/(r-p)}\right) > 0. \end{aligned} \tag{21}$$

On the other hand, the table of variation of the function m_φ is given by

t	0		t_0		∞
$m'_\varphi(t)$		$+$	0	$-$	
$m_\varphi(t)$			$m_\varphi(t_0)$		
		$\nearrow$		$\searrow$	
	0				$-\infty$

Now, since

$$0 < \int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z \leqslant \frac{\theta+r-1}{r-p} S_p^{(\theta-1)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta},$$

it follows by (21) that we can choose $\mu_1 > 0$ small enough so that for all $\mu \in (0,\mu_1)$, we have

$$\frac{\theta+r-1}{r-p} S_p^{(\theta-1)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta} < m_\varphi(t_0).$$

Therefore, for $\mu \in (0,\mu_1)$, we have

$$0 < \int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z < m_\varphi(t_0).$$

Hence, from the table of variation of m_φ we can deduce the existence of unique t_1 and t_2 such that $0 < t_1 < t_0 < t_2$ and

$$m_\varphi(t_1) = m_\varphi(t_2) = \int\limits_{\mathbb{R}^N} a(z)\varphi^{1-\theta}\,\mathrm{d}z.$$

Finally, from (19) and the table of variation of function m_φ we can see that t_1 and t_2 are the unique critical points of function $\phi_{\mu,u}$. More precisely, t_1 is a local minimum point, and t_2 is a local maximum point. Thus $t_1 u \in N_\mu^+$ and $t_2 u \in N_\mu^-$. This completes the proof of Lemma 7. □

Remark 2. It follows from Lemma 7 that $N_\mu^+ \neq \emptyset$ and $N_\mu^- \neq \emptyset$, provided that $0 < \mu < \mu_1$. Moreover, by Lemma 6, for every $0 < \mu < \mu_0$, we have

$$N_\mu = N_\mu^+ \cup N_\mu^-.$$

For the rest of the paper, we shall set

$$\mu^* = \min(\mu_0, \mu_1, \mu_2)$$

and define

$$\theta_\mu = \inf_{\varphi \in N_\mu} J_\mu(\varphi), \theta_\mu^+ = \inf_{\varphi \in N_\mu^+} J_\mu(\varphi) \quad \text{and} \quad \theta_\mu^- = \inf_{\varphi \in N_\mu^-} J_\mu(\varphi),$$

where

$$\mu_2 = \frac{(\theta + p - 1) S_r^{r/p}}{(\theta + r - 1) M} \left(\frac{(\theta + r - 1) p}{(1 - \theta)(r - p)} S_p^{(\theta - 1)/p} \|a\|_{p/(\theta + p - 1)} \right)^{(r-p)/(\theta+p-1)}.$$

Lemma 8. *If* $0 < \mu < \mu^*$*, then the following statements hold*:

(i) $$\theta_\mu \leqslant \theta_\mu^+ < 0.$$

(ii) *There exists* $C > 0$ *such that*

$$\theta_\mu^- \geqslant C > 0.$$

Proof. (i) Let $\varphi \in N_\mu^+$. Then from (13) we get

$$\frac{r - p}{\theta + r - 1} \|\varphi\|^p < \int_{\mathbb{R}^N} a(z) \varphi^{1-\theta} \, \mathrm{d}z.$$

So, combining the last inequality with (11), we obtain

$$\begin{aligned} J_\mu(\varphi) &= \frac{r - p}{pr} \|\varphi\|^p - \frac{\theta + r - 1}{r(1 - \theta)} \int_{\mathbb{R}^N} a(z) \varphi^{1-\theta} \, \mathrm{d}z \\ &\leqslant -\frac{(r - p)(\theta + p - 1)}{pr(1 - \theta)} \|\varphi\|^p < 0, \end{aligned}$$

so, we conclude that $\theta_\mu \leqslant \theta_\mu^+ < 0$.

(ii) Let $\varphi \in N_\mu^-$. Then by (5) and (14), we get

$$\|\varphi\| > \left(\frac{(\theta + p - 1) S_r^{\frac{r}{p}}}{(\theta + r - 1) \mu M} \right)^{1/(r-p)},$$

where M is the positive constant given by Eq. (2).

