European Journal of Combinatorics 120 (2024) 103995

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: https://www.sciencedirect.com/journal/european-journal-of-combinatorics

Mutual-visibility problems on graphs of diameter two

Serafino Cicerone [a,1], Gabriele Di Stefano [a,1], Sandi Klavžar [b,c,d,1], Ismael G. Yero [e,1]

[a] *Department of Information Engineering, Computer Science, and Mathematics, University of L'Aquila, Italy*
[b] *Faculty of Mathematics and Physics, University of Ljubljana, Slovenia*
[c] *Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[d] *Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia*
[e] *Departamento de Matemáticas, Universidad de Cádiz, Algeciras Campus, Spain*

A R T I C L E I N F O

Article history:
Received 4 January 2024
Accepted 30 April 2024
Available online xxxx

A B S T R A C T

The mutual-visibility problem in a graph G asks for the cardinality of a largest set of vertices $S \subseteq V(G)$ so that for any two vertices $x, y \in S$ there is a shortest x, y-path P so that all internal vertices of P are not in S. This is also said as x, y are visible with respect to S, or S-visible for short. Variations of this problem are known, based on the extension of the visibility property of vertices that are in and/or outside S. Such variations are called total, outer and dual mutual-visibility problems. This work is focused on studying the corresponding four visibility parameters in graphs of diameter two, throughout showing bounds and/or closed formulae for these parameters.

The mutual-visibility problem in the Cartesian product of two complete graphs is equivalent to (an instance of) the celebrated Zarankiewicz's problem. Here we study the dual and outer mutual-visibility problem for the Cartesian product of two complete graphs and all the mutual-visibility problems for the direct product of such graphs as well. We also study all the mutual-visibility problems for the line graphs of complete and complete bipartite graphs. As a consequence of this study, we present several relationships between the mentioned problems and some instances of the classical Turán problem. Moreover, we

E-mail addresses: serafino.cicerone@univaq.it (S. Cicerone), gabriele.distefano@univaq.it (G. Di Stefano), sandi.klavzar@fmf.uni-lj.si (S. Klavžar), ismael.gonzalez@uca.es (I.G. Yero).

[1] All authors contributed equally to this work.

https://doi.org/10.1016/j.ejc.2024.103995
0195-6698/© 2024 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

study the visibility problems for cographs and several non-trivial diameter-two graphs of minimum size.
© 2024 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

The mutual-visibility problem in graphs has recently appeared in [15], and has remarkably attracted the attention of several investigations, which can be seen in the series of articles [3,5–12,25,33]. Some reasons of such interest might come from the following facts.

- The problem has some origin in a computer science application related to situations arising in a framework of mobile entities in a network. That is, nodes of a network having some "mutual-visibility" properties can be seen as entities of a network requiring to communicate between themselves in a somehow confidential or private way. Namely, satisfying that for any exchanged information there should be a channel which does not pass through other entities. For some of these applied researches see for instance [1,2,6,14,30].
- A close relationship that exists between the mutual-visibility problem and the general position problem [26,35], which is also a distance related topic of high interest in the last recent years [20–22,24,29,34,37], see also [23,27] for the edge version of the general position problem in graphs.
- The connections that have appeared between the mutual-visibility problem with some classical topics in combinatorics. For instance, while studying the mutual-visibility problem in the Cartesian product of complete graphs, it has been noted that solving such a problem turns out to be equivalent to solve an instance of the well-known Zarankiewicz's problem (see [11]). Relatively similar to this, while considering the lower version of this problem, a closed relationship with a classical Bollobás-Wessel theorem was proved (see [3]). (The lower version of the problem is to find a smallest maximal mutual-visibility set of a graph.) Also, for the case of the total variant of the mutual-visibility, and the same families of graphs, it has been noted that it can be reformulated as a Turán-type problem on hypergraphs (see [5]).
- The standard mutual-visibility problem can be (and sometimes even needs to be) modified in several directions in order to consider different visibility situations. For instance, while studying the mutual-visibility problem in general Cartesian product graphs (see [11]), the notion of independent mutual-visibility was naturally required, thus defined, and their first basic properties identified. In the article [12], a total version of the mutual-visibility problem was needed, in order to study the strong product of graphs. This total notion was also a first step into the work [10], where this total version was further studied, together with two "partially" total ones that were introduced in order to close all the possible "visibility" situations that might exist between the vertices of a graph.

In a formal way, given a connected graph G and a set of vertices $X \subseteq V(G)$, two vertices $x, y \in V(G)$ are called to be X-*visible* if there is a shortest x, y-path (also called geodesic) whose interior vertices do not belong to X. With this idea in mind, for a given set $X \subseteq V(G)$ of a connected graph G, the following definitions are known from [10].

- *Mutual-visibility set*: if any two vertices of X are X-visible.
- *Outer mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x \in X$ and $y \in \overline{X}$ are X-visible.
- *Dual mutual-visibility set*: if any two vertices $x, y \in X$ and any two vertices $x, y \in \overline{X}$ are X-visible.
- *Total mutual-visibility set*: if any two vertices $x, y \in V(G)$ are X-visible.

Regarding such graph structures, the following parameters are defined as the cardinalities of the largest (respectively) mutual-visibility sets from the above ones.

mutual-visibility number	$\mu(G)$	dual mutual-visibility number	$\mu_{\rm d}(G)$
outer mutual-visibility number	$\mu_{\rm o}(G)$	total mutual-visibility number	$\mu_{\rm t}(G)$

If $\tau \in \{\mu, \mu_{\rm d}, \mu_{\rm o}, \mu_{\rm t}\}$, then we say that $X \subseteq V(G)$ in a *τ-set* if $|X| = \tau(G)$.

In the present investigation, we are focused on giving some contributions on these four mutual-visibility parameters on graphs of diameter two. Some motivations for this specific study are coming from already established results on such class of graphs. For instance, as already mentioned, the mutual-visibility problem in the Cartesian product of complete graphs is proved to be equivalent to solve an instance of the well-known Zarankiewicz's problem (see [11]), and such Cartesian products are of diameter two. In this sense, we continue this research direction on graphs of diameter two, which will indeed show that this problem, and the related variations, remain challenging while considering diameter-two graphs in general.

In the remaining of this section we give some preliminary terminologies and notations that shall be used throughout our exposition. In Section 2 we consider the Cartesian and direct products of complete graphs. Specifically, we give formulas for the dual and outer mutual-visibility numbers of the Cartesian product, which fulfills the existing gap for the visibility numbers of such graphs. These results allow us, among other things, to answer negatively a question in the literature regarding the relationship between the mutual-visibility number and the outer mutual-visibility number. We also compute all the mutual-visibility numbers of the direct product of complete graphs, showing that all of them achieve the same value. Section 3 focuses on the line graphs of complete and complete bipartite graphs. Through this study, we give several relationships between the mutual-visibility problems and some instances of the classical Turán problem. Among them, we for instance show that the mutual-visibility number of the line graph of complete graphs equals the number of edges of the Turán graph $T(n, 3)$, and that the total mutual-visibility number of such graphs equals the number of edges of the Turán related graph ${\rm ex}(n; C_4)$. Connections between the mutual-visibility problem on the line graphs of complete bipartite graphs and the Zarankiewicz's problem are also given in Section 3. Next, in Section 4 we consider the class of cographs, by studying those graphs G that have values in their mutual-visibility numbers equal to at least the order of G minus one. Section 5 is focused on non-trivial diameter-two graphs of minimum size. That is, we compute the values of the mutual-visibility parameters of the graphs belonging to this class. Finally, Section 6 gives some concluding remarks together with some future research lines that can be of interest as a continuation of this work.

1.1. Preliminaries

All graphs considered in this paper are finite and simple. The *distance* $d_G(u, v)$ between vertices u and v of a graph G is the length of a shortest u, v-path. The *degree* of a vertex v in G is denoted as $\deg_G(v)$. The *girth*, $g(G)$, of a graph G is the length of a shortest cycle of G. If G is a forest, then we set $g(G) = \infty$. For an integer $k \geq 1$, we shall write $[k] = \{1, \ldots, k\}$. The order and the size of G will be respectively denoted by $n(G)$ and $m(G)$. A vertex of G is *universal* if it is adjacent to all the other vertices of G.

