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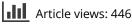
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Finite solvable groups with a rational skew-field of noncommutative real rational invariants

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ABSTRACT

We consider Noether's problem on the noncommutative rational functions invariant under a linear action of a finite group. For abelian groups the invariant skew-fields are always rational, for solvable group they are rational if the action is well-behaved – given by a so-called complete representation. We determine the groups that admit such representations and call them totally pseudo-unramified. We show that for a solvable group the invariant skew-field is finitely generated. Finally we study totally pseudo-unramified groups and classify totally pseudo-unramified p-groups of rank at most 5.

1. Introduction

Invariants of group actions are an important topic that appears in many mathematical areas and beyond mathematics in physics and chemistry. Classical invariant theory [33, 36] considers polynomials that are preserved under the action of a group G given by a homomorphism $G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[x_1, \ldots, x_n])$. A closely related topic are rational invariants, where one considers rational functions that are invariant under the action of a group G given by a homomorphism $G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}(x_1, \ldots, x_n))$. *Noether's problem* asks whether the subfield of invariants $\mathbb{F}(x_1, \ldots, x_n)^G$ is rational, i.e., purely transcendental over $\mathbb{F}$. The problem depends both on the group G and the base field $\mathbb{F}$ and topic is widely studied [12, 35, 37]. Noether's problem has a positive answer in one variable over any field (Lüroth's theorem), in two variables over $\mathbb{F} = \mathbb{C}$ and for linear actions of abelian groups over $\mathbb{C}$ [17]. There are groups with a negative answer; over $\mathbb{Q}$ this happens even for cyclic groups (one such is $\mathbb{Z}_{47}$ [28]) and there are examples over $\mathbb{C}$ as well [35].

We consider a noncommutative version of *Noether's problem*. We replace commutative rational functions with the *free skew-field* $\mathbb{F}\nleftarrow x_1, \ldots, x_n \nrightarrow$ also called the skew-field of *noncommutative rational functions* (in n variables). The free skew-field is the universal skew-field of fractions of the free associative algebra of noncommutative polynomials (see [2, 10]). The noncommutative Noether's problem then considers the rationality of the skew-field of invariants of a finite group G, i.e., whether the skew field

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of invariants $\mathbb{F}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle^G$ is isomorphic to the skew-field of noncommutative rational functions $\mathbb{F}\langle\!\langle y_1,\ldots,y_m\rangle\!\rangle$ for some $m\in\mathbb{N}$.

The polynomial noncommutative invariants are well studied [1, 11, 15, 39], with the main result being that the polynomial noncommutative invariants are always isomorphic to a free algebra that is usually not finitely generated. In [26] the authors consider noncommutative rational functions invariant under a faithful action of finite abelian group given by linear transformations of variables. In this case the skew-field of invariants $\mathbb{C}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle^A$ is rational in $|A|(n-1)+1$ variables [26, 4.1]. For example,

$$\mathbb{C}\langle\!\langle x,y\rangle\!\rangle^{S_2} = \mathbb{C}\langle\!\langle x+y,(x-y)^2,(x-y)(x+y)(x-y)\rangle\!\rangle \cong \mathbb{C}\langle\!\langle y_1,y_2,y_3\rangle\!\rangle.$$

In fact a closer inspection of the above example shows that we can replace $\mathbb{C}$ by $\mathbb{Q}$ or any other field of characteristic not equal to 2.

One would like to use the cited theorem recursively on solvable groups; given an abelian normal subgroup N of G first compute the N-invariants and proceed with the action of the quotient group G/N; however, the subsequent action might not be linear. In some cases the idea still works, for example for the action given by a *complete representation* as defined in [26] (or see Definition 2.1 below). If the action of a finite group G is given via a complete representation on the linear span of variables $x_1,\ldots,x_n$, then the skew-field of invariants $\mathbb{C}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle^G$ is rational in $|G|(n-1)+1$ variables [26, 5.1]. In fact, the proof of the cited theorem serves as an algorithm for expressing free generators of the skew-field of invariants in terms of the initial variables. Furthermore, [26] provides a class of groups called *totally unramified* such that their regular representation is complete, thus the noncommutative Noether's problem has a (partial) positive answer for them. Among totally unramified groups are the symmetric groups S_3 and S_4. Understanding their skew-fields of invariant noncommutative rational functions could shed some light on the theory of noncommutative symmetric rational functions which is, contrary to the commutative case, still far from complete [19].

The recursive method using normal abelian subgroups does not fail completely for a general finite solvable group G; using it we can show that $\mathbb{C}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle^G$ is finitely generated as a skew-field over $\mathbb{C}$ [26, 1.1]. In contrast, the ring of noncommutative polynomials invariant under a linear action of a finite group $\mathbb{F}\langle x_1,\ldots,x_n\rangle^G$ is almost never finitely generated [11, 6.8.4].

In this paper we study the noncommutative Noether's problem over $\mathbb{R}$ and prove the real versions of the main results of [26]. Then we study the reach of the cited results and our real counterparts. We introduce *totally pseudo-unramified groups*, a generalization of totally unramified groups, and show they are precisely the groups that admit complete representations, hence we harvest the full potential of [26, 5.1]. We also connect totally unramified groups and totally pseudo-unramified groups with established concepts from group theory and representation theory. Finally, we classify totally unramified and totally pseudo-unramified p-groups of rank at most 5. These groups are significant as there are p-groups of rank 5 whose fields of commutative rational invariants are not rational [9, 21, 31]. Unsurprisingly none of these groups are totally pseudo-unramified.

1.1. Main results and outline

Section 2 explains the notations and contains the main definitions followed by preliminaries on noncommutative rational functions, Malcev-Neumann series, Clifford theory and projective representations as needed for this paper.

In Section 3 we study real noncommutative rational functions invariant under a linear action of finite abelian and solvable. We obtain the following main results:

1. If the action of a group G on $\mathbb{F}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle$ is nontrivial on $\mathbb{F}$ and trivial on the variables, then $\mathbb{F}\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle^G = \mathbb{F}^G\langle\!\langle x_1,\ldots,x_n\rangle\!\rangle$ (Proposition 3.1).
2. The skew-field of noncommutative real rational functions invariant under a linear action of an abelian group is rational (Theorem 3.5).

3. The skew-field of noncommutative real rational functions invariant under an action of a group given via a complete representation is rational (Theorem 3.7).
4. The skew-field of noncommutative real rational functions invariant under an action of a finite solvable group is finitely generated. (Theorem 3.10).

In Section 4 we turn our attention to complete representations and totally pseudo-unramified groups and give the following results:

5. Totally pseudo-unramified groups are precisely the groups that admit complete representations (Theorem 4.4).
6. An example of a group that is not totally pseudo-unramified, yet it has a rational skew-field of noncommutative rational invariants (Example 4.7).
7. A semidirect product $A \rtimes G$ of an abelian group A and a totally pseudo-unramified group G is totally pseudo-unramified (Corollary 4.10).

In Section 5 we study totally unramified groups with main results as follows:

8. Totally unramified groups are closed under quotients (Proposition 5.4).
9. Metacyclic groups and semidirect products of abelian groups are totally unramified (Corollaries 5.11 and 5.12).
10. If G and H are isoclinic finite groups and G is totally unramified, then H is totally unramified (Proposition 5.15).

Finally, in Section 6 we study nilpotent totally unramified groups and establish:

11. A nilpotent totally unramified group is metabelian (Theorem 6.4).
12. In Section 6.1 we classify totally unramified and totally pseudo-unramified p-groups of rank up to 5.

Throughout the paper examples are given to demonstrate the strength of our results.

2. Definitions and preliminaries

2.1. Notation

Throughout the paper we aim to use standard notation. All considered fields have characteristic zero. All considered groups are assumed to be finite unless stated otherwise. We denote the set of complex irreducible characters of a group G by $\mathrm{Irr}(G)$ and the set of complex linear characters (characters of degree 1) by $\mathrm{Lin}(G)$. The trivial character of G is denoted by τ_G or simply τ. Complex class functions of G are endowed with a scalar product,

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

The irreducible characters are an orthonormal basis of class functions with respect to this scalar product. We denote the set of irreducible complex linear representations of a group G by $\mathrm{IRR}(G)$.

A complex character χ of a group G is *multiplicity free* if $\langle \chi, \mu \rangle \le 1$ for every irreducible character $\mu \in \mathrm{Irr}(G)$. A complex representation ρ of a group G is *multiplicity free* if its character χ_ρ is multiplicity free. Equivalently, ρ is multiplicity free if it is equivalent to a direct sum of pairwise nonequivalent irreducible representations.

For a subgroup N of G and an irreducible character $\mu \in \mathrm{Irr}(N)$ we denote the *irreducible characters of G lying over μ* by

$$\mathrm{Irr}_\mu(G) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi|_N, \mu \rangle > 0\} = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, \mathrm{Ind}_N^G \mu \rangle > 0\}.$$

By $\mathrm{IRR}_\mu(G)$ we denote the set of irreducible linear representations ρ of G such that their characters satisfy $\chi_\rho \in \mathrm{Irr}_\mu(G)$.

Let N be a normal subgroup of G, then G acts on the characters of N. The left action on a character μ of N is defined by ${}^g\mu(n) = \mu(g^{-1}ng)$. If μ is irreducible, then so is ${}^g\mu$.

2.2. Definitions

We proceed with our main definitions. The definitions of complete representations and totally unramified groups were introduced in [26]. The original definition of complete representation is missing the base cases of recursion; we correct this oversight here. Also the definition of $Q\pi$ is slightly changed – in [26] the summand $\pi_B \otimes \pi_B$ appears twice. Using Proposition 4.2 it is easy to see that the definitions are equivalent.

Definition 2.1. A complex linear representation π of a finite group G is *complete* if it decomposes as $\pi = \pi_B \oplus \pi_J$ and there is a nontrivial abelian normal subgroup $N \subseteq G$ such that:

1. $\pi_B|_N$ contains exactly the nontrivial irreducible linear representations of N as direct summands with multiplicity 1;
2. $G = N$ or the representation

$$Q\pi = \left[\pi \oplus (\pi_B \otimes \pi_B) \oplus (\pi_B \otimes \pi_J) \oplus (\pi_J \otimes \pi_B) \oplus (\pi_B \otimes \pi \otimes \pi_B)\right]_{N^\tau}$$

is a complete representation of G/N. Here, for a representation ρ, ρ_{N^τ} denotes the summands of ρ which are trivial on N and thus naturally gives rise to a representation of G/N.

A real linear representation of a finite group is *complete* if its complexification is complete.

Remark 2.2. Let $\rho\colon G \to \mathrm{GL}(V)$ be a complex linear representation and $N \subseteq G$ an abelian normal subgroup such that $\rho|_N = \bigoplus_{i=1}^m \mu_i$ is multiplicity free. Then V decomposes as a direct sum $\bigoplus_{i=1}^m V_{\mu_i}$ of one-dimensional subspaces such that $v \in V$ is in V_{μ_i} if and only if $\rho(n)v = \mu_i(n)v$ for every $n \in N$. Pick $v_i \in V_{\mu_i}$ then

$$\rho(g^{-1}ng)v_i = \mu_i(g^{-1}ng)v_i = {}^g\mu_i(n)v_i.$$

Rearranging yields

$$\rho(n)\left(\rho(g)v_i\right) = {}^g\mu_i(n)\left(\rho(g)v_i\right),$$

which shows that $\rho(g)v_i \in V_{{}^g\mu_i}$. Hence the matrix of $\rho(g)$ written in any basis $\{b_j \mid j = 1, \ldots, m\}$ such that $b_j \in V_{\mu_j}$ has precisely one nonzero entry in each row and each column.