Now, using the last inequality and (12), we get

$$
\begin{aligned}
J_\mu(\varphi) &\geqslant \frac{r-p}{pr}\|\varphi\|^p - \frac{\theta+r-1}{r(1-\theta)} S_p^{(1-\theta)/p}\|a\|_{p/(\theta+p-1)}\|\varphi\|^{1-\theta} \\
&\geqslant \|\varphi\|^{1-\theta}\left(\frac{r-p}{pr}\|\varphi\|^{\theta+p-1} - \frac{\theta+r-1}{r(1-\theta)} S_p^{(1-\theta)/p}\|a\|_{p/(\theta+p-1)}\right) \\
&> \left(\frac{(\theta+p-1)S_r^{r/p}}{(\theta+r-1)\mu M}\right)^{(1-\theta)/(r-p)}\left(\frac{r-p}{pr}\left(\frac{(\theta+p-1)S_r^{r/p}}{(\theta+r-1)\mu M}\right)^{(\theta+p-1)/(r-p)}\right. \\
&\quad \left. - \frac{\theta+r-1}{r(1-\theta)} S_p^{(\theta-1)/p}\|a\|_{p/(\theta+p-1)}\right).
\end{aligned}
$$

Since $0<\mu<\mu^*\leqslant\mu_2$ and $0<1-\theta\leqslant p<r$, it follows that $J_\mu>C$ for some $C>0$.

This completes the proof of Lemma 8. □

Next, we have the following results on the existence of minimizers in N_μ^+ and N_μ^- for $\mu\in(0,\mu^*)$.

Lemma 9. *If* $0<\mu<\mu^*$*, then there exists* $\varphi_\mu\in N_\mu^+$ *such that*

$$\theta_\mu^+ = J_\mu(\varphi_\mu).$$

That is, J_μ *attains its minimum on* N_μ^+.

Proof. Since J_μ is bounded from below on N_μ and hence also on N_μ^+, there exists $\{\varphi_k\}\subset N_\mu^+$ such that

$$\lim_{k\to\infty} J_\mu(\varphi_k) = \inf_{\varphi\in N_\mu^+} J_\mu(\varphi).$$

Since J_μ is coercive on N_μ, it follows that $\{\varphi_k\}$ is bounded on E. So, there exist φ_μ and a subsequence, again denoted by $\{\varphi_k\}$, such that as k tends to infinity, we have

$$
\begin{aligned}
&\varphi_k \rightharpoonup \varphi_\mu \quad \text{weakly in } E,\\
&\varphi_k \to \varphi_\mu \quad \text{strongly in } L^q(\mathbb{R}^N) \text{ for all } p<q<p^*,\\
&\varphi_k \to \varphi_\mu \quad \text{a.e. } \mathbb{R}^N.
\end{aligned}
$$

From Lemma 8 we know that $\inf_{u\in N_\mu^+} J_\mu(\varphi)<0$. On the other hand, since $\{\varphi_k\}\subset N_\mu$, we have

$$J_\mu(\varphi_k) = \frac{r-p}{pr}\|\varphi_k\|^p - \frac{\theta+r-1}{r(1-\theta)}\int\limits_{\mathbb{R}^N} a(z)\varphi_k^{1-\theta}(z)\,\mathrm{d}z,$$

so, we get

$$\frac{\theta+r-1}{r(1-\theta)}\int\limits_{\mathbb{R}^N} a(z)\varphi_k^{1-\theta}(z)\,\mathrm{d}z = \frac{r-p}{pr}\|\varphi_k\|^p - J_\mu(\varphi_k).$$

