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Best possible upper bounds on the restrained domination number of cubic graphs

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Abstract

A dominating set in a graph G is a set S of vertices such that every vertex in $V(G)\setminus S$ is adjacent to a vertex in S. A restrained dominating set of G is a dominating set S with the additional restraint that the graph $G-S$ obtained by removing all vertices in S is isolate-free. The domination number $\gamma(G)$ and the restrained domination number $\gamma_r(G)$ are the minimum cardinalities of a dominating set and restrained dominating set, respectively, of G. Let G be a cubic graph of order n. A classical result of Reed states that $\gamma(G) \le \frac{3}{8}n$, and this bound is best possible. To determine the best possible upper bound on the restrained domination number of G is more challenging, and we prove that $\gamma_r(G) \le \frac{2}{5}n$.

KEYWORDS

cubic graphs, domination, restrained domination

AMS CLASSIFICATION

05C69

1 | INTRODUCTION

A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S, where two vertices are neighbors in G if they are adjacent. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set S dominates a vertex v is $v \in S$ or if v has a neighbor in S. A *restrained dominating set* (RD-set), of

G is a dominating set S of G with the additional property that every vertex not in S has a neighbor not in S, that is, the subgraph of G induced by the set $V(G)\setminus S$ is isolate-free. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of an RD-set of G. A *γ_r-set of G* is an RD-set of G of minimum cardinality $\gamma_r(G)$. Restrained domination in graphs is well studied in the literature with over 100 publications, according to MathSciNet. We refer the reader to the excellent book chapter by Hattingh and Joubert in 2020 on restrained domination in graphs that gives the state of the art on the topic. For recent books on domination in graphs, we refer the reader to [13–15, 19].

A *cubic graph*, also called a 3-*regular graph*, is a graph in which every vertex has degree 3. A *subcubic graph* is a graph with maximum degree at most 3. Domination in cubic and subcubic graphs is very well studied in the literature (see, e.g., [1, 2, 4–6, 9–12, 18, 20, 21, 23–27]). We define a *special subcubic graph* as a subcubic graph G with minimum degree at least 2. In this paper, we continue the study of restrained domination in cubic graphs. We consider the following problem.

Problem 1. Determine the best possible constant c_{rdom} such that $\gamma_r(G) \le c_{\text{rdom}} \cdot n(G)$ for all cubic graphs G.

The best known upper bound to date, before this paper, on c_{rdom} is due to Hattingh and Joubert [11], who proved that $c_{\text{rdom}} \le \frac{5}{11}$. Their proof is nontrivial and uses intricate and ingenious counting arguments. We observe that the Petersen graph G, illustrated in Figure 1, has order $n(G) = 10$ and $\gamma_r(G) = 4 = \frac{2}{5}n(G)$, where the set consisting of the four shaded vertices is an example of a γ_r-set of G. This yields the trivial lower bound $c_{\text{rdom}} \ge \frac{2}{5}$.

Theorem 1 (Hattingh and Joubert [11]). $\dfrac{2}{5} \le c_{\text{rdom}} \le \dfrac{5}{11}$.

It is conjectured in [17] that the lower bound in Theorem 1 is the correct value of c_{rdom}. In this paper, we prove that this is indeed the case.

Theorem 2. $c_{\text{rdom}} = \dfrac{2}{5}$.

To prove Theorem 2, it suffices to show that if G is a cubic graph of order n, then $\gamma_r(G) \le \frac{2}{5}n$. However to prove this result, we relax the 3-regularity condition to allow vertices of degree 2 in the mix to make the inductive hypothesis easier to handle. If $n_2(G)$ and $n_3(G)$ denote the number of vertices of degree 2 and 3, respectively, in such a graph G, then we would like to prove that $10\gamma_r(G) \le 5n_2(G) + 4n_3(G)$ since if G is 3-regular this yields $\gamma_r(G) \le \frac{2}{5}n$.

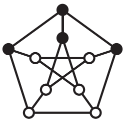

FIGURE 1 The Petersen graph G.

However, relaxing the 3-regularity condition results in a family $\mathcal{B}_{\rm rdom}$ of "troublesome graphs" for which the desired inequality $10\gamma_r(G) \le 5n_2(G) + 4n_3(G)$ does not hold. Therefore we add a function $\Omega(G)$ such that the statement becomes true even for these troublesome graphs. However, we try to keep $\Omega(G)$ as small as possible to establish a bound on $\gamma_r(G)$ that remains as strong as possible. The resulting bound will be the key result that will enable us to prove Theorem 2.

We proceed as follows. In Section 2, we formally state our key result, namely Theorem 3. In Section 2.1, we present the necessary graph theory notation. In Section 2.2, we introduce the concept of near-restrained dominating sets, which we will need when proving our key result. Known results are discussed in Section 2.3. In Section 3, we discuss properties of troublesome graphs that belong to the family $\mathcal{B}_{\rm rdom}$. A preliminary result is proven in Section 4. Proof of our key result is given in Section 5, and thereafter in Section 6, we deduce our main result.

2 | KEY RESULT

To prove our main result, namely Theorem 2, we identify a family $\mathcal{B}_{\rm rdom} = \{R_1, R_2, \ldots, R_{10}\}$ of 10 troublesome graphs G shown in Figure 2 that satisfy $10\gamma_r(G) > 5n_2(G) + 4n_3(G)$. Let $\mathcal{B}_{\rm rdom,1} = \{R_6, R_7, R_8, R_{10}\}$, $\mathcal{B}_{\rm rdom,2} = \{R_2, R_3\}$, $\mathcal{B}_{\rm rdom,3} = \{R_9\}$, $\mathcal{B}_{\rm rdom,4} = \{R_4, R_5\}$ and $\mathcal{B}_{\rm rdom,5} = \{R_1\}$. Let $f_i(G)$ denote the number of components of a special subcubic graph G that belong to $\mathcal{B}_{\rm rdom,\it i}$ for $i \in [5]$. We define

$$\Omega(G) = \sum_{i=1}^{5} i f_i(G).$$

We note that if G is a connected graph and $G \notin \mathcal{B}_{\rm rdom}$, then $\Omega(G) = 0$, while if $G \in \mathcal{B}_{\rm rdom}$, then $G \in \mathcal{B}_{\rm rdom,\it i}$ for some $i \in [5]$ in which case $\Omega(G) = i \le 5$. We define a weight function $\mathrm{w}(G)$ associated with G by

$$\mathrm{w}(G) = 5n_2(G) + 4n_3(G) + \Omega(G).$$

We define the *weight* $\mathrm{w}_G(v)$ of a vertex v in G as its contribution to the weight $5n_2(G) + 4n_3(G)$. Thus, if $\deg_G(v) = 2$, then $\mathrm{w}_G(v) = 5$, and if $\deg_G(v) = 3$, then $\mathrm{w}_G(v) = 4$. We

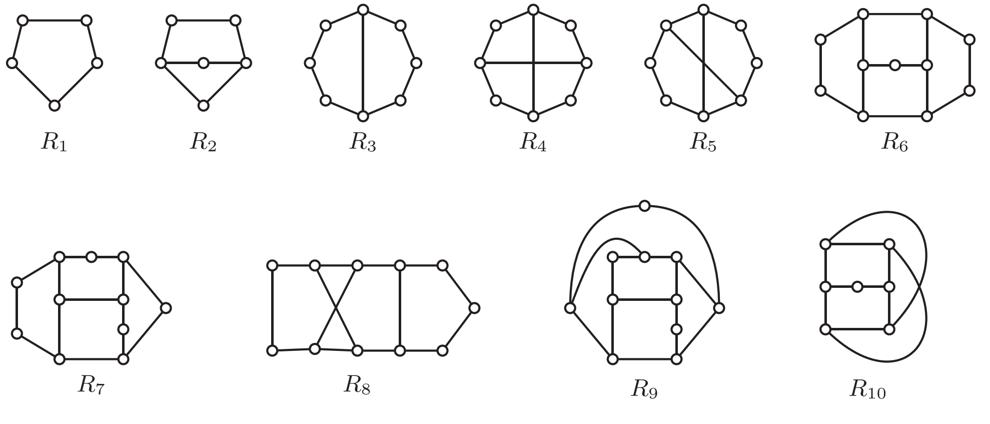

FIGURE 2 The family $\mathcal{B}_{\rm rdom}$.

define the *weight* $\mathrm{w}_G(S)$ of a set S of vertices in G as the sum of the weights of vertices in S, that is, $\mathrm{w}_G(S) = \sum_{v \in S} \mathrm{w}_G(v)$. We are now in a position to state our key result, a proof of which will be given in Section 5.

Theorem 3. *If G is a special subcubic graph, then* $10\gamma_r(G) \le \mathrm{w}(G)$.

2.1 | Notation

For notation and graph theory terminology, we in general follow [15]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. For a set of vertices $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Two vertices in G are *neighbors* if they are adjacent. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N(v)$. Two vertices are *open twins* if they have the same open neighborhood. We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree in G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* is a vertex of degree 0. A graph is *isolate-free* if it contains no isolated vertex.

We denote a *path*, a *cycle*, and a *complete* graph on n vertices by P_n, C_n, and K_n, respectively. A *diamond* is the graph $K_4 - e$, where e is an arbitrary edge of the K_4. A *domino* is a graph that can be obtained from a 6-cycle by adding an edge between two antipodal vertices of the 6-cycle. An F-*component* of a graph G is a component of G that is isomorphic to F. An edge-cut of a connected graph is a set of edges whose removal disconnected the graph. A k-edge-*cut* is an edge-cut of cardinality k. The *girth* of G is the length of the shortest cycle in G.

If G is a special subcubic graph, then we denote by $n_2(G)$ and $n_3(G)$ the number of vertices of degree 2 and 3, respectively, in G. For a special subcubic graph G, let $\mathcal{S}$ and $\mathcal{L}$ be the set of all vertices of degree 2 and 3 in G, respectively, that is, $\mathcal{L} = \{v \in V(G) : \deg_G(v) = 3\}$ and $\mathcal{S} = \{v \in V(G) : \deg_G(v) = 2\}$. We call a vertex in $\mathcal{L}$ a *large vertex*, and a vertex in $\mathcal{S}$ a *small vertex*. For $k \ge 3$, we define a k-*handle* to be a k-cycle that contains exactly one large vertex. For $k \ge 1$, a k-*linkage* is a path on $k + 2$ vertices that starts and ends at distinct large vertices and with k internal vertices of degree 2 in G. A *handle* is a k-handle for some $k \ge 3$, and a *linkage* is a k-linkage for some $k \ge 1$. We use the standard notation $[k] = \{1, \ldots, k\}$.

2.2 | Near restrained dominating sets

To prove our main result, we introduce the concept of a near-restrained dominating set. Given a graph G and a set S of vertices in G, we let $\overline{S}$ denote the complement of S, that is, $\overline{S} = V(G) \backslash S$. We define a *near restrained dominating set*, abbreviated NeRD-set, of G with respect to a subset X of vertices of G as a relaxed variant of an RD-set S of G such that either the vertices in X need not be dominated by S but every vertex in $\overline{S}$ is still required to have a neighbor in $\overline{S}$ or the vertices in X are dominated by S but need not have a neighbor in $\overline{S}$. Formally, a NeRD-set of G with respect to a specified subset X is a set $S \subseteq V(G)$ such that exactly one of the following two conditions hold:

(C1) The set S dominates the set $V(G) \backslash X$, and every vertex in $\overline{S}$ has a neighbor in $\overline{S}$.

(C2) The set S dominates the set $V(G)$, the set $X \subseteq \overline{S}$, and every vertex in $\overline{S} \setminus X$ has a neighbor in $\overline{S}$.

If condition (C1) holds, then we refer to the NeRD-set as a type-1 such set, while if condition (C2) holds, then we refer to the NeRD-set as a type-2 such set. We denote by $\gamma_{r,\text{ndom}}(G; X)$ the minimum cardinality of a type-1 NeRD-set with respect to the set X (where "ndom" stands for "not dominated" since the vertices in X are not required to be dominated), and we denote by $\gamma_{r,\text{dom}}(G; X)$ the minimum cardinality of a type-2 NeRD-set with respect to the set X (where "dom" stands for "dominated" since the vertices in X are dominated but not required to have a neighbor that is not dominated). If $X = \{v\}$, we simply write $\gamma_{r,\text{ndom}}(G; v)$ and $\gamma_{r,\text{dom}}(G; v)$ rather than $\gamma_{r,\text{ndom}}(G; \{v\})$ and $\gamma_{r,\text{dom}}(G; \{v\})$, respectively. Since every RD-set is also a NeRD-set, we note that $\gamma_{r,\text{ndom}}(G; X) \le \gamma_r(G)$ and $\gamma_{r,\text{dom}}(G; X) \le \gamma_r(G)$.

2.3 | Known bounds on restrained domination

Closed formulas for the restrained domination number of paths and cycles are given in [7], where it is shown that for $n \ge 1$, $\gamma_r(P_n) = n - 2\lfloor \frac{n-1}{3} \rfloor$ and for $n \ge 3$, $\gamma_r(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$. The following theorem summarizes classical results on bounds on the restrained domination number of a graph.

Theorem 4. *If G is a connected graph of order n, then the following hold.*

(a) [7] *If $\delta(G) \ge 1$, then $\gamma_r(G) \le n - 2$, unless G is a star $K_{1,n-1}$, in which case $\gamma_r(G) = n$.*
(b) [8] *If $\delta(G) \ge 2$ and $G \ne C_5$, then $\gamma_r(G) \le \frac{1}{2}n$.*
(c) [7, 16] *If $\delta(G) \ge 2$ and $n \ge 9$, then $\gamma_r(G) \le 12(n-1)$.*
(d) [11] *If* G *is a cubic graph, then $\gamma_r(G) \le \frac{5}{11}n$.*

3 | PROPERTIES OF GRAPH IN THE FAMILY $\mathcal{B}_{\text{rdom}}$

In this section, we present properties of graphs that belong to the family $\mathcal{B}_{\text{rdom}} = \{R_1, ..., R_{10}\}$. We note that there are no open twins in the graphs in the family $\mathcal{B}_{\text{rdom}}$ with the exception of R_2 which contains two vertices of degree 2 that have two common neighbors (of degree 3). We shall need the following properties of graphs in the family $\mathcal{B}_{\text{rdom}}$. These properties are straightforward to check (or can be checked by computer).

Observation 1. If $G \in \mathcal{B}_{\text{rdom}}$ and v is a vertex of degree 2 in G, then the following properties hold.

(a) $\gamma_r(R_i) = 3$ for $i \in \{1, 2, 10\}$, $\gamma_r(R_i) = 4$ for $i \in \{3, 4, 5\}$, and $\gamma_r(R_i) = 5$ for $i \in \{6, 7, 8, 9\}$.
(b) There exists a γ_r-set of G that contains v.
(c) There exists a γ_r-set of G that does not contains v.
(d) $\gamma_{r,\text{ndom}}(G; v) \le \gamma_r(G) - 1$.
(e) $\gamma_{r,\text{dom}}(G; v) \le \gamma_r(G) - 1$, unless v is an open twin of R_2.

(f) If X consists of two vertices of degree 2, then $\gamma_{r,\mathrm{dom}}(G; X) \le \gamma_r(G) - 1$.

Observation 2. Let $G \in \mathcal{B}_{\mathrm{rdom}}$ and let $e = xy$ be an arbitrary edge of G. If G^* is obtained from G by subdividing the edge e resulting in a new vertex v^* of degree 2 (with neighbors x and y), then $\gamma_r(G^*) \le \gamma_r(G)$. Furthermore, there exists a γ_r-set of G^* that contains v^* and contains neither x nor y.

Observation 3. Let $G \in \mathcal{B}_{\mathrm{rdom}}$ and let $e = xy$ be an arbitrary edge of G. If G^* is obtained from G by subdividing the edge e twice, resulting in a path xx_1y_1y, then $\gamma_r(G^*) \le \gamma_r(G)$. Furthermore, there exists a γ_r-set of G^* that contains x_1 but not y_1.

Observation 4. Let $G \in \mathcal{B}_{\mathrm{rdom}}$ and let $e = xy$ be an arbitrary edge of G. If G^* is obtained from G by subdividing the edge e three times resulting in a path $xv_1v_2v_3y$, then the following properties hold.

(a) $\gamma_{r,\mathrm{dom}}(G^*; v_1) \le \gamma_r(G)$ and $\gamma_{r,\mathrm{ndom}}(G^*; v_1) \le \gamma_r(G)$.
(b) If $G \in \{R_4, R_5, R_9\}$, then $\gamma_{r,\mathrm{dom}}(G^*; v_2) \le \gamma_r(G)$.

Observation 5. Let $G \in \mathcal{B}_{\mathrm{rdom}}$ and let $e = xy$ be an arbitrary edge of G. If G^* is obtained from G by subdividing the edge e four times resulting in a path $xv_1v_2v_3v_4y$, then there exists a RD-set S^* of G^* such that $S^* \cap \{v_1, v_2, v_3, v_4\} = \{v_1, v_4\}$ and the following properties hold.

(a) If $G \notin \{R_4, R_5\}$, then $|S^*| \le \gamma_r(G) + 1$.
(b) If $G \in \{R_4, R_5\}$, then $|S^*| \le \gamma_r(G)$.

Observation 6. Let $G \in \mathcal{B}_{\mathrm{rdom}}$ and let $e = xy$ be an arbitrary edge of G. If G^* is obtained from G by subdividing the edge e four times resulting in a path $xv_1v_2v_3v_4y$, then there exists a RD-set S^* of G^* such that $v_2 \in S^*$ and the following properties hold.

(a) If $G \ne R_2$ or if $G = R_2$ and neither x nor y is an open twin in G, then $|S^*| \le \gamma_r(G)$.
(b) If $G = R_2$ and x or y is an open twin in G, then $|S^*| \le \gamma_r(G) + 1$.

4 | PRELIMINARY RESULT

In this section, we present a preliminary result that we will need when proving our main result.

Lemma 1. *If G is a bipartite special subcubic graph with partite sets $\mathcal{S}$ and $\mathcal{L}$, then $\gamma_r(G) \le |\mathcal{L}|$.*

Proof. Let G be a bipartite subcubic graph with partite sets $\mathcal{S}$ and $\mathcal{L}$. Thus $\mathcal{S}$ and $\mathcal{L}$ are independent sets, and every vertex in $\mathcal{S}$ has degree 2 with two neighbors in $\mathcal{L}$ and every vertex in $\mathcal{L}$ has degree 3 with three neighbors in $\mathcal{S}$. Let $s = |\mathcal{S}|$ and $\ell = |\mathcal{L}|$.

Let F be the graph with $V(F) = \mathcal{L}$, where two vertices are adjacent in F if and only if they have a common neighbor (that belongs to $\mathcal{S}$) in the graph G. Let $\mathcal{L}_1$ be a maximal independent set in F, and let $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Let $\ell_1 = |\mathcal{L}_1|$ and let $\ell_2 = |\mathcal{L}_2|$. Let $\mathcal{S}_1$ be the set of vertices dominated by $\mathcal{L}_1$ in the graph G, and let $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. Possibly, $\mathcal{S}_2 = \emptyset$.

If a vertex in $\mathcal{S}_1$ has both its neighbors in $\mathcal{L}_1$, then the set $\mathcal{L}_1$ would contain two adjacent vertices in F, contradicting the fact that $\mathcal{L}_1$ is an independent set in F. Hence every vertex in $\mathcal{S}_1$ is adjacent to exactly one vertex of $\mathcal{L}_1$ and to exactly one vertex in $\mathcal{L}_2$. In particular, this implies that the subgraph $G[\mathcal{L}_1 \cup \mathcal{S}_1]$ of G induced by the set $\mathcal{L}_1 \cup \mathcal{S}_1$ consists of ℓ_1 vertex disjoint copies of $K_{1,3}$ where the central vertex of each star belongs to $\mathcal{L}_1$.

By the maximality of the independent set $\mathcal{L}_1$, the set $\mathcal{L}_1$ is a dominating set in F, implying that every vertex in $\mathcal{L}_2$ must have at least one neighbor in G that belongs to the set $\mathcal{S}_1$, that is, the set $\mathcal{S}_1$ dominates the set $\mathcal{L}_2$ in G. Let $\mathcal{L}_{2.i}$ be the set of vertices in $\mathcal{L}_2$ that have exactly i neighbors in $\mathcal{S}_1$ for $i \in [3]$. Further, let $\ell_{2.i} = |\mathcal{L}_{2.i}|$ for $i \in [3]$, and so $\ell_2 = \ell_{2.1} + \ell_{2.2} + \ell_{2.3}$.

Since each vertex in $\mathcal{S}_1$ has exactly one neighbor in $\mathcal{L}_2$, no two vertices in $\mathcal{L}_2$ have a common neighbor in $\mathcal{S}_1$. For each vertex v in $\mathcal{L}_{2.3}$, we select an arbitrary neighbor v' in $\mathcal{S}_1$ and let $\mathcal{S}_{1.1}$ be the resulting subset of vertices in $\mathcal{S}_1$, that is,

$$\mathcal{S}_{1.1} = \bigcup_{v \in \mathcal{L}_{2.3}} \{v'\}.$$

By our earlier observations, $|\mathcal{S}_{1.1}| = \ell_{2,3}$. Let $\mathcal{S}_{1.2} = \mathcal{S}_1 \setminus \mathcal{S}_{1.1}$. Each vertex in $\mathcal{L}_{2.3}$ has one neighbor in $\mathcal{S}_{1.1}$ and two neighbors in $\mathcal{S}_{1.2}$, while each vertex in $\mathcal{L}_{2.i}$ has i neighbors in $\mathcal{S}_{1.2}$ and $3 - i$ neighbors in $\mathcal{S}_2$ for $i \in \{1, 2\}$. Each vertex in $\mathcal{L}_2$ therefore has at least one neighbor in $\mathcal{S}_{1.2}$, and each vertex in $\mathcal{S}_{1.2}$ has exactly one neighbor in $\mathcal{L}_2$. Therefore, the subgraph of G induced by the set $\mathcal{S}_{1.2} \cup \mathcal{L}_2$ is isolate-free.

We now consider the set $D = \mathcal{L}_1 \cup \mathcal{S}_{1.1} \cup \mathcal{S}_2$. By construction, $V(G) \setminus D = \mathcal{S}_{1.2} \cup \mathcal{L}_2$. As observed earlier, the subgraph of G induced by the set $\mathcal{S}_{1.2} \cup \mathcal{L}_2$ is isolate-free. Moreover, every vertex in $\mathcal{S}_{1.2}$ is dominated by the set $\mathcal{L}_1 \subseteq D$ and every vertex of $\mathcal{L}_2$ is dominated by the set $\mathcal{S}_{1.1} \cup \mathcal{S}_2 \subseteq D$. Hence, D is indeed an RD-set. It remains for us to show that $|D| \le \ell$. Each vertex in $\mathcal{S}_2$ has no neighbor in $\mathcal{L}_1 \cup \mathcal{L}_{2.3}$, and therefore has both its neighbors in $\mathcal{L}_{2.1} \cup \mathcal{L}_{2.2}$. Counting edges between the set $\mathcal{S}_2$ and the sets $\mathcal{L}_{2.1} \cup \mathcal{L}_{2.2}$, we, therefore, have $2|\mathcal{S}_2| = 2\ell_{2.1} + \ell_{2.2} \le 2\ell_{2.1} + 2\ell_{2.2}$, and so $|\mathcal{S}_2| \le \ell_{2.1} + \ell_{2.2}$. Recall that $|\mathcal{S}_{1.1}| = \ell_{2,3}$. Hence, $|D| = |\mathcal{L}_1| + |\mathcal{S}_2| + |\mathcal{S}_{1.1}| \le \ell_1 + (\ell_{2.1} + \ell_{2.2}) + \ell_{2.3} = \ell_1 + \ell_2 = \ell$, as required. Therefore, $\gamma_r(G) \le |D| \le \ell$. □

5 | PROOF OF KEY RESULT

In this section, we present proof of our key result, namely Theorem 3. Recall its statement.

Theorem 3. *If G is a special subcubic graph, then* $10\gamma_r(G) \le \mathrm{w}(G)$.

Proof. Suppose, to the contrary, that there exists a counterexample to the theorem. Among all counterexamples, let G be chosen to have a minimum order. Thus if G' is a special subcubic graph of order less than $n(G)$, then G' is not a counterexample, that is, $10\gamma_r(G) > \mathrm{w}(G)$ and $10\gamma_r(G') \le \mathrm{w}(G')$ for all special subcubic graphs G' with $n(G') < n(G)$. The restrained domination number of a graph is the sum of the restrained domination numbers of its components. Hence by the minimality of G, the counterexample G is connected. For notational simplicity, we adopt the following notation throughout the proof. Let $n = n(G)$, $n_2 = n_2(G)$, and $n_3 = n_3(G)$. If G' is a special subcubic graph, then we let $n' = n(G')$, $n_2' = n_2(G')$, and $n_3' = n_3(G')$. Further, let k' be the number of components of G' that belong $\mathcal{B}_{\rm rdom}$, and let r' be the remaining components of G'. If G' is a connected graph, then we note that $k' + r' = 1$. Since $\delta(G) \geq 2$, we note that $n \geq 3$. If $G \in \mathcal{B}_{\rm rdom}$, then $10\gamma_r(G) = \mathrm{w}(G)$, contradicting the fact that G is a counterexample. Hence, $G \notin \mathcal{B}_{\rm rdom}$. If $n \in \{3, 4, 5\}$, then it is straightforward to check that $10\gamma_r(G) \leq \mathrm{w}(G)$, a contradiction. Hence, $n \geq 6$. In what follows, we present a series of claims describing some structural properties of G, which culminate in the implication of its nonexistence.

Claim 1. $\Delta(G) = 3$.

Proof. Suppose, to the contrary, that $\Delta(G) = 2$, and so G is a cycle C_n (and $n \geq 6$). In this case, $\mathrm{w}(G) = 5n$ and $\gamma_r(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$. Thus if $n \equiv 0 \pmod 3$, then $10\gamma_r(C_n) = 10n/3$. If $n \equiv 1 \pmod 3$, then $n \geq 7$ and $10\gamma_r(C_n) = 10(n + 2)/3$. If $n \equiv 2 \pmod 3$, then $n \geq 8$ and $10\gamma_r(C_n) = 10(n + 4)/3$. In all cases, $10\gamma_r(G) \leq \mathrm{w}(G)$, a contradiction. $\square$

Claim 2. The graph G does not contain a path on five vertices with the internal vertices all of degree 2 in G and such that either the two ends of the path are not adjacent or the two ends are adjacent and both have degree 3 in G.