Definition 2.3. A finite group G is *unramified* over a nontrivial normal abelian subgroup N if for every complex irreducible linear representation ρ of G, the restriction $\rho|_N$ is multiplicity free or trivial.

The group G is *totally unramified* if it is abelian or there exists a nontrivial abelian normal subgroup N such that G is unramified over N and G/N is totally unramified.

In the definition of a totally unramified group we can exchange representations for their characters. Using Frobenius reciprocity we observe that a group G is unramified over N if and only if for every nontrivial irreducible representation (character) μ of N the induced representation (character) $\mathrm{Ind}_N^G(\mu)$ is multiplicity free.

Remark 2.4. We can interpret the recursive condition in the definition of a totally unramified group as follows: a finite group G is totally unramified if there exists a series of normal subgroups

$$1 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = G$$

such that for every $j = 0, \ldots, n-1$ the quotient N_{j+1}/N_j is abelian and G/N_j is unramified over N_{j+1}/N_j.

The notion of "unramified over" is closely related to the so-called Gelfand triples (see [8]). A triple (G, H, ρ) consisting of a group G, subgroup H and an irreducible linear representation ρ of H is a *Gelfand triple* if the induced representation $\mathrm{Ind}_H^G \rho$ is multiplicity free. Thus a finite group G is unramified over an abelian normal subgroup N if and only if (G, N, μ) is a Gelfand triple for every nontrivial irreducible linear representation μ of N.

The next definition is a generalization of totally unramified groups and is, as we will see, tightly connected to complete representations.

Definition 2.5. A finite group G is *pseudo-unramified over* a nontrivial abelian normal subgroup N if for every irreducible character $\mu \in \mathrm{Lin}(N)$ there exists $\chi \in \mathrm{Irr}_\mu(G)$ such that $\chi|_N$ is multiplicity free.

A finite group G is *totally pseudo-unramified* if it is abelian or there exists a nontrivial abelian normal subgroup N such that G is pseudo-unramified over N and G/N is totally pseudo-unramified.

Again we can interchange representations with characters and a suitably modified version of Remark 2.4 holds for totally pseudo-unramified groups.

Clearly every totally pseudo-unramified group is solvable. If G is unramified over N, then clearly G is pseudo-unramified over N. Hence every totally unramified group is totally pseudo-unramified.

Remark 2.6. The definitions of totally unramified and totally pseudo-unramified groups allow for a straightforward checking of the properties using GAP [18]. We use it to work with examples; in the squeal the group with group ID $[n, m]$ refers to the group that is in GAP summoned by "SmallGroup(n, m)."

2.3. Noncommutative rational functions

The field of (commutative) rational functions is the field of fractions of (commutative) polynomials. The passage from noncommutative polynomials i.e., free associative algebra, to the skew-field of noncommutative rational function (also called free skew-field) is not as straightforward. We introduce terminology and basic concepts surrounding noncommutative rational functions. For a longer exposition we refer to [2, 10, 11, 24, 38].

A *noncommutative rational expression* is a syntactically valid combination of elements of the base field $\mathbb{F}$, variables, operations $+, \cdot$, inverse and parenthesis, for example:

$$\left(2x_1^3 x_2^4 x_1^5 - ((x_1 x_2 - x_2 x_1)^{-1} + 1)^2\right)^{-1} + x_3.$$

Such expressions can be evaluated on tuples of square matrices of equal size with coefficients in $\mathbb{F}$. An expression is *nondegenerate* if it is valid to evaluate it on some tuple of matrices. Two nondegenerate expressions are equivalent if they evaluate equally whenever both are defined. A *noncommutative rational function* is an equivalence class of a nondegenerate rational expression; these functions form the free skew-field $\mathbb{F}\,\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$. The free skew field $\mathbb{F}\,\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ is the universal skew-field of fractions of the free algebra $\mathbb{F}\langle x_1, \ldots, x_n \rangle$. It is universal in the sense that any epimorphism from $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ to a skew-field D extends to a specialization from $\mathbb{F}\,\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ to D.

Another way of constructing the free skew-field is as the universal localization of free algebra, i.e., we adjoin entries of inverses of all full matrices over the free algebra. Any noncommutative rational function $r \in \mathbb{F}\,\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ can be represented by a *linear realization*

$$r = c^* L^{-1} b$$

where $b, c \in \mathbb{F}^n$ and

$$L = A_0 + \sum_{i=1}^d A_i x_i$$

for some matrices $A_i \in \mathrm{M}_n(\mathbb{F})$.

We say that a skew-field is *rational* (or *free*) over $\mathbb{F}$ if it is isomorphic to the free skew-field $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ for some $n \in \mathbb{N}$. Variables $x_1,\ldots,x_n$ in $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ generate a free group Γ under multiplication. The skew-field $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ is also the universal field of fractions of the group algebra $\mathbb{F}\Gamma$, hence we also use notation $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\} = \mathbb{F}\,\{\!\langle \Gamma\rangle\!\}$.

2.4. Malcev-Neumann series and Connes operator

Let Γ be the free group generated by $X = \{x_1,\ldots,x_n\}$. A *formal (power) series* on Γ with coefficients in $\mathbb{F}$ is a function $S\colon \Gamma \to \mathbb{F}$. We denote the set of formal series by $\mathbb{F}^{\Gamma}$. Any formal series S can be uniquely presented by $\sum_{\omega\in\Gamma} S(\omega)\omega$. The *support* of a series S is $\operatorname{supp} S = \{\omega \mid S(\omega) \neq 0\}$.

Let $\leq$ be any total order of Γ compatible with the group structure. For an example of such an ordering we refer to [3, 34]. The *Malcev-Neumann series* $\mathbb{F}((\Gamma, \leq))$ (with respect to the given order) is the set of series $S \in \mathbb{F}^{\Gamma}$ such that their support is well ordered. Malcev-Neumann series form a skew-field under the pointwise addition and Cauchy product:

$$ab = \sum_{\omega\in\Gamma} \sum_{\substack{\alpha,\beta\in\Gamma \\ \alpha\beta=\omega}} a_\alpha b_\beta \omega.$$

For the sake of brevity we fix the ordering of Γ and denote $\mathbb{F}((\Gamma)) = \mathbb{F}((\Gamma, \leq))$. For more on Malcev-Neumann series we refer to [34].

The rational closure of the free algebra $\mathbb{F}<x_1,\ldots,x_n>$ or the group algebra $\mathbb{F}\Gamma$ in $\mathbb{F}((\Gamma))$ is isomorphic to $\mathbb{F}\,\{\!\langle \Gamma\rangle\!\}$ regardless of the ordering [29, 34]. We say that a series $r \in \mathbb{F}((\Gamma))$ is *rational* (over $\mathbb{F}$) if it belongs to the rational closure of $\mathbb{F}<x_1,\ldots,x_n>$, i.e., the smallest subring of $\mathbb{F}((\Gamma))$ that contains $\mathbb{F}<x_1,\ldots,x_n>$ and is closed under taking inverse.

Let M be the free monoid generated by X. Any rational function $r \in \mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ that is defined at 0 can be expanded to a series $r \in \mathbb{F}^M$. Conversely, a series $r \in \mathbb{F}^M$ represents a rational function if and only if its *Hankel matrix* has finite rank [5, 38]. Rationality in Malcev-Neumann series is a bit more intricate. Let $\mathcal{G} = \operatorname{Cay}(\Gamma, X)$ be the Cayley graph of Γ. Given a series $a \in \mathbb{F}((\Gamma))$ we have the *Connes operator* $[\mathfrak{F}, a]\colon \mathbb{F}\mathcal{G} \to \mathbb{F}^{\mathcal{G}}$. For the definition of the Connes operator we refer to [13, 16, 27]. A Malcev-Neumann series a is rational if and only if $[\mathfrak{F}, a]$ has finite rank ([16, 12], [27, 2.6]). Given a subfield $\mathbb{K} \subset \mathbb{F}$ and a series $a \in \mathbb{K}((\Gamma))$ the Connes operator $[\mathfrak{F}, a]\colon \mathbb{K}\mathcal{G} \to \mathbb{K}^{\mathcal{G}}$ is equal to the restriction of the Connes operator over $\mathbb{F}$.

2.5. Complex noncommutative rational invariants

We summarize the results and techniques from [26]. We say that a finite group G acts *linearly* on $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ if the action is defined by a linear representation $G \to \operatorname{GL}(V)$ where $V = \operatorname{span}_{\mathbb{F}}\{x_1,\ldots,x_n\}$. We say that a linear action is *diagonal* if each variable x_i spans an invariant subspace of V i.e., $g\cdot x_i = \chi_i(g)x_i$ where χ_i is a linear character of G. If G acts faithfully diagonally on $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ then G is abelian and $\mathbb{F}$ is a splitting field of G. Every linear representation of an abelian group A over a splitting field $\mathbb{F}$ is equivalent to a direct sum of representation of degree one, thus we can pass from a linear action of A on $\mathbb{F}\,\{\!\langle x_1,\ldots,x_n\rangle\!\}$ to a diagonal action via a linear transformation of variables.

Given a faithful diagonal action of a finite abelian group A on $\mathbb{F}\,\{\!\langle \Gamma\rangle\!\}$ we have a surjective group homomorphism $\Gamma \to A^*(= \operatorname{Hom}(A, \mathbb{F}^*) \cong A)$ defined by $x_i \mapsto \chi_i$. We denote the kernel of this homomorphism by Γ^A. By the Nielsen–Schreier formula, Γ^A is a free group of rank $|A|(n-1) + 1$.

Theorem 2.7. [26, 4.1] *If a finite abelian group A acts faithfully diagonally on $\mathbb{F}\,\{\!\langle \Gamma\rangle\!\}$ then $\mathbb{F}\,\{\!\langle \Gamma\rangle\!\}^A = \mathbb{F}\,\{\!\langle \Gamma^A\rangle\!\}$.*

The original statement of the theorem requires $\mathbb{F}$ to be algebraically closed but allows linear actions, yet algebraically closed field is only needed to pass from a linear action to a diagonal one.

We continue with invariants of complete representations. If the linear action of a group G on $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ is given via a complete representation $\pi = \pi_B \oplus \pi_J$ and N is an abelian normal subgroup from the definition, we can find a "good" set of free generators of N-invariants.

Let $\mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_n\} = V_B \oplus V_J$ be the decomposition with respect to $\pi = \pi_B \oplus \pi_J$ and let

$$\{b_\chi \mid \chi \in \mathrm{Irr}(N) \setminus \{\tau\}\} \quad \text{and} \quad \{v_k \mid k = 1, \ldots, \deg \pi_J\}$$

be bases of V_B and V_J, respectfully, such that for each $n \in N$ we have $\pi(n)b_\chi = \chi(n)b_\chi$ and $\pi(n)v_k = \mu_k(n)v_k$ for some $\mu_k \in \mathrm{Irr}(N)$. We also set $b_\tau = 1$. Then the free generators of N-invariants are

$$b_\chi b_\mu b_{(\chi\mu)^{-1}}, b_\theta v_k b_{(\theta\mu_k)^{-1}}$$

where χ and μ run through $\mathrm{Irr}(N) \setminus \{\tau\}$, θ runs through $\mathrm{Irr}(N)$ and $k = 1, \ldots, \deg \pi_J$ ([26, 4.2]). Using Remark 2.2 we show that G/N acts linearly on these generators via the representation $Q\pi$. The item (2) of the definition then allows us to continue recursively and conclude that the skew-field of invariants $\mathbb{C}\langle x_1, \ldots, x_n \rangle^G$ is rational ([26, 5.1]).