From (9), by letting $k \to \infty$ in the last equation, we obtain

$$\int\limits_{\mathbb{R}^N} a(z)\varphi_\mu^{1-\theta}(z)\,\mathrm{d}z > 0.$$

We now claim that φ_k converges strongly to φ_μ in E. If this were not true, then we would have

$$\|\varphi_\mu\|^p < \liminf_{k\to\infty} \|\varphi_k\|^p.$$

Since $\phi'_{\varphi_\mu}(t_1) = 0$, it would follow that $\phi'_{\varphi_k}(t_1) > 0$ for sufficiently large k. So, we must have $t_1 > 1$. However, $t_1\varphi_\mu \in N_\mu^+$, and therefore,

$$J_\mu(t_1\varphi_\mu) < J_\mu(\varphi_\mu) \leqslant \lim_{k\to\infty} J_\mu(\varphi_k) = \inf_{u\in N_\mu^+} J_\mu(\varphi),$$

which is a contradiction, that is, $\varphi_k \underset{k\to\infty}{\to} \varphi_\mu$.

Since $N_\mu^0 = \emptyset$, it follows that $\varphi_\mu \in N_\mu^+$. Finally, φ_μ is a minimizer for J_μ on N_μ^+. This completes the proof of Lemma 9. $\square$

Lemma 10. *If $0 < \mu < \mu^*$, then there exists $\psi_\mu \in N_\mu^-$ such that*

$$\theta_\mu^- = J_\mu(\psi_\mu).$$

That is, J_μ achieves its minimum on N_μ^-.

Proof. By Lemma 8, there exists $C > 0$ such that for all $\varphi \in N_\mu^-$, we have $J_\mu(\varphi) > C$. So, there exists a minimizing sequence $\{\varphi_k\} \subset N_\mu^-$ such that

$$\lim_{k\to\infty} J_\mu(\varphi_k) = \inf_{\varphi\in N_\mu^-} J_\mu(\varphi) > 0.$$

Since J_μ is coercive, we can deduce that $\{\varphi_k\}$ is bounded. So, for all $p \leqslant r < p^*$, there is a subsequence, still denoted by $\{\varphi_k\}$, and $\psi_\mu \in E$ such that if k tends to infinity, we get

$$\begin{aligned}
\varphi_k &\rightharpoonup \psi_\mu \quad \text{weakly in } E,\\
\varphi_k &\to \psi_\mu \quad \text{strongly in } L^r(\mathbb{R}^N),\\
\varphi_k &\to \psi_\mu \quad \text{a.e. } \mathbb{R}^N.
\end{aligned}$$

On the other hand, since $\{\varphi_k\} \subset N_\mu$, we have

$$J_\mu(\varphi_k) = \mu\frac{r+\theta-1}{r(1-\theta)} \int\limits_{\mathbb{R}^N} G\big(z, \varphi_k(z)\big)\,\mathrm{d}z - \frac{\theta+p-1}{p(1-\theta)}\|\varphi_k\|^p,$$

which implies

$$\mu\frac{r+\theta-1}{r(1-\theta)} \int\limits_{\mathbb{R}^N} G(z, \varphi_k)\,\mathrm{d}z = J_\mu(\varphi_k) + \frac{\theta+p-1}{p(1-\theta)}\|\varphi_k\|^p.$$

By letting $k \to \infty$ in last equation, we obtain

$$\int\limits_{\mathbb{R}^N} G(z, \psi_\mu)\,\mathrm{d}z > 0.$$

Hence, by Lemma 7 $\phi_{\mu,\varphi}$ has a maximum at some point t_2 and $t_2\psi_\mu \in N_\mu^-$. On the other hand, $\psi_k \in N_\mu^-$ implies that 1 is a global maximum point for ϕ_{μ,φ_k}, so, we get

$$J_\mu(t\varphi_k) = \phi_{\mu,\varphi_k}(t) \leqslant \phi_{\mu,\varphi_k}(1) = J_\mu(\varphi_k) \quad \text{for all } t > 0. \tag{22}$$