Proof. Suppose, to the contrary, that $P : uv_1v_2v_3w$ is a path in G, where $\deg_G(v_i) = 2$ for $i \in [3]$ and if uw is an edge, then $\deg_G(u) = \deg_G(w) = 3$. Since $\delta(G) = 2$ and $\Delta(G) = 3$, we can choose the path P so that $\deg_G(u) = 3$. Let G' be the graph of order $n' = n - 3$ obtained from G by deleting the set of vertices $\{v_1, v_2, v_3\}$. Further, if u and w are not adjacent, then we add the edge uw to G'. Let S' be a γ_r-set of G'. If $\{u, w\} \subseteq S'$, let $S = S' \cup \{v_1\}$. If $u \in S'$ and $w \notin S'$, let $S = S' \cup \{v_3\}$. If $u \notin S'$ and $w \in S'$, let $S = S' \cup \{v_1\}$. If $u \notin S'$ and $w \notin S'$, let $S = S' \cup \{v_2\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \leq \gamma_r(G') + 1$.

Suppose that u and w are not adjacent in G. In this case, the edge uw was added to G', implying that the degree of the vertices u and w remain unchanged. In particular, $\deg_{G'}(u) = 3$. The graph G' is a connected special subcubic and is not a counterexample, and so $10\gamma_r(G') \leq \mathrm{w}(G')$. Suppose that $G' \notin \mathcal{B}_{\rm rdom}$. In this case, $\mathrm{w}(G) = \mathrm{w}(G') + 15$, and so $10\gamma_r(G) \leq 10(\gamma_r(G') + 1) \leq \mathrm{w}(G') + 10 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\rm rdom}$. Thus, G is obtained from one of the graphs in $\mathcal{B}_{\rm rdom}$ by subdividing the (added) edge uw in G' three times, where as observed earlier $\deg_{G'}(u) = 3$ (and $\deg_{G'}(w) \in \{2, 3\}$). Since R_1 has no vertex of degree 3, we note that $G \neq R_1$. If $G' = R_2$, then $\gamma_r(G) \leq 4$ and $\mathrm{w}(G) = 43$. If $G' \in \{R_3, R_4, R_5\}$, then $\gamma_r(G) \leq 5$ and $\mathrm{w}(G) \geq 51$. If $G' \in \{R_6, R_7, R_8, R_9\}$, then $\gamma_r(G) \leq 6$ and $\mathrm{w}(G) \geq 64$. If $G' = R_{10}$, then $\gamma_r(G) \leq 4$ and $\mathrm{w}(G) = 44$. In all cases, $10\gamma_r(G) \leq \mathrm{w}(G)$, a contradiction.

Hence, u and w are adjacent in G. As before, the graph G' is a connected special subcubic graph and $10\gamma_r(G') \leq \mathrm{w}(G')$. By supposition, both u and w have degree 3 in G, and therefore have degree 2 in G'. Hence the weight of each of u and w decreases by 1 from weight 5 in G' to weight 4 in G. If $G' \notin \mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 15 - 2 = \mathrm{w}(G') + 13$, and so

$10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le \mathrm{w}(G') + 10 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Thus, G is obtained from one of the graphs in $\mathcal{B}_{\mathrm{rdom}}$ by adding an extra edge between two vertices of degree 2 in G', and then subdividing this added edge three times. Since none of R_4, R_9, and R_{10} has two adjacent vertices of degree 2, we note that $G' \neq \{R_4, R_9, R_{10}\}$. If $G' = R_1$, then $G = R_3$, while if $G' = R_5$, then $G = R_8$. In both cases, $G \in \mathcal{B}_{\mathrm{rdom}}$, a contradiction. If $G' = R_2$, then $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 41$. If $G' = R_3$, then $\gamma_r(G) = 5$ and $\mathrm{w}(G) = 51$. If $G' \in \{R_6, R_7, R_8\}$, then $\gamma_r(G) = 6$ and $\mathrm{w}(G) = 62$. In all cases, $10\gamma_r(G) \le \mathrm{w}(G)$, a contradiction. □

As a consequence of Claim 2, we have the following structure of handles and linkages.

Claim 3. The following properties hold in the graph G.

(a) If G contains a k-handle, then $k \in \{3, 4, 5\}$.
(b) If G contains a k-linkage, then $k \in \{1, 2\}$.

Claim 4. Let G be obtained from the disjoint union of a special subcubic graph G' of order less than n and a graph H by adding at least one edge between H and G'. If $\gamma_r(G) \le \gamma_r(G') + p$ for some integer $p \ge 0$, then $\mathrm{w}(G) < \mathrm{w}(G') + 10p$.

Proof. Suppose that $\gamma_r(G) \le \gamma_r(G') + p$ for some integer $p \ge 0$. Since G' is not a counterexample, no component of G' is a counterexample, implying by linearity that $10\gamma_r(G') \le \mathrm{w}(G')$. If $\mathrm{w}(G) \ge \mathrm{w}(G') + 10p$, then $10\gamma_r(G) \le 10(\gamma_r(G') + p) \le \mathrm{w}(G') + 10p \le \mathrm{w}(G)$, a contradiction. □

Claim 5. Let G be obtained from the disjoint union of a special subcubic graph G' of order less than n and a graph H by adding at least one edge between H and G'. If there exists a γ_r-set S_H of H such that every component of G' in $\mathcal{B}_{\mathrm{rdom}}$ has at least one neighbor that belongs to S_H in the graph G, then $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k'$.

Proof. If $k' \ge 1$, let $G_1, \ldots, G_{k'}$ denote the component of G' that belong to $\mathcal{B}_{\mathrm{rdom}}$. By supposition, there exists a γ_r-set S_H of H such that the component G_i contains a vertex v_i that is adjacent to a vertex in S_H for all $i \in [k']$. By Observation 1(d), $\gamma_{r,\mathrm{ndom}}(G_i; v_i) \le \gamma_r(G_i) - 1$ for all $i \in [k']$. If G' has $r' \ge 1$ components that do not belong to $\mathcal{B}_{\mathrm{rdom}}$, let $G_{k'+1}, \ldots, G_{k'+r'}$ denote these components of G'. Hence,

$$
\begin{aligned}
\gamma_r(G) &\le |S_H| + \left(\sum_{i=1}^{k'} \gamma_{r,\mathrm{ndom}}(G_i; v_i)\right) + \left(\sum_{i=k'+1}^{k'+r'} \gamma_r(G_i)\right) \\
&\le \gamma_r(H) + \left(\sum_{i=1}^{k'+r'} \gamma_r(G_i)\right) - k' \\
&= \gamma_r(H) + \gamma_r(G') - k' \\
&= \gamma_r(G') + p,
\end{aligned}
$$

where $p = \gamma_r(H) - k'$. By Claim 4, $\mathrm{w}(G) < \mathrm{w}(G') + 10p$. □

Claim 6. There is no 3-handle in G.

Proof. Suppose that $C: vv_1v_2v$ is a 3-handle, where $\deg_G(v) = 3$. Let v_3 be the third neighbor of v. Suppose that $\deg_G(v_3) = 3$. Let $G' = G - V(C)$. We note that G' is a connected special subcubic graph and $k' + r' = 1$. Applying Claim 5 with $H = C$ and $S_H = \{v\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$, where $p = \gamma_r(H) - k' = 1 - k'$. The weights of the vertices in G' remain unchanged in G, except for v_3 whose weight increases by 1 from weight 4 in G to weight 5 in G'. Moreover, if $k' = 1$ (i.e., if $G' \in \mathcal{B}_{\mathrm{rdom}}$), then there is an additional weight increase of at most 5 for creating the component G' that belongs to $\mathcal{B}_{\mathrm{rdom}}$. Hence, $\mathrm{w}(G) \geq \mathrm{w}_G(V(C)) + (\mathrm{w}(G') - 6k' - r') = 14 + (\mathrm{w}(G') - 6k' + (k' - 1)) = 14 + (\mathrm{w}(G') - 5k' - 1) = 14 + (\mathrm{w}(G') - 5(1 - p) - 1) = \mathrm{w}(G') + 5p + 8$. Therefore, $\mathrm{w}(G') + 5p + 8 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $8 < 5p$, implying that $p \geq 2$. However, $p = 1 - k' \leq 1$, a contradiction.

Hence, $\deg_G(v_3) = 2$. Let v_4 be the neighbor of v_3 different from v. Suppose that $\deg_G(v_4) = 3$. Let $G' = G - \{v, v_1, v_2, v_3\}$. Suppose that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = 19 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 18$. Let S' be a γ_r-set of G'. If $v_4 \in S'$, let $S = S' \cup \{v_1\}$, and if $v_4 \notin S'$, let $S = S' \cup \{v\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \leq \gamma_r(G') + 1$. Hence, $10\gamma_r(G) \leq 10(\gamma_r(G') + 1) \leq \mathrm{w}(G') + 10 = (\mathrm{w}(G) - 18) + 10 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$, and so the graph G is determined. If v_4 is an open twin of G', then $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 46$, and so $10\gamma_r(G) < \mathrm{w}(G)$, a contradiction. Hence, v_4 is not an open twin of G'. By Observation 1(e), $\gamma_r(G) \leq |\{v\}| + \gamma_{r,\mathrm{dom}}(G'; v_4) \leq 1 + (\gamma_r(G') - 1) = \gamma_r(G')$, and so $10\gamma_r(G) \leq 10\gamma_r(G') \leq \mathrm{w}(G')$. However, $\mathrm{w}(G) \geq 19 + (\mathrm{w}(G') - 6) = \mathrm{w}(G') + 13$, a contradiction.

Hence, $\deg_G(v_4) = 2$. Let v_5 be the neighbor of v_4 different from v_3. By Claim 3, we have $\deg_G(v_5) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4\}$ and let $G' = G - Q$. We note that G' is a connected special subcubic graph and $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_4\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$, where $p = \gamma_r(H) - k' = 2 - k'$. Hence, $\mathrm{w}(G) \geq \mathrm{w}_G(Q) + (\mathrm{w}(G') - 6k' - r') = 24 + (\mathrm{w}(G') - 6k' + (k' - 1)) = 24 + (\mathrm{w}(G') - 5k' - 1) = 24 + (\mathrm{w}(G') - 5(2 - p) - 1) = \mathrm{w}(G') + 5p + 13$. Therefore, $\mathrm{w}(G') + 5p + 13 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $13 < 5p$, implying that $p \geq 3$. However, $p = 2 - k' \leq 2$, a contradiction. □

By Claim 6, there is no 3-handle.

Claim 7. There is no 4-handle in G.

Proof. Suppose that $C: vv_1v_2v_3v$ is a 4-handle, where $\deg_G(v) = 3$. Let v_4 be the neighbor of v not on C.

Claim 7.1. $\deg_G(v_4) = 2$.

Proof. Suppose, to contrary, that $\deg_G(v_4) = 3$. Let x and y be the two neighbors of v_4 different from v_3. Suppose that x and y are both large vertices. Let $Q = \{v, v_1, v_2, v_3, v_4\}$ and let $G' = G - Q$. We note that G' has at most two components, and so $k' + r' \leq 2$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_2, v_4\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$, where $p = \gamma_r(H) - k' = 2 - k'$. On the other hand, $\mathrm{w}(G) \geq 23 + (\mathrm{w}(G') - 6k' - r') \geq 23 + (\mathrm{w}(G') - 6k' + (k' - 2)) = 23 + (\mathrm{w}(G') - 5k' - 2) = 23 + (\mathrm{w}(G') - 5(2 - p) - 2)$

$= \mathrm{w}(G') + 5p + 11$. Therefore, $\mathrm{w}(G') + 5p + 11 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $11 < 5p$, implying that $p \ge 3$. However, $p = 2 - k' \le 2$, a contradiction.

Hence, at least one of x and y is a small vertex. Renaming vertices if necessary, we may assume that $\deg_G(x) = 2$. Note that $\deg_G(y) \in \{2, 3\}$. Suppose $xy \in E(G)$. Since there is no 3-handle, the vertex y is large. Let z be the neighbor of y, different from x and v_4. Let G' be obtained from G by deleting x, y, and v_4, and adding the edge vz. The graph G' is a connected special subcubic graph of order less than n. Since no graph in $\mathcal{B}_{\mathrm{rdom}}$ contains a 4-handle, we note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = \mathrm{w}(G') + 13$. Let S' be a γ_r-set of G'. If $v \in S'$, let $S = S' \cup \{y\}$. If $v \notin S'$ and $z \in S'$, let $S = S' \cup \{v_4\}$. If $v \notin S'$ and $z \notin S'$, let $S = S' \cup \{x\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 1 = \gamma_r(G') + 1$. Hence, $10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le \mathrm{w}(G') + 10 = (\mathrm{w}(G) - 13) + 10 < \mathrm{w}(G)$, a contradiction.

Hence, $xy \notin E(G)$. Let $Q = \{v, v_1, v_2, v_3, v_4\}$ and let G' be obtained from $G - Q$ by adding the edge xy. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose $G' \notin \mathcal{B}_{\mathrm{rdom}}$. In this case, $\mathrm{w}(G) = 23 + \mathrm{w}(G')$. Let S' be a γ_r-set of G'. If $x \in S'$ or $y \in S'$, let $S = S' \cup \{v_1, v_4\}$. If $x \notin S'$ and $y \notin S'$, let $S = S' \cup \{v, v_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 2 = \gamma_r(G') + 2$. Therefore, $10\gamma_r(G) \le 10(\gamma_r(G') + 2) \le \mathrm{w}(G') + 20 < \mathrm{w}(G)$, a contradiction.

Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - V(C)$. We note that in this case, G^* is obtained from G' by subdividing the edge xy of G', where v_4 is the resulting vertex of degree 2 in G^*. By Observation 2, $\gamma_r(G^*) \le \gamma_r(G')$, and there exists a γ_r-set S^* of G^* that contains the vertex v_4. The set $S^* \cup \{v_2\}$ is a RD-set of G, and so $\gamma_r(G) \le 1 + |S^*| = 1 + \gamma_r(G^*) \le 1 + \gamma_r(G')$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le \mathrm{w}(G') + 10$. Moreover, noting that the degrees of the vertices in G' are the same as their degrees in G, we have $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 5) = \mathrm{w}(G') + 18$, a contradiction. □

By Claim 7.1, we have $\deg_G(v_4) = 2$. Let v_5 be the neighbor of v_4 different from v. Suppose that $\deg_G(v_5) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_2, v_4\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k = 2 - k$, implying $p \le 2$. On the other hand, using the same calculations as in the earlier proofs, we have $\mathrm{w}(G) \ge 24 + (\mathrm{w}(G') - 6k' - r') \ge \mathrm{w}(G') + 5p + 13$. Therefore, $\mathrm{w}(G') + 5p + 13 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $13 < 5p$, that is, $p \ge 3$. However, $p = 2 - k' \le 2$, a contradiction.

Hence, $\deg_G(v_5) = 2$. Let v_6 be the neighbor of v_5 different from v_4. By Claim 3, $\deg_G(v_6) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4, v_5\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. Let S' be a γ_r-set of G'. If $v_6 \in S'$, let $S = S' \cup \{v, v_1\}$. If $v_6 \notin S'$, let $S = S' \cup \{v_2, v_4\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 2 = \gamma_r(G') + 2$. We note that $k' + r' = 1$. Applying Claim 4 with $H = G[Q]$ and $p = 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) \ge 29 + (\mathrm{w}(G') - 6k' - r') \ge 29 + \mathrm{w}(G') - 6(k' + r') = 29 + \mathrm{w}(G') - 6 = \mathrm{w}(G') + 23$, a contradiction. This completes the proof of Claim 7. □

By Claim 7, there is no 4-handle.

Claim 8. There is no handle in G.

Proof. Suppose, to the contrary, that G contains a handle. By our earlier observations, it must be a 5-handle. Let $C : vv_1v_2v_3v_4v$ be a 5-handle, where $\deg_G(v) = 3$. Let v_5 be the third neighbor of v not on C.

Claim 8.1. $\deg_G(v_5) = 2$.

Proof. Suppose, to the contrary, that $\deg_G(v_5) = 3$. Let $G' = G - V(C)$. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose that $G' \in \{R_1, R_4, R_5\}$. If $G' = R_1$, then $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 48$. If $G' \in \{R_4, R_5\}$, then $\gamma_r(G) = 5$ and $\mathrm{w}(G) = 59$. If $G' = R_9$, then $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 52$. In all cases, $10\gamma_r(G) \le \mathrm{w}(G)$, a contradiction. Hence, $G' \notin \{R_1, R_4, R_5, R_9\}$. Let S' be a γ_r-set of G'. If $v_5 \in S'$, let $S = S' \cup \{v_2, v_3\}$. If $v_5 \notin S'$, let $S = S' \cup \{v_1, v_4\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 2 = \gamma_r(G') + 2$. We note that $k' + r' = 1$. Applying Claim 4 with $H = C$ and $p = 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 20$. Since $G' \notin \{R_1, R_4, R_5, R_9\}$, when reconstructing the graph G the contribution of the weight of G' to the weight of G decreases by at most $3k' + r'$. Thus, $\mathrm{w}(G) \ge 24 + (\mathrm{w}(G') - 3k' - r') \ge 24 + \mathrm{w}(G') - 3(k' + r') = 24 + \mathrm{w}(G') - 3 = \mathrm{w}(G') + 21$, a contradiction. □

By Claim 8.1, we have $\deg_G(v_5) = 2$. Let v_6 be the neighbor of v_5 different from v.

Claim 8.2. $\deg_G(v_6) = 2$.

Proof. Suppose, to the contrary, that $\deg_G(v_6) = 3$. Let x and y be the two neighbors of v_6 different from v_5. Suppose that x and y are both large vertices. Let $Q = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $G' = G - Q$. We note that G' has at most two components, and so $k' + r' \le 2$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_4, v_6\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, $\mathrm{w}(G) \ge 33 + (\mathrm{w}(G') - 6k' - r') \ge 33 + (\mathrm{w}(G') - 6k' + (k' - 2)) = 33 + (\mathrm{w}(G') - 5k' - 2) = 33 + (\mathrm{w}(G') - 5(3 - p) - 2) = \mathrm{w}(G') + 5p + 16$. Therefore, $\mathrm{w}(G') + 5p + 16 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $16 < 5p$, that is, $p \ge 4$. However, $p = 3 - k' \le 3$, a contradiction.

Hence at least one of x and y is a small vertex. Renaming vertices if necessary, we may assume that $\deg_G(x) = 2$. Note that $\deg_G(y) \in \{2, 3\}$. Suppose $xy \in E(G)$. Since there is no 3-handle, the vertex y is large. Let z be the neighbor of y different from x and v_6. Let G' be obtained from G by deleting x, y and v_6, and adding the edge v_5z. The graph G' is a connected special subcubic graph of order less than n. Since no graph in $\mathcal{B}_{\mathrm{rdom}}$ contains a 5-handle, we note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = \mathrm{w}(G') + 13$. Let S' be a γ_r-set of G'. If $v_5 \in S'$, let $S = S' \cup \{y\}$. If $v_5 \notin S'$ and $z \in S'$, let $S = S' \cup \{v_6\}$. If $v_5 \notin S'$ and $z \notin S'$, let $S = S' \cup \{x\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 1 = \gamma_r(G') + 1$. Hence, $10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le w(G') + 10 = (\mathrm{w}(G) - 13) + 10 < \mathrm{w}(G)$, a contradiction.

Hence, $xy \notin E(G)$. Let $Q = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and let G' be obtained from $G - Q$ by adding the edge xy. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose $G' \notin \mathcal{B}_{\mathrm{rdom}}$. In this case, $\mathrm{w}(G) = 33 + \mathrm{w}(G')$. Let S' be a γ_r-set of G'. If $x \in S'$ or $y \in S'$, let $S = S' \cup \{v_1, v_4, v_6\}$. If $x \notin S'$ and $y \notin S'$, let $S = S' \cup \{v_2, v_3, v_5\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 3 = \gamma_r(G') + 3$. Therefore, $10\gamma_r(G) \le 10(\gamma_r(G') + 3) \le \mathrm{w}(G') + 30 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - \{v, v_1, v_2, v_3, v_4, v_5\}$. We note that in this case, G^* is obtained from G' by subdividing the edge xy of G' where v_6 is the resulting vertex of degree 2 in G^*. By

Observation 2, $\gamma_r(G^*) \le \gamma_r(G')$ and there exists a γ_r-set S^* of G^* that contains the vertex v_6. The set $S^* \cup \{v_1, v_4\}$ is a RD-set of G, and so $\gamma_r(G) \le 2 + |S^*| = 2 + \gamma_r(G^*) \le \gamma_r(G') + 2$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le 10(\gamma_r(G') + 2) \le \mathrm{w}(G') + 20$. However noting that the degrees of the vertices in G' are the same as their degrees in G, we have $\mathrm{w}(G) \ge 33 + (\mathrm{w}(G') - 5) = \mathrm{w}(G') + 28$, a contradiction. □

By Claim 8.2, we have $\deg_G(v_6) = 2$. Let v_7 be the neighbor of v_6 different from v_5. By Claim 3, $\deg_G(v_7) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_4, v_6\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, $\mathrm{w}(G) \ge 33 + (\mathrm{w}(G') - 6k' - r') = 33 + (\mathrm{w}(G') - 6k' + (k' - 1)) = 33 + (\mathrm{w}(G') - 5k' - 1) = 33 + (\mathrm{w}(G') - 5(3 - p) - 1) = \mathrm{w}(G') + 15p + 17$. Therefore, $\mathrm{w}(G') + 15p + 17 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $17 < 5p$, that is, $p \ge 4$. However, $k' \ge 0$ and $p = 3 - k' \le 3$, a contradiction. This completes the proof of Claim 8. □

By Claim 8, there is no handle in G. In particular, the removal of a bridge cannot create a C_5-component. Recall that there is no k-linkage for any $k \ge 3$. Hence if $\delta(G) = 2$, then every vertex of degree 2 in G belongs to a k-linkage for some $k \in \{1, 2\}$.

Claim 9. If G contains a 2-linkage, then the two large vertices on the linkage are not adjacent.

Proof. Suppose, to the contrary, that G contains a 2-linkage $P : vv_1v_2u$ where u and v are adjacent. We note that $u, v \in \mathcal{L}$ and $v_1, v_2 \in \mathcal{S}$.

Claim 9.1. The vertices u and v have no common neighbor.

Proof. Suppose that u and v have a common neighbor, v_3. Since $n \ge 6$, the vertex v_3 is large. Let v_4 be the neighbor of v_3 not on P. Suppose that $\deg_G(v_4) = 3$. Let $Q = \{v, v_1, v_2, v_3, u\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 2 - k'$. Since G has no handle, we note that $G' \ne R_1$, implying that $\mathrm{w}(G) \ge 22 + (\mathrm{w}(G') - 5k' - r') = 22 + (\mathrm{w}(G') - 5k' + k' - 1) = 22 + (\mathrm{w}(G') - 4k' - 1) = 22 + (\mathrm{w}(G') - 4(2 - p) - 1) \ge \mathrm{w}(G') + 4p + 13$. Therefore, $\mathrm{w}(G') + 4p + 13 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $13 < 6p$, that is, $p \ge 3$. However, $p = 2 - k' \le 2$, a contradiction.

Hence, $\deg_G(v_4) = 2$. Let v_5 be the neighbor of v_4 different from v_3. Suppose that $\deg_G(v_5) = 3$. Let $G' = G - \{v, v_1, v_2, v_3, v_4, u\}$. Let S' be a γ_r-set of G'. If $v_5 \in S'$, let $S = S' \cup \{u, v\}$. If $v_5 \notin S'$, let $S = S' \cup \{v_1, v_3\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 2 = \gamma_r(G') + 2$. Applying Claim 4 with $p = 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 20$. Recall that G has no handle, and so $G' \ne R_1$. Therefore, $\mathrm{w}(G) \ge 27 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 22$, a contradiction.

Hence, $\deg_G(v_5) = 2$. Let v_6 be the neighbor of v_5 different from v_4. By Claim 3, $\deg_G(v_6) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4, v_5, u\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying

Claim 5 with $H = G[Q]$ and $S_H = \{u, v, v_5\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, noting that $G' \neq R_1$, we have $\mathrm{w}(G) \geq 32 + (\mathrm{w}(G') - 5k' - r') \geq 32 + (\mathrm{w}(G') - 5k' + k' - 1) = 32 + (\mathrm{w}(G') - 4k' - 1) = 32 + (\mathrm{w}(G') - 4(3 - p) - 1) = \mathrm{w}(G') + 4p + 19$. Therefore, $\mathrm{w}(G') + 4p + 19 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $19 < 6p$, that is, $p \geq 4$. However, $k' \geq 0$ and $p = 3 - k' \leq 3$, a contradiction. □

By Claim 9.1, the vertices u and v have no common neighbor. Let v_3 be the third neighbor of v not on P. Since u and v have no common neighbor, u and v_3 are not adjacent. Let G' be obtained from $G - \{v, v_1, v_2\}$ by adding the edge uv_3. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose that $G' \notin \mathcal{B}_{\mathrm{rdom}}$. In this case, $\mathrm{w}(G) = 14 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 13$. Let S' be a γ_r-set of G'. If $u \in S'$, let $S = S' \cup \{v\}$. If $u \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{v_2\}$. If $u \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 1 = \gamma_r(G') + 1$. Therefore, $\mathrm{w}(G) < 10\gamma_r(G) \leq 10(\gamma_r(G') + 1) \leq \mathrm{w}(G') + 10 < \mathrm{w}(G)$, a contradiction.

Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. If $G' = R_1$, then G would contain a 4-linkage, a contradiction. If $G \in \{R_4, R_5\}$, then $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 49$, and so $10\gamma_r(G) \leq \mathrm{w}(G)$, a contradiction. Hence, $G' \notin \{R_1, R_4, R_5\}$. Let $G^* = G - \{v_1, v_2\}$. Thus, G^* is obtained from G' by subdividing the edge uv_3 of G' where v is the resulting vertex of degree 2 in G^*. By Observation 2, $\gamma_r(G^*) \leq \gamma_r(G')$ and there exists a γ_r-set S^* of G^* that contains the vertex v and does not contain u or v_3. The set $S^* \cup \{v_1\}$ is a RD-set of G, and so $\gamma_r(G) \leq 1 + |S^*| = 1 + \gamma_r(G^*) \leq 1 + \gamma_r(G')$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \leq 10(\gamma_r(G') + 1) \leq \mathrm{w}(G') + 10$. We note that the degrees of the vertices in G' are the same as their degrees in G, except for the vertex u which has degree 3 in G and degree 2 in G'. As observed earlier, $G' \notin \{R_1, R_4, R_5\}$, implying that $\mathrm{w}(G) \geq 14 + (\mathrm{w}(G') - 1 - 3) = \mathrm{w}(G') + 10$, a contradiction. This completes the proof of Claim 9. □

Claim 10. If G contains a 1-linkage, then the two large vertices on the linkage are not adjacent.

Proof. Suppose, to the contrary, that G contains a 1-linkage $P : vv_1u$ where u and v are adjacent. We note that $u, v \in \mathcal{L}$ and $v_1 \in \mathcal{S}$.

Claim 10.1. The vertices u and v have no common neighbor.