2.6. Clifford theory

We give a short overview of Clifford theory. For a more thorough and general exposition we refer to [4, 14, 22].

Let G be a finite group and N an (abelian) normal subgroup. For $\mu \in \mathrm{Irr}(N)$ the *inertia subgroup* is $I_G(\mu) = \{g \in G \mid {}^g\mu = \mu\}$. In this subsection we use $H = I_G(\mu)$. Pick any left transversal $\{g_\alpha \mid \alpha \in G/H\}$ of H in G. Clifford theorem ([14, (11.1)], [22, (6.5)], [4, 7.3]) states that for any $\chi \in \mathrm{Irr}_\mu(G)$ we have (independently of the transversal)

$$\chi|_N = e_\chi \sum_{\alpha \in G/H} {}^{g_\alpha}\mu$$

where $e_\chi \in \mathbb{N}$ is called the *ramification* of χ over N. For any $\chi \in \mathrm{Irr}_\mu(G)$ we have a unique $\theta \in \mathrm{Irr}_\mu(H)$ such that

$$\chi|_H = \sum_{\alpha \in G/H} {}^{g_\alpha}\theta$$

and $\theta|_N = e_\chi \mu$ ([22, (6.11)], [4, 7.6]). By Frobenius reciprocity we get the dual statement:

$$\mathrm{Ind}_N^G(\mu) = \mathrm{Ind}_H^G(\mathrm{Ind}_N^H(\mu)) = \sum_{\theta \in \mathrm{Irr}_\mu(H)} e_\theta \, \mathrm{Ind}_H^G(\theta),$$

induction $\theta \mapsto \mathrm{Ind}_H^G(\theta)$ gives a bijection from $\mathrm{Irr}_\mu(H)$ to $\mathrm{Irr}_\mu(G)$ and the ramification over N $e_\theta = e_{\mathrm{Ind}_H^G(\theta)}$ is preserved. For our purposes we summarize the findings in the following proposition.

Proposition 2.8. *(1) A group G is unramified over a nontrivial abelian normal subgroup N if and only if for every nontrivial $\mu \in \mathrm{Irr}(N)$ the inclusion $\mathrm{Irr}_\mu(I_G(\mu)) \subseteq \mathrm{Lin}(I_G(\mu))$ holds.*

(2) A group G is pseudo-unramified over a nontrivial abelian normal subgroup N if and only if for every $\mu \in \mathrm{Irr}(N)$ the intersection $\mathrm{Irr}_\mu(I_G(\mu)) \cap \mathrm{Lin}(I_G(\mu))$ is non-empty.

Proof. (1) A group G is unramified over N if and only if for every nontrivial $\mu \in \mathrm{Irr}(N)$ and every $\chi \in \mathrm{Irr}_\mu(G)$ the ramification e_χ over N is equal to 1. The ramification e_χ is the same as the ramification e_θ of the unique $\theta \in \mathrm{Irr}_\mu(I_G(\mu))$ with the property $\mathrm{Ind}_{I_G(\mu)}^G(\theta) = \chi$. We get $\theta|_N = \theta(1)\mu = e_\chi \mu$, hence $e_\chi = \theta(1) = 1$ if and only if $\theta \in \mathrm{Lin}(I_G(\mu))$.

(2) A group G is pseudo-unramified over N if and only if for every $\mu \in \mathrm{Irr}(N)$ there exists $\chi \in \mathrm{Irr}_\mu(G)$ with ramification e_χ over N equal to 1. From here we reason as in the proof of (1). $\square$

2.7. Projective representations

To give a more palpable description of $\text{Irr}_\mu(I_G(\mu))$ we turn to projective representations. For a detailed discussion we refer to [4, Ch.6].

Let V be a finite dimensional (complex) vector space. *A (complex) projective representation* of a group G is a mapping $P\colon G \to \text{GL}(V)$ satisfying

$$\forall x, y \in G : P(x)P(y) = \pi(x, y)P(xy)$$

for some $\pi\colon G \times G \to \mathbb{C}^*$. We call π a *factor set* of P and P a *π-representation*. The *degree* of representation P is the dimension of V. A π-representation is *irreducible* if it does not have any nontrivial invariant subspaces. We denote the set of irreducible π-representations of G by $\text{IRR}^\pi(G)$.

Any factor set π satisfies the 2-cocycle condition:

$$\forall x, y, z \in G : \pi(x, y)\pi(xy, z) = \pi(x, yz)\pi(y, z).$$

Conversely any mapping $\pi\colon G \times G \to \mathbb{C}^*$ satisfying the 2-cocycle condition (2-cocycle) is a factor set of some projective representation. The factor sets equipped with pointwise multiplication form the abelian 2-cocycle group $Z^2(G, \mathbb{C}^*)$ (with trivial action on $\mathbb{C}^*$). The factor sets π, π' are *associated* if there exists a function $\lambda\colon G \to \mathbb{C}^*$ such that

$$\forall x, y \in G : \pi'(x, y) = \frac{\lambda(x)\lambda(y)}{\lambda(xy)}\pi(x, y).$$

The factor sets that are associated to the trivial factor set form the subgroup $B^2(G, \mathbb{C}^*)$ of 2-coboundaries. The second cohomology group

$$M(G) = H^2(G, \mathbb{C}^*) = Z^2(G, \mathbb{C}^*)/B^2(G, \mathbb{C}^*)$$

is also called the *Schur multiplier* of G. We denote the equivalence class of a factor set $\pi \in Z^2(G, \mathbb{C}^*)$ by $[\pi] \in M(G)$. We get another equivalent definition of the Schur multiplier using Hopf's formula $M(G) = H_2(G, \mathbb{Z}) \cong (R \cap [F, F])/[R, F]$, where F is a free group and R a normal subgroup such that $G \cong F/R$. For a thorough exposition on the Schur multiplier we refer to [25].

The next proposition follows directly from $M(G) \cong (R \cap [F, F])/[R, F]$.

Proposition 2.9. *The Schur multiplier of a finite cyclic group is trivial.*

Projective representations $P\colon G \to \text{GL}(V)$ and $P'\colon G \to \text{GL}(V')$ are *linearly equivalent* if there exists a linear isomorphism $S\colon V \to V'$ such that $P'(g) = SP(g)S^{-1}$ for every $g \in G$. Projective representations P and P' are *projectively equivalent* if there exists a function $\lambda\colon G \to \mathbb{C}^*$ such that λP and P' are linearly equivalent. We note that projectively equivalent representations have associated factor sets. Conversely there is a bijection between $\text{IRR}^\pi(G)$ and $\text{IRR}^{\pi'}(G)$ if π and π' are associated. Namely, if $\pi'(g, h) = \frac{\lambda(g)\lambda(h)}{\lambda(gh)}\pi(g, h)$, then the bijection is given by $P \mapsto \lambda P$.

Some properties of $\text{IRR}^\pi(G)$ are determined by the class $[\pi] \in M(G)$. One such property is described in the next lemma.

Lemma 2.10. *A group G has a π-representation of degree one if and only if $[\pi] = 1 \in M(G)$.*

Proof. If P is a π-representation of degree one, we get $\pi(g, h) = P(g)P(h)/P(gh)$. If $\pi(g, h) = \lambda(g)\lambda(h)/\lambda(gh)$, then λ is a π-representation of degree one. □

We now describe a connection between $\text{IRR}_\mu(I_G(\mu))$ for an irreducible character $\mu \in \text{Irr}(N)$ of an abelian normal subgroup N of G and some projective representations of $I_G(\mu)/N$. In the sequel we denote $H = I_G(\mu)$.

Let $\rho\colon H \to \text{GL}(V)$ be a linear representation of H of degree n with character $\chi_\rho \in \text{Irr}_\mu(H)$ for some $\mu \in \text{Irr}(N)$. Further we choose a left transversal $\{h_\alpha \mid \alpha \in H/N\}$ of N in H. The transversal defines a

2-cocycle $f \in Z^2(H/N, N)$ by $h_\alpha h_\beta = f(\alpha, \beta) h_{\alpha\beta}$. We define a mapping $P \colon H/N \to \mathrm{GL}(V)$ by

$$P(\alpha) = \rho(h_\alpha).$$

Then P is an irreducible projective representation of H/N of degree n with the factor set $\pi(\alpha, \beta) = \mu(f(\alpha, \beta))$. We say that P is a *descent* of ρ to H/N. A different choice of transversal defines an associated factor set and a projectively equivalent representation.

We can reverse the process. Let $P \colon H/N \to \mathrm{GL}(V)$ be an irreducible projective representation of H/N with a factor set $\pi(\alpha, \beta) = \mu(f(\alpha, \beta))$. We define a mapping $\rho \colon H \to \mathrm{GL}(V)$ by

$$\rho(nh_\alpha) = \mu(n)P(\alpha).$$

Then ρ is an irreducible linear representation of H of degree n and its character lies in $\mathrm{Irr}_\mu(H)$. We say that ρ is a *lift* of P to H.

We summarize the above discussion in a lemma.

Lemma 2.11. *Let N be an abelian normal subgroup of G and $\mu \in \mathrm{Irr}(N)$ an irreducible character. Further let $f \in Z^2(I_G(\mu)/N, N)$ be any 2-cocycle defining the extension from $I_G(\mu)/N$ to $I_G(\mu)$. Then there is a degree preserving bijection between $\mathrm{IRR}_\mu(I_G(\mu))$ and $\mathrm{IRR}^\pi(I_G(\mu)/N)$, where $\pi = \mu \circ f \in Z^2(I_G(\mu)/N, \mathbb{C}^*)$.*

2.8. Inflation-restriction exact sequence

Motivated by Lemma 2.11 we investigate the mapping $\mu \mapsto [\mu \circ f]$. This mapping appears in a case of the inflation-restriction exact sequence. The discussed sequence is thoroughly examined in [4, Ch.6.§5].

Let N be an (abelian) normal subgroup of H. We denote by

$$\mathrm{Lin}^H(N) = \{\mu \in \mathrm{Lin}(N) \mid {}^h\mu = \mu \text{ for any } h \in H\} = \{\mu \in \mathrm{Lin}(N) \mid I_H(\mu) = H\}$$

the linear characters of N invariant under the action of H. There is an exact sequence

$$1 \longrightarrow \mathrm{Lin}(H/N) \xrightarrow{\ \mathrm{Inf}\ } \mathrm{Lin}(H) \xrightarrow{\ \mathrm{Res}\ } \mathrm{Lin}^H(N) \xrightarrow{\ T_H\ } M(H/N) \tag{2.1}$$

where T_H is given by $T_H(\mu) = [\mu \circ f]$ with $f \in Z^2(H/N, N)$ being any 2-cocycle that describes the extension of H/N to H. We also note that $\mathrm{Lin}(H) \cong H/H'$ and $\mathrm{Lin}^H(N) \cong N/[N, H]$.

3. Real noncommutative rational invariants

In this section we show that the noncommutative real rational functions invariant under the linear action of an abelian group are rational over $\mathbb{R}$. Then we extend the result to actions given by complete representations. At the end we show that the skew-field of rational invariants of a finite solvable group is finitely generated.

We derive a technique that will allow us to pass from the invariants over $\mathbb{C}$ to invariants over $\mathbb{R}$ using the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$. For this we consider group actions on noncommutative rational functions that are nontrivial on the base field and trivial on the variables. We use Malcev-Neumann series to study functions invariant under such actions.