The *Cartesian product* $G \,\Box\, H$ and the *direct product* $G \times H$ of graphs G and H both have the vertex set $V(G) \times V(H)$. In $G \,\Box\, H$, vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. In $G \times H$, vertices (g, h) and (g', h') are adjacent if $gg' \in E(G)$ and $hh' \in E(H)$. In each of the two products, if $h \in V(H)$, then the set of vertices $\{(g, h) : g \in V(G)\}$ forms a *G-layer* which is denoted by G^h. For a given $g \in V(G)$, the *H-layer* ${}^gH$ is defined analogously. Note that in $G \,\Box\, H$, layers induce subgraphs isomorphic to G resp. H, while in $G \times H$ layers induce edgeless graphs.

The *union* $G \cup H$ of G and H is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The *join* $G + H$ of G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

A *cograph* is a graph which contains no induced path on four vertices. Cographs can be characterized in many different ways, see [13]. For instance, cographs are precisely the graphs that can be obtained from K_1 by means of a sequence of disjoint unions and joins of graphs.

2. Products of two complete graphs

In this section we consider the four invariants of interest on Cartesian and direct products of two complete graphs. The invariants μ and μ_t were already considered, here we add formulas for $\mu_{\rm d}$ and $\mu_{\rm o}$. Using these results we are able to answer in negative the question from [10], on whether $\mu(G) \le 2\mu_{\rm o}(G)$ holds for any graph G. We also prove the formula for the four invariants for the direct product of complete graphs.

In [11] it is proved that if $m, n \ge 2$, then $\mu(K_m \,\square\, K_n) = z(m, n; 2, 2)$, where $z(m, n; 2, 2)$ is the maximum number of 1s that an $m \times n$ binary matrix can have, provided that it contains no 2×2 submatrix of 1s. To determine $z(m, n; 2, 2)$ is a notorious open (instance of) Zarankiewicz's problem. When n is sufficiently large, the value $z(n, n; 2, 2)$ can be bounded as follows [4,17]:

$$n^{3/2} - n^{4/3} \le z(n, n; 2, 2) \le \frac{1}{2}n(1 + \sqrt{4n-3}),$$

which demonstrates that the growth is faster than linear.

For the total mutual-visibility number it was proved in [33] that if $n, m \ge 2$, then

$$\mu_{\rm t}(K_n \,\square\, K_m) = \max\{n, m\}.$$

For the dual and the outer mutual-visibility number, we have the following related respective results.

Theorem 2.1. *If $n, m \ge 3$, then*

(i) $\mu_{\rm d}(K_n \,\square\, K_m) = n + m - 1$,
(ii) $\mu_{\rm o}(K_n \,\square\, K_m) = n + m - 2$.

Proof. Let $V(K_k) = [k]$, so that $V(K_n \,\square\, K_m) = \{(i, j) : \ i \in [n], j \in [m]\}$. For the rest of the proof set $G = K_n \,\square\, K_m$.

(i) It is straightforward to check that the set $\{(i, 1) : \ i \in [n]\} \cup \{(1, j) : \ j \in [m]\}$ is a dual mutual-visibility set of cardinality $n + m - 1$, thus $\mu_{\rm d}(G) \ge n + m - 1$.

To prove the reverse inequality, suppose for a purpose of contradiction that there exists a dual mutual-visibility set X of G of cardinality at least $n + m$. Let $x = (i, j)$ be an arbitrary vertex from X. Since the union of the two layers containing x contains $n + m - 1$ vertices, there exists a vertex $x' = (i', j') \in X$, where $i \ne i'$ and $j \ne j'$. By the symmetry of G we may without loss of generality assume that $x' = (i + 1, j + 1)$. (Here and later below, indices are computed modulo n and m in the sense that $n + 1 = 1$.) As X is a dual mutual-visibility set, at least one of the vertices $(i + 1, j)$ or $(i, j + 1)$ must belong to X, for otherwise they are not X-visible. We may without loss of generality assume that $y = (i+1, j) \in X$. Then $(i, j+1) \notin X$, for otherwise x and x' are not X-visible. In Fig. 1(a) the situation so far is schematically presented, where we use the convention that the vertices from X are shown in black, and the vertices not from X in white.

Consider now the vertex $y = (i + 1, j)$. By the above argument used for x, there exists a vertex $z \in X$ which does not lie in the union of the two layers containing y. Assume first that $z = (i, j'')$, where $j'' \ne j$. Then clearly we also have $j'' \ne j + 1$. Now we must have $(i + 1, j'') \notin X$, for otherwise y and z are not X-visible. But then the two vertices $(i + 1, j'')$ and $(i, j + 1)$ do not belong to X and are not X-visible, a contradiction. This situation is shown in Fig. 1(b).

If $z = (i'', j + 1)$, then by symmetry, we also arrive to a contradiction.

Assume now that $z = (i'', j'')$, where $i'' \ne i, i + 1$ and $j'' \ne j, j + 1$. Considering the vertices $z = (i'', j'') \in X$ and $x' = (i + 1, j + 1) \in X$, we see that one of $(i'', j + 1)$ and $(i + 1, j'')$ belongs to X; for otherwise we get a contradiction, since these two vertices would not be X-visible. Suppose that $(i'', j + 1) \in X$, see Fig. 1(c). Consider the vertices $x = (i, j) \in X$, $(i'', j + 1) \in X$, and $(i, j + 1) \notin X$ to realize that $(i'', j) \in X$. But then (i'', j) and x' are not X-visible. This implies that $(i'', j + 1) \notin X$, and consequently $(i + 1, j'') \in X$, see Fig. 1(d). Now $(i, j'') \notin X$, for otherwise (i, j'') and y are not X-visible. Similarly, $(i'', j) \notin X$, otherwise it is not X-visible with $(i + 1, j)$, see Fig. 1(d) again. But now the vertices $(i, j'') \notin X$ and $(i'', j) \notin X$, are not X-visible. This final contradiction implies that $\mu_{\rm d}(G) \le n + m - 1$ which proves (i).

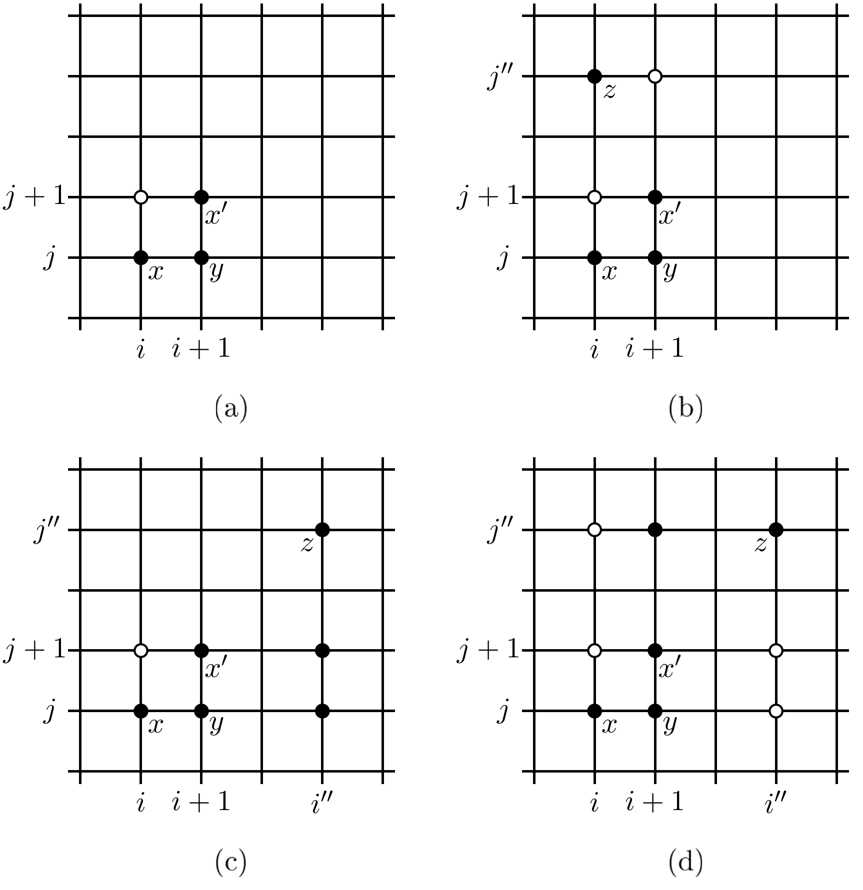

Fig. 1. Cases from the proof of Theorem 2.1(i).

(ii) It is straightforward to verify that the set $\{(i, 1) : \ 2 \le i \le n\} \cup \{(1, j) : \ 2 \le j \le m\}$ is an outer mutual-visibility set of cardinality $n + m - 2$, thus $\mu_o(G) \ge n + m - 2$.