Proof. Suppose that u and v have a common neighbor, v_2, and so $G[\{v, v_1, v_2, u\}]$ is a diamond. Since $n \geq 6$, the vertex v_2 is large. Let v_3 be the third neighbor of v_2 not on P. Suppose that $\deg_G(v_3) = 3$. Let $Q = \{v, v_1, v_2, u\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the vertex v, and so $\gamma_r(G) \leq \gamma_r(G') + 1$. Thus, $\mathrm{w}(G) < 10\gamma_r(G) \leq 10(\gamma_r(G') + 1) \leq \mathrm{w}(G') + 10$. Since there is no handle in G, we note that $G' \neq R_1$, implying that $\mathrm{w}(G) \geq 17 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 12$, a contradiction.

Hence, $\deg_G(v_3) = 2$. Let v_4 be the neighbor of v_3 different from v_2. Suppose that $\deg_G(v_4) = 3$. Let $Q = \{v, v_1, v_2, v_3, u\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n different from R_1. We note that $k' + r' = 1$.

Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 2 - k'$. On the other hand, $\mathrm{w}(G) \geq 22 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 13$. Therefore, $\mathrm{w}(G') + 5p + 13 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $13 < 5p$, that is, $p \geq 3$. However, $p = 2 - k' \leq 2$, a contradiction. Hence, $\deg_G(v_4) = 2$. Let v_5 be the neighbor of v_4 different from v_3. By Claim 3, $\deg_G(v_5) = 3$. Let $Q = \{v, v_1, v_2, v_3, v_4, u\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v, v_4\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 2 - k'$. On the other hand noting that $G' \neq R_1$, we have $\mathrm{w}(G) \geq 27 + (\mathrm{w}(G') - 5k' - r') \geq \mathrm{w}(G') + 4p + 18$. Therefore, $\mathrm{w}(G') + 4p + 18 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $18 < 6p$, that is, $p \geq 4$. However, $k' \geq 0$ and $p = 2 - k' \leq 2$, a contradiction. □

By Claim 10.1, the vertices u and v have no common neighbor. Let v_2 and u_2 be the third neighbors of v and u, respectively, not on P. Since u and v have no common neighbor, $u_1 \neq v_2$.

Claim 10.2. The vertices u_2 and v_2 are not adjacent.

Proof. Suppose that u_2 and v_2 are adjacent. Since $n \geq 6$, at least one of u_2 and v_2 is large. Renaming vertices if necessary, assume that $u_2 \in \mathcal{L}$. Suppose that $v_2 \in \mathcal{S}$ and $N(v_2) = \{v, u_2\}$. Let u_3 be the neighbor of u_2 different from u and v_2. Suppose that $u_3 \in \mathcal{L}$. Let $Q = \{v, v_1, v_2, u, u_2\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, u_2\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 2 - k'$. On the other hand, $\mathrm{w}(G) \geq 22 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 13$. Therefore, $\mathrm{w}(G') + 4p + 13 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $13 < 6p$, that is, $p \geq 3$. However, $p = 2 - k' \leq 2$, a contradiction. Hence, $u_3 \in \mathcal{S}$. Let u_4 be the neighbor of u_3 different from u_2.

Suppose that $u_4 \in \mathcal{L}$. Let $Q = \{v, v_1, v_2, u, u_2, u_3\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v, u_3\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 2 - k'$. On the other hand, $\mathrm{w}(G) \geq 27 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 18$. Therefore, $\mathrm{w}(G') + 4p + 18 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $18 < 6p$, that is, $p \geq 4$. However, $p = 2 - k' \leq 2$, a contradiction.

Hence, $u_4 \in \mathcal{S}$. Let u_5 be the neighbor of u_4 different from u_3. By Claim 3, $\deg_G(u_3) = 3$. Let $Q = \{v, v_1, v_2, u, u_2, u_3, u_4\}$ and let $G' = G - Q$. The resulting graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Let S' be a γ_r-set of G'. If $u_5 \in S'$, let $S = S' \cup \{u_2, v_1\}$. If $u_5 \notin S'$, let $S = S' \cup \{v, u_3\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 2 = \gamma_r(G') + 2$. Applying Claim 4 with $p = 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) \geq 32 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 27$, a contradiction.

Hence, $v_2 \in \mathcal{L}$. Recall that $u_2 \in \mathcal{L}$. Let $Q = \{v, v_1, u\}$ and let $G' = G - Q$. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 1 - k'$. Since there is no 3-linkage in G, we note that $G' \neq R_1$, implying that $\mathrm{w}(G) \geq 13 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 7$. Therefore, $\mathrm{w}(G') + 4p + 7 \leq \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $7 < 6p$, that is, $p \geq 2$. However, $p = 1 - k' \leq 1$, a contradiction. □

By Claim 10.2, the vertices u_2 and v_2 are not adjacent. Let G' be obtained from $G - \{v, v_1, u\}$ by adding the edge u_2v_2. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose that $G' \notin \mathcal{B}_{\rm rdom}$, implying that $\mathrm{w}(G) = 13 + \mathrm{w}(G')$. Let S' be a γ_r-set of G'. If $v_2 \in S'$, let $S = S' \cup \{u\}$. If $v_2 \notin S'$ and $u_2 \in S'$, let $S = S' \cup \{v\}$. If $u_2 \notin S'$ and $v_2 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 1 = \gamma_r(G') + 1$. Therefore, $\mathrm{w}(G) < 10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le \mathrm{w}(G') + 10 = \mathrm{w}(G) - 3 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\rm rdom}$. Let $G^* = G - v_1$. We note that in this case, G^* is obtained from G' by subdividing the edge u_2v_3 of G' twice where v_2vuu_2 is the resulting path in G^*. By Observation 3, $\gamma_r(G^*) \le \gamma_r(G')$ and there exists a γ_r-set S^* of G^* that contains the vertex v and does not contain u. The set S^* is an RD-set of G, and so $\gamma_r(G) \le |S^*| = \gamma_r(G^*) \le \gamma_r(G')$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le 10\gamma_r(G') \le \mathrm{w}(G')$. However, $\mathrm{w}(G) \ge 13 + (\mathrm{w}(G') - 5) = \mathrm{w}(G') + 9$, a contradiction. This completes the proof of Claim 10. □

Recall that G has no handle. By Claim 10, no small vertex belongs to a triangle. We state this formally.

Claim 11. No small vertex belongs to a triangle.

Claim 12. Two large vertices cannot be the ends of two common 2-linkages.

Proof. Suppose, to the contrary, that there are two large vertices u and v that belong to two common 2-linkages uv_1v_2v and vv_3v_4u in G. Thus, $C : uv_1v_2vv_3v_4u$ is a 6-cycle in G, where $u, v \in \mathcal{L}$ and $v_1, v_2, v_3, v_4 \in \mathcal{S}$.

Claim 12.1. The vertices u and v have no common neighbor.

Proof. Suppose that u and v have a common neighbor, v_5. If $v_5 \in \mathcal{S}$, then the graph G is determined and $\gamma_r(G) = 3$ and $\mathrm{w}(G) = 33$, a contradiction. Hence, $v_5 \in \mathcal{L}$. Let v_6 be the neighbor of v_5 different from u and v. Suppose that $v_6 \in \mathcal{L}$. Let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, v_5\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, $\mathrm{w}(G) \ge 32 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 19$. Therefore, $\mathrm{w}(G') + 4p + 19 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $19 < 6p$, that is, $p \ge 4$. However, $p = 3 - k' \le 3$, a contradiction.

Hence, $v_6 \in \mathcal{S}$. Let v_7 be the neighbor of v_6 different from v_5. Suppose that $v_7 \in \mathcal{L}$. Let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, v_6\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, $\mathrm{w}(G) \ge 37 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 24$. Therefore, $\mathrm{w}(G') + 4p + 24 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $24 < 6p$, that is, $p \ge 5$. However, $p = 3 - k' \le 3$, a contradiction.

Hence, $v_7 \in \mathcal{S}$. Let v_8 be the neighbor of v_7 different from v_6. By Claim 3, $\deg_G(v_8) = 3$. Let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and let $G' = G - Q$. The graph G' is a connected special subcubic graph of order less than n. Let S' be a γ_r-set of G'. If $v_8 \in S'$, let $S = S' \cup \{v_1, v_3, v_5\}$. If $v_5 \notin S'$, let $S = S' \cup \{v_1, v_3, v_6\}$. In both cases, S is an RD-set of G, and so

$\gamma_r(G) \le |S| = |S'| + 3 = \gamma_r(G') + 3$. Applying Claim 4 with $H = G[Q]$ and $p = 3$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) \ge 42 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 37$, a contradiction. □

By Claim 12.1, the vertices u and v have no common neighbor. Let x be the neighbor of u different from v_1 and v_4, and let y be the neighbor of v different from v_2 and v_3. Suppose that x and y are adjacent. If both x and y have degree 2, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 38$, a contradiction. Hence at least one of x and y are large. Renaming vertices if necessary, assume that y is large. An analogous proof as before shows that $x \in \mathcal{L}$. Let $Q = \{u, v, v_1, v_2, v_3, v_4\}$ and consider the graph $G' = G - \{u, v, v_1, v_2, v_3, v_4\}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{u, v\}$, and so $\gamma_r(G) \le \gamma_r(G') + 2$. Applying Claim 4 with $H = G[Q]$ and $p = 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p = \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) \ge 28 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 23$, a contradiction.

Hence, the vertices x and y are not adjacent. Let $Q = \{u, v, v_1, v_2, v_3, v_4\}$, and let G' be obtained from $G' = G - Q$ by adding the edge xy. The resulting graph G' is a connected special subcubic graph of order less than n. Suppose that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = 28 + \mathrm{w}(G')$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it u and v or v_1 and v_3, implying that $\gamma_r(G) \le \gamma_r(G') + 2$. Therefore, $\mathrm{w}(G) < 10\gamma_r(G) \le 10(\gamma_r(G') + 2) \le \mathrm{w}(G') + 20 = \mathrm{w}(G) - 8 < \mathrm{w}(G)$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - \{v_3, v_4\}$. Thus, G^* is obtained from G' by subdividing the edge xy of G' four times where xuv_1v_2vy is the resulting path in G^*. By Observation 5, there exists a RD-set S^* of G^* such that $|S^*| \le \gamma_r(G') + 1$ and $S^* \cap \{u, v_1, v_2, v\} = \{u, v\}$. The set S^* is an RD-set of G, and so $\gamma_r(G) \le |S^*| = \gamma_r(G') + 1$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le 10(\gamma_r(G') + 1) \le \mathrm{w}(G') + 10$. Since G has no 3-linkage, we note that $G' \neq R_1$, implying that $\mathrm{w}(G) \ge 28 + (\mathrm{w}(G') - 4) = \mathrm{w}(G') + 24$, a contradiction. This completes the proof of Claim 12. □

Claim 13. Two large vertices cannot be the ends of a common 1-linkage and a common 2-linkage.

Proof. Suppose, to the contrary, that there are two large vertices u and v such that uv_1v_2v is a 2-linkage and uv_3v is a 1-linkage in G. Thus, $C : uv_1v_2vv_3u$ is a 5-cycle in G, where $u, v \in \mathcal{L}$ and $v_1, v_2, v_3 \in \mathcal{S}$.

Claim 13.1. The vertex v_3 is the only common neighbor of u and v.

Proof. Suppose that u and v have two common neighbors. Let v_4 be the common neighbor of u and v different from v_3. If $v_4 \in \mathcal{S}$, then $G = R_2$, a contradiction. Hence, $v_4 \in \mathcal{L}$. Let v_5 be the neighbor of v_4 different from u and v.

Suppose that $v_5 \in \mathcal{S}$. Let v_6 be the neighbor of v_5 different from v_4. If $v_6 \in \mathcal{L}$, then let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5\}$ and let $G' = G - Q$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, v_5\}$ we obtain a contradiction. Hence, $v_6 \in \mathcal{S}$. Let v_7 be the neighbor of v_6 different from v_5. By Claim 3, we have $\deg_G(v_7) = 3$. Let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $G' = G - Q$. In this case, $\gamma_r(G) \le \gamma_r(G') + 3$, and applying Claim 4 with $H = G[Q]$ and $p = 3$ we obtain a contradiction.

Hence, $v_5 \in \mathcal{L}$. Let x and y be the two neighbors of v_5 different from v_4. Suppose that x and y are not adjacent. Let $Q = \{u, v, v_1, v_2, v_3, v_4, v_5\}$ and let G' be obtained from $G - Q$ by adding the edge xy. If $G' \notin \mathcal{B}_{\mathrm{rdom}}$, then $\gamma_r(G) \le \gamma_r(G') + 3$, and applying Claim 4 with $H = G[Q]$ and $p = 3$ we obtain a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. In this case, we let $Q^* = \{u, v, v_1, v_2, v_3, v_4\}$ and let $G^* = G - Q^*$. Thus, G^* is obtained from G' by subdividing the edge xy of G', where xv_5y is the resulting path in G^*. Applying Observation 2, $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However since G contains no 6-handle, $G' \ne R_1$, and so $\mathrm{w}(G) \ge 31 + (\mathrm{w}(G') - 4) = \mathrm{w}(G') + 27$, a contradiction.

Hence, $xy \in E(G)$. Since there is no 3-handle in G, at least one of x and y is a large vertex. Hence by Claim 11, $x \in \mathcal{L}$ and $y \in \mathcal{L}$. Let $w = v_5$, and so $G[\{w, x, y\}]$ is a triangle. Let x_1 and y_1 be the neighbors of x and y, respectively, different from w.

We show next that $x_1 \ne y_1$. Suppose that $x_1 = y_1$. Since no vertex of degree 2 belongs to a triangle, $x_1 \in \mathcal{L}$. Let x_2 be the neighbor of x_1 different from x and y. If $x_2 \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, y, x_1\}$ and $G' = G - Q$, and applying Claim 5 with $H = G[Q]$ and $S_H = \{v, v_2, v_4, x_1\}$ we obtain a contradiction. Hence, $x_2 \in \mathcal{S}$. Let x_3 be the neighbor of x_2 different from x_1. If $x_3 \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, y, x_1, x_2\}$ and $G' = G - Q$. In this case, $\gamma_r(G) \le \gamma_r(G') + 4$, and applying Claim 4 with $H = G[Q]$ and $p = 4$, we obtain a contradiction. Hence, $x_3 \in \mathcal{S}$. Let x_4 be the neighbor of x_3 different from x_2. By Claim 3, $\deg_G(x_4) = 3$. Thus, G contains the subgraph illustrated in Figure 3. We now let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, y, x_1, x_2, x_3\}$ and $G' = G - Q$. In this case, $\gamma_r(G) \le \gamma_r(G') + 4$, and applying Claim 4 with $H = G[Q]$ and $p = 4$, we obtain a contradiction.

Hence, $x_1 \ne y_1$, and so G contains the subgraph illustrated in Figure 4. Suppose that $x_1 \in \mathcal{L}$ and $y_1 \in \mathcal{L}$. Let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, y\}$ and let $G' = G - Q$. Let G_x and G_y be the components of G'. Possibly, $G_x = G_y$, in which case G' is connected. By our earlier observations, neither G_x nor G_y is an R_1-component. Applying Claim 4 with $H = G[Q]$ and $p = 3$ we have $\mathrm{w}(G) < \mathrm{w}(G') + 30$. If at most one component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 33$, a contradiction. Hence, $G_x \ne G_y$ and both G_x and G_y belong to $\mathcal{B}_{\mathrm{rdom}}$. If $G_x \notin \{R_4, R_5\}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 30$, a contradiction. Hence, $G_x \in \{R_4, R_5\}$. Analogously, $G_y \in \{R_4, R_5\}$. Let G_w be the component of $G - v_4w$ that contains v_4, and so $G_w = R_2$. We now take a NeRD-set of type-2 in G_x, and a NeRD-set of type-1 in each of G_y and G_w, and extend

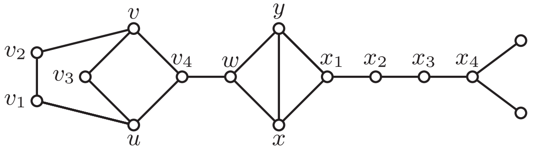

FIGURE 3 A subgraph in the proof of Claim 13.1.

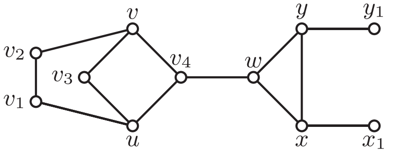

FIGURE 4 A subgraph in the proof of Claim 13.1.

these sets to an RD-set of G by adding to them the vertices w and y. By Observation 1, $\gamma_r(G) \le 2 + \gamma_{r,\mathrm{ndom}}(G_w; w) + \gamma_{r,\mathrm{dom}}(G_x; x) + \gamma_{r,\mathrm{ndom}}(G_y; y) \le 2 + (\gamma_r(G_w) - 1) + (\gamma_r(G_x) - 1) + (\gamma_r(G_y) - 1) = \gamma_r(G_w) + \gamma_r(G_x) + \gamma_r(G_y) - 1$. Thus, $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G_w) + \mathrm{w}(G_x) + \mathrm{w}(G_y) - 10$. However, $\mathrm{w}(G) \ge 12 + (\mathrm{w}(G_w) - 3) + (\mathrm{w}(G_x) - 5) + (\mathrm{w}(G_y) - 5) = \mathrm{w}(G_w) + \mathrm{w}(G_x) + \mathrm{w}(G_y) - 1$, a contradiction.

Hence, $x_1 \in \mathcal{S}$ or $y_1 \in \mathcal{S}$. Renaming vertices if necessary, we may assume that $x_1 \in \mathcal{S}$. Suppose that $x_1 y_1 \in E(G)$. By Claim 9, $y_1 \in \mathcal{L}$. Let y_2 be the neighbor of y_1 different from x_1 and y. Thus, G contains the subgraph illustrated in Figure 5. If $y_2 \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, x_1, y, y_1\}$ and $G' = G - Q$, and applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, w, y_1\}$ we obtain a contradiction. Hence, $y_2 \in \mathcal{S}$. Let y_3 be the neighbor of y_2 different from y_1. If $y_3 \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, x_1, y, y_1, y_2\}$ and $G' = G - Q$, and applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, w, x_1, y_2\}$ we obtain a contradiction. Hence, $y_3 \in \mathcal{S}$. Let y_4 be the neighbor of y_3 different from y_2. By Claim 3, $y_4 \in \mathcal{L}$. We now let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, x_1, y, y_1, y_2, y_3\}$ and let $G' = G - Q$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, w, x_1, y_3\}$, we obtain a contradiction.

Hence, $x_1 y_1 \notin E(G)$. Let z be the neighbor of x_1 different from x. Suppose that $y_1 z \notin E(G)$. In this case, let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, x_1, y\}$ and let G' be obtained from $G - Q$ by adding the edge $y_1 z$. If $G' \notin \mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = 44 + \mathrm{w}(G')$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - \{u, v, v_1, v_2, v_3, v_4, w\}$, and so G^* is obtained from G' by subdividing the edge $y_1 z$ of G' three times where zx_1xyy_1 is the resulting path in G^*. A NeRD-set of type-1 in G^* with respect to the vertex y can be extended to a RD-set by adding to it the set $\{v_1, v_3, w\}$, implying by Observation 4 that $\gamma_r(G) \le \gamma_{r,\mathrm{ndom}}(G^*; y) + 3 \le \gamma_r(G') + 3$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 30$. Since $G' \ne R_1$, we have $\mathrm{w}(G) \ge 44 + (\mathrm{w}(G') - 4) = \mathrm{w}(G) + 40$, a contradiction.

Hence, $y_1 z \in E(G)$. Thus G contains the subgraph illustrated in Figure 6, where $x_1 \in \mathcal{S}$. Since there is no 3-linkage, $y_1 \in \mathcal{L}$ or $z \in \mathcal{L}$. If $y_1 \in \mathcal{L}$ and $z \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, v_4, w, x, x_1, y\}$ and $G' = G - Q$, and applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, w, x_1\}$ we obtain a contradiction. Hence, either $y_1 \in \mathcal{S}$ and $z \in \mathcal{L}$ or $y_1 \in \mathcal{L}$ and $z \in \mathcal{S}$.

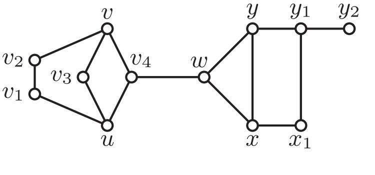

FIGURE 5 A subgraph in the proof of Claim 13.1.

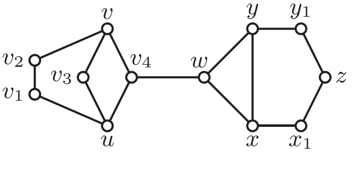

FIGURE 6 A subgraph in the proof of Claim 13.1.

Suppose that $y_1 \in \mathcal{S}$ and $z \in \mathcal{L}$. Let z_1 be the neighbor of z different from x_1 and y_1. If G' is obtained from $G - \{w, x, x_1, y, y_1, z\}$ by adding the edge $v_4 z_1$, then $\gamma_r(G) \le \gamma_r(G') + 2$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 20$. Since the graph G' contains a bridge, we note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = \mathrm{w}(G') + 26$, a contradiction. Hence, $y_1 \in \mathcal{L}$ and $z \in \mathcal{S}$. Let y_2 be the neighbor of y_1 different from y and z. If G' is obtained from $G - \{w, x, x_1, y, y_1, z\}$ by adding the edge $v_4 y_2$, then $\gamma_r(G) \le \gamma_r(G') + 2$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 20$. Since $G' \notin \mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 26$, a contradiction. This completes the proof of Claim 13.1. $\square$

By Claim 13.1, the vertex v_3 is the only common neighbor of u and v. Let x be the neighbor of u different from v_1 and v_3, and let y be the neighbor of v different from v_2 and v_3.

Claim 13.2. The vertices x and y are not adjacent.

Proof. Suppose that x and y are adjacent. By Claim 12, at least one of x and y is large. Renaming vertices if necessary, we assume that $y \in \mathcal{L}$. Let y_1 be neighbor of y different from x and v.

Suppose that $x \in \mathcal{S}$. If $y_1 \in \mathcal{L}$, then we let $Q = \{u, v, v_1, v_2, v_3, x, y\}$ and $G' = G - Q$. The graph G' is a connected subcubic graph. We note that $k' + r' = 1$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_1, v_3, y\}$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 10p$ where $p = \gamma_r(H) - k' = 3 - k'$. On the other hand, $\mathrm{w}(G) \ge 32 + (\mathrm{w}(G') - 5k' - r') = \mathrm{w}(G') + 4p + 19$. Therefore, $\mathrm{w}(G') + 4p + 19 \le \mathrm{w}(G) < \mathrm{w}(G') + 10p$, and so $19 < 6p$, that is, $p \ge 4$. However, $p = 3 - k' \le 3$, a contradiction. Hence, $y_1 \in \mathcal{S}$. Let y_2 be the neighbor of y_1 different from y. If $y_2 \in \mathcal{L}$, then let $G' = G - \{u, v, v_1, v_2, v_3, x, y, y_1\}$. In this case, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) \ge 37 - (\mathrm{w}(G') - 1 - 4) \ge \mathrm{w}(G') + 32$, a contradiction. Hence, $y_2 \in \mathcal{S}$. Let y_3 be the neighbor of y_2 different from y_1. By Claim 3, $y_3 \in \mathcal{L}$. In this case, let $Q = \{u, v, v_1, v_2, v_3, x, y, y_1, y_2\}$ and let $G' = G - Q$. Applying Claim 5 with $H = G[Q]$ and $S_H = \{v_2, v_3, x, y_2\}$, we obtain a contradiction.

Hence, $x \in \mathcal{L}$. We now consider the graph $G' = G - \{u, v, v_1, v_2, v_3\}$. Let S' be a γ_r-set of G'. In this case, $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 20$. If $G \notin \mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 2) = \mathrm{w}(G') + 21$, a contradiction. Hence, $G \in \mathcal{B}_{\mathrm{rdom}}$. We note that x and y are adjacent vertices of degree 2 in G'. Applying Observation 1(f) to the graph G' with $X = \{x, y\}$, we have $\gamma_{r,\mathrm{dom}}(G'; X) \le \gamma_r(G') - 1$. Let S'' be a minimum type-2 NeRD-set of G' with respect to the set X. The set $S'' \cup \{v_1, v_3\}$ is a RD-set of G, implying that $\gamma_r(G) \le |S''| + 2 \le \gamma_r(G') + 1$ and $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 2 - 4) = \mathrm{w}(G') + 15$, a contradiction. $\square$

By Claim 13.2, the vertices x and y are not adjacent. Let G' be obtained from $G - \{u, v, v_1, v_2, v_3\}$ by adding the edge xy. Suppose that $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) \ge 23 + \mathrm{w}(G')$. Let S' be a γ_r-set of G'. If $x \in S'$, let $S = \{v, v_2\}$. If $x \notin S'$ and $y \in S'$, let $S = \{u, v_1\}$. If $x \notin S'$ and $y \notin S'$, let $S = \{v_1, v_3\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 20$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - v_3$, and so G^* is obtained from G' by subdividing the added edge xy four times resulting in the path xuv_1v_2vy.

Suppose that $G' \neq R_2$ or $G' = R_2$ and neither x nor y is an open twin in G'. In this case, by Observation 6(a) there exists an RD-set S^* of G^* such that $v_2 \in S^*$ and $|S^*| \le \gamma_r(G)$. The set

$S^* \cup \{v_3\}$ is a RD-set of G, and so $\gamma_r(G) \le |S^*| + 1 \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 4) = \mathrm{w}(G') + 19$, a contradiction. Hence, $G' = R_2$ and x or y is an open twin in G. In this case, by Observation 6(b) there exists an RD-set S^* of G^* such that $v_2 \in S^*$ and $|S^*| \le \gamma_r(G) + 1$. The set $S^* \cup \{v_3\}$ is a RD-set of G, and so $\gamma_r(G) \le |S^*| + 1 \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However since $G' = R_2$, in this case $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 2) = \mathrm{w}(G') + 21$, a contradiction. This completes the proof of Claim 13. □

Claim 14. The removal of a bridge joining two large vertices cannot create a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

Proof. Let $e = xy$ be a bridge in G joining two adjacent large vertices x and y. Let G_x and G_y be the components of $G - e$ containing x and y, respectively. We note that both G_x and G_y are connected special subcubic graphs. Suppose, to the contrary, that at least one of G_x and G_y belongs to $\mathcal{B}_{\mathrm{rdom}}$. Renaming components if necessary, we may assume that $G_y \in \mathcal{B}_{\mathrm{rdom}}$.

Suppose that $G_x \in \mathcal{B}_{\mathrm{rdom}}$. Since there is no handle in G, we note that $G_x \ne R_1$ and $G_y \ne R_1$. Therefore, $\mathrm{w}(G) \ge (\mathrm{w}(G_x) - 1 - 4) + (\mathrm{w}(G_y) - 1 - 4) = \mathrm{w}(G_x) + \mathrm{w}(G_y) - 10$. By Observation 1(b) there exists a γ_r-set S_x of G_x that contains x. A type-1 NeRD-set of G_y with respect to the vertex y can be extended to a RD-set of G by adding to it the set S_x. Hence by Observation 1(d), $\gamma_r(G) \le \gamma_r(G_x) + \gamma_{r,\mathrm{ndom}}(G; y) \le \gamma_r(G_x) + \gamma_r(G_y) - 1$. Hence, $10\gamma_r(G) \le 10(\gamma_r(G_x) + \gamma_r(G_y) - 1) \le \mathrm{w}(G_x) + \mathrm{w}(G_y) - 10 \le \mathrm{w}(G)$, a contradiction.