Proposition 3.1. *1. Let the action of a (possibly infinite) group G on $\mathbb{F}(\!(\Gamma)\!)$ be given by an action on $\mathbb{F}$ and trivial action on Γ, then $\mathbb{F}(\!(\Gamma)\!)^G = \mathbb{F}^G(\!(\Gamma)\!)$.*

2. Let $\mathbb{K} \subset \mathbb{F}$ be fields, then $\mathbb{K}(\!(\Gamma)\!) \cap \mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle = \mathbb{K}\langle\!\langle\Gamma\rangle\!\rangle$.

3. Let the action of a (possibly infinite) group G on $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ be given by an action on $\mathbb{F}$ and a trivial action on Γ, then $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle^G = \mathbb{F}^G\langle\!\langle\Gamma\rangle\!\rangle$.

Proof. (1) If a series $\sum_{\omega \in \Gamma} a_\omega \omega \in \mathbb{F}((\Gamma))$ is invariant we get

$$g \cdot \sum_{\omega \in \Gamma} a_\omega \omega = \sum_{\omega \in \Gamma} (g \cdot a_\omega) \omega = \sum_{\omega \in \Gamma} a_\omega \omega$$

for every $g \in G$, hence the coefficients are invariant.

(2) Take a series $a \in \mathbb{K}((\Gamma)) \cap \mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$. The rank of Connes operator $[\mathfrak{F}, a]$ over $\mathbb{F}$ is finite, hence the rank of Connes operator over $\mathbb{K}$ is finite as well, thus a is rational over $\mathbb{K}$.

(3) We embed $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ into the skew-field of Malcev-Neumann series $\mathbb{F}((\Gamma))$. The action of G on $\mathbb{F}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ extends to an action on $\mathbb{F}((\Gamma))$ given by

$$g \cdot \sum_{\omega \in \Gamma} a_\omega \omega = \sum_{\omega \in \Gamma} (g \cdot a_\omega) \omega.$$

Now we apply (1) and (2). □

Given a rational expression or a realization of a noncommutative complex rational function it is not immediately obvious that the function can be written as a sum of real and imaginary part.

Corollary 3.2. *Any $r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ can be written as $r_R + ir_I$ with $r_R, r_I \in \mathbb{R}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$.*

Proof. Let $\mathbb{Z}_2$ act on $\mathbb{C}$ by the complex conjugation and trivially on the variables. We denote the action of the nontrivial element of $\mathbb{Z}_2$ on the function r by $\bar{r}$. We define $r_R = \frac{1}{2}(r + \bar{r})$ and $r_I = -i\frac{1}{2}(r - \bar{r})$. By Proposition 3.1, both functions are in $\mathbb{R}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$. □

We can always translate certain actions of $\mathbb{Z}_2$ on $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ to actions that fit the premise of Proposition 3.1.

Lemma 3.3. *Let the action of $\mathbb{Z}_2$ on $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ be defined by a group automorphism of Γ and a field automorphism of $\mathbb{F}$. There exist free generators of $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ such that $\mathbb{Z}_2$ acts diagonally on them. If the action on $\mathbb{F}$ is nontrivial, then there exist free generators of $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ such that $\mathbb{Z}_2$ acts trivially on them.*

Proof. Let $\theta \colon \Gamma \to \Gamma$ be a group automorphism of order two. By [30], there exist free generators $\{w_1, \ldots, w_n\}$ of Γ such that $\theta w_j = w_j^{-1}$ or $\theta w_j = A_j w_j B_j$ where A_j, B_j only depend on w_k for $k < j$ and $\theta A_j = A_j^{-1}, \theta B_j = B_j^{-1}$. We replace the generators w_j for which $\theta w_j = w_j^{-1}$ with $y_j = (w_j - 1)(w_j + 1)^{-1}$ and get $\theta y_j = -y_j$ and we replace generators w_j for which $\theta w_j = A_j w_j B_j$ by $y_j = (1 + A_j) w_j (1 + B_j)$ to get $\theta y_j = y_j$.

If the automorphism θ is nontrivial on $\mathbb{F}$ then there exists $c \in \mathbb{F}$ such that $\theta c = -c$. We replace the free generators y_j such that $\theta y_j = -y_j$ by $y_j' = cy_j$ and get $\theta y_j' = \theta c \theta y_j = y_j'$. □

Corollary 3.4. *Let the action of $\mathbb{Z}_2$ on $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle$ be given via automorphisms of Γ and $\mathbb{F}$. If the action is trivial on $\mathbb{F}$, then $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle^{\mathbb{Z}_2}$ is rational over $\mathbb{F}$. If the action is nontrivial of $\mathbb{F}$, then $\mathbb{F}\langle\!\langle\Gamma\rangle\!\rangle^{\mathbb{Z}_2}$ is rational over $\mathbb{F}^{\mathbb{Z}_2}$.*

Proof. First we apply Lemma 3.3. If the action on $\mathbb{F}$ is nontrivial we finish with Lemma 3.3, otherwise with Theorem 2.7. □

The method to determine real noncommutative invariants consists of expanding the action to rational functions over $\mathbb{C}$, computing complex invariants using [26, 4.1] and then pushing the results back to noncommutative rational functions over $\mathbb{R}$ using the above results.

Theorem 3.5. *Let a finite abelian group A act faithfully linearly on $\mathbb{R}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, then*

$$\mathbb{R}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle^A \cong \mathbb{R}\langle\!\langle w_1, \ldots, w_{|A|(n-1)+1} \rangle\!\rangle.$$

Proof. We extend the action of A to $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ and let $\mathbb{Z}_2$ act on $\mathbb{C}$ by the complex conjugation and trivially on the variables. The actions commute, hence we get an action of the group $A \times \mathbb{Z}_2$ and invariants $\mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^A = \mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^{A \times \mathbb{Z}_2}$.

We note that every representation of an abelian group over $\mathbb{R}$ is equivalent to a direct sum of irreducible representations of dimension at most 2. One dimensional representations are of the form $g \cdot x = \pm x$. Two dimensional irreducible representations are equivalent to a representation of the form

$$g \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and they are diagonalized over $\mathbb{C}$ in the variables $x + iy$ and $x - iy$, which are permuted by the action of $\mathbb{Z}_2$. Therefore we can choose a set of free generators X of $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ such that A acts diagonally on them and X is closed under the action of $\mathbb{Z}_2$. Let Γ be the group generated by X. Then Γ is closed under the action of $\mathbb{Z}_2$, hence Γ^A is also closed under the action of $\mathbb{Z}_2$. By Corollary 3.4 the skew-field of invariants $\mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^A = \mathbb{C}\langle\!\langle \Gamma^A\rangle\!\rangle^{\mathbb{Z}_2}$ is rational over $\mathbb{R}$ in $|A|(n-1)+1$ variables. $\qquad\square$

Example 3.6. Let $\mathbb{Z}_3$ act on $\mathbb{R}\langle\!\langle x, y, z\rangle\!\rangle$ via a cyclic permutation of variables $(x \mapsto y \mapsto z \mapsto x)$. Take $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. The action is diagonalized over $\mathbb{C}$ in variables

$$a = x + y + z, b = x + \omega y + \overline{\omega}z, c = x + \overline{\omega}y + \omega z$$

and the free generators of $\mathbb{C}\langle\!\langle x, y, z\rangle\!\rangle^{\mathbb{Z}_3}$ are

$$a, bc, bac, cb, cab, b^3, c^3.$$

We use a linear transformation to get the free generators of $\mathbb{R}\langle\!\langle x, y, z\rangle\!\rangle^{\mathbb{Z}_3}$:

$$w_1 = a, w_2 = bc + cb, w_3 = i(bc - cb), w_4 = bac + cab, \tag{3.1}$$
$$w_5 = i(bac - cab), w_6 = b^3 + c^3, w_7 = i(b^3 - c^3)$$

To express the generators in the initial variables x, y, z we first introduce some notation:

$$f_1(A, B, C) = AB + BC + CA, \quad f_2(A_1, A_2, A_3) = \sum_{\sigma \in S_3} A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)},$$

$$f_3(A, B, C) = A(A + B + C)B + B(A + B + C)C + C(A + B + C)A.$$

The free generators w_j are then expressed as follow:

$$w_1 = x + y + z,$$
$$w_2 = 2\left(x^2 + y^2 + z^2\right) - f_1(x, y, z) - f_1(x, z, y),$$
$$w_3 = \sqrt{3}(f_1(x, y, z) - f_1(x, z, y)),$$
$$w_4 = 3\left(x(x + y + z)x + y(x + y + z)y + z(x + y + z)z\right) - (x + y + z)^3$$
$$w_5 = \sqrt{3}\left(f_3(x, y, z) - f_3(x, z, y)\right)$$
$$w_6 = 2\left(x^3 + y^3 + z^3 + f_2(x, y, z)\right) -$$
$$\quad \frac{1}{2}\left(f_2(x, x, y) + f_2(x, x, z) + f_2(x, y, y) + f_2(x, z, z) + f_2(y, y, z) + f_2(y, z, z)\right),$$
$$w_7 = \frac{\sqrt{3}}{2}\left(f_2(x, x, z) + f_2(x, y, y) + f_2(y, z, z) - f_2(x, x, y) - f_2(x, z, z) - f_2(y, y, z)\right).$$

Next we consider the invariants of an action given via a complete representation.

Theorem 3.7. *Let a finite group G act on $\mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ via a complete representation, then*

$$\mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^G \cong \mathbb{R}\langle\!\langle w_1, \ldots, w_{|G|(n-1)+1}\rangle\!\rangle.$$

Proof. Denote the complete representation by π. We extend the action of G to $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$ and let $\mathbb{Z}_2$ act on $\mathbb{C}$ by the complex conjugation and trivially on the variables. Let N be the normal abelian subgroup of G and $\mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_n\} = V_B \oplus V_J$ the decomposition with respect to π according to the definition of complete representation. We get the basis of V

$$\{b_\chi \mid \chi \in \mathrm{Irr}(N)\backslash\{\tau\}\} \cup \{v_k \mid k = 1, \ldots, \deg \pi_J\}.$$

that diagonalize $\pi|_N$. We denote $b_\tau = 1$. Since every real irreducible character of an abelian group decomposes over $\mathbb{C}$ as a sum of conjugate linear characters we can choose a basis such that $\overline{b_\chi} = b_{\bar\chi} = b_{\chi^{-1}}$ and $\overline{v_k} = v_{k'}$ for some k'. By [26, 4.2],

$$\left\{b_\chi b_\mu b_{(\chi\mu)^{-1}}, b_\theta v_k b_{(\theta\mu_k)^{-1}} \mid \chi, \mu \in \mathrm{Irr}(N)\backslash\{\tau\}, \theta \in \mathrm{Irr}(N), k = 1, \ldots, \deg \pi_J\right\}$$

are the free generators of $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^N$ such that G/N acts linearly on them. We use a linear transformation to get free generators

$$b_\chi b_\mu b_{\overline{\chi\mu}} + b_{\bar\chi} b_{\bar\mu} b_{\chi\mu}, \quad i(b_\chi b_\mu b_{(\chi\mu)^{-1}} - b_{\bar\chi} b_{\bar\mu} b_{\chi\mu})$$

and

$$b_\theta v_k b_{\overline{\theta\mu_k}} + b_{\bar\theta} \overline{v_k} b_{\theta\mu_k}, \quad i(b_\theta v_k b_{\overline{\theta\mu_k}} - b_{\bar\theta} \overline{v_k} b_{\theta\mu_k}),$$

where χ and μ run through $\mathrm{Irr}(N)\backslash\{\tau\}$, θ runs through $\mathrm{Irr}(N)$ and $k = 1, \ldots, \deg \pi_J$. We note that some expressions appear twice and some are equal to 0; these we omit. Then $\mathbb{Z}_2$ acts trivially on these free generators, hence they are also free generators of $\mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^N$ by Proposition 3.1. The quotient G/N acts linearly on them via a representation that is equivalent to $Q\pi$, thus we can continue with recursion. $\square$

Example 3.8. Let the symmetric group S_3 act on $\mathbb{R}\langle\!\langle x, y, z\rangle\!\rangle$ by permuting the variables. The invariants of the cyclic normal subgroup $\langle(123)\rangle \cong \mathbb{Z}_3$ are computed in Example 3.6. The action of the quotient $\mathbb{Z}_2 \cong S_3/\langle(123)\rangle$ is given by the action of the cycle (12), that is defined by

$$a \mapsto a, \ b \mapsto \overline{\omega}c, \ c \mapsto \omega b,$$

hence the action on the free generators given by (3.1) is

$$w_1 \mapsto w_1, w_2 \mapsto w_2, w_3 \mapsto -w_3, w_4 \mapsto w_4, w_5 \mapsto -w_5, w_6 \mapsto w_6, w_7 \mapsto -w_7$$

and the free generators of $\mathbb{R}\langle\!\langle x, y, z\rangle\!\rangle^{S_3}$ are

$$w_1, w_2, w_4, w_6, w_3 w_3, w_3 w_5, w_3 w_7, w_5 w_3, w_7 w_3, w_3 w_1 w_3, w_3 w_2 w_3, w_3 w_4 w_3, w_3 w_6 w_3.$$

With a bit of further effort we can show that these are also the free generators of $\mathbb{Q}\langle\!\langle x, y, z\rangle\!\rangle^{S_3}$.