Let X be an arbitrary outer mutual-visibility set of G. We may assume without loss of generality that $(1, 1) \notin X$. Let $x = (i, j)$ be an arbitrary vertex of X. Then in one of the layers $(K_n)^j$ and ${}^i(K_m)$, the vertex x is the only vertex from X. Indeed, if we would have $(i, j') \in X$, $j' \ne j$, and $(i', j) \in X$, $i' \ne i$, then no matter whether (i', j') lies in X or not, the vertices x and (i', j') are not X-visible. We are now going to assign to each vertex $x \in X$ a unique variable as follows. If $x = (i, 1) \in X$, then assign to x the variable a_i and if $x = (1, j) \in X$, then assign to x the variable b_j. In addition, if $i, j \ge 2$ and $x = (i, j) \in X$, then in the case that $X \cap V((K_n)^j) = \{x\}$, we assign to x the variable b_j, and if $X \cap V({}^i(K_m)) = \{x\}$, then we assign to x a variable a_i. Note that if, say, $(i, 1) \in X$ and $(i, j) \in X$, $j \ne 1$, then $(i, 1)$ is assigned a_i and (i, j) is assigned b_j. Since to each vertex of X we assign a different variable and, having in mind that $(1, 1) \notin X$, the variables used are $a_2, \ldots, a_n$ and $b_2, \ldots, b_m$, we have $|X| \le n + m - 2$. We conclude that $\mu_o(G) \le n + m - 2$. $\square$

By Theorem 2.1 and the discussion before it, as soon as n and m are not small, we have

$$\mu_t(K_n \,\square\, K_m) < \mu_o(K_n \,\square\, K_m) < \mu_d(K_n \,\square\, K_m) < \mu(K_n \,\square\, K_m)\,. \tag{1}$$

In [10] a question was posed whether $\mu(G) \le 2\mu_o(G)$ is true in general. We can now answer this question in negative because $\mu_o(K_n \,\square\, K_n) = 2n - 2$ and $\mu(K_n \,\square\, K_n) \ge n^{3/2} - n^{4/3}$.

We now turn our attention to the direct product of complete graphs and prove the following result.

Theorem 2.2. *If $n, m \geq 5$, then $\mu_{\mathrm{t}}(K_n \times K_m) = \mu(K_n \times K_m) = nm - 4$.*

Proof. Let $V(K_k) = [k]$, so that $V(K_n \times K_m) = \{(i, j) : \ i \in [n], j \in [m]\}$. Set $G = K_n \times K_m$ for the rest of the proof.

We claim first that $\mu(G) \leq nm - 4$. Suppose on the contrary that there exists a larger mutual-visibility set X. Since a subset of a mutual-visibility set is a mutual-visibility set, we may assume that $|X| = nm - 3$. Let $V(G) \setminus X = \{x, y, z\}$. By the symmetry of G, it suffices to consider the following four cases.

Suppose that $x = (i, j)$, $y = (i', j')$, $z = (i'', j'')$, where $|\{i, i', i''\}| = 3$ and $|\{j, j', j''\}| = 3$. In this case we see that the vertices (i', j) and (i', j'') belong to X but are not X-visible. Hence this case is not possible.

In the second case, suppose that $x = (i, j)$, $y = (i', j)$, $z = (i', j')$, where $i \neq i'$ and $j \neq j'$. Now we infer that the vertices $(i - 1, j)$ and $(i - 1, j')$ belong to X but are not X-visible.

In the third case, suppose that $x = (i, j)$, $y = (i, j')$, $z = (i', j'')$, where $i \neq i'$ and $|\{j, j', j''\}| = 3$. Now consider the vertices (i', j) and (i', j') which both belong to X but are not X-visible.

In the last case, suppose that $x = (i, j)$, $y = (i', j)$, $z = (i'', j)$, where $|\{i, i', i''\}| = 3$. Now consider two vertices (k, j) and (k', j), where k and k' are selected in such a way that $|\{i, i', i'', k, k'\}| = 5$. (Such values k and k' exist since we have assumed that $n, m \geq 5$.) But now the vertices (k, j) and (k', j) belong to X, and they are not X-visible.

We can conclude that no matter how the set X lies in G, it cannot form a mutual-visibility set. This proves that $\mu(G) \leq nm - 4$.

Let $Y = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and let $X = V(G) \setminus Y$. We claim that X is a total mutual-visibility set of G. Let $x = (i, j)$ and $y = (i', j')$ be arbitrary vertices of G. Assume first that $x, y \in X$. If $i \neq i'$ and $j \neq j'$, then $xy \in E(G)$ and there is nothing to prove. Otherwise, $i = i'$ or $j = j'$. Assume without loss of generality that $i = i'$ and let $k \in [4]$ be such that $k \neq i, j, j'$. Then $(i, j)(k, k) \in E(G)$ and $(k, k)(i, j') \in E(G)$, hence x and y are X-visible. We proceed similarly in the case when $x \in X$ and $y \in Y$. Finally, if $x, y \in Y$, then $xy \in E(G)$. We have thus demonstrated that X is a total mutual-visibility set of G.

By the above, $\mu_{\mathrm{t}}(G) \geq mn - 4$. Combining this inequality with the earlier proved inequality $\mu(G) \leq nm - 4$, we have

$$nm - 4 \leq \mu_{\mathrm{t}}(G) \leq \mu(G) \leq nm - 4,$$

hence the equality holds everywhere and we are done. □

Corollary 2.3. *If $n, m \geq 5$, then*

$$\mu_{\mathrm{t}}(K_n \times K_m) = \mu_{\mathrm{o}}(K_n \times K_m) = \mu_{\mathrm{d}}(K_n \times K_m) = \mu(K_n \times K_m).$$

Proof. Combine Theorem 2.2 with the facts following directly from definitions that for any graph G we have $\mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{o}}(G) \leq \mu(G)$ and $\mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{d}}(G) \leq \mu(G)$, cf. [10]. □

Note that Corollary 2.3 is in sharp contrast to (1).

3. Line graphs

Given a graph G, the *line graph* $L(G)$ of G has vertex set $V(L(G)) = \{e_{uv} : \ uv \in E(G)\}$, and two vertices $e_{uv}, e_{u'v'}$ are adjacent in $L(G)$ if and only if the edges $uv, u'v'$ are incident in G. From now on, given a set of edges $F \subseteq E(G)$, we set $S_F = \{e_{uv} \in V(L(G)) : \ uv \in F\}$. Also, by G_F we represent the subgraph of G whose edges are those ones in F and vertices are those from the edges of F.

In this section we focus on the line graphs of complete graphs and of complete bipartite graphs. Notice that if $n \geq 4$, then $\mathrm{diam}(L(K_n)) = 2$, and if $m, n \geq 2$, then $\mathrm{diam}(L(K_{m,n})) = 2$. More generally, if $\mathrm{diam}(G) \leq 2$, then $\mathrm{diam}(L(G)) \leq 3$. We begin with a characterization of mutual-visibility sets in line graphs $L(G)$ for graphs G with $\mathrm{diam}(G) = 2$.

Lemma 3.1. *Let G be a graph of diameter 2 and $F \subseteq E(G)$. Then $S_F \subseteq V(L(G))$ is a mutual-visibility set of $L(G)$ if and only if for any two independent edges $uv, u'v' \in F$ one of the following conditions is satisfied.*

(i) *There is an edge $xy \notin F$ incident with both uv and $u'v'$, or*
(ii) *$d_{L(G)}(e_{uv}, e_{u'v'}) = 3$ and there is a vertex $z \in V(G)$ adjacent to (without loss of generality) u and u' in G, such that $uz, u'z \notin F$.*

Proof. ($\Rightarrow$) Assume S_F is a mutual-visibility set of $L(G)$, and let $uv, u'v' \in F$ be two independent edges. Since $\mathrm{diam}(G) = 2$, we have $d_{L(G)}(e_{uv}, e_{u'v'}) \in \{2, 3\}$. If $d_{L(G)}(e_{uv}, e_{u'v'}) = 2$, then because S_F is a mutual-visibility set, there exists a vertex $e_{xy} \notin S_F$ such that $e_{uv}, e_{xy}, e_{u'v'}$ is a geodesic. Thus, (i) holds as xy is incident to uv and to $u'v'$. If $d_{L(G)}(e_{uv}, e_{u'v'}) = 3$, then there must be a vertex $z \in V(G)$ adjacent to (without loss of generality) u and u' in G. Clearly, since S_F is a mutual-visibility set, there must be such vertex z with $e_{uz}, e_{u'z} \notin S_F$, hence (ii) holds.