Hence, $G_x \notin \mathcal{B}_{\mathrm{rdom}}$. By Claim 13 if $G_y = R_2$, then the vertex y cannot be one of the two open twins in R_2. Let S_x be a γ_r-set of G_x. If $x \in S_x$, then let S_y be a minimum type-1 NeRD-set of G_y with respect to the vertex y. In this case, the set $S_x \cup S_y$ is an RD-set of G, implying by Observation 1(d) that $\gamma_r(G) \le |S_x| + |S_y| \le \gamma_r(G_x) + \gamma_{r,\mathrm{ndom}}(G; y) \le \gamma_r(G_x) + \gamma_r(G_y) - 1$. If $x \notin S_x$, then let S_y is a minimum type-2 NeRD-set of G_y with respect to the vertex y. In this case, the set $S_x \cup S_y$ is an RD-set of G, implying by Observation 1(e) that $\gamma_r(G) \le |S_x| + |S_y| \le \gamma_r(G_x) + \gamma_{r,\mathrm{dom}}(G; y) \le \gamma_r(G_x) + \gamma_r(G_y) - 1$. In both cases, $\gamma_r(G) \le \gamma_r(G_x) + \gamma_r(G_y) - 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G_x) + \mathrm{w}(G_y) - 10$. However, $\mathrm{w}(G) \ge (\mathrm{w}(G_x) - 1) + (\mathrm{w}(G_y) - 1 - 4) = \mathrm{w}(G_x) + \mathrm{w}(G_y) - 6$, a contradiction. □

Claim 15. The removal of the two small vertices on a 2-linkage cannot create a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

Proof. Let $P : vv_1v_2u$ be a 2-linkage, and so $u, v \in \mathcal{L}$ and $v_1, v_2 \in \mathcal{S}$. By Claim 9, $uv \notin E(G)$. Suppose, to the contrary, that $G' = G - \{v_1, v_2\}$ creates a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$. Let G_u and G_v be the components of $G - e$ containing u and v, respectively, where we may assume renaming vertices, if necessary, that $G_v \in \mathcal{B}_{\mathrm{rdom}}$. Suppose that $G_u = G_v$, and so the graph G' is connected. In this case, let S_v be a minimum type-1 NeRD-set of G_v with respect to the vertex v. The set $S_v \cup \{v_1\}$ is an RD-set of G, implying by Observation 1 that $\gamma_r(G) \le 1 + \gamma_{r,\mathrm{ndom}}(G; v) \le \gamma_r(G_v)$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G_v)$. However, $\mathrm{w}(G) = 10 + (\mathrm{w}(G_v) - 2 - 4) = \mathrm{w}(G_v) + 4$, a contradiction. Hence, $G_u \ne G_v$, and so G' is disconnected with two components G_u and G_v.

Let S_u be a γ_r-set of G_v. Suppose that $u \in S_u$. In this case, the set S_u can be extended to an RD-set of G by adding to it a γ_r-set of G_v that contains v, which exists by Observation 1(d), implying that $\gamma_r(G) \le \gamma_r(G_u) + \gamma_r(G_v)$. Suppose that $u \notin S_u$. By Observation 1(d),

$\gamma_{r,\mathrm{ndom}}(G_v; v) \le \gamma_r(G_v) - 1$. In this case, the set S_u can be extended to an RD-set of G by adding to it the vertex v_2 and a minimum type-1 NeRD-set of G_v with respect to the vertex v, implying that $\gamma_r(G) \le |S_u| + 1 + \gamma_{r,\mathrm{ndom}}(G_v; v) \le \gamma_r(G_u) + \gamma_r(G_v)$. Thus in both cases, $\gamma_r(G) \le \gamma_r(G_u) + \gamma_r(G_v)$, implying that $\mathrm{w}(G) < \mathrm{w}(G_u) + \mathrm{w}(G_v)$. However, $\mathrm{w}(G) \ge 10 + (\mathrm{w}(G_u) - 1 - 4) + (\mathrm{w}(G_v) - 1 - 4) = \mathrm{w}(G_u) + \mathrm{w}(G_v)$, a contradiction. □

Claim 16. The removal of the small vertex on a 1-linkage cannot create a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

Proof. Let $P : vv_1u$ be a 1-linkage, and so $u, v \in \mathcal{L}$ and $v_1 \in \mathcal{S}$. By Claim 9, $uv \notin E(G)$. Suppose, to the contrary, that $G' = G - v_1$ creates a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$. Let G_u and G_v be the components of $G - v_1$ containing u and v, respectively, where we may assume renaming vertices if necessary, that $G_v \in \mathcal{B}_{\mathrm{rdom}}$. Let u_1 and u_2 be the two neighbors of u different from v_1. Let S_v^1 be a minimum type-1 NeRD-set of G_v with respect to the vertex v. By Observation 1(d), $|S_v^1| = \gamma_{r,\mathrm{ndom}}(G_v; v) \le \gamma_r(G_v) - 1$. Let S_v^2 be a minimum type-2 NeRD-set of G_v with respect to the vertex v. By Claim 13, if $G_v = R_2$, then the vertex v is not one of the open twins in G_v, implying by Observation 1(e) that $|S_v^2| = \gamma_{r,\mathrm{dom}}(G_v; v) \le \gamma_r(G_v) - 1$.

Suppose that $u_1u_2 \in E(G)$. By Claim 11, $u_1, u_2 \in \mathcal{L}$. Let $Q = V(G_v) \cup \{u, v_1\}$ and let $G' = G - Q$. Suppose that $G' \in \mathcal{B}_{\mathrm{rdom}}$. In this case, let S' be a minimum type-1 NeRD-set of G' with respect to the vertex u_1. By Observation 1(d), $|S'| \le \gamma_{r,\mathrm{ndom}}(G'; u_1) \le \gamma_r(G') - 1$. The set $S' \cup \{u\} \cup S_v^2$ is a RD-set of G, and so $\gamma_r(G) \le |S'| + 1 + |S_v^2| \le (\gamma_r(G') - 1) + 1 + (\gamma_r(G_v) - 1) = \gamma_r(G') + \gamma_r(G_v) - 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + \mathrm{w}(G_v) - 10$. However, $\mathrm{w}(G) \ge 9 + (\mathrm{w}(G') - 2 - 4) + (\mathrm{w}(G_v) - 1 - 4) = \mathrm{w}(G') + \mathrm{w}(G_v) - 2$, a contradiction. Hence, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Thus, $\mathrm{w}(G) \ge 9 + (\mathrm{w}(G') - 2) + (\mathrm{w}(G_v) - 1 - 4) = \mathrm{w}(G') + \mathrm{w}(G_v) + 2$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $S_v^2 \cup \{u\}$, implying that $\gamma_r(G) \le \gamma_r(G') + |S_v^2| + 1 \le \gamma_r(G') + (\gamma_r(G_v) - 1) + 1 = \gamma_r(G') + \gamma_r(G_v)$. Thus, $\mathrm{w}(G) < \mathrm{w}(G') + \mathrm{w}(G_v) \le \mathrm{w}(G) - 2 < \mathrm{w}(G)$, a contradiction.

Hence, $u_1u_2 \notin E(G)$. Let $Q = V(G_v) \cup \{u, v_1\}$ and let G' be obtained from $G - Q$ by adding the edge u_1u_2. The resulting graph G' is a connected subcubic graph. Suppose that $G' \in \mathcal{B}_{\mathrm{rdom}}$. In this case, let $Q^* = Q \backslash \{u\}$, and let $G^* = G - Q^*$, and so G^* is obtained from G' by subdividing the added edge u_1u_2 where u is the resulting new vertex of degree 2 in G^*. By Observation 2, $\gamma_r(G^*) \le \gamma_r(G')$ and there exists a γ_r-set S^* of G^* that contains u. The set $S^* \cup S_v^2$ is a RD-set of G, and so $\gamma_r(G) \le |S^*| + |S_v^2| \le \gamma_r(G') + \gamma_r(G_v) - 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + \mathrm{w}(G_v) - 10$. However, $\mathrm{w}(G) \ge 9 + (\mathrm{w}(G') - 4) + (\mathrm{w}(G_v) - 1 - 4) = \mathrm{w}(G') + \mathrm{w}(G_v)$, a contradiction. Hence, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Thus, $\mathrm{w}(G) \ge 9 + \mathrm{w}(G') + (\mathrm{w}(G_v) - 1 - 4) = \mathrm{w}(G') + \mathrm{w}(G_v) + 4$. Let S' be a γ_r-set of G'. If at least one of u_1 and u_2 belongs to S', let $S = S' \cup \{u\} \cup S_v^2$. If $u_1 \notin S'$ and $u_2 \notin S'$, let $S = S' \cup \{v_1\} \cup S_v^1$. In both cases, S is an RD-set of G and $|S| \le |S'| + 1 + \gamma_r(G_v) - 1 = \gamma_r(G') + \gamma_r(G_v)$. Thus, $\mathrm{w}(G) < \mathrm{w}(G') + \mathrm{w}(G_v) \le \mathrm{w}(G) - 4 < \mathrm{w}(G)$, a contradiction. □

By our earlier observations, every edge of G either joins two large vertices or belongs to a 2-linkage or belongs to a 1-linkage. Hence as an immediate consequence of Claims 14, 15, and 16, we have the following property of the graph G.

Claim 17. The removal of a bridge cannot create a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

As a consequence of Claim 17, we have the following claim.

Claim 18. The graph G does not contain R_{10} as a subgraph.

Proof. Suppose, to the contrary, that R' is a subgraph of G, where $R' = R_{10}$. Let v be small vertex (of degree 2) in R'. Since $G \notin \mathcal{B}_{\mathrm{rdom}}$, we note that $R' \neq G$, implying that v is a large vertex in G. Let v' be the vertex adjacent to v that does not belong to R'. The edge vv' is a bridge of G whose removal creates a R_{10}-component, contradicting Claim 17. □

We are now in a position to prove that there is no 2-linkage in G.

Claim 19. There is no 2-linkage in G.

Proof. Suppose, to the contrary, that G contains a 2-linkage. Let $P: uv_1v_2v$ be a 2-linkage, where x and y are the two neighbors of v not on P.

Claim 19.1. At least one of ux and uy is not an edge.

Proof. Suppose, to the contrary, that u is adjacent to both x and y. By Claim 13, $x, y \in \mathcal{L}$. If xy is an edge, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 26$, a contradiction. Hence, xy is not an edge. Let x_1 and y_1 be the neighbors of x and y, and let $G' = G - \{u, v_1, v_2\}$. Suppose $G' \in \mathcal{B}_{\mathrm{rdom}}$. In this case, xvy is a path in G' where x, v, and y all have degree 2 in G', implying that $G' \in \{R_1, R_3, R_8\}$. If $G' = R_1$, then $G = R_5$, a contradiction. If $G' = R_3$, then G could contain a 3-linkage, a contradiction. If $G' = R_8$, then G is determined and $\gamma_r(G) \leq 6$ and $\mathrm{w}(G) = 60$, a contradiction. Hence, $G' \notin \mathcal{B}_{\mathrm{rdom}}$, implying that $\mathrm{w}(G) = 14 + (\mathrm{w}(G') - 3) = \mathrm{w}(G') + 11$. Let S' be a γ_r-set of G'. If $v \notin S$, then either $x \in S'$ and $y \notin S'$ or $x \notin S'$ and $y \in S'$. In this case, we let $S = S' \cup \{v_2\}$. If $v \in S'$ and neither x not y belongs to S', then we let $S = S' \cup \{u\}$. If $v \in S'$ and at least one of x and y belongs to S', then we let $S = S' \cup \{v_2\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 1 = \gamma_r(G') + 1$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 10$. This contradicts our earlier observation that $\mathrm{w}(G) = \mathrm{w}(G') + 11$. □

Claim 19.2. Neither ux nor uy is an edge.

Proof. Suppose, to the contrary, that u is adjacent to exactly one of x and y. We may assume that uy is an edge. By Claim 13, $y \in \mathcal{L}$. By Claim 19.1, ux is not an edge. Let G' be obtained from $G - \{v, v_1, v_2\}$ by adding the edge ux. The graph G' is a connected special subcubic graph of order less than n. Let S' be a γ_r-set of G'. If $u \in S$, then let $S = S' \cup \{v\}$. If $u \notin S'$ and $x \in S'$, then we let $S = S' \cup \{v_1\}$. If $u \notin S'$ and $x \notin S'$, then we let $S = S' \cup \{v_2\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 1 = \gamma_r(G') + 1$, and so $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If $G' \notin \{R_1, R_4, R_5\}$, then $\mathrm{w}(G) \geq 14 + (\mathrm{w}(G') - 1 - 3) = \mathrm{w}(G') + 10$, a contradiction. Hence, $G' \in \{R_1, R_4, R_5\}$. Since u is a vertex of degree 3 in G', we note that $G' \neq R_1$. If $G' = R_4$, then $G = R_7$, a contradiction. If $G' = R_5$, then $G = R_6$, a contradiction. □

By Claim 19.2, the vertex u is adjacent to neither x nor y. Renaming vertices if necessary, we may assume that $\deg_G(x) \le \deg_G(y)$.

Claim 19.3. $x, y \in \mathcal{S}$.

Proof. Suppose that $y \in \mathcal{L}$. Let G' be obtained from $G - \{v, v_1, v_2\}$ by adding the edge ux. As in the proof of Claim 19.2, $\gamma_r(G) \le |S| = |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If no component of G' belongs to $\mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 13$, a contradiction. Hence, at least one component in G' belongs to $\mathcal{B}_{\rm rdom}$. Suppose that G' is connected. Since u is a vertex of degree 3 in G', we note that $G' \neq R_1$. If $G' \in \{R_4, R_5\}$, then the graph G is determined and in all cases, $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 49$, a contradiction. Hence, $G \notin \{R_1, R_4, R_5\}$, implying that $\mathrm{w}(G) \ge 14 + (\mathrm{w}(G') - 1 - 3) = \mathrm{w}(G') + 10$, a contradiction.

Hence, G' is disconnected with two components. Let G_x be the component of G' containing the vertices u and x, and let G_y be the component containing the vertex y. Both G_x and G_y are connected special subcubic graphs. Further, we note that the edge vy is a bridge in G, implying by Claim 17 that $G_y \notin \mathcal{B}_{\rm rdom}$ and therefore $G_x \in \mathcal{B}_{\rm rdom}$. Let G_v be the component of $G - vy$ that contains the vertex v. Thus, G_v is obtained from the graph G' by subdividing the edge ux three times, resulting in the path uv_1v_2vx.

Let S_v^1 be a minimum type-1 NeRD-set of G_v with respect to the vertex v, and let S_v^2 be a minimum type-2 NeRD-set of G_v with respect to the vertex v. By Observation 4, $|S_v^1| = \gamma_{r,\rm ndom}(G^*; v_1) \le \gamma_r(G')$ and $|S_v^2| = \gamma_{r,\rm dom}(G^*; v_1) \le \gamma_r(G')$. Let S_y be a γ_r-set of G_y. If $y \in S_y$, let $S = S_y \cup S_v^1$. If $y \notin S_y$, let $S = S_y \cup S_v^2$. In both cases, S is an RD-set of G, implying that $\gamma_r(G) \le \gamma_r(G_y) + \gamma_r(G')$, and so $\mathrm{w}(G) < \mathrm{w}(G_y) + \mathrm{w}(G')$. However, $\mathrm{w}(G) \ge 14 + (\mathrm{w}(G') - 4) + (\mathrm{w}(G_y) - 1) = \mathrm{w}(G_y) + \mathrm{w}(G') + 9$, a contradiction. Hence, $y \in \mathcal{S}$. By our choice of the vertex x, this implies that $x \in \mathcal{S}$. □

By Claim 19.3, $x \in \mathcal{S}$ and $y \in \mathcal{S}$. Thus, all three neighbors of v are small vertices. Interchanging the roles of u and v, analogous arguments show that all three neighbors of u are small vertices. Recall that $ux \notin E(G)$ and $uy \notin E(G)$, and so u and v do not have a common neighbor. By Claim 11, no small vertex belongs to a triangle, implying that $xy \notin E(G)$. Let x_1 and y_1 be the neighbors of x and y, respectively, different from v. Possibly, $x_1 = y_1$.

Claim 19.4. $x_1 \neq y_1$.

Proof. Suppose, to the contrary, that $x_1 = y_1$. In this case, we let $z = x_1$. Since G has no handle, $z \in \mathcal{L}$. Thus, $C : vxzyv$ is a 4-cycle in G, where $v, z \in \mathcal{L}$ and $x, y \in \mathcal{S}$. Let z_1 be the neighbor of z different from x and y. Since all three neighbors of u belong to $\mathcal{S}$, we note that $uz \notin E(G)$. Thus, $u \neq z_1$. Let G' be obtained from $G - \{v, v_2, x, y, z\}$ by adding the edge v_1z_1. The resulting graph G' is a connected subcubic graph.

Suppose that $G' \in \mathcal{B}_{\rm rdom}$. Let $G^* = G - y$, that is, G^* is obtained from G' by subdividing the added edge v_1z_1 four times resulting in the path $v_1v_2vxzz_1$. By Observation 5, there exists a RD-set S^* of G^* such that $|S^*| \le \gamma_r(G') + 1$ and $S^* \cap \{v_2, v, x, z\} = \{v_2, z\}$. The set S^* is an RD-set of G, and so $\gamma_r(G) \le |S^*| = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. Noting that $G' \neq R_1$ and the degrees of the vertices in G' are the same as their degrees in G, we have $\mathrm{w}(G) \ge 23 + (\mathrm{w}(G') - 4) = \mathrm{w}(G') + 19$, a contradiction. Hence, $G' \notin \mathcal{B}_{\rm rdom}$, implying that $\mathrm{w}(G) = 22 + \mathrm{w}(G')$. Let S' be a γ_r-set of G'. If at least one of v_1 and z_1 belongs to S', let $S = S' \cup \{v_2, z\}$. If $u_1 \notin S'$ and $z_1 \notin S'$, let $S = S' \cup \{v, x\}$. In both cases, S is an RD-set of G and $|S| \le |S'| + 2 = \gamma_r(G') + 2$. Thus, $\mathrm{w}(G) < \mathrm{w}(G') + 20$, a contradiction. □

By Claim 19.4, $x_1 \neq y_1$. By Claim 12 if $x_1 \in \mathcal{S}$, then $ux_1 \notin E(G)$ and if $y_1 \in \mathcal{S}$, then $uy_1 \notin E(G)$.

Claim 19.5. $x_1 y_1 \in E(G)$.

Proof. Suppose that $x_1 y_1 \notin E(G)$. Let G' be obtained from $G - \{v, v_1, v_2, x, y\}$ by adding the edge $x_1 y_1$. The resulting graph G' is a subcubic graph. We note that either G' is connected or has two components. Let G_{xy} be the component of G' containing the added edge $x_1 y_1$. If G' is disconnected, then let G_u be the second component of G' which necessarily contains the vertex u. In this case, the edge uv_1 is a bridge in G, implying by Claim 17 that $G_u \notin \mathcal{B}_{\rm rdom}$. Therefore, the component G_{xy} is the only possible component of G' that belongs to $\mathcal{B}_{\rm rdom}$.

Let S' be a γ_r-set of G'. If $x_1 \in S'$, let $S = S' \cup \{v_1, y\}$. If $x_1 \notin S'$ and $y_1 \in S'$, let $S = S' \cup \{v_1, x\}$. If $x_1 \notin S', y_1 \notin S'$ and $u \in S'$, let $S = S' \cup \{v\}$. If $x_1 \notin S', y_1 \notin S'$ and $u \notin S'$, let $S = S' \cup \{v, v_2\}$. In all cases, S is an RD-set of G and $|S| \le |S'| + 2 = \gamma_r(G') + 2$. Thus, $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 20$. If G' has no component in $\mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) \ge 24 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 23$, a contradiction. Hence by our earlier observations, G' has exactly one component in $\mathcal{B}_{\rm rdom}$, namely the component G_{xy}. By our earlier properties of the graph G, we note that $G_{xy} \neq R_1$. If $G_{xy} \notin \{R_4, R_5\}$, then $\mathrm{w}(G) \ge 24 + (\mathrm{w}(G') - 1 - 3) = \mathrm{w}(G') + 20$, a contradiction. Hence, $G_{xy} \in \{R_4, R_5\}$, implying that $\mathrm{w}(G) = 24 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') - 19$.

If G' is connected, then $G' = G_{xy}$ and we let $G^* = G - \{v_1, v_2\}$. If G' is disconnected, then G' consists of the two components G_u and G_{xy} and we let $G^* = G - (V(G_u) \cup \{v_1, v_2\})$. In both cases, G^* is the graph obtained from G_{xy} by subdividing the added edge $x_1 y_1$ three times resulting in the path $x_1 xvyy_1$. Recall that $G_{xy} \in \{R_4, R_5\}$. Applying Observation 4 we have $\gamma_{r,\rm dom}(G^*; v) \le \gamma_r(G_{xy})$. Thus, there exists a type-2 NeRD-set S^* in G^* with respect to the vertex v such that $|S^*| \le \gamma_r(G_{xy})$. If G' is connected, then let $S = S^*$, and note that in this case, $|S| \le \gamma_{r,\rm dom}(G^*; v) \le \gamma_r(G_{xy}) = \gamma_r(G')$. If G' is disconnected, let $S = S^* \cup S_u$ where S_u is a γ_r-set of G_u, and note that in this case, $|S| \le \gamma_{r,\rm dom}(G^*; v) + \gamma_r(G_u) \le \gamma_r\,(G_{xy} + \gamma_r(G_u) = \gamma_r(G')$. In both cases, $|S| \le \gamma_r(G')$. Further, in both cases $S \cup \{v_1\}$ is a RD-set of G, implying that $\gamma_r(G) \le |S| + 1 \le \gamma_r(G') + 1$. Hence, $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 10$, a contradiction. □

By Claim 19.5, $x_1 y_1 \in E(G)$. Since G has no handle, at least one of x_1 and y_1 is large. If exactly one of x_1 and y_1 is large, then we would contradict Claim 13. Hence, $x_1 \in \mathcal{L}$ and $y_1 \in \mathcal{L}$. Let $G' = G - \{v, v_1, v_2, x, y\}$. The resulting graph G' is a special subcubic graph. Let G_{xy} be the component of G' containing the edge $x_1 y_1$, and let G_u be the component of G' containing the vertex u. If G' is connected, then $G_u = G_{xy}$. If G' is disconnected, then G_{xy} and G_u are the two components of G'. Further, in this case, uv_1 is a bridge in G, implying by Claim 17 that $G_u \notin \mathcal{B}_{\rm rdom}$. Therefore, the component G_{xy} is the only possible component of G' that belongs to $\mathcal{B}_{\rm rdom}$.

Let S' be a γ_r-set of G'. If $\{u, x_1, y_1\} \subseteq S'$, let $S = S' \cup \{v_1, v_2\}$. If $S' \cap \{u, x_1, y_1\} = \{u, y_1\}$, let $S = S' \cup \{v_1, x\}$. If $S' \cap \{u, x_1, y_1\} = \{u, y_1\}$, let $S = S' \cup \{v_1, y\}$. If $S' \cap \{u, x_1, y_1\} = \{u\}$, let $S = S' \cup \{v\}$. If $S' \cap \{u, x_1, y_1\} = \{x_1, y_1\}$, let $S = S' \cup \{v_2\}$. If $S' \cap \{u, x_1, y_1\} = \{x_1\}$, let $S = S' \cup \{v_2, y\}$. If $S' \cap \{u, x_1, y_1\} = \{y_1\}$, let $S = S' \cup \{v_2, x\}$. If $S' \cap \{u, x_1, y_1\} = \emptyset$, let $S = S' \cup \{v, v_2\}$. In all cases, S is an RD-set of G and $|S| \le |S'| + 2 = \gamma_r(G') + 2$. Thus, $\mathrm{w}(G) < 10\gamma_r(G) \le \mathrm{w}(G') + 20$.

If $G_{xy} \notin \mathcal{B}_{\mathrm{rdom}}$ or if $G_{xy} \in \mathcal{B}_{\mathrm{rdom},1}$, then $\mathrm{w}(G) \ge 24 + (\mathrm{w}(G') - 3 - 1) = \mathrm{w}(G') + 20$, a contradiction. Hence, $G_{xy} \in \{R_1, R_2, R_3, R_4, R_5, R_9\}$. We note that x_1 and y_1 are adjacent vertices of degree 2 in G_{xy}, implying that $G_{xy} \notin \{R_4, R_9\}$. If $G_{xy} = R_1$, then necessarily $G' = G_{xy}$ and the graph G is determined. In this case, $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 46$, a contradiction. Hence, $G_{xy} \ne R_1$. If $G_{xy} = R_3$, then since G has no 3-linkage, $G' = G_{xy}$ and the graph G is determined. In this case, $\gamma_r(G) = 5$ and $\mathrm{w}(G) = 59$, a contradiction. Hence, $G_{xy} \ne R_3$. Therefore, $G_{xy} \in \{R_2, R_5\}$. By our earlier observations, the vertex u is adjacent to neither x_1 nor y_1. Further, we note that the vertex u and its two neighbors in G', as well as x_1 and y_1, all have degree 2 in G'. Moreover, $x_1 y_1$ in an edge. These properties implies that G' is disconnected. Thus, G' has two components, namely G_{xy} and G_u. As observed earlier, $G_u \notin \mathcal{B}_{\mathrm{rdom}}$. Let x_2 and y_2 be the neighbors of x_1 and y_1, respectively, in G_{xy}.

Suppose that $G_{xy} = R_2$. We note that x_2 and y_2 are the two large vertices in R_2. Let z_1 and z_2 be the two common neighbors of x_2 and y_2 in G_{xy}. Thus, the graph in Figure 7 is a subgraph of G. Let S_u be a γ_r-set of G_u. If $u \in S_u$, let $S = \{v, x, y_2, z_1\}$. If $u \notin S_u$, let $S = \{v_2, x_1, x_2, y_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S_u| + 4 = \gamma_r(G_u) + 4$. Hence, $\mathrm{w}(G) < \mathrm{w}(G_u) + 40$. However, $\mathrm{w}(G) = 50 + (\mathrm{w}(G_u) - 1) = \mathrm{w}(G_u) + 49$, a contradiction.