Now, we claim that $\varphi_k \to \psi_\mu$ as $k \to \infty$. Suppose that this is were not true, then we would get

$$\|\psi_\mu\|^p < \liminf_{k\to\infty} \|\varphi_k\|^p.$$

So, from Eq. (22) and the Fatou lemma we would obtain

$$\begin{aligned}
J_\mu(t_2\psi_\mu) &= \frac{t_2^p}{p}\|\psi_\mu\|^p - \frac{t_2^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\psi_\mu^{1-\theta}\,\mathrm{d}z - \frac{\mu t_2^r}{r}\int\limits_{\mathbb{R}^N} G\big(z, \psi_\mu(z)\big)\,\mathrm{d}z \\
&< \liminf_{k\to\infty}\left(\frac{t_2^p}{p}\|\varphi_k\|^p - \frac{t_2^{1-\theta}}{1-\theta}\int\limits_{\mathbb{R}^N} a(z)\varphi_k^{1-\theta}\,\mathrm{d}z - \frac{\mu t_2^r}{r}\int\limits_{\mathbb{R}^N} G\big(z, \varphi_k(z)\big)\,\mathrm{d}z\right) \\
&\leqslant \lim_{k\to\infty} J_\mu(t_2\varphi_k) \leqslant \lim_{k\to\infty} J_\mu(\varphi_k) = \inf_{\varphi\in N_\mu^-} J_\mu(\varphi),
\end{aligned}$$

which is a contradiction. Hence, $\varphi_k \to \psi_\mu$ as $k \to \infty$.

Since $N_\mu^0 = \emptyset$, it follows that $\psi_\mu \in N_\mu^-$. Finally, ψ_μ is a minimizer for J_μ on N_μ^-. This completes the proof of Lemma 10. □

5 The proof of Theorem 2

We shall need the following two auxiliary lemmas to prove that the local minimum of the functional energy is a weak solution for problem (1).

Lemma 11. *Assume that hypotheses of Theorem* 2 *are satisfied and* $\mu \in (0, \mu^*)$. *Then the following statements hold*:

(i) *There exist* $r_1 > 0$ *and a continuous function* $\rho_1 : B(0, r_1) \to (0, \infty)$ *such that*

$$\rho_1(0) = 1 \quad \textit{and} \quad \rho_1(\varphi)(\varphi_\mu + \varphi) \in N_\mu^+ \quad \textit{for all } \varphi \in B(0, r_1).$$

(ii) *There exist* $r_2 > 0$ *and a continuous function* $\rho_2 : B(0, r_2) \to (0, \infty)$ *such that*

$$\rho_2(0) = 1 \quad \textit{and} \quad \rho_2(\varphi)(\psi_\mu + \varphi) \in N_\mu^- \quad \textit{for all } \varphi \in B(0, r_2).$$

Proof. We give the proof only for assertion (i) since the proof for assertion (ii) is similar. So, let $\Phi : E \times (0,\infty)$ be a function defined by

$$\Phi(\varphi,t) = t^{\theta+p-1}\|\varphi_\mu+\varphi\|^p - t^{\theta+r-1}\int\limits_{\mathbb{R}^N} G(z,\,\varphi_\mu+\varphi)\,\mathrm{d}z - \int\limits_{\mathbb{R}^N} a(z)|\varphi_\mu+\varphi|^{1-\theta}\,\mathrm{d}z.$$

Since $\varphi_\mu \in N_\mu^+ \subset N_\mu$, we have $\Phi(0,1)=0$. On the other hand, $\varphi_\mu \in N_\mu^+$ implies that

$$\frac{\partial \Phi}{\partial t}(0,1) = (\theta+p-1)\|\varphi_\mu\|^p - (\theta+r-1)\int\limits_{\mathbb{R}^N} G(z,\varphi_\mu)\,\mathrm{d}z > 0.$$

So, by the Implicit function theorem, there exist $r_1>0$ and a continuous function $\rho_1 : B(0,r_1) \to (0,\infty)$ such that

$$\rho_1(0)=1 \quad \text{and} \quad \rho_1(\varphi)(\varphi_\mu+\varphi) \in N_\mu^+ \quad \text{for all } \varphi \in B(0,r_1).$$

This completes the proof of Lemma 11. □

Lemma 12. *Assume that hypotheses of Theorem* 2 *are satisfied and* $\mu \in (0,\mu^*)$. *Then for every* $\varphi \in E$, *the following statements hold*:

(i) *There exists* $T_1 > 0$ *such that*

$$J_\mu(\varphi_\mu) \leqslant J_\mu(\varphi_\mu + t\varphi) \quad \textit{for all } t \in (0,T_1).$$

(ii) *There exists* $T_2 > 0$ *such that*

$$J_\mu(\psi_\mu) \leqslant J_\mu(\psi_\mu + t\varphi) \quad \textit{for all } t \in (0,T_2).$$

Proof. We shall give the proof only for assertion (i) since the proof for assertion (ii) is similar. So, let $\varphi \in E$ and $\delta_\varphi : [0,\infty) \to \mathbb{R}$ be a function defined by

$$\delta_\varphi(t) = (p-1)\|\varphi_\mu+t\varphi\|^p + \theta\int\limits_{\mathbb{R}^N} a(z)|\varphi_\mu+t\varphi|^{1-\theta}\,\mathrm{d}z - (r-1)\int\limits_{\mathbb{R}^N} G(z,\varphi_\mu+t\varphi)\,\mathrm{d}z.$$

Since $\varphi_\mu \in N_\mu^+ \subset N_\mu$, we obtain

$$\theta\int\limits_{\mathbb{R}^N} a(z)|\varphi_\mu|^{1-\theta}\,\mathrm{d}z = \theta\|\varphi_\mu\|^p + (r-1)\int\limits_{\mathbb{R}^N} G(z,\varphi_\mu)\,\mathrm{d}z \tag{23}$$

and

$$(\theta+p-1)\|\varphi_\mu\|^p - (\theta+r-1)\int\limits_{\mathbb{R}^N} G(z,\varphi_\mu+t\varphi)\,\mathrm{d}z > 0. \tag{24}$$

By combining Eqs. (23) and (24) with the definition of the function δ_φ, we get $\delta_\varphi(0) > 0$. So, the continuity of the function δ_φ implies the existence of $T_0 > 0$ such that

$$\delta_\varphi(t) > 0 \quad \text{for all } t \in [0, T_0].$$

On the other hand, by Lemma 11, for every $t \in [0, r_1]$, there exists $\overline{\rho_1}(t)$ such that

$$\overline{\rho_1}(t)(\varphi_\mu + t\varphi) \in N_\mu^+ \quad \text{and} \quad \lim_{t \to 0^+} \overline{\rho_1}(t) = 1. \tag{25}$$

Moreover, by Lemma 9, we have

$$\theta_\mu^+ = J_\mu(\varphi_\mu) \leqslant J_\mu\big(\overline{\rho_1}(t)(\varphi_\mu + t\varphi)\big) \quad \text{for all } t \in (0, T_0).$$

Now, from that fact that $\Phi''_{\mu,\varphi_\mu}(1) > 0$ and the continuity in t we get

$$\Phi''_{\mu,\varphi_\mu + t\varphi}(1) > 0 \quad \text{for all } t \in [0, T_1] \text{ and for some small enough } T_1 \in (0, T_0).$$

So, using Eq. (25), we can get small enough $T_1 \in (0, T_0)$ such that

$$\theta_\mu^+ = J_\mu(\varphi_\mu) \leqslant J_\mu(\varphi_\mu + t\varphi) \quad \text{for all } t \in [0, T_1).$$

This completes the proof of Lemma 12. □

Now we are ready to present the proof of Theorem 2.

Proof of Theorem 2. As a direct consequence of Lemmas 9 and 10, we can deduce that J_μ has minimizers $\varphi_\mu \in N_\mu^+$ and $\psi_\mu \in N_\mu^-$. Moreover, $N_\mu^+ \cap N_\mu^- = \emptyset$ implies that φ_μ and ψ_μ are distinct.