($\Leftarrow$) We need to show that any two vertices $e_{uv}, e_{u'v'} \in S_F$ are S_F-visible in $L(G)$. There is nothing to prove if $e_{uv}e_{u'v'} \in E(L(G))$. Assume that $d_{L(G)}(e_{uv}, e_{u'v'}) = 2$. Then (ii) does not apply, hence by (i) there is an edge xy incident with both $uv, u'v'$ such that $e_{xy} \notin S_F$. Thus, $e_{uv}, e_{xy}, e_{u'v'}$ is a geodesic whose internal vertices are not in S_F, and so, $e_{uv}, e_{u'v'} \in S_F$ are S_F-visible. Assume next that $d_{L(G)}(e_{uv}, e_{u'v'}) = 3$. Then (ii) applies, so that there is a vertex $z \in V(G)$ adjacent to u and u' in G, such that $e_{uz}, e_{u'z} \notin S_F$. Hence, $e_{uv}, e_{uz}, e_{u'z}, e_{u'v'}$ is a geodesic in $L(G)$ whose interior vertices are not in S_F. Hence $e_{uv}, e_{u'v'}$ are S_F-visible in $L(G)$ in this case as well. Since $\mathrm{diam}(L(G)) \leq 3$ we are done. $\square$

Lemma 3.1 reduces the verification whether a set of vertices of $L(G)$, where $\mathrm{diam}(G) \leq 2$, is a mutual-visibility set to the search for the set of edges of largest cardinality in G satisfying the conditions of the lemma. This can be interpreted as an instance of a Turán-type problem. The first striking example of this claim is the following result. For its statement recall that the *Turán graph* $T(n, r)$ is a complete r-partite graph of order n in which sizes of the r parts are as equal as possible.

Theorem 3.2. *Let $n \geq 3$ be an integer and $F \subseteq E(K_n)$. Then $S_F \subseteq V(L(K_n))$ is a μ-set of $L(K_n)$ if and only if $(K_n)_F \cong T(n, 3)$.*

Proof. For $n = 3$ we have $L(K_3) = K_3$ and the assertion is clear. Suppose in the rest that $n \geq 4$. Then $\mathrm{diam}(L(K_n)) = 2$ and thus Lemma 3.1 implies that if S_F is a mutual-visibility set of $L(K_n)$, then for any two independent edges $uv, u'v' \in F$ there is an edge xy incident with both $uv, u'v'$ such that $xy \notin F$. This can be equivalently reformulated by saying that S_F is a mutual-visibility set of $L(K_n)$ if and only if $(K_n)_F$ does not contain a K_4. Turán's theorem (see [36, Theorem 11.1.3]) completes the argument. $\square$

Since the Turán graph $T(n, r)$ has $(1 - \frac{1}{r} + o(1))\,\frac{n^2}{2}$ edges, we deduce the following consequence of Theorem 3.2.

Corollary 3.3. *If $n \geq 3$, then $\mu(L(K_n)) = (\frac{2}{3} + o(1))\,\frac{n^2}{2}$.*

Now, with respect to the remaining mutual-visibility parameters of the graph $L(K_n)$, we note the following facts. If F is a set of edges of K_n, then the corresponding set S_F in $L(K_n)$ has the (total, outer or dual) mutual-visibility properties based on the existence of certain structures obtained from pairs of not incident edges from $E(K_n)$, F, or $E(K_n) \setminus F$. Recall that a pair of not incident edges $uv, u'v' \in E(K_n)$ are S_F-visible in $L(K_n)$ whenever there is an edge $xy \notin F$ such that (without loss of generality) $x = u$ and $y = u'$. These facts, the definitions of (total, outer or dual) mutual-visibility sets and the structure of $L(K_n)$ allow to readily observe the following result, whose proof is rather simple and left to the reader.

Lemma 3.4. *Let $n \geq 3$ be an integer and let $F \subseteq E(K_n)$. Then,*

(i) *S_F is a total mutual-visibility set of $L(K_n)$ if and only if for any two not incident edges $uv, u'v' \in E(K_n)$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$.*

(ii) *S_F is a dual mutual-visibility set of $L(K_n)$ if and only if*

- *for any two not incident edges $uv, u'v' \in E(K_n) \setminus F$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$, and*
- *for any two not incident edges $xy, x'y' \in F$ the subgraph induced by x, y, x', y' has at least one edge not in F different from xy and $x'y'$.*

(iii) *S_F is an outer mutual-visibility set of $L(K_n)$ if and only if*

- *for any two not incident edges $uv, u'v'$ with $uv \in E(K_n) \setminus F$ and $u'v' \in F$ the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$, and*
- *for any two not incident edges $xy, x'y' \in F$ the subgraph induced by x, y, x', y' has at least one edge not in F different from xy and $x'y'$.*

By using Lemma 3.4, we can give the following conclusions on the (total, outer or dual) mutual-visibility number of $L(K_n)$.

Proposition 3.5. *For any integer $n \geq 3$, $\mu_t(L(K_n)) \geq n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor$.*

Proof. Let $w \in V(K_n)$ and let $A = \{vw \in E(K_n) : v \in V(K_n) \setminus \{w\}\}$. Also, let G' be the complete graph induced by $V(K_n) \setminus \{w\}$, and let M be a maximum matching in G'. Now, consider the set of edges $F = A \cup M$ of K_n. Observe that S_F satisfies the properties of Lemma 3.4(i). Thus S_F is a total mutual-visibility set of $L(K_n)$, and the bound follows since $|S_F| = n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor$. □

By using a computer we have checked that the bound of Proposition 3.5 is tight for $n \in \{4, 5, 6, 7\}$. On the other hand, the equality does not hold in general. For instance, the set $\{01, 12, 23, 34, 45, 56, 67, 78, 89, 90, 04, 19, 26, 38, 57, 79\}$ of vertices of $L(K_{10})$, or equivalently, edges of K_{10}, where we have taken $V(K_{10}) = \{0, 1, \dots, 9\}$, is a total mutual-visibility set of $L(K_{10})$ of cardinality 16. However, Proposition 3.5 only yields $\mu_t(L(K_{10})) \geq 13$.

The total mutual-visibility number of $L(K_n)$ has an interesting relation with the extension of the Turán problem to forbidden generic graphs.

Definition 3.6 (*[36, Page 479]*)**.** The *Turán number* of a graph H, written $\mathrm{ex}(n; H)$, is the maximum number of edges in an n vertex graph not containing H.

Theorem 3.7. *For any integer $n \geq 3$, $\mu_t(L(K_n)) = \mathrm{ex}(n; C_4)$.*

Proof. By Lemma 3.4(i), given a set $F \subseteq E(G)$, the set S_F is a total mutual-visibility set of $L(K_n)$ if and only if for any two not incident edges $uv, u'v' \in E(K_n)$, the subgraph induced by u, v, u', v' has at least one edge not in F different from uv and $u'v'$. Consider any four vertices u, v, u', v' of K_n. They induce a graph $G' = K_4$ with three pairs of not incident edges. Since for each pair at least one edge (not belonging to the pair) is not in F, it holds that at least two incident edges of G' are not in F. Equivalently, this happens if and only if the edges of F in G' does not form a cycle C_4. Hence, in order to find a μ_t-set in $L(K_n)$ we need to find a largest set of edges of K_n that does not induce any C_4. By definition, its size is $\mathrm{ex}(n; C_4)$. □

Corollary 3.8. *For any large enough integer n, $\frac{1}{2}(n^{3/2} - n^{4/3}) \leq \mu_t(L(K_n)) \leq \frac{1}{4}n(1 + \sqrt{4n-3})$.*

Proof. Given Theorem 3.7 on the equivalence of $\mu_t(L(K_n))$ and $\mathrm{ex}(n; C_4)$, the upper bound was first proved by Reiman in [31]. The lower bound (and a rediscovery of the upper bound) can be found in [4,17]. □

We next proceed with finding similar results as the above ones for the other two remaining mutual-visibility parameters (outer and dual).