Hence, $G_{xy} \ne R_2$, and so $G_{xy} = R_5$. Let x_3 and y_3 be the two common neighbors of x_2 and y_2 in G_{xy}, and let x_4 and y_4 be the remaining vertices in G_{xy}, where $x_3 x_4 y_4 y_3$ is a path. Thus, the graph in Figure 8 is a subgraph of G. Let S_u be a γ_r-set of G_u, and let $S = S_u \cup \{v_1, x, y, x_3, y_3\}$. The set S is an RD-set of G, and so $\gamma_r(G) \le |S_u| + 5 = \gamma_r(G_u) + 5$. Hence, $\mathrm{w}(G) < \mathrm{w}(G_u) + 50$. However, $\mathrm{w}(G) = 58 + (\mathrm{w}(G_u) - 1) = \mathrm{w}(G_u) + 57$, a contradiction. This completes the proof of Claim 19. □

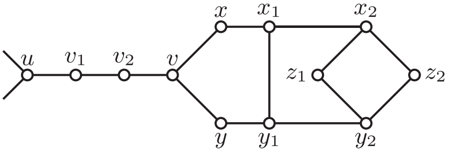

FIGURE 7 A subgraph in the proof of Claim 19 when $G_{xy} = R_2$.

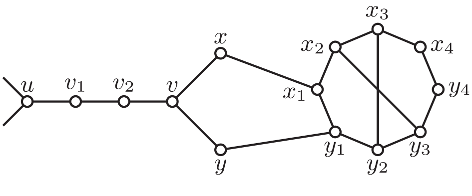

FIGURE 8 A subgraph in the proof of Claim 19 when $G_{xy} = R_5$.

By Claim 19, there is no 2-linkage in G. By our earlier observations, every vertex of degree 2 in G, if any, therefore belongs to a 1-linkage.

Claim 20. Two large vertices cannot be the ends of two common 1-linkages.

Proof. Suppose, to the contrary, that there are two large vertices u and v such that uv_1v and vv_2u are 1-linkages in G. Thus, $C: uv_1vv_2u$ is a 4-cycle in G, where $u, v \in \mathcal{L}$ and $v_1, v_2 \in \mathcal{S}$.

Claim 20.1. The vertices v_1 and v_2 are the only two common neighbors of u and v.

Proof. Suppose that u and v have a third common neighbor v_3. If $v_3 \in \mathcal{S}$, then $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 23$, a contradiction. Hence, $v_3 \in \mathcal{L}$. Let v_4 be the neighbor of v_3 different from u and v. if $v_4 \in \mathcal{L}$, then let $G' = G - \{u, v, v_1, v_2, v_3\}$. By Claim 17, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it v and v_1, and so $\gamma_r(G) \leq \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = 22 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 21$, a contradiction. Hence, $v_4 \in \mathcal{S}$. Let v_5 be the neighbor of v_4 different from v_3. If $v_5 \in \mathcal{L}$, then let $G' = G - \{u, v, v_1, v_2, v_3, v_4\}$. By Claim 17, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Let S' be a γ_r-set of G'. If $v_5 \in S'$, let $S = S' \cup \{v, v_1\}$. If $v_5 \notin S'$, let $S = S' \cup \{v, v_3\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = 27 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 26$, a contradiction. Hence, $v_5 \in \mathcal{S}$. Let v_6 be the neighbor of v_5 different from v_4. By Claim 3, $v_6 \in \mathcal{L}$. In this case, let $G' = G - \{u, v, v_1, v_2, v_3, v_4, v_5\}$. By Claim 17, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it $\{v, v_1, v_5\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = 32 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 31$, a contradiction. □

By Claim 20.1, the vertices v_1 and v_2 are the only two common neighbors of u and v. Let x be the neighbor of u different from v_1 and v_2, and let y be the neighbor of v different from v_1 and v_2.

Claim 20.2. The vertices x and y are not adjacent.

Proof. Suppose that x and y are adjacent. If $x \in \mathcal{S}$ and $y \in \mathcal{S}$, then $G = R_2$, a contradiction. Hence at least one of x and y are large. Renaming vertices if necessary, assume that $y \in \mathcal{L}$. Let y_1 be neighbor of y different from x and v. If $x \in \mathcal{S}$, then the edge yy_1 is a bridge whose removal creates an R_2-component, contradicting Claim 17. Hence, $x \in \mathcal{L}$. Let x_1 be neighbor of x different from u and y.

Suppose that $x_1 \neq y_1$. In this case, let G' be obtained from $G' - \{u, v, v_1, v_2, y\}$ by adding the edge xy_1. Let S' be a γ_r-set of G'. If $x \in S'$, let $S = S' \cup \{v, y\}$. If $x \notin S'$ and $y_1 \in S$, let $S = S' \cup \{u, v_1\}$. If $x \notin S'$ and $y_1 \notin S$, let $S = S' \cup \{v, v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| = |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G' \notin \mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 21$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. Let $G^* = G - v_2$, that is, G^* is obtained from G' by subdividing the edge xy_1 four times resulting in the path xuv_1vyy_1. By Observation 5, there exists an RD-set S^* of G^* such that $S^* \cap \{u, v_1, v, y\} = \{u, y\}$

and $|S^*| \le \gamma_r(G') + 1$. The set S^* is an RD-set of G, and so $\gamma_r(G) \le |S^*| \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \le 22 + (\mathrm{w}(G') - 1 - 4) = \mathrm{w}(G') + 17$, a contradiction.

Hence, $x_1 = y_1$. In this case, we let $z = x_1$. Since no small vertex belongs to a triangle, we note that $z \in \mathcal{L}$. Let z_1 be the neighbor of z different from x and y. If $z_1 \in \mathcal{L}$, then we let $G' = G - \{u, v, v_1, v_2, x, y, z\}$. By Claim 17, we note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Since $\gamma_r(G) \le \gamma_r(G') + 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = 30 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 29$, a contradiction. Hence, $z_1 \in \mathcal{S}$. Let z_2 be the neighbor of z_1 different from z. Since every vertex of degree 2 belongs to a 1-linkage, we note that $z_2 \in \mathcal{L}$. We now let $G' = G - \{u, v, v_1, v_2, x, y, z, z_1\}$. By Claim 17, we note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Since $\gamma_r(G) \le \gamma_r(G') + 3$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = 35 + (\mathrm{w}(G') - 1) = \mathrm{w}(G') + 34$, a contradiction. □

By Claim 20.2, the vertices x and y are not adjacent.

Claim 20.3. $x \in \mathcal{L}$ and $y \in \mathcal{L}$.

Proof. Suppose that at least one of x and y is small. Renaming vertices if necessary, we may assume that $x \in \mathcal{S}$. Let z be the neighbor of x different from u. Necessarily, $z \in \mathcal{L}$. Suppose that $yz \notin E(G)$. In this case, let G' be the connected subcubic graph obtained from $G - \{u, v, v_1, v_2, x\}$ by adding the edge yz. Let S' be a γ_r-set of G'. If at least one of z and y belongs to S', let $S = S' \cup \{v, x\}$. If $z \notin S'$ and $y \notin S'$, let $S = S' \cup \{u, v_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$. Thus, $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G' \notin \{R_1, R_4, R_5\}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 20$, a contradiction. Hence, $G' \in \{R_1, R_4, R_5\}$. Since every vertex of degree 2 in G belongs to a 1-linkage, $G' \notin \{R_1, R_5\}$, and so $G' = R_4$. Let $G^* = G - v_2$, and so G^* is obtained from G' by subdividing the added edge yz four times resulting in the path $zxuv_1vy$. By Observation 5, there exists an RD-set S^* of G^* such that $S^* \cap \{x, u, v_1, v\} = \{x, v\}$ and $|S^*| \le \gamma_r(G)$. The set S^* is an RD-set of G, and so $\gamma_r(G) \le |S^*| \le \gamma_r(G')$, implying that $\mathrm{w}(G) < 10\gamma_r(G') \le \mathrm{w}(G')$. However, $w(G) = \mathrm{w}(G') + 19$, a contradiction. Hence, $yz \in E(G)$. Recall that $x \in \mathcal{S}$ and $z \in \mathcal{L}$.

Suppose that $y \in \mathcal{L}$. In this case, let $G' = G - \{u, v, v_1, v_2, x\}$. We note that $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G' \notin \mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 21$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom}}$. A type-1 NeRD-set of G' with respect to the vertex y can be extended to an RD-set of G by adding to it the set $\{v, x\}$. Therefore by Observation 1, $\gamma_r(G) \le \gamma_{r,\mathrm{ndom}}(G'; y) + 2 \le (\gamma_r(G) - 1) + 2 = \gamma_r(G) + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 17$, a contradiction. Hence, $y \in \mathcal{S}$. Let z_1 be the neighbor of z different from x and y.

If $z_1 \in \mathcal{L}$, then let $G' = G - \{u, v, v_1, v_2, x, y, z\}$. By Claim 17, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. We note that $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 31$, a contradiction. Hence, $z_1 \in \mathcal{S}$. Let z_2 be the neighbor of z_1 different from z. Necessarily, $z_2 \in \mathcal{L}$. We now let $G' = G - \{u, v, v_1, v_2, x, y, z, z_1\}$. By Claim 17, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v, x, z_1\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 36$, a contradiction. □

By Claim 20.3, $x \in \mathcal{L}$ and $y \in \mathcal{L}$. Recall that x and y are not adjacent. Let x_1 and x_2 be the two neighbors of x different from u.

Claim 20.4. $x_1x_2 \in E(G)$.

Proof. Suppose that $x_1x_2 \notin E(G)$. Let G' be the subcubic graph obtained from $G - \{u, v, v_1, v_2, x\}$ by adding the edge x_1x_2. Let G_x be the component of G' containing the added edge x_1x_2. If G' is disconnected, then let G_y be the second component of G' which necessarily contains the vertex y. In this case, the edge vy is a bridge in G, implying by Claim 17 that $G_y \notin \mathcal{B}_{\rm rdom}$. Therefore, the component G_x is the only possible component of G' that belongs to $\mathcal{B}_{\rm rdom}$. Let S' be a γ_r-set of G'. If at least one of x_1 and x_2 belongs to S', let $S = S' \cup \{v, x\}$. If $x_1 \notin S'$ and $x_2 \notin S'$, let $S = S' \cup \{u, v_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| \leq |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G_x \notin \mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) \geq \mathrm{w}(G') + 21$, a contradiction. Hence, $G_x \in \mathcal{B}_{\rm rdom}$. Our earlier properties of the graph G imply that $G_x \neq R_1$. Let G^* be obtained from G_x by subdividing the added edge x_1x_2 resulting in the path x_1xx_2. Applying Observation 2, there exists a γ_r-set S^* of G^* such that $x \in S^*$ and $|S^*| = \gamma_r(G^*) \leq \gamma_r(G_x)$. If G' is connected, then let $S = S^* \cup \{v\}$. In this case, $|S| \leq |S^*| + 1 \leq \gamma_r(G_x) + 1 = \gamma_r(G') + 1$. If G' is disconnected, let $S = S^* \cup S_y \cup \{v\}$, where S_y is a γ_r-set of G_y. In this case, $|S| \leq |S^*| + |S_y| + 1 \leq \gamma_r(G_x) + \gamma_r(G_y) + 1 = \gamma_r(G') + 1$. In both cases, S is an RD-set of G and $|S| \leq \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \geq \mathrm{w}(G') - 17$, a contradiction. $\square$

We now return to the proof of Claim 20. By Claim 20.4, $x_1x_2 \in E(G)$. Since no vertex of degree 2 belongs to a triangle in G, we note that $x_1, x_2 \in \mathcal{L}$. Recall that $y \in \mathcal{L}$. If y is adjacent to both x_1 and x_2, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 34$, a contradiction. Hence renaming vertices if necessary, we may assume that $x_1y \notin E(G)$. Let G' be the connected subcubic graph obtained from $G - \{u, v, v_1, v_2, x\}$ by adding the edge x_1y. Let S' be a γ_r-set of G'. If at least one of x_1 and y belongs to S', let $S = S' \cup \{v, x\}$. If $x_1 \notin S'$ and $y \notin S'$, let $S = S' \cup \{u, v_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S| \leq |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G' \notin \mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) \geq \mathrm{w}(G') + 21$, a contradiction. Hence, $G' \in \mathcal{B}_{\rm rdom}$. Let $G^* = G - v_2$, that is, G^* is obtained from G' by subdividing the edge x_1y four times resulting in the path x_1xuv_1vy. By Observation 5, there exists an RD-set S^* of G^* such that $S^* \cap \{x, u, v_1, v\} = \{x, v\}$ and $|S^*| \leq \gamma_r(G') + 1$. The set S^* is an RD-set of G, and so $\gamma_r(G) \leq |S^*| \leq \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \geq \mathrm{w}(G') + 17$, a contradiction. This completes the proof of Claim 20. $\square$

By Claim 20, there is no 4-cycle in G that contains two small (nonadjacent) vertices.

Claim 21. No small vertex in G belongs to a 4-cycle.

Proof. Suppose, to the contrary, that there is a vertex $v \in \mathcal{S}$ that belongs to a 4-cycle $C : vv_1v_2v_3v$. By our earlier observations, $v_i \in \mathcal{L}$ for $i \in [3]$. Let u_i be the neighbor of v_i that does not belong to C for $i \in [3]$.

Claim 21.1. $u_1 \in \mathcal{L}$ and $u_3 \in \mathcal{L}$.

Proof. Suppose that at least one of u_1 and u_3 is small. Renaming vertices if necessary, we may assume that $u_1 \in \mathcal{S}$. Since no small vertex belongs to a triangle, $u_1v_2 \notin E(G)$. Since no 4-cycle contains two small vertices, $u_1v_3 \notin E(G)$. Let u be the neighbor of u_1 different from v_1. By our earlier observations, $u \in \mathcal{L}$ and $u \notin \{v_2, v_3\}$. If u is adjacent to both v_2 and v_3, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 26$, a contradiction. Hence, u is not adjacent to at least one of v_2 and v_3.

Suppose that $uv_3 \notin E(G)$. In this case, let G' be the connected subcubic graph obtained from $G - \{v, v_1, u_1\}$ by adding the edge uv_3. Let S' be a γ_r-set of G'. If $u \in S'$, let $S = S' \cup \{v\}$. If $u \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{u_1\}$. If $u \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If $G' \notin \{R_1, R_4, R_5\}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 10$, a contradiction. Hence, $G' \in \{R_1, R_4, R_5\}$. We note that u and v_3 are adjacent vertices of degree 3 in G', and so $G' \ne R_1$. Since there is no 2-linkage in G, we note that $G' \ne R_5$. Hence, $G' = R_4$. The graph G is now determined and satisfies $\gamma_r(G) = 4$ or $\mathrm{w}(G) = 49$, a contradiction.

Hence, $uv_3 \in E(G)$, that is, $u = u_3$. We now let $G' = G - \{v, v_1, u_1\}$. The graph G' is a connected subcubic graph. Let S' be a γ_r-set of G'. If $u \in S'$, let $S = S' \cup \{v\}$. If $u \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{u_1\}$. If $u \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If $G' \notin \mathcal{B}_{\mathrm{rdom}}$ or if $G' \in \mathcal{B}_{\mathrm{rdom},1}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 10$, a contradiction. Hence, $G' \in \mathcal{B}_{\mathrm{rdom},i}$ for some $i \in \{2, 3, 4, 5\}$. Since uv_3v_2 is a path in G', and u, v_3, and v_2 all have degree 2 in G', either $G' = R_1$ or $G' = R_3$. If $G' = R_1$, then G would contain a 2-linkage, and if $G' = R_3$, then G would contain a 3-linkage. Both cases produce a contradiction. □

By Claim 21.1, $u_1 \in \mathcal{L}$ and $u_3 \in \mathcal{L}$. If $u_1 = u_2 = u_3$, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 21$, a contradiction. Renaming vertices if necessary, we may assume that $u_2 \ne u_3$. In this case, let G' be the subcubic graph obtained from $G - \{v, v_1, v_2\}$ by adding the edge u_2v_3. Let G_1 be the component of G' containing the vertex u_1 and let G_2 be the component of G' containing the added edge u_2v_3. If G' is connected, then $G' = G_1 = G_2$. If disconnected, then the edge u_1v_1 is a bridge in G, implying by Claim 17 that $G_1 \notin \mathcal{B}_{\mathrm{rdom}}$. Therefore, the component G_2 is the only possible component of G' that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

Let S' be a γ_r-set of G'. If $u_2 \in S'$, let $S = S' \cup \{v\}$. If $u_2 \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{v_2\}$. If $u_2 \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. By our earlier properties of the graph G, we note that $G_2 \ne R_1$. If $G_2 \notin \{R_4, R_5, R_9\}$, then $\mathrm{w}(G) \ge \mathrm{w}(G) + 10$, a contradiction. Hence, $G_2 \in \{R_4, R_5, R_9\}$. If G' is connected, then the graph G is determined and either $G_2 \in \{R_4, R_5\}$, in which case $\gamma_r(G) \le 4$ and $\mathrm{w}(G) = 47$, or $G_2 = R_9$, in which case $\gamma_r(G) \le 5$ and $\mathrm{w}(G) = 58$. In both cases, we have a contradiction. Hence, G' is disconnected.

Since every small vertex in G belongs to a 1-linkage, the case $G_2 = R_5$ cannot occur, and so $G_2 \in \{R_4, R_9\}$. Let G_v be the component of $G - v_1u_1$ that contains the vertex v. Thus, G_v is obtained from G_2 by subdividing the added edge u_2v_3 three times resulting in the path $u_2v_2v_1vv_3$ and adding the edge v_2v_3. If $G_2 = R_4$, then $\gamma_r(G_v) \le 4$, and so $\gamma_r(G) \le \gamma_r(G_1) + \gamma_r(G_v) \le \gamma_r(G_1) + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G_1) + 40$. However in this case, $\mathrm{w}(G) = \mathrm{w}(G_1) + 47$, a contradiction. If $G_2 = R_9$, then $\gamma_r(G_v) \le 5$, and so

$\gamma_r(G) \le \gamma_r(G_1) + \gamma_r(G_v) \le \gamma_r(G_1) + 5$, implying that $w(G) < w(G_1) + 50$. However in this case, $w(G) = w(G_1) + 58$, a contradiction. This completes the proof of Claim 21. □

Recall that no small vertex belongs to a triangle. By Claim 21, no small vertex in G belongs to a 4-cycle. Hence every cycle that contains a small vertex in G has at least five vertices.

Claim 22. No large vertex has two small neighbors and one large neighbor.

Proof. Suppose, to the contrary, that $v \in \mathcal{L}$ and $N(v) = \{v_1, v_2, v_3\}$ where $v_1, v_2 \in \mathcal{S}$ and $v_3 \in \mathcal{L}$. Let u_1 and u_2 be the neighbors of v_1 and v_2, respectively, different from v. Since no vertex of degree 2 belongs to a triangle or a 4-cycle, $\{u_1, u_2\} \cap N(v) = \emptyset$. Further, u_1v_1 and u_2v_2 are the only edges between $\{u_1, u_2\}$ and $N(v)$. Let u_3 and w_3 be the neighbors of v_3 different from v_3. The graph illustrated in Figure 9 is a subgraph of G.

Let $G = G - \{v, v_1, v_2\}$. The graph G' is a subcubic graph with at most three components. Let S' be a γ_r-set of G'. If $u_1 \in S'$, let $S = S' \cup \{v_2\}$. If $u_1 \notin S'$ and $u_2 \in S'$, let $S = S' \cup \{v_1\}$. If $u_1 \notin S'$ and $u_2 \notin S'$, let $S = S' \cup \{v\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 1 = \gamma_r(G') + 1$, implying that $w(G) < w(G') + 10$. If no component belongs to $\mathcal{B}_{\rm rdom}$, then $w(G) \ge w(G') + 11$, a contradiction. Hence at least one component of G' belongs to $\mathcal{B}_{\rm rdom}$. Let H be such a component of G'. Possibly, $G' = H$. Since the removal of a bridge cannot create a component that belongs to $\mathcal{B}_{\rm rdom}$, the component H necessarily contains at least two vertices from the set $\{u_1, u_2, v_3\}$. We note that each of u_1, u_2, and v_3 has degree 2 in G'. Thus, at least one of u_1 and u_2 belong to the component H.

Suppose that exactly one of u_1 and u_2 belong to H. Let G^* be obtained from G' by adding the edge u_1u_2. The resulting graph G^* is a connected subcubic graph that contains a bridge, namely the added edge u_1u_2. Since no graph in $\mathcal{B}_{\rm rdom}$ contains a bridge, $G^* \notin \mathcal{B}_{\rm rdom}$. Let S^* be a γ_r-set of G^*. If $u_1 \in S^*$, let $S = S^* \cup \{v_2\}$. If $u_1 \notin S^*$ and $u_2 \in S^*$, let $S = S^* \cup \{v_1\}$. If $u_1 \notin S^*$ and $u_2 \notin S^*$, let $S = S^* \cup \{v\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S^*| + 1 = \gamma_r(G^*) + 1$, implying that $w(G) < w(G^*) + 10$. However, since $G^* \notin \mathcal{B}_{\rm rdom}$, we have $w(G) = w(G^*) + 13$, a contradiction. Hence, $\{u_1, u_2\} \subset V(H)$.

Let $X = \{u_1, u_2\}$, and so $X \subset V(H)$. As observed earlier, u_1 and u_2 have degree 2 in G'. Let S_H be a minimum type-2 NeRD-set in H with respect to the set X. By Observation 1(f), we have $|S_H| = \gamma_{r,\rm dom}(H; X) \le \gamma_r(H) - 1$. Suppose that $G' = H$. In this case, the set $S_H^* = S_H \cup \{v\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_H^*| + 1 \le (\gamma_r(H) - 1) + 1 = \gamma_r(G')$, implying that $w(G) < w(G')$. However, $w(G) \ge w(G') + 7$, a contradiction. Hence, $G' \ne H$. Let G_3 be the component of G' containing the vertex v_3, and so $G' = H \cup G_3$. We note that the removal of the bridge vv_3 creates the component G_3, implying that $G_3 \notin \mathcal{B}_{\rm rdom}$. Let S_3 be a γ_r-set of G_3. In this case, the set $S_H^* = S_H \cup S_3 \cup \{v\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_H^*| + |S_3| + 1 \le (\gamma_r(H) - 1) + \gamma_r(G_3) + 1 = \gamma_r(G')$. Therefore, $w(G) < w(G')$.

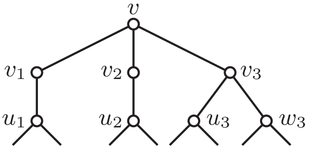

FIGURE 9 A subgraph in the proof of Claim 22.

However, $\mathrm{w}(G) \geq 14 + (\mathrm{w}(H) - 2 - 4) + (\mathrm{w}(G_3) - 1) = (\mathrm{w}(H) + \mathrm{w}(G_3)) + 7 = \mathrm{w}(G') + 7$, a contradiction. This completes the proof of Claim 22. □

By Claim 22, no large vertex has two small neighbors and one large neighbor. Hence if a large vertex has a small neighbor, then it has either one small neighbor or three small neighbors.

Claim 23. No large vertex has exactly one small neighbor.

Proof. Suppose, to the contrary, that $v \in \mathcal{L}$ has exactly one small neighbor. Let $N(v) = \{v_1, v_2, v_3\}$, where $v_1 \in \mathcal{S}$ and $v_2, v_3 \in \mathcal{L}$. Let u be the neighbor of v_1 different from v. Necessarily, $u \in \mathcal{L}$. Let u_1 and u_2 be the two neighbors of u different from v_1. Since no vertex of degree 2 belongs to a 3-cycle or 4-cycle, the vertices u_1, u_2, v_2, v_3 are pairwise distinct.

Claim 23.1. $\{u_1, u_2\} \subset \mathcal{S}$.

Proof. Suppose, to the contrary, that at least one of u_1 and u_2 is large. By Claim 22 this implies that both u_1 and u_2 are large. Let $G' = G - \{u, v, v_1\}$. Suppose that G' contains a component F such that $F \in \mathcal{B}_{\mathrm{rdom}}$. Since the removal of a bridge cannot create a component that belongs to $\mathcal{B}_{\mathrm{rdom}}$, the component F contains at least two vertices from the set $\{u_1, u_2, v_2, v_3\}$.

Suppose that F contains a vertex from both $\{u_1, u_2\}$ and $\{v_2, v_3\}$. By symmetry, and renaming vertices if necessary, we may assume that $\{u_2, v_3\} \subset V(F)$. We note that both u_2 and v_3 have degree 2 in F. Applying Observation 1(f) to the graph F with $X = \{u_2, v_3\}$, we have $\gamma_{r,\mathrm{dom}}(F; X) \leq \gamma_r(F) - 1$. Let S_F be a minimum type-2 NeRD-set of F with respect to the set X, and so $|S_F| = \gamma_{r,\mathrm{dom}}(F; X) \leq \gamma_r(F) - 1$.

Suppose that F is the only component of G' that belongs to $\mathcal{B}_{\mathrm{rdom}}$. If G' is connected, then the set $S_F \cup \{v_1\}$ is an RD-set of G. If G' is disconnected, then the set $S_F \cup \{v_1\}$ can be extended to an RD-set by adding to it a γ_r-set from the component(s) of G' different from F. In both cases, we infer that $\gamma_r(G) \leq 1 + (\gamma_r(G') - 1) = \gamma_r(G')$, implying that $\mathrm{w}(G) < \mathrm{w}(G')$. Since G' has exactly one component that belongs to $\mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) \geq \mathrm{w}(G') + 4$, a contradiction. Hence, the graph G' contains a component H, different from F, that belongs to $\mathcal{B}_{\mathrm{rdom}}$. In this case, $\{u_1, v_2\} \subset V(H)$ and, analogously as with the component F, there exists a minimum type-2 NeRD-set of H with respect to the set $\{u_1, v_2\}$ satisfying $|S_H| \leq \gamma_r(H) - 1$. The set $S_F \cup S_H \cup \{v_1\}$ is a RD-set of G, and so $\gamma_r(G) \leq 1 + |S_F| + |S_H| \leq 1 + (\gamma_r(F) - 1) + (\gamma_r(H) - 1) = \gamma_r(G') - 1$, noting that $G' = F \cup H$. This implies that $\mathrm{w}(G) \leq \mathrm{w}(G') - 10$. However, $\mathrm{w}(G) \geq 13 + (\mathrm{w}(G') - 4 - 5 - 5) = \mathrm{w}(G') - 1$, a contradiction. Hence, if the graph G' contains a component C in $\mathcal{B}_{\mathrm{rdom}}$, then either $\{u_1, u_2\} \subseteq V(C)$ and $\{v_2, v_3\} \cap V(C) = \emptyset$ or $\{v_2, v_3\} \subseteq V(C)$ and $\{u_1, u_2\} \cap V(C) = \emptyset$. If all edges are present between $\{u_1, u_2\}$ and $\{v_2, v_3\}$, then $G = R_{10}$, a contradiction. Hence renaming vertices if necessary, we may assume that $u_1 v_2 \notin E(G)$.