The computing method fails for a general linear action of a solvable group, however we can still use it to show that the skew-field of noncommutative real rational invariants of a solvable group is finitely generated (as a skew-field over $\mathbb{R}$), whereas the ring of noncommutative polynomials invariant under a linear action of a finite group is almost never finitely generated [11, 6.8.4].

We begin with abelian groups.

Lemma 3.9. *Let D be a finitely generated skew-field over $\mathbb{R}$ and let a finite abelian group A act on D via a homomorphism $A \to \mathrm{Aut}_{\mathbb{R}}(D)$. Then D^A is a finitely generated skew-field over $\mathbb{R}$.*

Proof. Let $d_1, \ldots, d_m$ be the generators of D. We define an action of A on the free skew-field $E = \mathbb{R}\langle\!\langle x_{ig} \mid i = 1, \ldots, m, g \in A\rangle\!\rangle$ by $gx_{ih} = x_{i(gh)}$. Then the specialization $\psi: E \to D$ defined by $\psi(x_{ig}) = gd_i$ satisfies $\psi(gy) = g\psi(y)$ and its domain is closed under the action of A. By Theorem 3.5, E^A is rational in finitely many variables. Further, the action of A on E is given by a direct sum of copies of the regular representation of A, hence it contains every representation of A. Thus we proceed as in the proof of

Theorem 3.7 to find free generators of E^A that are polynomial in the initial variables, hence they are in the domain of ψ.

We restrict the specialization ψ to E^A. Any $d \in D^A$ is the image of some $r \in F$, but then it is also the image of $1/|A| \sum_{g \in A} gr \in E^A$ which shows that $\psi|_{E^A}$ is surjective. Hence, D^A is generated by the images of the generators of E^A.   □

We apply the lemma inductively to deduce the result for solvable groups.

Theorem 3.10. *Let D be a finitely generated skew-field over $\mathbb{R}$ and let a finite solvable group G act on D via a homomorphism $G \to \mathrm{Aut}_{\mathbb{R}}(D)$. Then D^G is a finitely generated skew-field over $\mathbb{R}$. In particular this holds for $D = \mathbb{R}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle^G$.*

Proof. Let N be a nontrivial abelian subgroup of G. By Lemma 3.9, the skew-field D^N is finitely generated. We continue with the action of G/N on D^N and conclude the proof by induction.   □

4. Complete representations and totally pseudo-unramified groups

In this section we study complete representations and totally pseudo-unramified groups. Our first nonabelian examples of totally pseudo-unramified (and totally unramified) groups are dihedral groups.

Example 4.1. The dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = abab = e \rangle$ $(n \geq 3)$ has irreducible representations of degree one and two. The representatives of two-dimensional irreducible representations are given by

$$\pi_\omega \colon a \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}, \; b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ω is a n-th root of unity such that $\omega \neq \omega^{-1}$. The restriction of π_ω to the normal subgroup $\langle a \rangle$ is equivalent to $\pi_\omega|_{\langle a \rangle} \cong \mu_\omega \oplus \mu_{\omega^{-1}}$, where $\mu_x(a) = x$. Clearly the restriction is multiplicity free, therefore D_{2n} is unramified over $\langle a \rangle$. The quotient $D_{2n}/\langle a \rangle$ is abelian, hence, D_{2n} is totally unramified and also totally pseudo-unramified. If $n = 3$, the representation π_ω is complete, otherwise it is not.

The above example shows that the standard representation of $D_6 \cong S_3$ is complete. We will show that the same holds for the standard representation of S_4. First we prove an easy proposition.

Proposition 4.2. *(1) Let π be a subrepresentation of ρ and assume π is complete, then ρ is complete.*
(2) Let ρ be a subrepresentation of π and suppose $\pi \oplus \rho$ is complete, then π is complete.

Proof. (1) Write $\pi = \pi_B \oplus \pi_J$ as guaranteed in the definition of a complete representation, further write $\rho = \pi \oplus \pi_0$. Then we can decompose $\rho = \rho_B \oplus \rho_J$ where $\rho_B = \pi_B$ and $\rho_J = \pi_J \oplus \pi_0$. Clearly this decomposition satisfies item (1) from the definition. The item (2) is clearly true if we are at the last step of the recursion. Otherwise $Q\pi$ is a subrepresentation of $Q\rho$ and we finish by recursion.

(2) We can assume that ρ is irreducible. Write $\pi \oplus \rho = \pi_B \oplus \pi'_J$. Item (1) of the definition of a complete representation shows that ρ appears in π_B with multiplicity at most 1, hence we can write $\pi'_J = \pi_J \oplus \rho$ and $\pi = \pi_B \oplus \pi_J$. Then we compute

$$Q(\pi \oplus \rho) = Q\pi \oplus [\rho \oplus (\pi_B \otimes \rho) \oplus (\rho \otimes \pi_B) \oplus (\pi_B \otimes \rho \otimes \pi_B)]_{N^\tau},$$

since every summand of

$$[\rho \oplus (\pi_B \otimes \rho) \oplus (\rho \otimes \pi_B) \oplus (\pi_B \otimes \rho \otimes \pi_B)]_{N^\tau}$$

is contained in $Q\pi$ as a subrepresentation we can finish with recursion.   □

Example 4.3. The symmetric group S_4 contains an abelian normal subgroup
$$V = \{e, (12)(34), (13)(24), (14)(23)\},$$
which is isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The character tables of S_4 and V are:

S_4	{}	{2}	{2, 2}	{3}	{4}
τ_{S_4}	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	3	1	-1	0	-1
χ_3	3	-1	-1	0	1
χ_4	2	0	2	-1	0

V				
τ_V	1	1	1	1
μ_1	1	-1	1	-1
μ_2	1	1	-1	-1
μ_3	1	-1	-1	1

The character of the standard representation ρ of S_4 is χ_2. Its restriction to V is $\chi_2|_V = \mu_1 + \mu_2 + \mu_3$, thus ρ satisfies item (1) of the definition of a complete representation. It remains to show that $Q\rho$ is complete. For this we compute $\chi_2 \otimes \chi_2 = \tau + \chi_2 + \chi_3 + \chi_4$. Since χ_4 is induced by the character of the standard representation of $S_3 \cong S_4/V$, the representation $Q\rho$ contains the standard representation of S_3 as a subrepresentation and is therefore complete.

Tracing through the character table it is not hard to see that S_4 is unramified over V, thus S_4 is totally unramified. We cannot extend these examples to S_5 as it is not solvable.

As promised in we connect complete representations and totally pseudo-unramified groups in the next theorem.

Theorem 4.4. *(1) The regular representation of a totally pseudo-unramified group is complete.*
(2) If a group admits a complete representation, then it is totally pseudo-unramified.
(3) A group is totally pseudo-unramified if and only if it admits a complete representation.

Proof. (1) Let G be totally pseudo-unramified and let R_G be the regular representation. Let G be pseudo-unramified over an abelian normal subgroup N. Partition $\mathrm{Irr}(N)$ into equivalence classes of the form $[\mu] = \{{}^g\mu \mid g \in G\}$. For each such class of nontrivial characters pick $\chi_{[\mu]} \in \mathrm{Irr}(G)$ such that $\langle \chi_{[\mu]}|_N, \theta \rangle = 1$ for any (and hence all) $\theta \in [\mu]$ as guaranteed in the definition of "pseudo-unramified over". Write χ_B for the sum of these characters and let π_B be a subrepresentation of R_G with character χ_B. Then write $R_G = \pi_B \oplus \pi_J$. By construction, $\pi_B|_N$ contains all nontrivial representations of N with multiplicity one. Notice that π_J contains a regular representation of G/N as a subrepresentation and therefore the representation $Q\pi$ contains a regular representation of G/N. By Proposition 4.2 we can proceed with recursion.

(2) Let π be a complete representation of G and let N be the abelian normal subgroup from the definition of complete representation. Then the characters of the irreducible summands of π_B are the characters required to show that G is pseudo-unramified over N. We conclude the proof with recursion.

(3) Follows directly from (1) and (2). □

The smallest group that is not totally pseudo-unramified is $\mathrm{SL}_2(\mathbb{F}_3)$ of order 24. The smallest examples of groups that are totally pseudo-unramified but not totally unramified are four groups of order 48 with GAP group IDs [48, 15], [48, 16], [48, 17], and [48, 18]. There exist p-groups (for every prime p) that are not totally pseudo-unramified. We provide examples in Section 6.1.

The next example shows that (normal) subgroups and quotients of a totally pseudo-unramified group are not necessarily totally pseudo-unramified.

Example 4.5. The group G with the structure description $C_2 \times ((C_8 \rtimes C_4) \rtimes C_2)$ and group ID [128, 254] is totally pseudo-unramified and the group H with the structure description $(C_8 \rtimes C_4) \rtimes C_2$ and group ID [64, 10] is not totally pseudo-unramified. However, we have $G \cong C_2 \times H$, hence the quotient $G/C_2 \cong H$ and the subgroup $1 \times H$ of G are not totally pseudo-unramified.

Totally pseudo-unramified groups behave well under the direct product.

Proposition 4.6. *(1) Let G and H be finite groups and suppose G is pseudo-unramified over N, then $G \times H$ is pseudo-unramified over $N \times 1$.*

(2) If G and H are totally pseudo-unramified groups, then $G \times H$ is totally pseudo-unramified.

Proof. (1) Any irreducible representation of $N \times 1$ is of the form $\mu \otimes \tau_1$ where $\mu \in \mathrm{Irr}(N)$. If $\chi \in \mathrm{Irr}_\mu(G)$ has multiplicity free restriction $\chi|_N$, then $\chi \otimes \tau_H \in \mathrm{Irr}_{\mu \otimes \tau_1}(G \times H)$ has multiplicity free restriction $(\chi \otimes \tau_H)|_{N \times 1} = \chi|_N \otimes \tau_1$.