Theorems 3.2 and 3.7 provide us with a way to calculate the values of $\mu(L(K_n))$ and $\mu_t(L(K_n))$. They are based on the analysis of forbidden subgraphs for $(K_n)_F$ where S_F is a mutual-visibility or

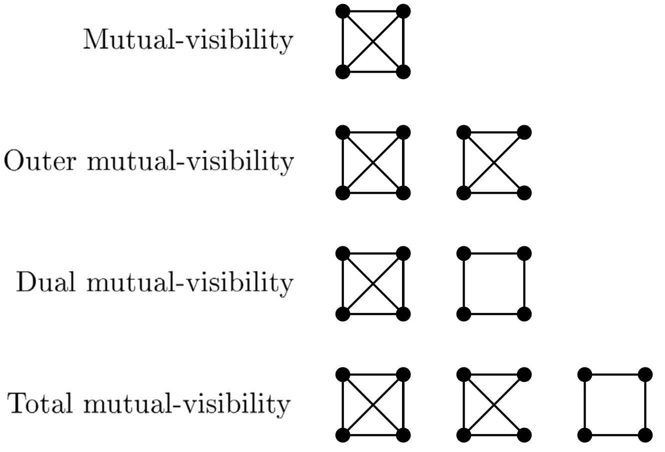

Fig. 2. Forbidden induced subgraphs for $(K_n)_F$, where F is such that S_F is a (outer, dual, total) mutual-visibility set of $L(K_n)$.

a total mutual-visibility sets of $L(K_n)$, respectively. By using Lemma 3.4, an analysis of the *induced* forbidden subgraphs of $(K_n)_F$ for the (dual, outer, total) mutual-visibility set S_F of $L(K_n)$ shows that only three forbidden graphs are involved: K_4, K_4^-, and C_4 (see Fig. 2). As proved in Theorem 3.2, K_4 is the only forbidden subgraph of $(K_n)_F$ for any mutual-visibility sets S_F of $L(K_n)$. As for the total mutual-visibility, all the three induced graphs are forbidden, but, since C_4 is a subgraph of both K_4 and K_4^-, it is sufficient to forbid only this graph, and then $\mu_{\rm t}(L(K_n)) = {\rm ex}(n, C_4)$, as stated by Theorem 3.7.

Similarly, for the outer mutual-visibility, the induced forbidden subgraphs are K_4 and K_4^-, and since K_4^- is a subgraph of K_4, we have the following result.

Theorem 3.9. *For any integer $n \geq 3$, $\mu_{\rm o}(L(K_n)) = {\rm ex}(n; K_4^-)$.*

Based on this relationship above, and using the next known result, we are able to give the exact value of $\mu_{\rm o}(L(K_n))$.

Theorem 3.10 *([32, Theorem 1(a)]). If F has chromatic number k and a critical edge, and n is large enough, then ${\rm ex}(n, F) = |E(T(n, k-1))|$. Moreover, $T(n, k-1)$ is the unique extremal graph.*

Since the graph K_4^- has chromatic number 3 and a critical edge, we deduce that the edges of K_n that form an outer mutual-visibility set of the largest cardinality in $L(K_n)$, together with the vertices in such edges, form a graph isomorphic to the Turán graph $T(n, 2)$. Recall that $T(n, 2)$ is the bipartite graph of order n with partite sets of cardinality $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. Thus, the following result holds.

Corollary 3.11. *For any large enough integer n, $\mu_{\rm o}(L(K_n)) = \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor$.*

Now, for the dual mutual-visibility, the two induced forbidden subgraphs are K_4 and C_4. Then the following result holds.

Theorem 3.12. *Let $F \subseteq E(K_n)$. Then $S_F \subseteq V(L(K_n))$ is a dual mutual-visibility set of $L(K_n)$ if and only if $(K_n)_F$ is a (K_4, C_4)-free graph.*

In contrast with the cases which appear along with standard, total and outer mutual-visibility sets, there is a lack (to the best of our knowledge) of results concerning the largest number of edges in a (K_4, C_4)-free graph of order n. This made that the result above for the dual mutual-visibility sets does not lead to a bound or formulae for the dual mutual-visibility number of $L(K_n)$. By computer checking, we only know that for $n \in [10]$ the largest graphs have 0, 1, 3, 5, 7, 10, 12, 15, 18, 21 edges, respectively. On the positive side, this means that it is worthy of considering studying this problem independently.

We next continue with the line graphs of a complete bipartite graph $K_{m,n}$, $m, n \geq 2$. Then we recall that Palmer [28] proved that the line graph of a connected graph G is a nontrivial Cartesian product if and only if $G = K_{n,m}$, $n, m \geq 2$, see [19, Proposition 1.2]. So, $L(K_{m,n}) \cong K_m \,\Box\, K_n$. It is already known from [11] that $\mu(K_m \,\Box\, K_n) = z(m, n; 2, 2)$, where $z(m, n; 2, 2)$ is the Zarankiewicz number, that can also be seen as the maximum number of edges in a complete bipartite graph $K_{m,n}$ that has no 4-cycle. We now state that the same conclusion can be also obtained by using Lemma 3.1. The proof of it runs along the lines of the proof of Theorem 3.2, and thus it is left to the reader.

Theorem 3.13. *Let $n, m \geq 2$ and $F \subseteq E(K_{m,n})$. Then $S_F \subseteq V(L(K_{m,n}))$ is a μ-set of $L(K_{m,n})$ if and only if S_F is a largest set of vertices of $L(K_{m,n})$ such that $(K_{m,n})_F$ contains no 4-cycle.*

The following consequence of Theorem 3.13 then follows from the above-mentioned observations from [11].

Corollary 3.14. *For any two integers $n, m \geq 2$, $\mu(L(K_{m,n})) = z(m, n; 2, 2)$.*

We close this section by again using the fact that the line graph of a complete bipartite graph $K_{n,m}$ is isomorphic to $K_n \,\Box\, K_m$. Hence, a result from [33], and Theorem 2.1 lead to the following consequence.

Corollary 3.15. *For any two integers $n, m \geq 2$, $\mu_{\rm t}(L(K_{m,n})) = \max\{n, m\}$, $\mu_{\rm d}(L(K_{m,n})) = n + m - 1$, and $\mu_{\rm o}(L(K_{m,n})) = n + m - 2$.*

4. Visibility in cographs

In this section, we consider the mutual-visibility in cographs. In the main result we prove that if G is a cograph, then either $\mu(G) = \mu_{\rm t}(G)$ $(= \mu_{\rm o}(G) = \mu_{\rm d}(G))$, or $\mu(G) = \mu_{\rm d}(G) = n(G) - 1$ and $\mu_{\rm t}(G) = \mu_{\rm o}(G) = n(G) - 2$. For this, several additional definitions are required.

A cograph is a graph all of whose connected induced subgraphs have diameter at most 2. Moreover, each cograph can also be built up from a single vertex by adding a sequence of twins. Two vertices u and v of G are *false twins* if $N_G(u) = N_G(v)$ and are *true twins* if $N_G[u] = N_G[v]$, where $N_G(u)$ is the *open neighborhood* of u and $N_G[u] = N_G(u) \cup \{u\}$. We say that u and v are *twins* it they are either false twins or true twins. Given a graph G and $v \in V(G)$, $G - v$ denotes the subgraph of G induced by $V(G) \setminus \{v\}$. A graph G such that $\mu(G) = \mu_{\rm t}(G)$ is called a *$(\mu, \mu_{\rm t})$-graph*.

We start by recalling the following characterizations from [12,15].

Lemma 4.1 (*[15, Lemma 4.8]*). *Given a graph G, then $\mu(G) \geq n(G) - 1$ if and only if there exists a vertex v adjacent to each vertex u such that $\deg_{G-v}(u) < n(G) - 2$.*

In [12], any vertex v of G fulfilling the condition in the above lemma was called *enabling*.

Proposition 4.2 (*[12, Proposition 3.5]*). *A cograph G is a $(\mu, \mu_{\rm t})$-graph if and only if it has a universal vertex or no enabling vertices.*

The following definition aims to reformulate the previous characterizations in terms of graph structure.

Definition 4.3. A *big-μ* graph is any graph G defined as $G = (K_1 \cup K_t) + H$, where K_1, K_t, and H are three distinct graphs such that $t \geq 0$ (i.e., K_t can be an empty graph).

From this definition, it follows that each non-trivial clique is a big-μ graph (it is sufficient to take K_t empty and H as a clique). Consequently, observe that if G is a big-μ graph, then $\mu(G) = n(G)$ when K_t is empty and H isomorphic to a clique, and $\mu(G) = n(G) - 1$ otherwise. This observation explains the term big-μ.

The two characterizations recalled above will be reformulated by using the following lemma.