Let G'' be the graph obtained from G' by adding the edge $u_1 v_2$. The resulting graph G' is a subcubic graph with at most three components. Let G_1 be the component of G' containing the added edge $u_1 v_2$, and let G_2 and G_3 be the components of G' containing v_3 and u_2, respectively. If G' is connected, then $G_1 = G_2 = G_3$. Let S' be a γ_r-set of G'. If

$u_1 \in S'$, let $S = S' \cup \{v\}$. If $u_1 \notin S'$ and $v_2 \in S'$, let $S = S' \cup \{u\}$. If $u_1 \notin S'$ and $v_2 \notin S'$, let $S = S' \cup \{v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If no component of G' belongs to $\mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) \ge \mathrm{w}(G') + 11$, a contradiction. Hence at least one component of G' belongs to $\mathcal{B}_{\rm rdom}$. Let H be such a component of G. If $H \ne G_1$, then since the removal of a bridge in G cannot create a component in $\mathcal{B}_{\rm rdom}$, this implies that $\{u_2, v_3\} \subset V(H)$ and $\{u_1, v_2\} \cap V(H) = \emptyset$. However, such a component H is a component in G', contradicting our earlier properties of a component of G' that belongs to $\mathcal{B}_{\rm rdom}$. Hence, $H = G_1$ and H is the only component of G' that belongs to $\mathcal{B}_{\rm rdom}$.

Suppose that G' is connected, and so $G' = G_1 \in \mathcal{B}_{\rm rdom}$ and the graph G is determined. We note that the vertices u_1 and v_2 are adjacent vertices of degree 3 in G', and so $G' \notin \{R_1, R_2\}$. Further, we note that u_2 and v_3 have degree 2 in G'. Reconstructing the graph G from $G' \in \mathcal{B}_{\rm rdom}$ it can be readily checked that $10\gamma_r(G) \le \mathrm{w}(G)$, a contradiction. Hence, G' is disconnected, and so $G_1 \ne G_2$ or $G_1 \ne G_3$. By symmetry and renaming vertices if necessary, we may assume that $G_1 \ne G_3$. Let G_u be obtained from G_1 by subdividing the added edge u_1v_2 of G_1 three times resulting path in the path $u_1uv_1vv_2$. Let S_u^1 be a minimum type-1 NeRD-set of G_u with respect to the vertex u, and let S_v^2 be a minimum type-2 NeRD-set of G_u with respect to the vertex u. By Observation 4, $|S_u^1| = \gamma_{r,\rm ndom}(G_u; u) \le \gamma_r(G_1)$ and $|S_u^1| = \gamma_{r,\rm dom}(G_u; u) \le \gamma_r(G_1)$. Recall that $G_1 \ne G_3$. Let S_3 be a γ_r-set of G_3. If $u_2 \in S_3$, then let $S = S_u^1 \cup S_3$, while if $u_2 \notin S_3$, then let $S = S_u^2 \cup S_3$. In both cases, $|S| \le \gamma_r(G_1) + \gamma_r(G_3)$.

If $G_2 = G_1$ or $G_2 = G_3$, then $\gamma(G') = \gamma_r(G_1) + \gamma_r(G_3)$ and S is a RD-set of G. If $G_2 \ne G_1$ or $G_2 \ne G_3$, then $\gamma(G') = \gamma_r(G_1) + \gamma_r(G_2) + \gamma_r(G_3)$ and S can be extended to a RD-set of G by adding to it a γ_r-set of G_2. In both cases, we have that $\gamma_r(G) \le |S| \le \gamma_r(G')$, and we infer that $\mathrm{w}(G) < \mathrm{w}(G')$. As observed earlier, G_1 is the only component of G' that belongs to $\mathcal{B}_{\rm rdom}$. Hence, $\mathrm{w}(G) \ge 13 + (\mathrm{w}(G') - 2 - 4) = \mathrm{w}(G') + 7$, a contradiction. □

By Claim 23.1, $u_1 \in \mathcal{S}$ and $u_2 \in \mathcal{S}$.

Claim 23.2. There is no edge between $\{u_1, u_2\}$ and $\{v_2, v_3\}$.

Proof. Suppose that there is an edge between $\{u_1, u_2\}$ and $\{v_2, v_3\}$. Renaming vertices if necessary, we may assume that $u_1v_2 \in E(G)$. Since no small vertex belongs to a 4-cycle, we note that $u_2v_2 \notin E(G)$. Suppose that $u_2v_3 \in E(G)$. If $v_2v_3 \in E(G)$, then the graph G is determined and $\gamma_r(G) = 3$ and $\mathrm{w}(G) = 31$, a contradiction. Hence, $v_2v_3 \notin E(G)$. In this case, let G' be the connected subcubic graph obtained from $G - \{u, v, v_1, u_1, u_2\}$ by adding the edge v_2v_3. Let S' be a γ_r-set of G'. If $v_2 \in S'$, let $S = S' \cup \{u_2, v\}$. If $v_2 \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{v, u_1\}$. If $v_2 \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{u, v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. We note that v_2 and v_3 are adjacent vertices of degree 2 in G'. By our earlier properties of the graph G, we infer that $G' \notin \{R_1, R_4, R_5\}$. Let G^* be obtained from G' by subdividing the edge v_2v_3 four times resulting in the path $v_2u_1uv_1vv_3$. By Observation 6(a), there exists an RD-set S^* in G^* such that $v_1 \in S^*$ and $|S^*| \le \gamma_r(G')$. The set $S^* \cup \{u_2\}$ is a RD-set of G, and so $\gamma_r(G) \le |S^*| + 1 \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G) + 18$, a contradiction. Hence, $u_2v_3 \notin E(G)$. Let x be the neighbor of u_2 different from u. By our earlier observations, $x \in \mathcal{L}$ and $x \notin \{v_2, v_3\}$.

We show next that $xv_2 \in E(G)$. Suppose, to the contrary, that $xv_2 \notin E(G)$. In this case, let G' be the subcubic graph obtained from $G - \{u, v, v_1, u_1, u_2\}$ by adding the edge xv_2. Let G_x be the component containing the added edge xv_2, and let G_3 be the component containing the vertex v_3. If G' is connected, then $G_x = G_3$. If G' is disconnected, then it has two components, G_x and G_3. In this case, since the removal of a bridge cannot create a component in $\mathcal{B}_{\rm rdom}$, we note that $G_3 \notin \mathcal{B}_{\rm rdom}$.

Let S' be a γ_r-set of G'. If $v_2 \in S'$, let $S = S' \cup \{u_2, v\}$. If $v_2 \notin S'$ and $x \in S'$, let $S = S' \cup \{v, u_1\}$. If $v_2 \notin S'$ and $x \notin S'$, let $S = S' \cup \{u, v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. We note that v_2 and x are adjacent vertices of degree 2 and degree 3, respectively, in G_x. In particular, $G_x \ne R_1$. If $G_x = R_5$, then $G' = G_x$, and the graph G is determined and $\gamma_r(G) \le 5$ and $\mathrm{w}(G) = 57$, a contradiction. Hence, $G_x \ne R_5$.

Suppose that $G_x = R_4$. If $G' = G_x$, then the graph G is determined and $\gamma_r(G) \le 5$ and $\mathrm{w}(G) = 57$, a contradiction. Hence, G' is disconnected. In this case, let G_v be the component of $G - vv_3$ that contains the vertex v. We infer from the structure of the graph G_v (using the structure of G_x) that a γ_r-set of G_3 can be extended to an RD-set of G by adding to it five vertices from G_v, and so $\gamma_r(G) \le 5 + \gamma_r(G_3)$. This implies that $\mathrm{w}(G) < \mathrm{w}(G_3) + 50$. However, $\mathrm{w}(G) = \mathrm{w}(G_3) + 56$, a contradiction. Hence, $G_x \ne R_4$.

Suppose that $G_x = R_9$. If $G' = G_x$, then the graph G is determined and $\gamma_r(G) \le 6$ and $\mathrm{w}(G) = 60$, a contradiction. Hence, G' is disconnected. In this case, let G_v be the component of $G - vv_3$ that contains the vertex v. We infer from the structure of the graph G_v (using the structure of G_x) that a γ_r-set of G_3 can be extended to an RD-set of G by adding to it six vertices from G_v, and so $\gamma_r(G) \le 6 + \gamma_r(G_3)$. This implies that $\mathrm{w}(G) < \mathrm{w}(G_3) + 60$. However, $\mathrm{w}(G) = \mathrm{w}(G_3) + 60$, a contradiction. Hence, $G_x \ne R_9$.

Hence, $G_x \notin \{R_1, R_4, R_5, R_9\}$. Recall that if $G_3 \ne G_x$, then $G_3 \notin \mathcal{B}_{\rm rdom}$. Thus there is at most one bad component in G', and such a component does not belong to $\{R_1, R_4, R_5, R_9\}$. Hence, $\mathrm{w}(G) \ge \mathrm{w}(G') + 20$, a contradiction. Hence, $xv_2 \in E(G)$. The graph G therefore contains the subgraph shown in Figure 10A. Let C be the cycle $vv_1uu_1v_2v$, and let G' be the connected special subcubic graph obtained from $G - V(C)$ by adding the edge u_2v_3. Let S' be a γ_r-set of G'. If $u_2 \in S'$, let $S = S' \cup \{v, v_2\}$. If $u_2 \notin S'$ and $v_3 \in S'$, let $S = S' \cup \{u, u_1\}$. If $u_2 \notin S'$ and $v_3 \notin S'$, let $S = S' \cup \{v_1, v_2\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G' \notin \mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 21$, a contradiction. Hence, $G' \in \mathcal{B}_{\rm rdom}$. We note that $P : xu_2v_3$ is a path in G' where the vertices x, u_2 and v_3 have degrees 2, 2, and 3, respectively in G'. Our earlier properties of the graph G, together with the existence of the path P in G', imply that $G' = R_7$. Reconstructing the graph G from G' now yields the graph shown in Figure 10B that satisfies $\gamma_r(G) = 6$ and $\mathrm{w}(G) = 70$, a contradiction. (The six shaded vertices, e.g., shown in Figure 10B form a γ_r-set in G.) This completes the proof of Claim 23.2. $\square$

Let x and y be the neighbors of u_1 and u_2, respectively, different from u. By Claim 23.2, there is no edge between $\{u_1, u_2\}$ and $\{v_2, v_3\}$, implying that $\{x, y\} \cap \{v_2, v_3\} = \emptyset$. Hence, the graph illustrated in Figure 11 is a subgraph of G, where possibly edges between $\{x, y\}$ and $\{v_2, v_3\}$ may exist. By our earlier observations, $\{v_1, u_1, u_2\} \subseteq \mathcal{S}$ and $\{u, v, v_2, v_3, x, y\} \subseteq \mathcal{L}$.

Claim 23.3. $xy \notin E(G)$.

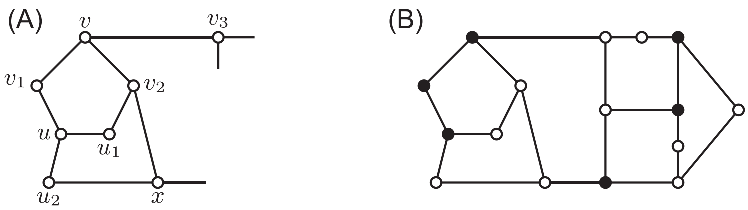

FIGURE 10 (A) A subgraphs in the proof of Claim 23.2. (B) The graph G in the proof of Claim 23.2.

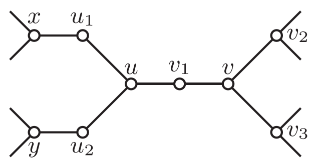

FIGURE 11 A subgraph in the proof of Claim 23.

Proof. Suppose that $xy \in E(G)$. Thus, $C : xu_1uu_2yx$ is a 5-cycle in G. Let x_1 and y_1 be the neighbors of x and y, respectively, that do not belong to the 5-cycle C. (Possibly, $x_1 = y_1$.) By Claim 22, $x_1, y_1 \in \mathcal{L}$. Let G' be the special subcubic graph obtained from $G - V(C)$ by adding the edge v_1x_1. Let G_x be the component of G' containing the added edge v_1x_1, and let G_y be the component of G' containing y_1. If G' is connected, then $G_x = G_y$. If G' is disconnected, then G' has two components, G_x and G_y. In this case, since the removal of a bridge cannot create a component in $\mathcal{B}_{\rm rdom}$, we note that $G_y \notin \mathcal{B}_{\rm rdom}$.

Let S' be a γ_r-set of G'. If $x_1 \in S'$, let $S = S' \cup \{u, u_2\}$. If $x_1 \notin S'$ and $v_1 \in S'$, let $S = S' \cup \{x, y\}$. If $x_1 \notin S'$ and $v_1 \notin S'$, let $S = S' \cup \{u_1, y\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If $G_x \notin \mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = w(G') + 21$, a contradiction. Hence, $G_x \in \mathcal{B}_{\rm rdom}$. Let G_x^* be the component of $G - \{u_2, y\}$ that contains the vertex x. Thus, G_x^* is obtained from the graph G_x by subdividing the added edge v_1x_1 three times resulting in the path $v_1uu_1xx_1$. Let S_x^* be a minimum type-2 NeRD-set of G_x^* with respect to the vertex x. Thus the set S_x^* is a dominating set in G_x^*. Further, $x \notin S_x^*$ and the vertex x is the only possible vertex in G_x^* with all its neighbors in S_x^*. By Observation 4, we have $|S_x^*| = \gamma_{r,\rm dom}(G_x^*; x) \le \gamma_r(G_x)$. Let $S^* = S_x^* \cup \{u_2\}$. If G' is connected, then $\gamma_r(G_x) = \gamma_r(G')$ and S^* is an RD-set of G. In this case, $\gamma_r(G) \le |S^*| = |S_x^*| + 1 \le \gamma_r(G') + 1$. If G' is disconnected, then $G_x \neq G_y$ and $S^* \cup S_y$ is an RD-set of G, where S_y is a γ_r-set of G_y. In this case, $\gamma_r(G) \le |S^*| + |S_y| = |S_x^*| + 1 + |S_y| \le \gamma_r(G_x) + 1 + \gamma_r(G_y) = \gamma_r(G') + 1$. In both cases, $\gamma_r(G) \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 17$, a contradiction. □

We now return to the proof of Claim 23. By Claim 23.3, the vertices x and y are not adjacent in G. Let G' be the special subcubic graph obtained from $G - \{u, u_1, u_2, v, v_1\}$ by adding the edge xy. Let G_x be the component of G' containing the added edge xy, and let G_2 and G_3 be the components of G' containing v_2 and v_3, respectively. If G' is connected, then $G_x = G_2 = G_3$. Let S' be a γ_r-set of G'. If $x \in S'$, let $S = S' \cup \{u_2, v\}$. If $x \notin S'$ and $y \in S'$, let $S = S' \cup \{u_1, v\}$. If $x \notin S', y \notin S'$ and $v_2 \in S$, let $S = S' \cup \{u\}$. If $x \notin S', y \notin S'$

and $v_2 \notin S$, let $S = S' \cup \{u, v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| \le |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 21$, a contradiction. Hence, there is a component in G' that belongs to $\mathcal{B}_{\mathrm{rdom}}$.

Suppose that G_2 or G_3 is different from G_x and belongs to $\mathcal{B}_{\mathrm{rdom}}$. Renaming vertices if necessary, by symmetry, we may assume that $G_2 \neq G_x$ and $G_2 \in \mathcal{B}_{\mathrm{rdom}}$. Since the removal of a bridge cannot create a component in $\mathcal{B}_{\mathrm{rdom}}$, we infer that $G_2 = G_3$. Further, both v_2 and v_3 have degree 2 in G_2. Applying Observation 1(f) to the graph G_2 with $X = \{v_2, v_3\}$, we have $\gamma_{r,\mathrm{dom}}(G_2; X) \le \gamma_r(G_2) - 1$. Let S^* be a minimum type-2 NeRD-set of G_2 with respect to the set X. Let S_x be a γ_r-set of G_x. If $x \in S_x$, let $S = S_x \cup S^* \cup \{u_2, v_1\}$. If $x \notin S_x$ and $y \in S_x$, let $S = S_x \cup S^* \cup \{u_1, v_1\}$. If $x \notin S_x$ and $y \notin S_x$, let $S = S_x \cup S^* \cup \{u, v_1\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S_x| + |S^*| + 2 \le \gamma_r(G_x) + (\gamma_r(G_2) - 1) + 2 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 13$, a contradiction. Hence if $G_2 \neq G_x$, then $G_2 \notin \mathcal{B}_{\mathrm{rdom}}$, and if $G_3 \neq G_x$, then $G_3 \notin \mathcal{B}_{\mathrm{rdom}}$.

Since there is a component in G' that belongs to $\mathcal{B}_{\mathrm{rdom}}$, we infer that G_x is the only such component of G'. If $G_x \in \mathcal{B}_{\mathrm{rdom},1}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, $G_x \notin \{R_6, R_7, R_8\}$. We note that x and y are adjacent vertices of degree 3 in G_x, implying that $G_x \neq \{R_1, R_2\}$. If $G_x = R_3$, then our properties of the graph G imply that G' is connected and v_2 and v_3 are the vertices of degree 2 in R_3 that have no degree 3 neighbor. In this case, the graph G is determined and $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 59$, a contradiction. Hence, $G_x \neq R_3$, implying that $G_x \in \{R_4, R_5, R_9\}$.

Let G^* be obtained from G_x by subdividing the added edge xy three times resulting in the path xu_1uu_2y. Applying Observation 4(b) we have $\gamma_{r,\mathrm{dom}}(G^*; u) \le \gamma_r(G_x)$. Thus, there exists a type-2 NeRD-set S^* in G^* with respect to the vertex u such that $|S^*| \le \gamma_r(G_x)$. The set S^* is a dominating set in G^*. Further, $u \notin S^*$ and the vertex u is the only possible vertex in G^* with all its neighbors in S^*. Let $S = S^* \cup \{v\}$. If G' is connected, then $S^* \cup \{v\}$ is an RD-set of G, and so $\gamma_r(G) \le |S^*| + 1 \le \gamma_r(G_x) + 1 = \gamma_r(G') + 1$. If G' is disconnected, then every γ_r-set of $G' - V(G_x)$ can be extended to an RD-set of G by adding to it the set $S^* \cup \{v\}$, implying once again that $\gamma_r(G) \le \gamma_r(G') + 1$. Hence in both cases we infer that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) = w(G') + 17$, a contradiction. This completes the proof of Claim 23. □

Claim 24. The graph G is a cubic graph.

Proof. Suppose, to the contrary, that G contains a small vertex. By Claim 22, no large vertex has exactly two small neighbors. By Claim 23, no large vertex has exactly one small neighbor. Hence if a large vertex has a small neighbor, then all three of its neighbors are small. Thus the three neighbors of every large vertex are either all small or all large. Since G is connected and contains at least one small vertex, this implies that G is a bipartite subcubic graph with partite sets $\mathcal{S}$ and $\mathcal{L}$. Thus, by Lemma 1, $\gamma_r(G) \le |\mathcal{L}|$, and so $\mathrm{w}(G) < 10\gamma_r(G) \le 10|\mathcal{L}|$. However in this case, $3|\mathcal{L}| = 2|\mathcal{S}|$, and so $\mathrm{w}(G) = 5|\mathcal{S}| + 4|\mathcal{L}| = 5 \times \frac{3}{2}|\mathcal{L}| + 4|\mathcal{L}| > 10|\mathcal{L}$, a contradiction. □

By Claim 24, G is a (connected) cubic graph. Recall by Claim 18 that R_{10} is not a subgraph of G. We note that R_9 contains three small vertices, and every graph in $\mathcal{B}_{\mathrm{rdom}} \setminus \{R_9, R_{10}\}$ contains at least four small vertices. Our earlier observations therefore yield the following properties of graph G.

Claim 25. If E' is a k-edge-cut in G and G' is a component of $G - E'$ that belongs to $\mathcal{B}_{\rm rdom}$, then $k \ge 3$ and the following properties hold.

(a) If $k = 3$, then $G' = R_9$.
(b) If $k = 4$, then $G' \in \{R_2, R_4, R_5, R_9\}$.
(c) If $k = 5$, then $G' \in \{R_1, R_6, R_7, R_8\}$.

Claim 26. If $G' \in \mathcal{B}_{\rm rdom}$ is a special subcubic component of $G - S$ where $S \subset V(G)$, then G' contains at least three vertices of degree 2.

Claim 27. The graph G contains no diamond.

Proof. Suppose, to the contrary, that G contains a diamond D, where $V(D) = \{v_1, v_2, v_3, v_4\}$ and where v_1v_2 is the missing edge in D. Let u_i be the neighbor of v_i not in D for $i \in [2]$. Suppose that $u_1 = u_2$. Let u be the neighbor of u_1 different from v_1 and v_2, and let $G' = G - (V(D) \cup \{u, u_1\})$. The graph G' is a special subcubic graph that contains exactly two small vertices, and so by Claim 26 no component of G' belongs to $\mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_3, u\}$, and so $\gamma_r(G) \le \gamma_r(G) + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = \mathrm{w}(G) + 22$, a contradiction. Hence, $u_1 \neq u_2$. In this case, let $G' = G - V(D)$. The graph G' is a special subcubic graph that contains exactly two small vertices, and so no component of G' belongs to $\mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the vertex v_3, and so $\gamma_r(G) \le \gamma_r(G) + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. In this case, $\mathrm{w}(G) = \mathrm{w}(G) + 14$, a contradiction. □

Claim 28. The graph G contains no triangle.

Proof. Suppose, to the contrary, that T is a triangle in G where $V(T) = \{v_1, v_2, v_3\}$. Let x_i be the third neighbor of v_i that does not belong to T for $i \in [3]$. By Claim 27, the graph G contains no diamond, and so the vertices x_1, x_2 and x_3 are pairwise distinct. Let $X = \{x_1, x_2, x_3\}$. Suppose that $G[X]$ contains a vertex of degree 2. Renaming vertices if necessary, we may assume that $\{x_1x_2, x_2x_3\} \subset E(G)$. If $x_1x_3 \in E(G)$, then G is the 3-prism $C_3 \,\square\, K_2$, and so $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 24$, a contradiction. Hence, $x_1x_3 \notin E(G)$. Let y_i be the neighbor of x_i different from x_2 and v_i for $i \in \{1, 3\}$. If $y_1 = y_3$, then we let $Q = V(T) \cup X \cup \{y_1\}$ and $G' = G - Q$. In this case, G' is a special connected subcubic graph that contains exactly one small vertex, and so, by Claim 26, $G' \notin \mathcal{B}_{\rm rdom}$. Since $\gamma_r(G) \le \gamma_r(G) + 2$, we have $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = \mathrm{w}(G) + 27$, a contradiction. Hence, $y_1 \neq y_3$. We now let $Q = V(T) \cup X$ and $G' = G - Q$. In this case, G' is a special subcubic graph that contains exactly two small vertices, and so, by Claim 26, no component of G' belongs to $\mathcal{B}_{\rm rdom}$. Once again $\gamma_r(G) \le \gamma_r(G) + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = 24 + (\mathrm{w}(G') - 2) = \mathrm{w}(G) + 22$, a contradiction.

Hence, $G[X]$ contains no vertex of degree 2, implying that $G[X]$ contains at least one isolated vertex. By symmetry, we may assume that x_1 is isolated in $G[X]$, that is, x_1 is adjacent to neither x_2 nor x_3. Let y_1 and y_2 be the two neighbors of x_1 different from v_1. We now let $Q = V(T) \cup \{x_1\}$ and let $G' = G - Q$. The graph G' is a special subcubic graph. We note that $k' + r' \le 4$.

Let S' be a γ_r-set of G'. If $y_1 \in S'$, let $S = S' \cup \{v_2\}$. If $y_1 \notin S'$, let $S = S' \cup \{v_1\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = |S'| + 1 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 12$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\mathrm{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. Thus, G_1 contains at least three small vertices. Let $X_1 \subset V(G_1) \cap \{y_1, y_2, x_2\}$ be chosen so that $|X_1| = 2$. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the set X_1. By Observation 1(f), $|S_1| = \gamma_{r,\mathrm{dom}}(G_1; X_1) \le \gamma_r(G_1) - 1$.

Suppose that $G_1 \in \{R_2, R_4, R_5\}$. By Claim 25, $G' = G_1$, and so $k' = 1$ and $r' = 0$. In this case, the set $S_1 \cup \{v_1\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_1| + 1 \le \gamma_r(G_1) = \gamma_r(G')$. Suppose that $G_1 = R_9$, implying that $k' = r' = 1$. In this case, we let G_2 be the second component of G', and so $G_2 \notin \mathcal{B}_{\mathrm{rdom}}$. Let S_2 be a γ_r-set of G_2. The set S_2 can be extended to an RD-set of G by adding to it the set $S_1 \cup \{v_1\}$, and so $\gamma_r(G) \le |S_1| + 1 + |S_2| \le \gamma_r(G_1) + \gamma_r(G_2) = \gamma_r(G')$. In both cases, $\gamma_r(G) \le \gamma_r(G')$, implying that $\mathrm{w}(G) < \mathrm{w}(G')$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 8$, a contradiction. □

Claim 29. The graph G contains no $K_{2,3}$ as a subgraph.

Proof. Suppose, to the contrary, that H is a subgraph of G, where $H \cong K_{2,3}$. Let X and Y be the partite sets of H where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$. Since G is triangle-free, the sets X and Y are independent. Let v_i be the neighbor of x_i not in H for $i \in [3]$. If $v_1 = v_2 = v_3$, then $G = K_{3,3}$. In this case, $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 24$, a contradiction. Hence renaming vertices if necessary, we may assume that $v_1 \ne v_2$.

Claim 29.1. The vertices v_1, v_2, v_3 are pairwise distinct.

Proof. Suppose, to the contrary, that the vertices v_1, v_2, v_3 are not pairwise distinct, and so $v_1 = v_3$ or $v_2 = v_3$. Renaming vertices if necessary, we may assume that $v_2 = v_3$. Suppose that $v_1v_2 \in E(G)$. In this case, let v denote the neighbor of v_1 different from x_1 and v_2. Thus, vv_1 is a bridge in G. Let G' be the component of $G - vv_1$ that contains the vertex v. By Claim 25, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Let S' be a γ_r-set of G'. If $v \in S'$, let $S = \{y_1, x_2\}$. If $v \notin S'$, let $S = \{x_1, v_2\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = w(G') + 27$, a contradiction. Hence, $v_1v_2 \notin E(G)$. We now let $G' = G - (V(H) \cup \{v_2\})$. The graph G' is a special subcubic graph that contains exactly two small vertices. By Claim 25, no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{y_1, x_2\}$, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 22$, a contradiction. □

By Claim 29.1, the vertices v_1, v_2, v_3 are pairwise distinct.

Claim 29.2. The graph $G[\{v_1, v_2, v_3\}]$ is isolate-free.

Proof. Suppose, to the contrary, that $G[\{v_1, v_2, v_3\}]$ contains an isolated vertex. Renaming vertices if necessary, we may assume that the vertex v_1 is adjacent to neither v_2 nor v_3. Let $G' = G - (V(H) \cup \{v_1\})$. Thus, G' is a special subcubic graph that contains exactly four small vertices. Let S' be a γ_r-set of G'. Let u_1 and u_2 be two neighbors of v_1 in G different from x_1. If $u_1 \notin S'$, let $S = S' \cup \{x_1, y_1\}$. If $u_1 \in S'$, let $S = S' \cup \{y_1, x_2\}$. In both cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\mathrm{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$.