(2) The base case is $A \times H$ where A is abelian. By (1), $A \times H$ is pseudo-unramified over $A \times 1$ and $(A \times H)/(A \times 1) \cong H$ is totally pseudo-unramified. We reduce the general case to the base case recursively using (1). □

We return to Example 4.5 and consider it from the point of view of the noncommutative rational invariants.

Example 4.7. We have $G \cong C_2 \times H$ where G is the group with GAP group ID [128, 254] that is totally pseudo-unramified and H is the group with GAP group ID [64, 10] that is not totally pseudo-unramified. Let G act on the free skew-field $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ via a complete representation. Then the skew-field of rational invariants $\mathbb{C}\langle x_1, \ldots, x_n \rangle^G \cong \mathbb{C}\langle y_1, \ldots, y_{|G|(n-1)+1} \rangle$ is rational by [26, 5.1]. However, we can take an intermediate step and first compute the skew-field of C_2-invariants $\mathbb{C}\langle x_1, \ldots, x_n \rangle^{C_2} \cong \mathbb{C}\langle z_1, \ldots, z_{2n-1} \rangle$ that is rational, hence the skew-field $\mathbb{C}\langle z_1, \ldots, z_{2n-1} \rangle^H \cong \mathbb{C}\langle x_1, \ldots, x_n \rangle^G$ is also rational. Thus we have an example of a skew-field of rational invariants of a finite group that is rational yet the group is not totally pseudo-unramified. Furthermore, G is pseudo-unramified over C_2, by Proposition 4.6, thus tracing trough the proofs of Theorem 4.4 and [26, 5.1] we can show that the considered action of H is linear.

We proceed with a cohomological characterization of the notion of "pseudo-unramified over".

Theorem 4.8. *Let N be a nontrivial abelian normal subgroup of G. Then G is pseudo-unramified over N if and only if for all irreducible characters $\mu \in \mathrm{Irr}(N)$ and for any cocycle $f \in Z^2(I_G(\mu)/N, N)$ defining the extension from $I_G(\mu)/N$ to $I_G(\mu)$, the class $[\mu \circ f] \in M(I_G(\mu)/N)$ is trivial.*

Proof. By (2) of Proposition 2.8, the group G is pseudo-unramified over N if and only if $\mathrm{IRR}_\mu(I_G(\mu))$ contains a representation of degree 1. By Lemma 2.11, instead of the degrees of $\mathrm{IRR}_\mu(I_G(\mu))$ we can consider the degrees of irreducible projective representations $\mathrm{IRR}^\pi(I_G(N)/N)$, where π is any representative of $[\mu \circ f]$. By Lemma 2.10, $\mathrm{IRR}^\pi(I_G(N)/N)$ contains a representation of degree 1 if and only if $[\mu \circ f]$ is trivial. □

We can reformulate Theorem 4.8 using T_H from the exact sequence (2.1).

Corollary 4.9. *A group G is pseudo-unramified over a nontrivial abelian normal subgroup N if and only if for every subgroup H of G containing N the map $T_H \colon \mathrm{Lin}^H(N) \to M(H/N)$ is trivial.*

Proof. Suppose that G is unramified over N. If the subgroup H is of the form $I_G(\mu)$ for some $\mu \in \mathrm{Irr}(N)$, we can directly use Theorem 4.8. Now take an arbitrary subgroup H of G containing N any $\mu \in \mathrm{Lin}^H(N)$, then a direct computation shows $H \subseteq I_G(\mu)$. It remains to show that $T_H(\mu)$ is trivial, which is true since $T_H(\mu)$ is equal to the restriction of $T_{I_G(\mu)}(\mu)$ to H/N.

The backwards implication follows from applying the assumptions to the inertia subgroups. □

We give two more corollaries to Theorem 4.8.

Corollary 4.10. *A semidirect product $A \rtimes G$ of a totally pseudo-unramified group G with an abelian group A is totally pseudo-unramified.*

Proof. The inertia subgroups of the characters of A are of the form $A \rtimes H$, for some subgroup H of G. The 2-cocycle defining the extension from H to $A \rtimes H$ is trivial, hence $A \rtimes G$ is pseudo-unramified over A by Theorem 4.8. The quotient $(A \rtimes G)/A \cong G$ is totally pseudo-unramified, hence $A \rtimes G$ is totally pseudo-unramified as well. □

Example 4.5 shows that a semidirect product of an abelian group with a totally pseudo-unramified group need not be totally pseudo-unramified.

Corollary 4.11. *If the commutator subgroup G' is abelian and $G' \neq [G, G, G]$, then G is not pseudo-unramified over G'.*

Proof. We consider the exact sequence (2.1). The invariant characters $\mathrm{Lin}^G(G') \cong G'/[G', G]$ are nontrivial and the restriction Res: $\mathrm{Lin}(G) \to \mathrm{Lin}^G(G')$ in (2.1) is trivial, therefore the map $T_G \colon \mathrm{Lin}^G(G') \to M(G/G')$ is nontrivial. We now apply Corollary 4.9. □

5. Totally unramified groups

In this section we focus on totally unramified groups. As mentioned in the previous section symmetric groups S_3 and S_4 and all dihedral groups are totally unramified. The smallest example of a group that is not totally unramified is $\mathrm{SL}_2(\mathbb{F}_3)$, same as in the totally pseudo-unramified case.

We draw attention to the theorem that addresses a concept similar to "unramified over."

Theorem 5.1. [32] *Let H be a subgroup of G. For every irreducible character $\chi \in \mathrm{Irr}(G)$ the restriction $\chi|_H$ is multiplicity free if and only if the centralizer $C_{\mathbb{C}G}(\mathbb{C}H)$ of the group algebra $\mathbb{C}H$ is commutative.*

We get the following corollary.

Corollary 5.2. *Let N be an abelian normal subgroup of G. If the centralizer $C_{\mathbb{C}G}(\mathbb{C}H)$ is commutative, then G is unramified over N.*

The converse does not hold, the reason being that in the definition of "unramified over" we allow restrictions to be multiples of the trivial character. Any totally unramified group that is not metabelian (such as S_4) provides a concrete example where the converse fails.

As in the case of totally pseudo-unramified groups a (normal) subgroup of a totally unramified group is not necessarily totally unramified.

Example 5.3. The group $D_{16} \times S_3$ (with GAP group ID [96, 117]) is totally unramified and contains a normal subgroup with structure description $(C_3 \times D_8) \rtimes C_2$ (group ID [48, 15]), that is not totally unramified.

Totally pseudo-unramified groups are closed under direct products but not closed under quotients. In the case of totally unramified groups the things are reversed.

Proposition 5.4. *Let $\varphi \colon G \to H$ be a surjective group homomorphism.*

1. *If G is unramified over a nontrivial abelian normal subgroup N and $M = \varphi(N)$ is nontrivial, then H is unramified over M.*
2. *If G is totally unramified, then H is totally unramified.*

Proof. (1) There are bijections given by inflation $\mathrm{Inf}\colon \mathrm{Irr}(H) \to \{\chi \in \mathrm{Irr}(G) \mid \ker \varphi \subseteq \ker \chi\}$ and $\mathrm{Inf}\colon \mathrm{Irr}(M) \to \{\chi \in \mathrm{Irr}(N) \mid \ker \varphi|_N \subseteq \ker \chi\}$. We note that $\mathrm{Inf}(\chi|_M) = (\mathrm{Inf}\,\chi)|_N$.

For any $\chi \in \mathrm{Irr}(H)$ we consider its inflation $\mathrm{Inf}\,\chi \in \mathrm{Irr}(G)$. The restriction $(\mathrm{Inf}\,\chi)|_N$ is multiplicity free or a multiple of the trivial character, whence the same must hold for $\mathrm{Inf}(\chi|_M)$ and $\chi|_M$.

(2) Let G be totally unramified and

$$1 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1} \subsetneq N_n = G$$

a series of normal subgroups such that N_{j+1}/N_j is abelian and G_1/N_j is unramified over N_{j+1}/N_j. Consider the series

$$1 = \varphi(N_0) \subseteq \varphi(N_1) \subseteq \cdots \subseteq \varphi(N_{n-1}) \subseteq \varphi(N_n) = H.$$

We can assume that all the inclusions are strict, otherwise we can remove the redundant elements. By $\bar{\varphi}\colon G/N_j \to H/\varphi(N_j)$ we denote the induced homomorphism and note that it is surjective. The group $\varphi(N_{j+1})/\varphi(N_j) \cong \bar{\varphi}(N_{j+1}/N_j)$ is abelian. The group $H/\varphi(N_j) \cong \bar{\varphi}(G/N_j)$ is unramified over $\varphi(N_{j+1})/\varphi(N_j) \cong \bar{\varphi}(N_{j+1}/N_j)$ by (1). $\qquad\square$

An example of a direct product of totally unramified groups that is not totally unramified is $S_4 \times S_4$.

The next results narrow down the candidates for abelian normal subgroups N of G over which G is potentially unramified.

Proposition 5.5. *Let G be a finite group and N a nontrivial central subgroup $(1 \subsetneq N \subseteq Z(G))$. Then G is unramified over N if and only if G is abelian.*

Proof. Because N is central we have $I_G(\mu) = G$ for every $\mu \in \mathrm{Irr}(N)$. By (1) of Proposition 2.8, we have $\mathrm{Irr}_\mu(G) \subset \mathrm{Lin}(G)$ for any nontrivial $\mu \in \mathrm{Irr}(N)$. Assume that G is not abelian, then there exists $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. We note that $\chi \in \mathrm{Irr}_\tau(G)$. Next take any $\theta \in \mathrm{Irr}_\mu(G)$ for some nontrivial $\mu \in \mathrm{Irr}(N)$. Then $\theta\chi$ is an irreducible character of G, $(\theta\chi)(1) = \chi(1) > 1$ and $(\theta\chi)|_N = \chi(1)\mu$, which is a contradiction.

The backwards implication is obvious. $\qquad\square$

Corollary 5.6. *If a group G is unramified over N, then either $N = [N, G]$ or $[N, G] = G'$. In particular $N \subseteq G'$ or $G' \subseteq N$.*

Proof. If $[N, G] \neq N$, then $G/[N, G]$ is unramified over $N/[N, G]$ by Proposition 5.4. The subgroup $N/[N, G]$ is central in $G/[N, G]$, hence $G/[N, G]$ is abelian by Proposition 5.5. $\qquad\square$

To reduce the number of recursive steps in the definition of a totally unramified group we would like to take an abelian normal subgroup as big as possible, however, there are limitations.

Proposition 5.7. *Let G be unramified over N and let $M \supsetneq N$ be an abelian normal subgroup of G. Then G is unramified over M if and only if G/N is unramified over M/N.*

Proof. Assume G/N is unramified over M/N. For any $\chi \in \mathrm{Irr}(G)$ we have two options: $\chi|_N$ is a sum of distinct irreducible characters or $\chi|_N$ is a multiple of the trivial character. In the first case $\chi|_M$ must be a sum of distinct irreducible characters of M otherwise an irreducible character would also appear multiple times in $(\chi|_N) = (\chi|_M)|_N$. In the second case χ is induced from a character $\bar{\chi} \in \mathrm{Irr}(G/N)$. We have $\chi|_M = \mathrm{Ind}(\bar{\chi}|_{M/N})$. Since $\bar{\chi}|_{M/N}$ is a multiple of the trivial character or multiplicity free, so is $\chi|_M$.