Lemma 4.4. *Let G be an arbitrary graph. Then G is a big-μ graph if and only if G has a vertex v adjacent to each vertex u such that $\deg_{G-v}(u) < n(G) - 2$.*

Proof. ($\Rightarrow$) Assume that G is a big-μ graph. That is, there exists three distinct graphs K_1, K_t, and H such that $G = (K_1 \cup K_t) + H$. Let v be the vertex forming the graph K_1. Since the vertices u of H are the only vertices such that $\deg_{G-v}(u) < n(G) - 2$, and since v is adjacent to all of them, the thesis follows.

($\Leftarrow$) Assume now G has a vertex v adjacent to each vertex u such that $\deg_{G-v}(u) < n(G) - 2$. To show that G is a big-μ graph, take K_1 formed by v, G' as the graph induced by $N_G(v)$, and G'' as the graph induced by $V(G) \setminus N_G[v]$. Let u'' be a vertex of G''. Since u'' is not adjacent to v in G, it holds that $\deg_{G-v}(u'') \geq n(G) - 1$. This implies that G'' is a clique, and there exists an edge in G between each pair of vertices $u' \in V(G')$ and $u'' \in V(G'')$. Hence, we deduce that $G = (K_1 \cup G'') + G'$, where G' is an arbitrary graph and G'' is a clique. □

Corollary 4.5. *Let G be an arbitrary graph. Then $\mu(G) \geq n(G) - 1$ if and only if G is a big-μ graph.*

Proof. This is an immediate consequence of Lemmas 4.1 and 4.4. □

Corollary 4.6. *Let G be a cograph. Then $\mu(G) > \mu_t(G)$ if and only if G is a big-μ graph $G = (K_1 \cup K_t) + H$ with no universal vertices.*

Proof. ($\Rightarrow$) From Proposition 4.2 we have $\mu(G) > \mu_t(G)$ if and only if G has an enabling vertex v and G has no universal vertices. By Lemma 4.4, we have that $G = (K_1 \cup K_t) + H$ is a big-μ graph.

($\Leftarrow$) As $G = (K_1 \cup K_t) + H$ has no universal vertices, G is not a clique and then $\mu(G) < n(G)$. Hence, by Corollary 4.5, $\mu(G) = n(G) - 1$. We claim that $\mu_t(G) < n(G) - 1$. Assume, on the contrary, that $\mu_t(G) = n(G) - 1$. Let S be a μ_t-set of G and let u be the only vertex of $V(G)$ not in S and let v be the only vertex in K_1. Since G has no universal vertices we deduce, (1) K_t is not empty and (2) H has at least two not adjacent vertices x and y. Then, by (1), u cannot be v, otherwise u is not in mutual-visibility with any vertex in K_t. Moreover, u cannot be a vertex of K_t, otherwise u is not in mutual-visibility with v. By (2), u cannot be a vertex of H, otherwise x and y are not in mutual-visibility. Hence $\mu_t(G) < n(G) - 1$. □

This corollary implies that the smallest cograph G which is not a (μ, μ_t)-graph corresponds to the cycle $C_4 = (K_1 \cup K_t) + H$, with $t = 1$ and $H = K_1 \cup K_1$. If v is the vertex forming K_1, h_1, h_2 are the vertices forming H, and k is the unique vertex of K_t, it can be observed that each cograph which is not a (μ, μ_t)-graph can be obtained from this initial cycle by applying no split operations to v, any possible split operation to h_1 and h_2 and only true-twin operations to k.

In [15] it is shown that $\mu(G) \geq n(G) - 2$ for each cograph G, and that the exact value of $\mu(G)$ can be computed in polynomial time. The following statement extends the analysis to the other visibility parameters.

Theorem 4.7. *If G is a cograph, then either $\mu(G) = \mu_t(G)$, or $\mu(G) = \mu_d(G) = n(G) - 1$ and $\mu_t(G) = \mu_o(G) = n(G) - 2$.*

Proof. If $\mu(G) \neq \mu_t(G)$ then, by Corollary 4.6, G is a big-μ cograph $G = (K_1 \cup K_t) + H$ without universal vertices. By the proof of the same corollary, we have $\mu(G) = n(G) - 1$. Concerning $\mu_d(G) = n(G) - 1$, it easily follows by observing that $V(G) \setminus \{u\}$, where u is the unique vertex of K_1, is a dual mutual-visibility set of G.

By using again the proof of Corollary 4.6, we have $\mu_t(G) < n(G) - 1$. Notice that the same arguments can be used to show that $\mu_o(G) < n(G) - 1$ also holds. Hence, to conclude the proof, it is sufficient to show there exists a total mutual-visibility set of G with $n(G) - 2$ elements. To this end, let $S = V(G) \setminus \{u, v\}$ where u is the unique vertex of K_1 and v is a vertex in H. Let us show that S is a total mutual-visibility set of G. The vertices in S are in mutual-visibility, since any two of them are adjacent or they are adjacent vertices of u. The vertices u and v are adjacent. Vertex u is in mutual-visibility with all the vertices in S, as u is adjacent to them or in mutual-visibility through vertex v. Analogously, vertex v is in mutual-visibility with all the vertices in S as v is adjacent to them or in mutual-visibility through vertex u. Then, S is a total mutual-visibility set of G with $n(G) - 2$ vertices. □

A well-known superclass of cographs is that formed by *distance-hereditary graphs*. In fact, these graphs can be generated by using true twins, false twins, and pendant vertices. Concerning the problem of characterizing all the distance-hereditary graphs G in which $\mu(G) > \mu_t(G)$, we conjecture that the following holds:

Conjecture 4.8. *If G is a distance-hereditary graph but not a big-μ cograph without universal vertices, then $\mu(G) = \mu_t(G)$.*

We must remark that this conjecture is also supported by numerous computer-assisted simulations.

5. Non-trivial diameter-two graphs of minimum size

Let G be a diameter-two graph with no universal vertex. Then Erdős and Rényi proved that $m(G) \geq 2n(G) - 5$. More than two decades later, Henning and Southey characterized the graphs which achieve the bound. In this section we determine μ, μ_o, μ_d, and μ_t for these extremal diameter-two graphs.

Let us restate the mentioned classical result of Erdős and Rényi on the minimum size of a diameter-two graph with no universal vertex.

Theorem 5.1 (*[16, Theorem 3 and discussion at page 633]*)**.** *If G is a diameter-two graph with no universal vertex, then $m(G) \geq 2n(G) - 5$.*

We will also use the following auxiliary result, which could find other uses.

Lemma 5.2. *If G is a connected graph of order at least 3 and with $g(G) \geq 5$, then an outer mutual-visibility set is an independent set.*

Proof. Let X be an outer mutual-visibility set of G and suppose that X contains vertices x and y such that $xy \in E(G)$. Since G has at least three vertices and is connected, we may assume that z is a neighbor of y different from x. As $g(G) \geq 5$, we must have $z \in X$, for otherwise $x \in X$ and $z \notin X$ are not X-visible. But then $x, z \in X$ are not X-visible, a contradiction. □

Let P be the Petersen graph and note that P attains the bound of Theorem 5.1. It is already known that $\mu(P) = 6$, see [11], and that $\mu_t(P) = 0$, see [33]. By a case analysis, we also get that $\mu_d(P) = 0$. On the other hand, Lemma 5.2 implies that $\mu_o(P) \leq 4$, and it can be easily checked that an independent set of P of cardinality 4 is an outer mutual-visibility set. In summary,

$$\mu_t(P) = \mu_d(P) = 0, \quad \mu_o(P) = 4, \quad \mu(P) = 6.$$

Let G_7 (cf. Fig. 3) be the graph obtained from the cycle C_3 by adding a pendant edge to each vertex of the cycle and then adding a new vertex and joining it to the three degree-one vertices. In [18], the following family $\mathcal{G}$ of graphs has been defined:

(i) $\mathcal{G}$ contains C_5, G_7, and the Petersen graph; and
(ii) $\mathcal{G}$ is closed under degree-two vertex duplication (cf. Fig. 3).

The graphs that achieve equality in the bound of Theorem 5.1 are characterized as follows.