At least one of u_1 and u_2, and at least one of v_2 and v_3 belong to G_1. Renaming vertices if necessary, we may assume that $\{u_1, v_2\} \subset V(G_1)$. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the set $X_1 = \{u_1, v_2\}$. By Observation 1(f), $|S_1| = \gamma_{r,\mathrm{dom}}(G_1; X_1) \le \gamma_r(G_1) - 1$. Suppose that $G_1 \in \{R_2, R_4, R_5\}$. By Claim 25, $G' = G_1$, and so $k' = 1$ and $r' = 0$. In this case, the set $S_1 \cup \{x_1, y_1\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_1| + 2 \le \gamma_r(G_1) + 1 = \gamma_r(G') + 1$. Suppose that $G_1 = R_9$, implying that $k' = r' = 1$. In this case, we let G_2 be the second component of G', and so $G_2 \notin \mathcal{B}_{\mathrm{rdom}}$. Let S_2 be a γ_r-set of G_2. The set S_2 can be extended to an RD-set of G by adding to it the set $S_1 \cup \{x_1, y_1\}$, and so $\gamma_r(G) \le |S_1| + 2 + |S_2| \le \gamma_r(G_1) + \gamma_r(G_2) + 1 = \gamma_r(G') + 1$. In both cases, $\gamma_r(G) \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 16$, a contradiction. □

By Claim 29.2, the graph $G[\{v_1, v_2, v_3\}]$ is isolate-free. Renaming vertices if necessary, we may assume that v_1v_2 and v_2v_3 are edges. Since G is triangle-free, we note that v_1v_3 is not an edge. Let u_1 be the neighbor of v_1 different from x_1 and v_2, and let u_3 be the neighbor of v_3 different from x_3 and v_2. Suppose that $u_1 \ne u_3$. Hence, the graph illustrated in Figure 12A is a subgraph of G. In this case, let $Q = V(H) \cup \{v_1, v_2, v_3\}$ and let $G' = G - Q$. We note that G' is a special subcubic graph and is obtained by deleting the edges of a 2-edge-cut in G. By Claim 25, no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_2, x_2, y_2\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction.

Hence, $u_1 = u_3$ and let us rename this common neighbor of v_1 and v_3 by u. Let w be the third neighbor of u different from v_1 and v_3. Thus, the graph illustrated in Figure 12B is a subgraph of G. In this case, let $Q = V(H) \cup \{v_1, v_2, v_3, u\}$ and let $G' = G - Q$. We note that G' is a connected special subcubic graph and is obtained by deleting the cut-edge uw in G. By Claim 25, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{u, x_2, y_2\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 35$, a contradiction. This completes the proof of Claim 29. □

Claim 30. The graph G contains no domino as a subgraph.

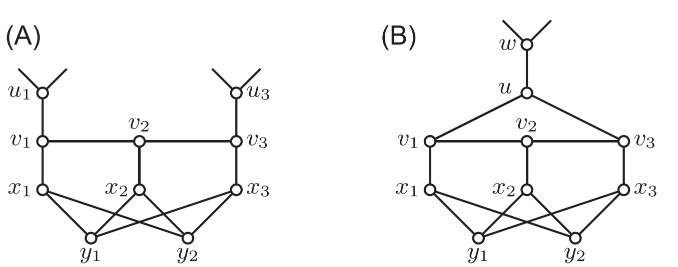

FIGURE 12 Subgraphs in the proof of Claim 29. (A) $u_1 \ne u_3$. (B) $u_1 = u_3 = u$.

Proof. Suppose, to the contrary, that G contains a domino F as a subgraph. Let $V(F) = \{v_1, v_2, \ldots, v_6\}$, where $v_1v_2 \ldots v_6v_1$ is a 6-cycle and v_2v_5 is an edge. Since G is triangle-free and $K_{2,3}$-free, we note that F is an induced subgraph of G. Let x_i be the neighbor of v_i that does not belong to F for $i \in \{1, 3, 4, 6\}$. Since G is triangle-free, $x_1 \neq x_6$ and $x_3 \neq x_4$.

Claim 30.1. $x_1 \neq x_3$ and $x_4 \neq x_6$.

Proof. Suppose, to the contrary, that $x_1 = x_3$ or $x_4 = x_6$. Renaming vertices if necessary, we may assume by symmetry that $x_1 = x_3$. Thus, x_1 is a common neighbor of v_1 and v_3 different from v_2. Let us rename the vertex x_1 by x for notational simplicity.

Suppose firstly that $x_4 = x_6$, and so x_4 is a common neighbor of v_4 and v_6 different from v_5. Let us rename the vertex x_4 by y for notational simplicity. If $xy \in E(G)$, then the graph G is determined and $\gamma_r(G) = 2$ and $\mathrm{w}(G) = 32$, a contradiction. Hence, $xy \notin E(G)$. Let x_1 and y_1 be the neighbors of x and y, respectively, that do not belong to F. Suppose that $x_1 = y_1$, and let us rename this common neighbor of x and y by w. Let z be the third neighbor of w different from x and y. In this case, let G' be the component of $G - wz$ that contains the vertex z. We note that G' is a connected special subcubic graph and the vertex z is the only vertex of degree 2 in G', and so $G' \notin \mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_1, v_4, w\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 35$, a contradiction. Hence, $x_1 \neq y_1$. In this case, we let $G' = G - (V(F) \cup \{x, y\})$. We note that G' is a special subcubic graph that contains exactly two vertices of degree 2. By Claim 25, no component of G' belongs to $\mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_1, v_4\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. However, $\mathrm{w}(G) = w(G') + 30$, a contradiction.

Hence, $x_4 \neq x_6$, that is, v_5 is the only common neighbor of v_4 and v_6. Since G is triangle-free, $x \neq x_4$ and $x \neq x_6$, that is, the vertices x, x_4, x_6 are pairwise distinct. Suppose that x is adjacent to x_4 or x_6. Renaming vertices if necessary, we may assume $xx_6 \in E(G)$. Suppose that $x_4x_6 \in E(G)$. In this case, let y be the neighbor of x_4 different from v_4 and x_6. Hence, the graph illustrated in Figure 13A is a subgraph of G. Let G' be the component of $G - x_4y$ that contains the vertex y. We note that $G' \notin \mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x, x_4, v_5\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 35$, a contradiction. Hence, $x_4x_6 \notin E(G)$. In this case, let w be the neighbor of x_6 different from x and v_6. Hence, the graph illustrated in Figure 13B is a subgraph of G. We now let $G' = G - (V(F) \cup \{x, x_6\})$. The special subcubic graph G' contains exactly two vertices of degree 2, and so by Claim 25 no component of G' belongs to $\mathcal{B}_{\rm rdom}$. Every γ_r-set of G' can

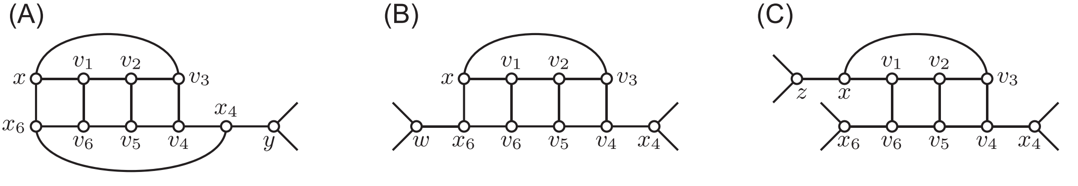

FIGURE 13 Subgraphs in the proof of Claim 30.1. (A) $x_4x_6 \in E(G)$. (B) $x_4x_6 \notin E(G)$. (C) $xx_6 \notin E(G)$.

be extended to an RD-set of G by adding to it the set $\{v_1, v_4, x_6\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction.

Hence, x is adjacent to neither x_4 nor x_6. In this case, let z be the neighbor of x different from v_1 and v_3. Hence, the graph illustrated in Figure 13C is a subgraph of G (where the edge x_4x_6 may or may not exist). We now let $G' = G - (V(F) \cup \{x\})$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_1, v_4\}$, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. Since G' is obtained from G by deleting the edges in a 3-edge-cut, by Claim 25 either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = R_9$. Hence, $\mathrm{w}(G) \ge \mathrm{w}(G') + 22$, a contradiction. □

Claim 30.2. $x_1 \ne x_4$ and $x_3 \ne x_6$.

Proof. Suppose, to the contrary, that $x_1 = x_4$ or $x_3 = x_6$. Renaming vertices if necessary, we may assume by symmetry that $x_1 = x_4$. Thus, x_1 is a common neighbor of v_1 and v_4. Let us rename the vertex x_1 by x for notational simplicity. Suppose firstly that $x_3 = x_6$, and so x_3 is a common neighbor of v_3 and v_6. Let us rename the vertex x_3 by y for notational simplicity. If $xy \in E(G)$, then the graph G is determined and $\gamma_r(G) = 3$ and $\mathrm{w}(G) = 32$, a contradiction. Hence, $xy \notin E(G)$. Let x_1 and y_1 be the neighbors of x and y, respectively, that do not belong to F. Suppose that $x_1 = y_1$, and let us rename this common neighbor of x and y by w. Let z be the third neighbor of w different from x and y. In this case, let G' be the component of $G - wz$ that contains the vertex z, and so G' is a connected special subcubic graph. Further, the vertex z is the only vertex of degree 2 in G', and so $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_1, v_4, y\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 35$, a contradiction. Hence, $x_1 \ne y_1$. We now let $G' = G - (V(F) \cup \{x, y\})$, and so G' is a special subcubic graph that contains exactly two vertices of degree 2. By Claim 25, no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x, y, v_2\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction.

Hence, $x_3 \ne x_6$, that is, the vertices x, x_3, x_6 are pairwise distinct. Suppose that x is adjacent to neither x_3 nor x_6. Let x' be the neighbor of x different from v_1 and v_4. In this case, we let $G' = G - (V(F) \cup \{x\})$. Let S' be a γ_r-set of G'. If $x' \in S'$, let $S = S' \cup \{v_3, v_6\}$. If $x' \notin S'$, let $S = S' \cup \{v_1, v_4\}$. In both cases, the set S is an RD-set of G, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. Since G' is obtained from G by deleting the edges in a 3-edge-cut, by Claim 25 either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = R_9$. Hence, $\mathrm{w}(G) \ge \mathrm{w}(G') + 22$, a contradiction. Thus, either $xx_3 \in E(G)$ or $xx_6 \in E(G)$.

Suppose that $xx_3 \in E(G)$. If $x_3x_6 \in E(G)$, then let y be the neighbor of x_6 different from x_3 and v_6, and let G' be the component of $G - x_6y$ that contains the vertex y. Thus, $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_6, v_1, v_4\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 35$, a contradiction. Hence, $x_3x_6 \notin E(G)$. In this case, we let $G' = G - (V(F) \cup \{x, x_3\})$. The special subcubic graph G' contains exactly two vertices of degree 2, and so by Claim 25 no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_3, v_6, x_3\}$, and so

$\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction.

Hence, $xx_3 \notin E(G)$, implying that $xx_6 \in E(G)$. If $x_3x_6 \in E(G)$, then let y be the neighbor of x_3 different from v_3 and x_6, and let G' be the component of $G - x_3y$ that contains the vertex y. We note that $G' \notin \mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_3, v_1, v_4\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = w(G') + 35$, a contradiction. Hence, $x_3x_6 \notin E(G)$. In this case, we let $G' = G - (V(F) \cup \{x, x_6\})$. The special subcubic graph G' contains exactly two vertices of degree 2, and so by Claim 25 no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_3, v_6, x_6\}$, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction. □

By Claim 30.1, $x_1 \ne x_3$ and $x_4 \ne x_6$. By Claim 30.2, $x_1 \ne x_4$ and $x_3 \ne x_6$. Thus the vertices x_1, x_3, x_4, x_6 are pairwise distinct. Let $G' = G - V(F)$. The graph G' is a special subcubic graph with exactly four vertices of degree 2. Every γ_r-set of G' can be extended to a RD-set of G by adding to it the set $\{v_1, v_4\}$, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $w(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\mathrm{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. Necessarily, G_1 contains at least three vertices from the set $\{x_1, x_3, x_4, x_6\}$. In particular, $\{x_1, x_4\} \subset V(G_1)$ or $\{x_3, x_6\} \subset V(G_1)$. Renaming vertices if necessary, we may assume by symmetry that $\{x_3, x_6\} \subset V(G_1)$. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the set $X_1 = \{x_3, x_6\}$. We note that $X_1 \cap S_1 = \emptyset$. By Observation 1(f), $|S_1| = \gamma_{r,\mathrm{dom}}(G_1; X_1) \le \gamma_r(G_1) - 1$. If G' is connected, then $G' = G_1$ and the set $S_1 \cup \{v_1, v_4\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_1| + 2 \le (\gamma_r(G') - 1) + 2 = \gamma_r(G') + 1$. If G' is disconnected, then let G_2 be the component of G' different from G_1 which yields $G_1 = R_9$. In this case, let S_2 be a γ_r-set of G_2 and note that the set $S_1 \cup S_2 \cup \{v_1, v_4\}$ is a RD-set of G, implying that $\gamma_r(G) \le |S_1| + |S_2| + 2 \le (\gamma_r(G_1) - 1) + \gamma_r(G_2) + 2 = \gamma_r(G') + 1$. Thus in both cases, $\gamma_r(G) \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 17$, a contradiction. This completes the proof of Claim 30. □

By Claim 30, the graph G contains no domino as a subgraph.

Claim 31. If the graph G contains a 4-cycle C, then the subgraph of G induced by $V(C)$ and all neighbors in G of vertices in $V(C)$ is isomorphic to the corona $C \circ K_1$ of the 4-cycle C.

Proof. Suppose that G contains a 4-cycle $C : v_1v_2v_3v_4v_1v_4$. Since G is triangle-free, the cycle C is an induced cycle. Let x_i be the neighbor of v_i that does not belong to C for $i \in [4]$. Since G has no triangle and no $K_{2,3}$-subgraph, the vertices x_1, x_2, x_3, x_4 are pairwise distinct. Let $X = \{x_1, x_2, x_3, x_4\}$. To prove the claim, it suffices to show that the set X is independent. Suppose, to the contrary, that X is not an independent set. Since G contains no domino as a subgraph, $x_ix_{i+1} \notin E(G)$ for all $i \in [4]$, where indices are taken modulo 4. Hence, $x_ix_{i+2} \in E(G)$ for some $i \in [4]$, where indices are taken modulo 4. Renaming vertices if necessary, we may assume that $x_1x_3 \in E(G)$. Let y_1 be the third neighbor of x_1 different from v_1 and x_3, and let y_3 be the third neighbor of x_3 different from v_3 and x_1. Since G is triangle-free, $y_1 \ne y_3$. Further, since G contains no domino as a subgraph,

$\{y_1, y_3\} \cap \{x_2, x_4\} = \emptyset$. Thus, the vertices x_2, x_4, y_1, y_3 are pairwise distinct and the graph illustrated in Figure 14 is a subgraph of G. Let $G' = G - (V(C) \cup \{x_1, x_3\})$.

Let S' be a γ_r-set of G'. If $y_1 \in S'$, let $S = S' \cup \{v_3, v_4\}$. If $y_1 \notin S'$, let $S = S' \cup \{v_1, x_3\}$. In both cases, the set S is an RD-set of G, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If no component of G' belongs to $\mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = 24 + (\mathrm{w}(G') - 4) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\rm rdom}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. Necessarily, G_1 contains at least three vertices from the set $\{x_2, x_4, y_1, y_3\}$. At least one of y_1 and y_3 belong to G_1. By symmetry, we may assume that $y_1 \in V(G_1)$. Further, at least one of x_2 and x_4 belongs to G_1. By symmetry, we may assume that $x_2 \in V(G_1)$.

Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the set $X_1 = \{y_1, x_2\} \subset V(G_1)$. We note that $X_1 \cap S_1 = \emptyset$. By Observation 1(f), $|S_1| = \gamma_{r,\rm dom}(G_1; X_1) \le \gamma_r(G_1) - 1$. If G' is connected, then $G' = G_1$ and the set $S_1 \cup \{v_1, x_3\}$ is a RD-set of G, and so $\gamma_r(G) \le |S_1| + 2 \le (\gamma_r(G') - 1) + 2 = \gamma_r(G') + 1$. If G' is disconnected, then let G_2 be the component of G' different from G_1. In this case, let S_2 be a γ_r-set of G_2. The set $S_1 \cup S_2 \cup \{v_1, x_3\}$ is a RD-set of G, implying that $\gamma_r(G) \le |S_1| + |S_2| + 2 \le (\gamma_r(G_1) - 1) + \gamma_r(G_2) + 2 = \gamma_r(G') + 1$. Thus in both cases, $\gamma_r(G) \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 16$, a contradiction. Hence, the set X is an independent set. □

Claim 32. If G contains a 4-cycle $C : v_1v_2v_3v_4v_1v_4$ where x_i is the neighbor of v_i that does not belong to C for $i \in [4]$, then $|N(x_i) \cap N(x_{i+2})| \le 1$ for $i \in [2]$.

Proof. Let the cycle C and the vertices x_1, x_2, x_3, x_4 be as in the statement of the claim. By Claim 31 and our earlier observations, the graph illustrated in Figure 15 is a subgraph of G where $\{x_1, x_2, x_3, x_4\}$ is an independent set. Suppose, to the contrary, that $|N(x_i) \cap N(x_{i+2})| = 2$ for some $i \in [2]$. By symmetry, we may assume that $|N(x_1) \cap N(x_3)| = 2$. Let u and z be the two common neighbors of x_1 and x_3. Since G is triangle-free, the vertices u and z are not adjacent. Let u' and z' be the third neighbors of u and z, respectively, different from x_1 and x_3. Since G has no $K_{2,3}$-subgraph, we note that $u' \neq z'$.

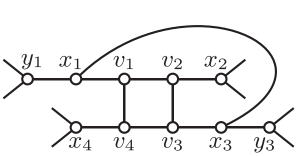

FIGURE 14 A subgraph in the proof of Claim 31.

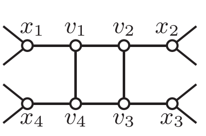

FIGURE 15 A subgraph in the proof of Claim 32.

Claim 32.1. $\{u', z'\} \neq \{x_2, x_4\}$.

Proof. Suppose that $\{u', z'\} = \{x_2, x_4\}$. Renaming vertices if necessary, we may assume by symmetry that $u' = x_2$ and $z' = x_4$. Suppose that x_2 and x_4 have a common neighbor x. Let y be the third neighbor of x. Let G' be the component of $G - xy$ that contains the vertex y, and so G' is a connected special subcubic graph that contains exactly one vertex of degree 2. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_1, x_4, v_2, v_3\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$. Since $G' \notin \mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) = w(G') + 43$, a contradiction. Hence, x_2 and x_4 have no common neighbor. Let y_2 be the neighbor of x_2 different from u and v_2, and let y_4 be the neighbor of x_4 different from z and v_4. By our earlier observations, $y_2 \neq y_4$. If $y_2 y_4 \in E(G)$, then let $Q = V(C) \cup \{x_1, x_2, x_3, x_4\} \cup \{u, z, y_2, y_4\}$ and $G' = G - Q$. Thus, G' is a special subcubic graph with exactly two vertices of degree 2, implying that no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_2, v_3, x_1, y_4\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$. However, $\mathrm{w}(G) = \mathrm{w}(G') + 46$, a contradiction.

Hence, $y_2 y_4 \notin E(G)$. We now let $Q = V(C) \cup \{x_1, x_2, x_3, x_4\} \cup \{u, z, y_2\}$ and $G' = G - Q$. Thus, G' is a special subcubic graph with exactly three vertices of degree 2, implying that either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = R_9$. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{v_3, v_4, x_1, y_2\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 41$, a contradiction. Hence, $G' = G_9$. In this case the set $\{v_3, v_4, x_1, y_2\}$ can be extended to a RD-set of G by adding to it $\gamma_r(G') - 1$ vertices from G' applying Observation 1(d) with respect to the vertex x_4, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. However since $G' = G_9$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 38$, a contradiction. □

Claim 32.2. $\{u', z'\} \cap \{x_2, x_4\} = \varnothing$.

Proof. Suppose that $\{u', z'\} \cap \{x_2, x_4\} \neq \varnothing$, implying by Claim 32.1 that $|\{u', z'\} \cap \{x_2, x_4\}| = 1$. Renaming vertices if necessary, we may assume by symmetry that $u' = x_2$. Thus, $z' \neq x_4$. Let v be the neighbor of u' different from u and v_2. Since the set $\{x_1, x_2, x_3, x_4\}$ is independent, $v \neq x_4$. Since G contains no $K_{2,3}$ as a subgraph, $u' \neq z'$, that is, $v \neq z$. If $v \neq z'$, then we let $Q = V(C) \cup \{x_1, x_2, x_3, u, z\}$ and let $G' = G - Q$. Thus, G' is a special subcubic graph with exactly three vertices of degree 2. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_1, x_2, v_3\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. Since either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = G_9$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 30$, a contradiction. Hence, $v = z'$. If $vx_4 \in E(G)$, then we let w be the neighbor of x_4 different from v and v_4. Further we let $Q = V(C) \cup \{x_1, x_2, x_3, x_4, u, v, z\}$ and $G' = G - Q$. Thus, G' is a connected special subcubic graph with exactly one vertex of degree 2. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_1, v, v_3\}$, and so $\gamma_r(G) \leq \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. Since $G' \notin \mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 43$, a contradiction. Hence, $vx_4 \notin E(G)$. We now let $Q = V(C) \cup \{x_1, x_2, x_3, u, v, z\}$ and $G' = G - Q$. Thus, G' is a special subcubic graph with exactly two vertices of degree 2. Once again, every γ_r-set of G' can be extended to a RD-set of G by adding to it the set $\{x_1, v, v_3\}$, and so

$\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$. Since no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 38$, a contradiction. □

By Claim 32.2, $\{u', z'\} \cap \{x_2, x_4\} = \emptyset$. Let $Q = V(C) \cup \{x_1, x_3, u, z\}$ and let $G' = G - Q$. Thus, G' is a special subcubic graph with exactly four vertices of degree 2. Every γ_r-set of G' can be extended to an RD-set of G by adding to it the set $\{x_1, v_3\}$, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. Since at most one component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, we have $\mathrm{w}(G) = \mathrm{w}(G') + 24$, a contradiction. This completes the proof of Claim 32. □

Claim 33. If G contains a 4-cycle $C : v_1v_2v_3v_4v_1v_4$ where x_i is the neighbor of v_i that does not belong to C for $i \in [4]$, then $N(x_i) \cap N(x_{i+2}) = \emptyset$ for $i \in [2]$.

Proof. Let the cycle C and the vertices x_1, x_2, x_3, x_4 be as in the statement of the claim. Thus the graph illustrated in Figure 15 is a subgraph of G where $\{x_1, x_2, x_3, x_4\}$ is an independent set. Suppose, to the contrary, that $|N(x_i) \cap N(x_{i+2})| \ge 1$ for some $i \in [2]$. By symmetry, we may assume that $|N(x_1) \cap N(x_3)| \ge 1$. By Claim 32, $|N(x_1) \cap N(x_3)| = 1$. Let z be the common neighbor of x_1 and x_3, and let z' be the third neighbor of z.

Claim 33.1. The vertex z is adjacent to neither x_2 nor x_4.

Proof. Suppose, to the contrary, that the vertex z is adjacent to x_2 or x_4, that is, $z' = x_2$ or $z' = x_4$. By symmetry, we may assume that $z' = x_4$. Let $Q = V(C) \cup \{x_1, x_2, x_3, x_4, z\}$. Let y_i be the neighbor of x_i not in Q for $i \in \{1, 3, 4\}$. Since the vertex z is the only common neighbor of x_1 and x_3, we note that $y_1 \ne y_3$.

Claim 33.1.1. The vertices y_1, y_3, y_4 are pairwise distinct.

Proof. Suppose that the vertices y_1, y_3, y_4 are not pairwise distinct. By symmetry, we may assume that $y_1 = y_4$. Suppose firstly that $y_1 = y_3 = y_4$. In this case, let $Q' = V(C) \cup \{x_1, x_3, x_4, y_1, z\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph with exactly small vertex, and so $G' \notin \mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 35$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. Hence, the vertices y_1 and y_3 are distinct.

Let z_1 be the neighbor of y_1 different from x_1 and x_4. Suppose that x_2, y_3, z_1 are pairwise distinct. In this case, we let $Q' = V(C) \cup \{x_1, x_3, x_4, y_1, z\}$ and $G' = G - Q'$. Thus, G' is a special subcubic graph with three small vertices, and so by Claim 25 either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = R_9$. Hence, $\mathrm{w}(G) \ge \mathrm{w}(G') + 30$. Moreover, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. Hence, x_2, y_3, z_1 are not pairwise distinct vertices. Since x_2 and x_3 are not adjacent, $x_2 \ne y_3$. Hence either $z_1 = x_2$ or $z_1 = y_3$.

Suppose that $z_1 = x_2$. In this case, let y_2 be the neighbor of x_2 different from v_2 and y_1. If $y_2 \ne y_3$, then let $Q' = V(C) \cup \{x_1, x_2, x_3, x_4, y_1, z\}$ and $G' = G - Q'$. Thus, G' is a special subcubic graph with two small vertices, and so no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, whence $\mathrm{w}(G) = \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. If $y_2 = y_3$, then in this case let $Q' = V(C) \cup \{x_1, x_2, x_3, x_4, y_1, y_2, z\}$ and $G' = G - Q'$. Thus, G' is a connected special subcubic graph with one small vertex, and so $G' \notin \mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 43$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction.

Hence, $z_1 = y_3$. Suppose that $x_2 y_3 \in E(G)$. In this case, we let $Q' = V(C) \cup \{x_1, x_2, x_3, x_4, y_1, y_3, z\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph with one small vertex, and so $G' \notin \mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 43$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. Hence, $x_2 y_3 \notin E(G)$. In this case, we let $Q' = V(C) \cup \{x_1, x_3, x_4, y_1, y_3, z\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph with two small vertices, and so no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, yielding $\mathrm{w}(G) = \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, a contradiction. □

Claim 33.1.2. The vertex x_2 is adjacent to at most one of y_1 and y_4.

Proof. Suppose that x_2 is adjacent to both y_1 and y_4. In this case, we let $Q' = V(C) \cup \{x_1, x_2, x_3, x_4, y_1, y_4, z\}$ and let $G' = G - Q'$. Suppose that G' is a special subcubic graph, and so G' contains exactly three small vertices. By Claim 25 either no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ or G' is connected and $G' = R_9$. If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 41$, while if $G' = R_9$, then $\mathrm{w}(G) = \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. Hence, G' is not a special subcubic graph.