Conversely, if G is unramified over M, then by Proposition 5.4, G/N is unramified over M/N. $\qquad\square$

Corollary 5.8. *Let G be unramified over N and assume $Z(G) \not\subset N$. Then G is unramified over $Z(G)N$ if and only if $G' \subseteq N$.*

Proof. By Propositionsrefprop-cent, G is unramified over $Z(G)N$ if and only if G/N is unramified over $Z(G)N/N$. We notice that $Z(G)N/N$ is nontrivial and central in G/N, therefore, by Proposition 5.7, G is unramified over $Z(G)N$ if and only if G/N is abelian. □

There indeed exist examples of totally unramified groups that are not unramified over any abelian normal subgroup that contains the center. One such group is $C_3 \rtimes ((C_{10} \times C_2) \rtimes C_2)$ with GAP group ID [120, 11].

We proceed with a cohomological characterization of the notion of "unramified over".

Theorem 5.9. *Let N be a nontrivial abelian normal subgroup of G. Then G is unramified over N if and only if for all nontrivial $\mu \in \mathrm{Irr}(N)$ the following conditions are satisfied:*

1. $I_G(\mu)/N$ is abelian;
2. for any 2-cocycle $f \in Z^2(I_G(\mu)/N, N)$ defining the extension from $I_G(\mu)/N$ to $I_G(\mu)$, the class of the factor set $[\mu \circ f] \in M(I_G(\mu)/N)$ is trivial.

Proof. By (1) of Proposition 2.8, G is unramified over N if and only if for every nontrivial character $\mu \in \mathrm{Irr}(N)$, every representation in $\mathrm{IRR}_\mu(I_G(\mu))$ is of degree one. By Lemma 2.11, instead of the degrees of $\mathrm{IRR}_\mu(I_G(\mu))$ we can consider the degrees of irreducible projective representations $\mathrm{IRR}^\pi(I_G(N)/N)$, where π is any representative of $[\mu \circ f]$.

If $[\mu \circ f]$ is trivial, then we consider the degrees of linear representations $\mathrm{IRR}(I_G(\mu)/N)$. We note that $I_G(\mu)/N$ is abelian if and only if all its irreducible linear representations are of degree one. If $[\mu \circ f]$ is not trivial, then by Lemma 2.10, none of the irreducible $(\mu \circ f)$-representations of $I_G(\mu)/N$ is of degree one. Hence $\mathrm{IRR}_\mu(I_G(\mu))$ contains only representations of degree 1 if and only if $I_G(\mu)/N$ is abelian and $[\mu \circ f]$ trivial. □

Next we combine Theorem 5.9 with the exact sequence (2.1).

Corollary 5.10. *A group G is unramified over N if and only if for every subgroup H of G containing N such that $\mathrm{Lin}^H(N)$ is nontrivial, the map $T_H \colon \mathrm{Lin}^H(N) \to M(H/N)$ is trivial and H/N is abelian.*

Proof. We begin with the forward implication. If H is of the form $I_G(\mu)$ for some $\mu \in \mathrm{Irr}(N)$, we can directly use Theorem 5.9. Next take an arbitrary subgroup H containing N and a nontrivial character $\mu \in \mathrm{Lin}^H(N)$, then $H \subseteq I_G(\mu)$, therefore H/N is abelian. Also $T_H(\mu)$ is equal to the restriction of $T_{I_G(\mu)}(\mu)$ to H/N, hence it is trivial.

To prove the backwards implication we just apply the assumptions to the inertia subgroups. □

Using Theorem 5.9 we provide two classes of totally unramified groups.

Corollary 5.11. *(1) A group G is unramified over an abelian normal subgroup N if $I_G(\mu)/N$ is cyclic for all $\mu \in \mathrm{Irr}(N)$. In particular, if G/N is cyclic, then G is unramified over N.*
(2) Metacyclic groups are totally unramified.

Proof. Since the Schur multiplier of a cyclic group is trivial we can apply Theorem 5.9. □

Corollary 5.12. *Semidirect products of abelian groups are totally unramified.*

Proof. Let $G = A \rtimes B$ be a semidirect product of abelian groups. We show that G is unramified over A. The inertia subgroups of the characters of A are of the form $A \rtimes C$, for some subgroup C of B. Then the 2-cocycle f defining the extension from A to $A \rtimes C$ is associated to the trivial one and hence $\lambda \circ f$ is associated to the trivial 2-cocycle. We now apply Theorem 5.9. □

5.1. Isoclinism

The notion of isoclinism was introduced by Hall [20]. For any group G we have the induced commutator map $G/Z(G) \times G/Z(G) \to G'$ defined by

$$(g_1 Z(G), g_2 Z(G)) \mapsto [g_1, g_2].$$

Groups G and H are *isoclinic* if there exist isomorphisms $\varphi \colon G/Z(G) \to H/Z(H)$ and $\psi \colon G' \to H'$ that commute with the commutator map, that is, if $\varphi(g_i Z(G)) = h_i Z(H)$ for $i = 1, 2$, then $\psi[g_1, g_2] = [h_1, h_2]$. An equivalence class with respect to isoclinism is called an *(isoclinism) family*, a group of the smallest order in a family is called *a stem group*.

Groups from the same family share some properties concerning their representations and we will make use of this in our study of totally unramified groups.

Lemma 5.13. *Let G and H be finite groups and let $\varphi \colon G/Z(G) \to H/Z(H)$ and $\psi \colon G' \to H'$ satisfy the isoclinism conditions and let N be a nontrivial normal subgroup of G containing $Z(G)$ and let M be the biggest normal subgroup of H satisfying $\varphi(N/Z(G)) = \varphi(M/Z(H))$.*

1. *If N is abelian, then M is abelian.*
2. *G/N is isomorphic to H/M.*
3. *Let ρ_H be an irreducible linear representation of H, P a descent of ρ_H to $H/Z(H)$ and ρ_G any lift of $\varphi^* P$ to a linear representation of G. The restriction $\rho_H|_M$ is multiplicity free (trivial) if and only if $\rho_G|_N$ is multiplicity free (trivial).*
4. *If N is abelian, then G is unramified over N if and only if H is unramified over M.*

Proof. (1) For any $h_1, h_2 \in M$ take $g_1, g_2 \in N$ such that $\varphi(g_i Z(G)) = h_i Z(H)$, then $[h_1, h_2] = \psi[g_1, g_2] = \psi(1) = 1$.

(2) By Noether's isomorphism theorems we have

$$G/N \cong \frac{G/Z(G)}{N/Z(G)} \cong \frac{H/Z(H)}{M/Z(H)} \cong H/M.$$

(3) It is enough to prove the statement in one direction. The other follows from the symmetry of isoclinism. Let ρ_H be an irreducible linear representation of H of degree n and let $\rho_H|_M = \bigoplus_{i=1}^n \sigma_H^i$ be the decomposition into irreducible summands. Suppose the restriction is multiplicity free or trivial. Let P be the projective representation obtained by descent of ρ to $H/Z(H)$. Then $P|_{M/Z(H)}$ is a direct sum of descents of σ_H^i, $i = 1, \dots, n$. Let the linear representation ρ_G of G be a lift of $\varphi^* P$. Then $\varphi^*(P|_{M/Z(H)})$ is also descent of $\rho_G|_N$, hence, $\rho_G|_N$ is multiplicity free (trivial) as well.

(4) By [6, Ch. 3, Cor. 2.5], an irreducible projective representation of $G/Z(G) \cong H/Z(G)$ lifts to a linear representation of G if and only if it lifts to a linear representation of H, therefore, we can use (3) on each irreducible representation of H. □

We move to a case where G is unramified over a normal abelian subgroup N with $N \subset G'$.

Lemma 5.14. *Let G and H be finite groups and let $\varphi \colon G/Z(G) \to H/Z(H)$ and $\psi \colon G' \to H'$ satisfy the isoclinism conditions.*

1. *For any $g \in G$ and $h \in H$, such that $\varphi(g Z(G)) = h Z(H)$, and any $n \in G'$ we have $\psi(g n g^{-1}) = h \psi(n) h^{-1}$. In particular, for any normal subgroup N of G contained in G' the image $\psi(N)$ is a normal subgroup of H.*
2. *For any normal subgroup N of G contained in G' the quotient G/N is isoclinic to $H/\psi(N)$.*
3. *For any projective representation P of $H/Z(H)$, its lift to a linear representation ρ_H of group the H and any lift of a representation $\varphi^* P$ to a linear representation ρ_G of group the G, we have $\rho_G|_N = \psi^*(\rho_H|_M)$.*
4. *If G in unramified over a nontrivial normal abelian subgroup N contained in G' then H is unramified over $\psi(N)$.*

Proof. (1) It is enough to consider commutators. Let $n = [g_1, g_2]$ and $m = \psi(n)$. Take $h_1, h_2 \in H$ such that $\varphi(g_i Z(G)) = h_i Z(H)$, then we have $m = [h_1, h_2]$. Take any $h \in H$ and let $g \in G$ be such that $\varphi(gZ(G)) = hZ(H)$. We get

$$hmh^{-1} = [hh_1 h^{-1}, hh_2 h^{-1}] \quad \text{and} \quad gng^{-1} = [gg_1 g^{-1}, gg_2 g^{-1}],$$

which shows that $\psi(gng^{-1}) = hmh^{-1}$.

(2) Denote $M = \psi(N)$. For $(gN)Z(G/N) \in (G/N)/Z(G/N)$ we define $\bar{\varphi}((gN)Z(G/N)) = (hM)Z(H/M)$, where $h \in H$ is any element satisfying $\varphi(gZ(G)) = hZ(H)$. This yields a well defined isomorphism. Then $\bar{\varphi} : (G/N)/Z(G/N) \rightarrow (H/M)/Z(H/M)$ and $\bar{\psi} : G'/N \rightarrow H'/M$, where $\bar{\psi}(xN) = \psi(x)M$, satisfy the isoclinism conditions.

(3) Pick transversals $\{h_\alpha \mid \alpha \in H/Z(H)\}$ and $\{g_\alpha \mid \alpha \in G/Z(G)\}$. Representations ρ_H and ρ_G are given by $\rho_H(zh_\alpha) = \lambda(z)P(\alpha)$ and $\rho_G(wg_\alpha) = \mu(w)P(\varphi(\alpha))$, for some linear characters λ and μ of $Z(H)$ and $Z(G)$, respectively.

It is enough to show the desired equality $\rho_G(n) = \psi^*(\rho_H|_M)(n)$ for a commutator $n = [g_\alpha, g_\beta]$ where g_α and g_β are from the transversal. We show the identity by expanding

$$\rho_G(n) = [\rho_G(g_\alpha), \rho_G(g_\beta)] = [P(\varphi(\alpha)), P(\varphi(\beta))]$$

and

$$\rho_H(\psi(n)) = \rho_H([h_{\varphi(\alpha)}, h_{\varphi(\beta)}]) = [P(\varphi(\alpha)), P(\varphi(\beta))].$$

(4) Any irreducible linear representation of G is a lift of an irreducible projective representation of $G/Z(G)$. By [6, Ch. 3, Cor. 2.5], an irreducible projective representation of $G/Z(G) \cong H/Z(H)$ lifts to a linear representation of G if and only if it lifts to a linear representation of H. The restriction of lifts to G' and H', respectively, yield equivalent representations of, by (3), and therefore, restrictions to $N \cong \psi(N)$ are equivalent as well. Hence, the restriction of every irreducible linear representation of G to N is multiplicity free or trivial if and only if restriction of every irreducible linear representation of H to $\psi(N)$ is multiplicity free or trivial. □

Finally we combine the two cases.