Next, we shall prove that φ_μ and ψ_μ are weak solutions for problem (1). To this end, let $\varphi \in E$. Then by the assertion (i) of Lemmas 11, 12, we obtain

$$0 \leqslant J_\mu(\varphi_\mu + t\varphi) - J_\mu(\varphi_\mu) \quad \text{for all } t \in (0, T_1).$$

Dividing the last inequality by t and letting t tend to zero, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\Delta\varphi_\mu|^{p-2}\Delta\varphi_\mu\Delta\varphi - \lambda\frac{|\varphi_\mu|^{p-2}\varphi_\mu\varphi}{|z|^{2p}} + |\nabla\varphi_\mu|^{p-2}\nabla\varphi_\mu\nabla\varphi \right) \mathrm{d}z \\
&\quad - \int_{\mathbb{R}^N} a(z)\varphi_\mu^{-\theta}\varphi \,\mathrm{d}z - \mu\int_{\mathbb{R}^N} f(z)h(\varphi_\mu)\varphi \,\mathrm{d}z \geqslant 0.
\end{aligned}$$

Since φ is arbitrary in E, it follows that in the last inequality we can replace φ by $-\varphi$. So, for all $\varphi \in E$, we get

$$\begin{aligned}
0 = &\int_{\mathbb{R}^N} \left(|\Delta\varphi_\mu|^{p-2}\Delta\varphi_\mu\Delta\varphi - \lambda\frac{|\varphi_\mu|^{p-2}\varphi_\mu\varphi}{|z|^{2p}} + |\nabla\varphi_\mu|^{p-2}\nabla\varphi_\mu\nabla\varphi \right) \mathrm{d}z \\
&- \int_{\mathbb{R}^N} a(z)\varphi_\mu^{-\theta}\varphi \,\mathrm{d}z - \mu\int_{\mathbb{R}^N} f(z)h(\varphi_\mu)\varphi \,\mathrm{d}z.
\end{aligned}$$

That is, φ_μ is a weak solution of problem (1). Moreover, from Eq. (10) we see that φ_μ is nontrivial.

Finally, if we proceed as above using assertion (ii) of Lemmas 11 and 12, we can prove that ψ_μ is also a nontrivial weak solution of problem (1). This completes the proof of Theorem 2. □

6 An application

As an application of our main results, we shall consider the following problem:

$$\Delta_p^2\varphi - \lambda\frac{|\varphi|^{p-2}\varphi}{|z|^{2p}} + \Delta_p\varphi = \frac{a(z)}{\varphi^\theta} + \mu f(z)|\varphi|^{r-2}\varphi \quad \text{in } \mathbb{R}^N, \tag{26}$$

where $\mu > 0$, $1 < p < N/2$, $0 < \theta < 1$, and λ satisfies Eq. (4).

We note that problems of type (26) describe, e.g., the deformations of an elastic beam. Also, they give a model for studying traveling waves in suspension bridges.

First, let us assume that $1 < r < p$, f is a positive function in

$$L^{p^*/(p^*-r)}(\mathbb{R}^N) \cap L^s_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for some } s \in \left(\frac{p^*}{p^*-r}, \frac{p}{p^*-r}\right),$$

which implies that the first part of hypothesis (H1) is satisfied.

On the other hand, it is easy that the function $h(z) = |\varphi|^{r-2}\varphi$ satisfies the second part of hypothesis (H1). Moreover, a simple calculation shows that

$$0 < rf(z)H(\varphi) = f(z)h(\varphi)\varphi,$$

so, hypothesis (H2) is also satisfied.

Finally, if

$$a \in L^{p^*/(p^*+\theta-1)}(\mathbb{R}^N) \cap L^\beta_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for some } \beta \in \left(\frac{p^*}{p^*+\theta-1}, \frac{p}{\theta+p-1}\right),$$

then Theorem 1 ensures the existence of nontrivial solution for problem (26).

Next, we assume that $p < r < p^*$ and a is a positive function in $L^{p/(\theta+p-1)}(\mathbb{R}^N)$, that is, hypothesis (H5) is satisfied. It is not difficult to see that if

$$g(z,\varphi) = f(z)|\varphi|^{r-2}\varphi,$$

then

$$G(z,\varphi) = f(z)|\varphi|^r,$$

so, hypothesis (H4) is also satisfied. Hence, Theorem 2 now ensures the existence of two nontrivial solutions for problem (26).

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