Theorem 5.3 (*[18, Theorem 1]*)**.** *If G is a diameter-two graph of order n and size m with no universal vertex, then $m = 2n - 5$ if and only if $G \in \mathcal{G}$.*

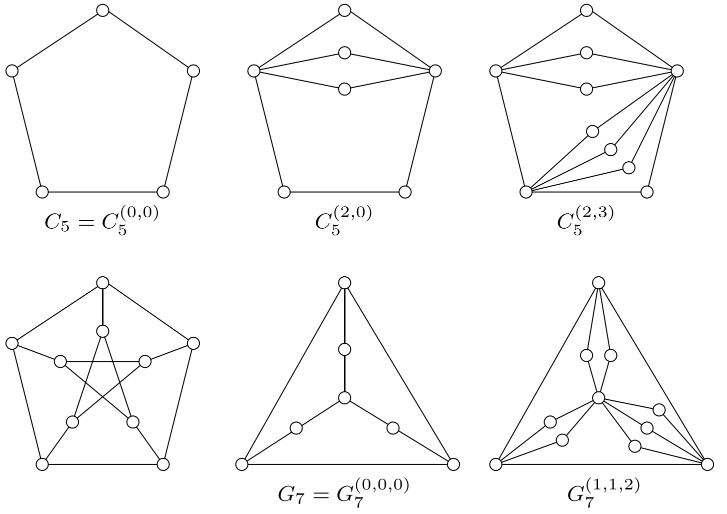

Fig. 3. Some examples of graphs from the family $\mathcal{G}$.

In what follows, we compute $\tau(G)$ for each $G \in \mathcal{G}$ and for each $\tau \in \{\mu, \mu_o, \mu_d, \mu_t\}$. To this aim, denote by $C_5^{(i,j)}$ the graph obtained from the cycle C_5 with one vertex duplicated $i \geq 0$ times, and another vertex duplicated $j \geq 0$ times. For instance, in Fig. 3 the graphs $C_5 = C_5^{(0,0)}$, $C_5^{(2,0)}$, and $C_5^{(2,3)}$ are shown.

Lemma 5.4. *If X is a mutual-visibility (resp. outer, dual, or total mutual-visibility) set of a graph G and $x \in X$, then $X \setminus \{x\}$ is a mutual-visibility (resp. outer, dual, or total mutual-visibility) set of $G - x$.*

Proof. Let X be a mutual-visibility set. For any two vertices u, v in $X \setminus \{x\}$, there is at least one shortest u, v-path P not passing through x. Then the removal of x from G does not destroy P in $G - x$. Hence, for the generality of u, v, the set $X \setminus \{x\}$ is a mutual-visibility set of G. Similar arguments work for the outer, dual or total mutual-visibility cases. □

Proposition 5.5. *For integers $i, j \geq 0$,*

- $\mu_t(C_5^{(i,j)}) = i + j$,
- $\mu_o(C_5^{(i,j)}) = \mu_d(C_5^{(i,j)}) = i + j + 2$,
- $\mu(C_5^{(i,j)}) = i + j + 3$.

Proof. In [10] it is shown that the four formulae are correct when $i = j = 0$. Notice, however, that although it is $\mu_o(C_5^{(0,0)}) = \mu_d(C_5^{(0,0)}) = 2$, two vertices in a dual mutual-visibility set are adjacent, whereas two vertices in an outer mutual-visibility are not. Fig. 4 shows a visibility set for each of the four variants. In each case, it is clear that each time a degree-two vertex duplication is made, the new vertex can be included in the visibility set and hence the corresponding parameter is increased by one.

Concerning the optimality, consider now all the four cases of mutual-visibility sets for $G = C_5^{(i,j)}$. We prove by induction that $\mu_t(G) = i + j$, $\mu_o(G) = i + j + 2$, $\mu_t(G) = i + j + 2$, $\mu_t(G) = i + j + 3$. As already observed, the statement holds for the initial case in which $i = j = 0$. Assume it holds for $k = i + j > 0$ and consider the case $G = C_5^{(i,j)}$ obtained with $k + 1$ degree-two vertices duplications. Assume, by contradiction, that there exists a μ_t-set (μ_o-set, μ_d-set, μ-set) X of G with $|X| > n(G) - 5$ ($|X| > n(G) - 3$, $|X| > n(G) - 3$, $|X| > n(G) - 2$). Then we can consider any vertex $v \in X$ and set

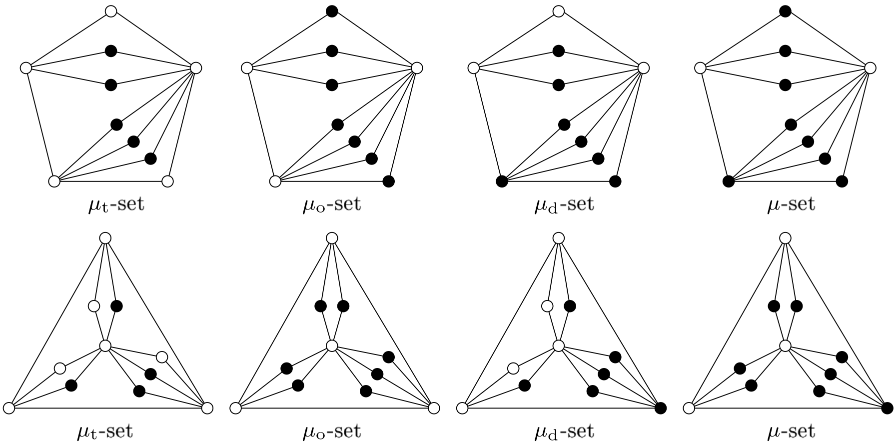

Fig. 4. Variations of mutual-visibility sets in some graphs of the family $\mathcal{G}$.

$X' = X \setminus \{v\}$ and $G' = G - v$. By Lemma 5.4, X' is a total mutual-visibility (outer mutual-visibility, dual mutual-visibility, mutual-visibility) set of G', with a size larger than that assumed by induction. □

It is worth to remark that the case concerning $\mu(G)$ in Proposition 5.5 can be also proved by simply observing that the mutual-visibility set of G, shown in Fig. 4, contains $i + j + 3 = n(G) - 2$ elements, and that Corollary 4.5 implies that this set is indeed a μ-set, since G is not a big-μ graph.

Similarly as above, denote by $G_7^{(i,j,k)}$ the graph obtained from G_7 when the three degree-two vertices have been respectively duplicated i, j, and k times. For instance, Fig. 3 shows the graphs $G_7 = G_7^{(0,0,0)}$ and $G_7^{(1,1,2)}$.

Proposition 5.6. *For the graph $G_7^{(i,j,k)}$ we have the following formulae:*

- $\mu_t(G_7^{(i,j,k)}) = i + j + k$,
- $\mu_o(G_7^{(i,j,k)}) = i + j + k + 3$,
- $\mu_d(G_7^{(i,j,k)}) = \begin{cases} 3; & i + j + k = 0, \\ i + j + k + 2; & i + j + k \geq 1. \end{cases}$
- $\mu(G_7^{(i,j,k)}) = i + j + k + 4$.

Proof. The statement can be proved by using the same inductive approach used in the proof of Proposition 5.5. The correctness of the base cases can be easily verified thanks to the limited size of the graph. We just remark that for the dual mutual-visibility case, when $i = j = k = 0$, the μ_d-set is composed of the three vertices of the C_3 cycle; when the first degree-two vertex duplication is made (i.e., $i = 1$ and $j = k = 0$), a μ_d-set can be identified by selecting the two "twin" degree-two vertices along with their adjacent vertex in the C_3 cycle. Extending this graph further leads to identifying the μ_d-set as represented in Fig. 4. □

6. Concluding remarks

In this paper we considered graphs of diameter two and their values for the (classic, total, dual and outer) mutual-visibility parameters. We next comment some possible open questions that might be of interest to continue exploring.