Let z_1 and z_4 be the neighbors of y_1 and y_4, respectively, in G that do not belong to Q. Since G' is not a special subcubic graph, the vertices y_3, z_1, z_4 are not pairwise distinct. If y_3 is adjacent to both y_1 and y_4, then the graph G is determined and $\gamma_r(G) = 3$ and $\mathrm{w}(G) = 48$, a contradiction. If y_3 is adjacent to exactly one of y_1 and y_4, then by symmetry we may assume that $y_3 y_4$ is an edge. In this case, we let $Q' = V(C) \cup \{x_1, x_2, x_3, x_4, y_1, y_3, y_4, z\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph with two small vertices, and so no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. Hence, y_3 is adjacent to neither y_1 nor y_4, that is, $y_3 \neq z_1$ and $y_3 \neq z_4$, implying that $z_1 = z_4$.

If $z_1 y_3 \in E(G)$, then we let $Q' = Q \cup \{y_1, y_3, y_4, z_1\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph with one small vertex, and so $G' \notin \mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 51$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. If $z_1 y_3 \notin E(G)$, then let z' be the neighbor of z_1 different from y_1 and y_4, and in this case let $Q' = Q \cup \{y_1, y_4, z_1\}$ and $G' = G - Q'$. Thus, G' is a special subcubic graph with two small vertices, and so no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. □

By Claim 33.1.2, the vertex x_2 is adjacent to at most one of y_1 and y_4. By symmetry, we may assume that $x_2 y_1 \notin E(G)$. Let $Q' = Q \setminus \{x_2\}$ and let G' be obtained from $G - Q'$ by adding the edge $x_2 y_1$. Thus, G' is a subcubic graph with exactly two small vertices, namely y_3 and y_4, and so no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 30$. Let S' be a γ_r-set of G'. If $x_2 \in S'$, let $S = S' \cup \{v_3, v_4, x_1\}$. If $x_2 \notin S'$ and $y_1 \in S'$, let $S = S' \cup \{v_2, v_3, x_4\}$. If $x_2 \notin S'$ and $y_1 \notin S'$, let $S = S' \cup \{v_1, x_3, x_4\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. This completes the proof of Claim 33. □

Claim 34. The graph G has no 4-cycle.

Proof. Suppose, to the contrary, that G contains a 4-cycle $C: v_1v_2v_3v_4v_1v_4$. Let x_i be the neighbor of v_i that does not belong to C for $i \in [4]$. Thus the graph illustrated in Figure 15 is a subgraph of G where $\{x_1, x_2, x_3, x_4\}$ is an independent set. By Claim 33, $N(x_i) \cap N(x_{i+2}) = \emptyset$ for $i \in [2]$. Thus, x_1 and x_3 have no common neighbor, and x_2 and x_4 have no common neighbor. Let y_i and z_i be the two neighbors of x_i different from v_i for $i \in \{1, 3\}$. By our earlier observations, the vertices $x_2, x_4, y_1, y_3, z_1, z_3$ are pairwise distinct.

If x_2 is adjacent to both y_3 and z_3, then $C': x_2y_3x_3z_3x_2$ is a 4-cycle. However, in this case the neighbors v_2 and v_3 of the vertices x_2 and x_3, respectively, that do not belong to the cycle C' are adjacent, contradicting Claim 31. Hence, the vertex x_2 is not adjacent to at least one of y_3 and z_3. Renaming vertices, if necessary, we may assume that x_2 is not adjacent to y_3. Let $Q = V(C) \cup \{x_1, x_3\}$ and let G' be obtained from $G - Q$ by adding the edge x_2y_3. The resulting graph G' is a special subcubic that contains exactly four small vertices, namely x_4, y_1, z_1, z_3. Thus at most one component of G' belongs to $\mathcal{B}_{\rm rdom}$.

Let S' be a γ_r-set of G', and let S be the set defined as follows. If $y_3 \in S'$ and $y_1 \notin S'$, let $S = S' \cup \{v_1, v_2\}$. If $y_3 \in S'$ and $y_1 \in S'$, let $S = S' \cup \{v_2, v_4\}$. If $y_3 \notin S', x_2 \in S'$ and $y_1 \notin S'$, let $S = S' \cup \{v_1, x_3\}$. If $y_3 \notin S', x_2 \in S'$ and $y_1 \in S'$, let $S = S' \cup \{v_4, x_3\}$. If $x_2 \notin S'$ and $y_3 \notin S'$, let $S = S' \cup \{x_1, v_3\}$. The resulting set S is an RD-set of G, and so $\gamma_r(G) \le \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If G' has no component in $\mathcal{B}_{\rm rdom}$, then $\mathrm{w}(G) = 24 + (w(G') - 4) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\rm rdom}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$.

Suppose that the added edge x_2y_3 belongs to G_1. In this case, G_1 contains two adjacent vertices of degree 3, and so $G_1 \in \{R_4, R_5, R_9\}$. Let G_1^* be the graph obtained from G_1 by subdividing the edge x_2y_3 three times resulting in the path $x_2v_2v_3x_3y_3$. Let S_1^* be a minimum type-2 NeRD-set of G_1^* with respect to the vertex v_3. By Observation 4(b), $|S_1^*| = \gamma_{r,\rm dom}(G_1^*; v_3) \le \gamma_r(G_1)$. By our earlier observations, at least one of y_1 and z_1 belong to the graph G_1. Renaming vertices if necessary, we may assume that $y_1 \in V(G_1)$. If $y_1 \in S_1^*$, then let $S = S_1^* \cup \{v_3\}$. If $y_1 \notin S_1^*$, then let $S = S_1^* \cup \{v_1\}$. If $G' = G_1$, then the set S is a RD-set of G, and so $\gamma_r(G) \le |S_1^*| + 1 \le \gamma_r(G_1) + 1 = \gamma_r(G') + 1$. If $G' \ne G_1$, then $G_1 = R_9$. In this case, G_1 contains three vertices of degree 2 in G', and so G' is disconnected and contains a second component G_2. Since G_2 contains exactly one small vertex, the component G_2 does not belong to $\mathcal{B}_{\rm rdom}$. Every γ_r-set of G_2 can be extended to an RD-set of G by adding to it the set S, and so in this case $\gamma_r(G) \le |S| + |S_2| = |S_1^*| + 1 + |S_2| \le (\gamma_r(G_1) + 1) + \gamma_r(G_2) = \gamma_r(G') + 1$. In both cases, $\gamma_r(G) \le \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 16$, a contradiction.

Hence, the added edge x_2y_3 does not belong to G_1, implying that G' is disconnected. Let $G_2 = G' - V(G_1)$. We note that G_2 contains the added edge x_2y_3 and contains at most two components and contains at most one small vertex. Thus, no component of G_2 belongs to $\mathcal{B}_{\rm rdom}$. Let S_2 be a γ_r-set of G_2. We note that $\gamma_r(G') = \gamma_r(G_1) + \gamma_r(G_2)$. Recall that G_1 contains at least three vertices from the set $\{x_4, y_1, z_1, z_3\}$.

Suppose that $y_3 \in S_2$ or $x_2 \in S_2$. In this case, we let $X_1 \subset V(G_1)$ such that $|X_1| = 2$ and $X_1 \subset \{x_4, y_1, z_1\}$ noting that at least two vertices in $\{x_4, y_1, z_1\}$ belong to the component G_1. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the vertex X_1. By Observation

1(f), $|S_1| = \gamma_{r,\text{dom}}(G_1; X_1) \le \gamma_r(G_1) - 1$. If $y_3 \in S_2$, let $S^* = S_1 \cup S_2 \cup \{v_1, v_2\}$. If $y_3 \notin S_2$ and $x_2 \in S_2$, let $S^* = S_1 \cup S_2 \cup \{v_1, x_3\}$. Suppose that $y_3 \notin S_2$ and $x_2 \notin S_2$. At least one of y_1 and z_1 belong to the graph G_1. Renaming vertices if necessary, we may assume that $y_1 \in V(G_1)$. In this case, we let S_1 be a minimum type-1 NeRD-set of G_1 with respect to the vertex y_1. By Observation 1(d), $|S_1| = \gamma_{r,\text{ndom}}(G_1; y_1) \le \gamma_r(G_1) - 1$. Let $S^* = S_1 \cup S_2 \cup \{x_1, v_3\}$. In all cases, $|S_1| \le \gamma_r(G_1) - 1$ and the set S^* is a RD-set of G, and so $\gamma_r(G) \le |S_1| + |S_2| + 2 \le (\gamma_r(G_1) - 1) + \gamma_r(G_2) + 2 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 16$, a contradiction. This completes the proof of Claim 34. □

Claim 35. The graph G has no 5-cycle.

Proof. Suppose to the contrary that G contains a 5-cycle $C : v_1v_2v_3v_4v_5v_1$. Let x_i be the neighbor of v_i that does not belong to C for $i \in [5]$. Let $X = \{x_1, \ldots, x_5\}$ and let $Q = V(C) \cup X$.

Claim 35.1. Each vertex $x \in X$ has at least one neighbor in X.

Proof. Suppose, to the contrary, that there is a vertex in X with no neighbor in X. Renaming vertices if necessary that x_1 has no neighbor in X. Let y_1 and z_1 be the two neighbors of x_1 different from v_1. If x_2 is adjacent to both y_1 and z_1, then $x_1y_1x_2z_1x_1$ is a 4-cycle in G, a contradiction. Hence we may assume that $x_2z_1 \notin E(G)$. Let $Q' = V(C) \cup \{x_1\}$ and let G' be obtained from $G - Q'$ by adding the edge $e = x_2z_1$. Thus, G' is a special subcubic graph that contains exactly four small vertices. By Claim 25, at most one component of G' belongs to $\mathcal{B}_{\text{rdom}}$. Let S' be a γ_r-set of G'. If $x_2 \in S'$, let $S = S' \cup \{x_1, v_4\}$. If $x_2 \notin S'$ and $z_1 \in S'$, let $S = S' \cup \{v_2, v_5\}$. If $x_2 \notin S'$ and $z_1 \notin S'$, let $S = S' \cup \{v_1, v_3\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \le |S| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$. If no component of G' belongs to $\mathcal{B}_{\text{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\text{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. If $G_1 = R_2$, then since G contains no 4-cycle the added edge e belongs to G_1, implying that G_1 contains two adjacent vertices of degree 3, a contradiction. Hence, $G_1 \in \{R_4, R_5, R_9\}$.

Suppose that $e \in E(G_1)$. If $G_1 \in \{R_4, R_5\}$, then the graph G is determined (in the sense that $V(G) = V(C) \cup V(G_1)$) and $\gamma_r(G) \le 4$ and $\mathrm{w}(G) = 56$, a contradiction. Hence, $G_1 = R_9$. In this case, G' is disconnected and contains two components. Let G_2 be the second component of G', and so G_2 contains exactly one small vertex and $G_2 \notin \mathcal{B}_{\text{rdom}}$. Let G^* be the subgraph of G or order 17 induced by $V(C) \cup V(G_1)$. Every γ_r-set of G^* can be extended to an RD-set of G by adding to it a γ_r-set of G_2, and so $\gamma_r(G) \le \gamma_r(G_2) + \gamma_r(G^*) \le \gamma_r(G_2) + 6$, implying that $\mathrm{w}(G) < \mathrm{w}(G_2) + 60$. However, $\mathrm{w}(G) = 17 \times 4 + (\mathrm{w}(G_2) - 1) = \mathrm{w}(G_2) + 67$, a contradiction. Hence, $e \notin E(G_1)$. Let $G_2 = G' - V(G_1)$, and so $e \in E(G_2)$ and $\gamma_r(G') = \gamma_r(G_1) + \gamma_r(G_2)$. Since G_1 contains at least three small vertices, the graph G_2 contains at most one small vertex. Further, G_2 has at most two components, and so no component of G_2 belongs to $\mathcal{B}_{\text{rdom}}$. Let S_2 be a γ_r-set of G_2. We now define an RD-set S in G as follows.

Suppose that $x_2 \in S_2$. We note that at least one of x_4 and y_1 belongs to G_1. Let $v \in \{x_4, y_1\} \cap V(G_1)$. Let S_1 be a minimum type-1 NeRD-set of G_1 with respect to the vertex v. By Observation 1(d), $|S_1| = \gamma_{r,\text{ndom}}(G_1; v) \le \gamma_r(G_1) - 1$. Let $S = S_1 \cup S_2 \cup \{x_1, v_4\}$.

Suppose that $x_2 \notin S_2$ and $z_1 \in S_2$. We note that at least one of x_3 and x_4 belongs to G_1. Let $v \in \{x_3, x_4\} \cap V(G_1)$. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the vertex v. By Observation 1(e), $|S_1| = \gamma_{r,\text{dom}}(G_1; v) \le \gamma_r(G_1) - 1$. Let $S = S_1 \cup S_2 \cup \{v_2, v_5\}$.

Suppose that $x_2 \notin S_2$ and $z_1 \notin S_2$. We note that at least one of x_4 and x_5 belongs to G_1. Let $v \in \{x_4, x_5\} \cap V(G_1)$. Let S_1 be a minimum type-2 NeRD-set of G_1 with respect to the vertex v. By Observation 1(e), $|S_1| = \gamma_{r,\text{dom}}(G_1; v) \le \gamma_r(G_1) - 1$. Let $S = S_1 \cup S_2 \cup \{v_1, v_3\}$.

In all cases, $|S_1| \le \gamma_r(G_1) - 1$ and the set S is an RD-set of G. Therefore, $\gamma_r(G) \le |S_1| + |S_2| + 2 \le (\gamma_r(G_1) - 1) + \gamma_r(G_2) + 2 = \gamma_r(G') + 1$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 10$. However, $\mathrm{w}(G) \ge \mathrm{w}(G') + 16$, a contradiction. □

By Claim 35.1, each vertex $x \in X$ has at least one neighbor in X. Hence, $G[X]$ contains at least three edges. Since G has no 4-cycles, we infer that $G[X]$ contains at most five edges. Using symmetry, we may assume without loss of generality that $\{x_1x_4, x_2x_4, x_3x_5\} \subset E(G)$ noting that G contains no 4-cycles.

Claim 35.2. $G[X]$ contains at least four edges.

Proof. Suppose, to the contrary, that $G[X]$ contains exactly three edges. Let y_i be neighbor of x_i not in Q for $i \in \{1, 2, 3, 5\}$. Since G has no 4-cycles, we note that $y_1 \ne y_2$, and since G has no triangles, we note that $y_3 \ne y_5$. We show firstly that $y_2 \ne y_3$. Suppose, to the contrary, that $y_2 = y_3$, and let z be the third neighbor of y_2.

Suppose that $y_1 = y_5$. Let z' be the neighbor of y_1 different from x_1 and x_5. Suppose that $z = z'$. In this case, let $Q' = Q \cup \{y_1, y_2, z\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph that contains exactly one small vertex, and so $G' \notin \mathcal{B}_{\text{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 51$. However, $\gamma_r(G) \le \gamma_r(G') + 5$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 50$, a contradiction. Hence, $z \ne z'$. In this case, let $Q' = Q \cup \{y_1, y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains exactly two small vertices, and so no component of G' belongs to $\mathcal{B}_{\text{rdom}}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, a contradiction.

Hence, $y_1 \ne y_5$. Since G contain no 4-cycle, $y_2y_5 \notin E(G)$. Suppose that $y_1y_2 \notin E(G)$, and so z is distinct from both y_1 and y_5. In this case, let $Q' = Q \cup \{y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains exactly three small vertices, and so either no component of G' belongs to $\mathcal{B}_{\text{rdom}}$ or G' is connected and $G' = R_9$. Therefore, $\mathrm{w}(G) \ge \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction.

Hence, $y_1y_2 \in E(G)$. If, in addition, $y_1y_5 \in E(G)$, then let $Q' = Q \cup \{y_1, y_2, y_5\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph that contains exactly one small vertex and so $G' \notin \mathcal{B}_{\text{rdom}}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 51$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. On the other hand, if $y_1y_2 \in E(G)$ and $y_1y_5 \notin E(G)$, then let $Q' = Q \cup \{y_1, y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains two small vertices and so no component of G' is in $\mathcal{B}_{\text{rdom}}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. Hence, $y_2 \ne y_3$.

We show next that $y_2 \ne y_5$. Suppose, to the contrary, that $y_2 = y_5$, and let z be the third neighbor of y_2. Suppose that $y_1 = y_3$. Let z' be the neighbor of y_1 different from x_1 and x_3. Suppose that $z = z'$. In this case, let $Q' = Q \cup \{y_1, y_2, z\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph that contains exactly one small vertex, and so

$G' \notin \mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 51$. However, $\gamma_r(G) \le \gamma_r(G') + 5$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 50$, a contradiction. Hence, $z \ne z'$. In this case, let $Q' = Q \cup \{y_1, y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains exactly two small vertices, and so no component of G' belongs to $\mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, a contradiction.

Hence, $y_1 \ne y_3$. Since G contains no 4-cycle, vertex y_2 is not adjacent to y_3, that is, $z \ne y_3$. Suppose that $z \ne y_1$. In this case, let $Q' = Q \cup \{y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains exactly three small vertices, and either no component of G' belongs to $\mathcal{B}_{\rm rdom}$ or G' is connected and $G' = R_9$. Thus, $\mathrm{w}(G) = \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, a contradiction.

Hence, $z = y_1$, that is, $y_1y_2 \in E(G)$. If, in addition, $y_1y_3 \in E(G)$, then let $Q' = Q \cup \{y_1, y_2, y_3\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph that contains exactly one small vertex and so $G' \notin \mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 51$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. On the other hand, if $y_1y_2 \in E(G)$ and $y_1y_3 \notin E(G)$, then let $Q' = Q \cup \{y_1, y_2\}$ and let $G' = G - Q'$. Thus, G' is a special subcubic graph that contains two small vertices and so no component of G' is in $\mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 46$. However, $\gamma_r(G) \le \gamma_r(G') + 4$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 40$, a contradiction. Hence, $y_2 \ne y_5$.

By our earlier observations, the vertices y_1, y_2, y_3, and y_5 are pairwise distinct. Let $G' = G - Q$. Thus, G' is a special subcubic graph that contains exactly four small vertices. At most one component of G' belongs to $\mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 32$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. This completes the proof of Claim 35.2. □

By Claim 35.2, the graph $G[X]$ contains at least four edges. Hence at least one of x_1x_3 and x_2x_5 is an edge. By symmetry, we may assume that $x_1x_3 \in E(G)$. If $x_2x_5 \in E(G)$, then the graph G is determined and is isomorphic to the Petersen graph shown in Figure 1. In this case, $\gamma_r(G) = 4$ and $\mathrm{w}(G) = 40$, a contradiction. Hence, $x_2x_5 \notin E(G)$. Let y_i be the neighbor of x_i not in Q for $i \in \{2, 5\}$. Suppose that $y_2 = y_5$. In this case, let $Q' = Q \cup \{y_2\}$ and let $G' = G - Q'$. Thus, G' is a connected special subcubic graph that contains exactly one small vertex, and so $G' \notin \mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) = \mathrm{w}(G') + 41$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. Hence, $y_2 \ne y_5$. We now let $G' = G - Q$. Thus, G' is a special subcubic graph that contains exactly two small vertices, and so no component of G' belongs to $\mathcal{B}_{\rm rdom}$ and $\mathrm{w}(G) \ge \mathrm{w}(G') + 38$. However, $\gamma_r(G) \le \gamma_r(G') + 3$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 30$, a contradiction. This completes the proof of Claim 35. □

Claim 36. The graph G has no 6-cycle.

Proof. Suppose, to the contrary, that G contains a 6-cycle $C: v_1v_2v_3v_4v_5v_6v_1$. Thus, G has girth equal to 6. In particular, C is an induced cycle in G. Let x_i be the neighbor of v_i that does not belong to C for $i \in [6]$. The girth condition implies that $x_i \ne x_j$ for $1 \le i < j \le 6$. Let $X = \{x_1, \ldots, x_6\}$. The girth condition implies that the only possible edges in $G[X]$ are the edges x_1x_4, x_2x_5 and x_3x_6. Let G' be the special subcubic graph obtained from $G - V(C)$ by adding the edge x_1x_2. Thus, G' contains exactly four small vertices, namely x_3, x_4, x_5, x_6. By Claim 25, at most one component of G' belongs to $\mathcal{B}_{\rm rdom}$. Let S' be a γ_r-set of G'. If $x_1 \in S'$, let $S = S' \cup \{v_2, v_5\}$. If $x_1 \notin S'$ and $x_2 \in S'$, let $S = S' \cup \{v_1, v_4\}$. If $x_1 \notin S'$

and $x_2 \notin S'$, let $S = S' \cup \{v_3, v_6\}$. In all cases, S is an RD-set of G, and so $\gamma_r(G) \leq |S'| + 2 = \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$.

If no component of G' belongs to $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\mathrm{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. The set $X_1 = \{x_3, x_4, x_5, x_6\}$ of small vertices in G_1 is either independent or induces a graph that contains exactly one edge, namely the edge x_3x_6. Further, every cycle of length less than 6 in G_1 must contain the added edge x_1x_2 since graph G contains no cycles of length 3, 4 or 5. If G_1 contains the edge x_1x_2, then G_1 contains two adjacent vertices of degree 3. From these properties of the graph G' we infer that $G_1 \notin \{R_2, R_9\}$. Since R_5 contains two pairs of small vertices that are adjacent while the set X_1 contains at most one pair of small vertices that are adjacent, $G_1 \neq R_5$, implying that $G_1 = R_4$ and $X_1 \subset V(G_1)$. The structure of R_4 implies that in this case, every small vertex in R_4 is at distance 2 from two other small vertices. In particular, the vertex x_4 is at distance 2 from at least one of x_3 or x_5 in G'. If x_3 and x_4 are at distance 2 in G' and w denotes their common neighbor in G', then $x_4v_4v_3x_3wx_4$ is a 5-cycle in G. If x_4 and x_5 are at distance 2 in G' and z denotes their common neighbor in G', then $x_4v_4v_5x_5zx_4$ is a 5-cycle in G. In both cases, we contradict the girth at least 6 condition in G. □

By Claim 36, the graph G has no 6-cycle. Let u and v be adjacent vertices in G, and let $N(u) = \{u_1, u_2, v\}$ and $N(v) = \{u, v_1, v_2\}$. Further, let $N(u_i) = \{u, u_{i1}, u_{i2}\}$ and let $N(v_i) = \{v, v_{i1}, v_{i2}\}$ for $i \in [2]$. Thus, G contains the subgraph shown in Figure 16. Let $X = \{u_{11}, u_{12}, u_{21}, u_{22}, v_{11}, v_{12}, v_{21}, v_{22}\}$. Since the graph G has girth at least 7, the set X is an independent set. The subgraph shown in Figure 16 is therefore an induced subgraph of G.

Let $Q = \{u, u_1, u_2, v, v_1, v_2\}$ and let G' be obtained from $G - Q$ be adding the edges $e = u_{12}u_{21}$ and $f = v_{12}v_{21}$. Thus, G' is a special subcubic graph that contains exactly four small vertices, namely the vertices in the set $X' = \{u_{11}, u_{22}, v_{11}, v_{22}\}$. Let S' be a γ_r-set of G', and let $S = S' \cup \{u^*, v^*\}$ where the vertices u^* and v^* are defined as follows. If $u_{12} \in S'$, let $u^* = u_2$. If $u_{12} \notin S'$ and $u_{21} \in S'$, let $u^* = u_1$. If $u_{12} \notin S'$ and $u_{21} \notin S'$, let $u^* = u$. If $v_{12} \in S'$, let $v^* = v_2$. If $v_{12} \notin S'$ and $v_{21} \in S'$, let $v^* = v_1$. If $v_{12} \notin S'$ and $v_{21} \notin S'$, let $v^* = v$. The resulting set S is an RD-set of G, and so $\gamma_r(G) \leq \gamma_r(G') + 2$, implying that $\mathrm{w}(G) < \mathrm{w}(G') + 20$.

If G' has no component in $\mathcal{B}_{\mathrm{rdom}}$, then $\mathrm{w}(G) = \mathrm{w}(G') + 20$, a contradiction. Hence, G' contains a component G_1 that belongs to $\mathcal{B}_{\mathrm{rdom}}$. By Claim 25, there is only one such component and $G_1 \in \{R_2, R_4, R_5, R_9\}$. Necessarily, G_1 contains at least three vertices from the set X'. As observed earlier, the set X is an independent set, and therefore so too is the subset X' of X, implying that $G_1 \notin \{R_2, R_5\}$. Every cycle of length less than 7 in G_1 must contain at least one of the added edges e and f since the graph G has girth at least 7. If G_1 contains the edge e or f, then both ends of the added edge have degree 3 in G_1. From these properties of the graph G', we deduce that if $G_1 = R_4$, then $G' = G_1$. But this would

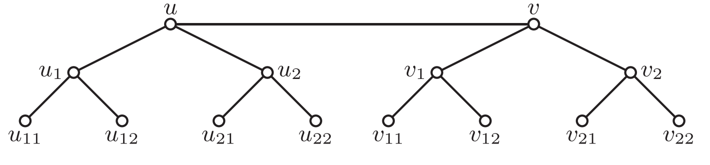

FIGURE 16 A subgraph in the graph G.

imply that $G[X] = C_8$, contradicting our earlier observation that X is an independent set. Hence, $G_1 = R_9$. In this case, both added edges e and f must belong to G_1. However, removing any two edges from R_9 creates a graph which still contains a 5-cycle. This implies that G itself contains a 5-cycle, which is a contradiction. This final contradiction concludes the proof of Theorem 3. □

6 | PROOF OF MAIN RESULT

In this section, we prove our main result, namely Theorem 2. As a consequence of key result, namely Theorem 3, we have the following upper bound on the restrained domination number of a cubic graph.

Theorem 5. *If G is a cubic graph of order n, then $\gamma_r(G) \le \frac{2}{5}n$.*

Proof. Let G be a cubic graph of order n. Thus, $n_2(G) = 0$ and $n_3(G) = n$. Since every graph in the family $\mathcal{B}_{\mathrm{rdom}}$ contains a vertex of degree 2, no component of G belongs to the family $\mathcal{B}_{\mathrm{rdom}}$. The weight of G is therefore $\mathrm{w}(G) = 4n$. Hence by Theorem 3, $10\gamma_r(G) \le \mathrm{w}(G) = 4n$, or, equivalently, $\gamma_r(G) \le \frac{2}{5}n$. □

By Theorem 5, $c_{\mathrm{rdom}} \le \frac{2}{5}$. As observed earlier, the Petersen graph shows that $c_{\mathrm{rdom}} \ge \frac{2}{5}$. Consequently, $c_{\mathrm{rdom}} = \frac{2}{5}$, yielding the result of Theorem 2. We remark that a classical result in domination theory due to Blank [3] and McCuaig and Shepherd [22] states that if G is a connected graph of order $n \ge 8$ with $\delta(G) \ge 2$, then $\gamma(G) \le \frac{2}{5}n$. Hence by Theorem 5 this $\frac{2}{5}$-bound for domination also holds for restrained domination if we replace the minimum degree at least 2 requirement with a 3-regularity condition.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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