Theorem 5.15. *If an isoclinic family contains a totally unramified group, then every finite group in this family is totally unramified.*

Proof. All finite groups in the family of abelian groups are totally unramified. Consider finite representatives G and H of a nonabelian isoclinic family. If G is totally unramified then there exists a normal abelian group N such that G is unramified over N, G/N is totally unramified and $G' \subseteq N$ or $N \subseteq G'$ by Corollary 5.6. In the case $G' \subseteq N$ we can always assume $Z(G) \subseteq N$, otherwise use Corollary 5.8 to replace N with $Z(G)N$. Then we use Lemma 5.13 to find a normal abelian subgroup M of H such that H is unramified over M and $H/M \cong G/N$ is abelian, hence, totally unramified. In the case $N \subseteq G'$ we use Lemma 5.14 to find a normal abelian subgroup M of H such that H is unramified over M and H/M is isoclinic to G/N and finish the proof by recursion. □

Although a direct product of totally unramified group need not be totally unramified we have the following weaker result.

Corollary 5.16. *A direct product of a totally unramified group and an abelian group is totally unramified.*

Proof. A group G is isoclinic to $G \times A$ for any abelian group A, hence, the result follows from Theorem 5.15. □

6. Nilpotent totally unramified groups

In this section we prove some stronger results for nilpotent totally unramified groups.

Corollary 6.1. *If G is metabelian nilpotent, then it is not unramified over G'.*

Proof. Directly from Corollary 4.11. □

This yields a new proof of a well-known result.

Corollary 6.2. *If G is a metabelian nilpotent group that is not abelian, then G/G' is not cyclic.*

Proof. By Corollary 6.1, G is not unramified over G', hence the quotient G/G' is not cyclic by Corollary 5.11. □

We can improve Corollary 5.6 for nilpotent groups.

Corollary 6.3. *If a nilpotent group G is unramified over N, then $[N, G] = G'$.*

Proof. Follows directly from Corollary 5.6, as a nilpotent group G does not contain any nontrivial abelian subgroup N with property $[N, G] = N$. □

Being nilpotent totally unramified is a very restrictive property.

Theorem 6.4. *If a nilpotent group G is unramified over an abelian normal subgroup N, then $G' \subsetneq N$. In particular every totally unramified nilpotent group is metabelian.*

Proof. By Corollary 6.3, we get $G' \subseteq N$ and the equality is excluded since $[G', G] \subsetneq G'$. □

Contrary to the general case nilpotent totally unramified groups are closed under the direct product.

Proposition 6.5. *If G_1 and G_2 are totally unramified nilpotent groups, then $G_1 \times G_2$ is totally unramified.*

Proof. Let G_i be unramified over N_i, we show that $G_1 \times G_2$ is unramified over $N_1 \times N_2$, then the quotient is abelian and we are done.

Every irreducible character of $G_1 \times G_2$ is of the form $\chi_1 \otimes \chi_2$ where χ_i is an irreducible character of G_i. We get $(\chi_1 \otimes \chi_2)|_{N_1 \times N_2} = \chi_1|_{N_1} \otimes \chi_2|_{N_2}$ and each $\chi_i|_{N_i}$ is multiplicity free or a multiple of the trivial character. We consider two cases; either both of $\chi_i|_{N_i}$ are multiplicity free or at least one of them is a multiple of the trivial character.

Let $\chi_1|_{N_1} = \sum_{i=1}^m \mu_i$ and $\chi_2|_{N_2} = \sum_{j=1}^n \theta_j$, where μ_i and θ_j are pairwise distinct characters of N_1 and N_2 respectively. Then $\chi_1|_{N_1} \otimes \chi_2|_{N_2} = \sum_{i,j} \mu_i \otimes \theta_j$ and $\mu_i \otimes \theta_j$ are pairwise distinct.

Let $\chi_1|_{N_1}$ be a multiple of the trivial character. By Theorem 6.4, N_1 contains G_1' and therefore $\ker(\chi_1)$ contains G_1'. This forces χ_1 to be linear. The restriction $\chi_1|_{N_1} \otimes \chi_2|_{N_2} = \tau \otimes \chi_2|_{N_2}$ is clearly multiplicity free or a multiple of the trivial character. The case where $\chi_2|_{N_2}$ is a multiple of the trivial character is symmetric. □

If G is unramified over N and M is an abelian normal subgroup containing N, G is not necessarily unramified over M. This obstacle does not apply to nilpotent groups.

Proposition 6.6. *If a nilpotent group G is unramified over a nontrivial abelian normal subgroup N, then G is unramified over any abelian normal subgroup containing N.*

Proof. Let M be an abelian normal subgroup of G containing N. By Theorem 6.4, G/N is abelian. An abelian group is unramified over any nontrivial subgroup. By Proposition 5.7, G is unramified over M. □

6.1. Totally unramified p-groups

Every nilpotent group is a direct product of p-groups. Thus Propositions 5.4 and 6.5 show that a nilpotent group is totally unramified if and only if its Sylow p-subgroups are totally unramified. It is therefore of interest to understand totally unramified p-groups. We classify totally unramified p-group of rank at most 5 starting with rank at most 4.

Proposition 6.7. *Every p-group of rank at most 4 is totally unramified.*

Proof. Groups of order p and p^2 are abelian and every group of order p^3 is metacyclic, hence totally unramified by Corollary 5.11. Every group G of order p^4 has an abelian normal subgroup N of order p^3 [7]. Then G/N is cyclic and by Proposition 5.11, G is totally unramified. □

We continue with groups of order p^5. We separate the cases $p = 2$ and $p \geq 3$. For $p = 2$ we use GAP for direct computation. All groups of order 2^5 are totally unramified. There are groups of order 2^6 that are not totally unramified but are totally pseudo-unramified and also groups that are not totally pseudo-unramified.

To classify the totally unramified p-groups it is enough, by Theorem 5.15, to classify isoclinism families that contain a totally unramified group. For $p \geq 3$ we use the classification of p-groups by [23], where we also refer to for the explanation of the classification and notation. We just mention that isoclinism families are denoted by $\Phi_s, s = 1, 2, \ldots$ and if the word $[g_i, g_j]$ where g_i and g_j are generators does not appear among relations of the group presentation, the relation $[g_i, g_j] = 1$ should be assumed.

There are 10 families of p-groups ($p \geq 3$) that contain a stem group of rank at most 5; $\Phi_1, \Phi_2, \ldots, \Phi_{10}$. The family Φ_1 contains all abelian groups. The families Φ_2 and Φ_3 have a stem group of rank 3 and 4 respectively which are totally unramified by Proposition 6.7. The remaining families with a stem group of rank 5 are $\Phi_4, \Phi_5, \ldots, \Phi_{10}$. We deal with each family separately.

By Proposition 6.6 and Theorem 6.4, to check whether a p-group G is totally unramified we only have to consider maximal abelian normal subgroups of G that strictly contain G'. We use Corollaries 5.11 and 5.12 to give positive answers.

(Φ_4)   We consider

$$\Phi_4(1^4) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \alpha_i^p = \beta_i^p = 1 \text{ for } i = 1, 2 \rangle.$$

The subgroup $\langle \alpha, \beta_1, \beta_2 \rangle$ is abelian normal and has the abelian subgroup $\langle \alpha_1, \alpha_2 \rangle$ as a complement, i.e., the group $\Phi_4(1^4)$ is a semidirect product of abelian groups and therefore totally unramified.

(Φ_5)   We consider

$$\Phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_2, \alpha_3] = \beta, \alpha_i^p = \beta^p = 1 \text{ for } i = 1, 2, 3, 4 \rangle.$$

The subgroup $\langle \alpha_1, \alpha_3, \beta \rangle$ is abelian normal and has the abelian subgroup $\langle \alpha_2, \alpha_4 \rangle$ as a complement, i.e., the group $\Phi_5(1^5)$ is a semidirect product of abelian groups and therefore totally unramified.

(Φ_6)   We consider

$$\Phi_6(1^5) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta^p = \beta_i^p = 1 \text{ for } i = 1, 2 \rangle.$$

The commutator subgroup $\langle \beta, \beta_1, \beta_2 \rangle$ is a maximal abelian normal subgroup, therefore the group $\Phi_6(1^5)$ is not totally unramified.

(Φ_7) We consider

$$\Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \text{ for } i = 1, 2 \rangle.$$

The subgroup $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is abelian normal and has the abelian subgroup $\langle \alpha, \beta \rangle$ as a complement, i.e., the group $\Phi_7(1^5)$ is a semidirect product of abelian groups and therefore totally unramified.

(Φ_8) We consider

$$\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^{p^2} = \alpha_2^{p^2} = 1 \rangle.$$

The subgroup $\langle \alpha_1, \beta \rangle$ is abelian normal and has a cyclic quotient $\langle \alpha_2 \rangle$. Therefore the group $\Phi_9(1^5)$ is totally unramified.

(Φ_9) We consider

$$\Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \text{ for } i = 1, 2, 3 \rangle.$$

The subgroup $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is abelian normal and has a cyclic quotient $\langle \alpha \rangle$. Therefore the group $\Phi_9(1^5)$ is totally unramified.

(Φ_{10}) We consider

$$\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \text{ for } i = 1, 2, 3 \rangle.$$

The commutator subgroup $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ is a maximal abelian normal subgroup, therefore the group $\Phi_6(1^5)$ is not totally unramified.

Remark 6.8. There exist totally unramified groups (of order at least p^4) that are not semidirect products of abelian groups. Additionally, $\Phi_7(2111)$ is totally unramified but is not a semidirect product of abelian groups and does not contain any abelian normal subgroup with a cyclic quotient.

The classification of totally unramified p-groups of rank at most 5 also classifies totally pseudo-unramified p-groups of rank at most 5.

Proposition 6.9. *A p-group of rank at most 5 is totally unramified if and only if it is totally pseudo-unramified.*

Proof. We only have to prove the backwards implication for $p \geq 3$. If G is p-group of rank at most 5 and not totally unramified, then it is from the isoclinism family Φ_6 or Φ_{10}. We note that these groups have $|G'| = p^3$ and hence $|\operatorname{Lin}(G)| = |G/G'| = p^2$.

Assume G is pseudo-unramified over N and not totally unramified. Then G/N is not abelian, otherwise G would be unramified over N, thus totally unramified. However, G/N is metabelian, therefore $|\operatorname{Lin}(G/N)| = |(G/N)/(G/N)'| \geq p^2$ by Corollary 6.2. Then the inf: $\operatorname{Lin}(G/N) \to \operatorname{Lin}(G)$ is surjective. Restriction and T_G from the exact sequence (2.1) are trivial maps, hence $\operatorname{Lin}^G(N) \cong N/[N, G]$ must be trivial which is a contradiction since $N \neq [N, G]$. □

We give reasoning for the family Φ_{11} with a stem group of rank 6 which provides examples of groups of nilpotency class 2 that are not totally unramified and an example of a totally pseudo-unramified p-group that is not totally unramified.

(Φ_{11}) We consider

$$\Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_3,$$
$$\alpha_i^p = \beta_i^p = 1 \text{ for } i = 1, 2, 3 \rangle.$$

The commutator subgroup $\langle \beta_1, \beta_2, \beta_3 \rangle$ is also the center. There are three maximal abelian normal subgroups $N_i = \langle \alpha_i, \beta_1, \beta_2, \beta_3 \rangle$ for $i = 1, 2, 3$. We have $\beta_i \notin [N_i, G]$, therefore G is not unramified over N_i by Corollary 6.3, hence not totally unramified.

Furthermore, the decomposition $\Phi_{11}(1^6) = \langle \alpha_1, \beta_2, \beta_3 \rangle \rtimes \langle \beta_1, \alpha_2, \alpha_3 \rangle$ shows that $\Phi_{11}(1^6)$ is totally pseudo-unramified by Corollary 4.10. Hence, we cannot extend the Proposition 6.9 to groups of order p^6.

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