- The class of graphs of diameter two is very wide. We have studied here a few of them, but some other non-trivial classes might be of interest as well. Among them, we remark the Kneser

graph $K(n,2)$, which is indeed the complement of $L(K_n)$, and the line graphs of complete multipartite graphs (with at least three partite sets). For this latter ones, since we do not have an exact solution for $\mu(L(K_{m,n}))$ (and it seems to be beyond the reach of existing methods), we cannot, of course, expect an exact result for the mutual-visibility number of general complete multipartite graphs. Suppose $F \subseteq E(K_{n_1,\ldots,n_k})$, $k \geq 3$, $n_1, \ldots, n_k \geq 2$, is a μ-set of $L(K_{n_1,\ldots,n_k})$. As $\mathrm{diam}(L(K_{n_1,\ldots,n_k})) = 2$, we can again apply Lemma 3.1 to see that if $uv, u'v' \in F$ are independent edges, then there is an edge xy incident with both uv and $u'v'$ such that $xy \notin F$. If the edges uv and $u'v'$ are only between two of the multipartite sets, then in $(K_{n_1,\ldots,n_k})_F$ we have C_4 as a forbidden subgraph. If the edges uv and $u'v'$ lie in three multipartite sets, then in $(K_{n_1,\ldots,n_k})_F$ we have K_4^- as a forbidden subgraph, while if the end vertices of uv and $u'v'$ lie in four multipartite sets, then in $(K_{n_1,\ldots,n_k})_F$ we have K_4 as a forbidden subgraph. However, all these facts are not exactly related to each other, since the situations are somehow not comparable. Consequently, it would be interest to continue the study of the (dual, outer, total) mutual-visibility number of these line graphs.
- Theorem 2.1 completes the studies on the mutual-visibility variants of 2-dimensional Hamming graphs (those of diameter two). For higher dimensions, the total version was studied in [5], and the problem seems to be very challenging due to its connection with some Turán-type problems in hypergraphs. This makes natural to consider the remaining variants (dual and outer) for Hamming graphs of higher dimension.
- Based on the fact that finding the value of any of the studied mutual-visibility parameters of graphs of diameter two seems to be a hard task, we consider the following question of interest. Which is the computational complexity of computing the (outer, dual, total and classical) mutual-visibility number of graphs of diameter two?
- In connection with Theorem 3.12, as we already mentioned, it seems there is a lack of results concerning the largest number of edges in a (K_4, C_4)-free graph of order n. Based on this fact, it might be of interest to separately study this problem from a combinatorial point of view. A consequence of such study will clearly give some knowledge on the dual mutual-visibility number of $L(K_n)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used in this investigation.

Acknowledgments

S. Cicerone and G. Di Stefano were partially supported by the European project "Geospatial based Environment for Optimisation Systems Addressing Fire Emergencies" (GEO-SAFE), contract no. H2020-691161, and by the Italian National Group for Scientific Computation (GNCS-INdAM). S. Klavžar was partially supported by the Slovenian Research Agency (ARIS) under the grants P1-0297, N1-0218, and N1-0285. I.G. Yero has been partially supported by the Spanish Ministry of Science and Innovation through the grant PID2019-105824GB-I00. Moreover, this investigation was completed while this author (I.G. Yero) was making a temporary stay at "Universitat Rovira i Virgili" supported by the program "Ayudas para la recualificación del sistema universitario español para 2021–2023, en el marco del Real Decreto 289/2021, de 20 de abril de 2021".

References

[1] A. Aljohani, G. Sharma, Complete visibility for mobile robots with lights tolerating faults, Int. J. Netw. Comput. 8 (2018) 32–52.
[2] S. Bhagat, Optimum algorithm for the mutual visibility problem, Lect. Notes Comput. Sci. 12049 (2020) 31–42.
[3] B. Brešar, I.G. Yero, Lower (total) mutual visibility in graphs, Appl. Math. Comput. 465 (2024) Paper 128411.
[4] W.G. Brown, On graphs that do not contain a Thomsen graph, Canad. Math. Bull. 9 (1966) 281–285.
[5] C. Bujtás, S. Klavžar, J. Tian, Total mutual-visibility in Hamming graphs, 2023, arXiv:2307.05168[math.CO].
[6] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, The geodesic mutual visibility problem: Oblivious robots on grids and trees, Pervasive Mob. Comput. 95 (2023) Paper 101842.
[7] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, Time-optimal geodesic mutual visibility of robots on grids within minimum area, in: Proc. of 5th Int. Symp. on Stabilization, Safety, and Security of Distributed Systems, SSS, in: Lecture Notes in Computer Science, vol. 14310, Springer, 2023, pp. 385–399.
[8] S. Cicerone, A. Di Fonso, G. Di Stefano, A. Navarra, F. Piselli, Mutual visibility in hypercube-like graphs, 2023, arXiv:2308.14443[math.CO].
[9] S. Cicerone, G. Di Stefano, Mutual-visibility in distance-hereditary graphs: a linear-time algorithm, Procedia Comput. Sci. 223 (2023) 104–111.
[10] S. Cicerone, G. Di Stefano, L. Drožđek, J. Hedžet, S. Klavžar, I.G. Yero, Variety of mutual-visibility problems in graphs, Theoret. Comput. Sci. 974 (2023) Paper 114096.
[11] S. Cicerone, G. Di Stefano, S. Klavžar, On the mutual-visibility in Cartesian products and in triangle-free graphs, Appl. Math. Comput. 438 (2023) Paper 127619.
[12] S. Cicerone, G. Di Stefano, S. Klavžar, I.G. Yero, Mutual-visibility in strong products of graphs via total mutual-visibility, 2024, arXiv:2210.07835v2[math.CO].
[13] D.G. Corneil, H. Lerchs, L.S. Burlingham, Complement reducible graphs, Discrete Appl. Math. 3 (1981) 163–174.
[14] G.A. Di Luna, P. Flocchini, S.G. Chaudhuri, F. Poloni, N. Santoro, G. Viglietta, Mutual visibility by luminous robots without collisions, Inform. and Comput. 254 (2017) 392–418.
[15] G. Di Stefano, Mutual visibility in graphs, Appl. Math. Comput. 419 (2022) Paper 126850.
[16] P. Erdős, A. Rényi, On a problem in the theory of graphs, Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962) 623–641.
[17] P. Erdős, A. Rényi, V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[18] M.A. Henning, J. Southey, A characterization of the non-trivial diameter two graphs of minimum size, Discrete Appl. Math. 187 (2015) 91–95.
[19] W. Imrich, S. Klavžar, D.F. Rall, Topics in Graph Theory: Graphs and their Cartesian Product, A K Peters, Wellesley, MA, 2008.
[20] S. Klavžar, A. Krishnakumar, J. Tuite, I.G. Yero, Traversing a graph in general position, Bull. Aust. Math. Soc. 108 (2023) 353–365.
[21] S. Klavžar, P.K. Neethu, S.V. U. Chandran, The general position achievement game played on graphs, Discrete Appl. Math. 317 (2022) 109–116.
[22] S. Klavžar, B. Patkós, G. Rus, I.G. Yero, On general position sets in Cartesian products, Results Math. 76 (2021) Paper 123.
[23] S. Klavžar, E. Tan, Edge general position sets in Fibonacci and Lucas cubes, Bull. Malays. Math. Sci. Soc. 46 (2023) Paper 120.
[24] D. Korže, A. Vesel, General position sets in two families of Cartesian product graphs, Mediterr. J. Math. 20 (2023) Paper 203.
[25] D. Kuziak, J.A. Rodríguez-Velázquez, Total mutual-visibility in graphs with emphasis on lexicographic and Cartesian products, Bull. Malays. Math. Sci. Soc. 46 (2023) Paper 197.
[26] P. Manuel, S. Klavžar, A general position problem in graph theory, Bull. Aust. Math. Soc. 98 (2018) 177–187.
[27] P. Manuel, R. Prabha, S. Klavžar, The edge general position problem, Bull. Malays. Math. Sci. Soc. 45 (2022) 2997–3009.
[28] E.M. Palmer, Prime line-graphs, Nanta Math. 6 (1973) 75–76.
[29] B. Patkós, On the general position problem on Kneser graphs, Ars Math. Contemp. 18 (2020) 273–280.
[30] P. Poudel, A. Aljohani, G. Sharma, Fault-tolerant complete visibility for asynchronous robots with lights under one-axis agreement, Theoret. Comput. Sci. 850 (2021) 116–134.
[31] I. Reiman, Über ein Problem von K. Zarankiewicz, Acta. Math. Acad. Sci. Hungar. 9 (1958) 269–273.
[32] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq, Tihany, Academic Press, New York, 1968, 1966, pp. 279–319.
[33] J. Tian, S. Klavžar, Graphs with total mutual-visibility number zero and total mutual-visibility in Cartesian products, Discuss. Math. Graph Theory (2023) http://dx.doi.org/10.7151/dmgt.2496.
[34] J. Tian, K. Xu, The general position number of Cartesian products involving a factor with small diameter, Appl. Math. Comput. 403 (2021) Paper 126206.
[35] S.V. U. Chandran, G.J. Parthasarathy, The geodesic irredundant sets in graphs, Int. J. Math. Combin. 4 (2016) 135–143.
[36] D.B. West, Combinatorial Mathematics, Cambridge University Press, Cambridge, 2021.
[37] Y. Yao, M. He, S. Ji, On the general position number of two classes of graphs, Open Math. 20 (2022) 1021–1029.