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Isotropy groups of the action of orthogonal *congruence on Hermitian matrices ☆

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We present a procedure which enables the computation and the description of structures of isotropy subgroups of the group of complex orthogonal matrices with respect to the action of *congruence on Hermitian matrices. A key ingredient in our proof is an algorithm giving solutions of a certain rectangular block (complex-alternating) upper triangular Toeplitz matrix equation.

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1. Introduction

We denote by Her_n the real vector space of all n-by-n Hermitian matrices; A is Hermitian if and only if $A = A^* := (\overline{A})^T$. Let further $O_n(\mathbb{C})$ be the group of complex orthogonal n-by-n matrices. A matrix Q is orthogonal precisely when $Q^{-1} = Q^T$. The action of *orthogonal *congruence* on Her_n is defined as follows:

$$\Phi\colon O_n(\mathbb{C}) \times \mathrm{Her}_n \to \mathrm{Her}_n, \qquad (Q, A) \mapsto Q^*AQ. \tag{1.1}$$

The study of Hermitian matrices under *congruence is indeed quite general, as can be concluded from Hua's fundamental result [13, Theorem 12] on the geometry of Hermitian matrices (extended by Wan [24, Theorem 6.4]); see also the paper by Radjavi and Šemrl [19]. On the other hand (1.1) can be seen as a representation of $O_n(\mathbb{C})$ as a real classical group (e.g. see the monograph [25]).

The *isotropy group* at $A \in \mathrm{Her}_n$ with respect to the action (1.1) is denoted by

$$\Sigma_A := \{Q \in O_n(\mathbb{C}) \mid Q^*AQ = A\}. \tag{1.2}$$

Isotropy groups provide an important information about a group action (see textbooks [6,18]). In a generic case (on a complement of a real analytic subset of codimension 1) isotropy groups for (1.1) are clearly trivial (Proposition 2.1), while in general the situation is more involved. The main purpose of this paper is to give an inductive procedure that enables the computation and the description of a group structure of an isotropy group (1.2) in a nongeneric case (Theorem 2.3 and Theorem 2.7). Analoguous results for skew-Hermitian matrices under orthogonal *conjugation are valid as well. Key ingredients in the proof are Lemma 4.1 and Lemma 4.2. They provide solutions of certain block rectangular (complex-alternating) upper triangular Toeplitz matrix equations. These equations characterize orthogonality of a solution Q of the equation $A\overline{Q} = QA$ (or equivalently $A = QAQ^*$, i.e. $Q^* \in \Sigma_A$); for a general Q this equation was solved by Bevis, Hall and Hartwig [2].

In contrast to the complex case, the situation in the real case is simple. Each real symmetric matrix is real orthogonally similar to $\Lambda = \oplus_{r=1}^N(\oplus_{j=1}^{m_r}\lambda_j)$ with $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ pairwise distinct. Since $Q^T\Lambda Q = \Lambda$ for real orthogonal Q transforms to the Sylvester equation $\Lambda Q = Q\Lambda$, the isotropy group at Λ with respect to real orthogonal similarity consists of matrices $Q = \oplus_{r=1}^N Q_r$ with Q_r real orthogonal of size $m_r \times m_r$.

Pairs (A, B) with A arbitrary and B symmetric (i.e. $B = B^T$) with respect to transformations (cP^*AP, P^TBP) for a nonsingular matrix P and $c \in \mathbb{C} \setminus \{0\}$ are studied in CR-geometry in the theory of CR-singularities of codimension 2. Normal forms under this action for 2×2 matrices were obtained by Coffman [4]. In higher dimensions the isotropy groups of (1.1) are expected to some extend to be applied to tackle this problem as well as a closely related problem of simultaneous reduction of (A, B) under transformations (P^*AP, P^TBP) with P nonsingular. By applying Autonne-Takagi factorization we first reduce (A, B) to (A', I) with the identity I. Next, we write $A' = H_1 + iH_2$ with H_1, H_2 Hermitian. We put H_1 into Hong's orthogonal *congruence normal form [8] and then simplify H_2 by using matrices from the isotropy group Σ_{H_1}, as they keep H_1, I intact. We add that a reduction of Hermitian-symmetric pairs was considered by Hua [14], Hong [8], Hong, Horn and Johnson [11], among others.

2. The main results

Isotropy groups corresponding to elements of $\mathrm{Orb}(A) := \{Q^*AQ \mid Q \in O_n(\mathbb{C})\}$ (i.e. the orbit of A with respect to (1.1)) are conjugate, thus it suffices to compute them for representatives of orbits. Hong [8, Theorem 2.7] proved that each Hermitian matrix A is orthogonally *congruent to a matrix of the form

$$\mathcal{H}^\varepsilon(A) = \bigoplus_j \varepsilon_j H_{\alpha_j}(\lambda_j) \oplus \bigoplus_k K_{\beta_k}(\mu_k) \oplus \bigoplus_l L_{\gamma_l}(\xi_l), \tag{2.1}$$

in which $\lambda_j \geq 0$, $\mu_k > 0$, $\mathrm{Im}(\xi_l) > 0$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$, all $\varepsilon_j \in \{1, -1\}$ with $\varepsilon_j = 1$ if $\lambda_j = 0$ and α_j odd, and where λ_j^2, $-\mu_k^2$ and ξ_l^2 are nonnegative, positive and nonreal eigenvalues of $A\overline{A}$, respectively;

$$H_n(z) := \frac{1}{2}\left(\begin{bmatrix} 0 & & 1 & 2z \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 1 \\ 1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 2z & 1 & & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{bmatrix}\right) \qquad (n\text{-by-}n), \tag{2.2}$$

$$K_n(z) := \begin{bmatrix} 0 & -iH_n(z) \\ iH_n(z) & 0 \end{bmatrix}, \qquad L_n(z) := \begin{bmatrix} 0 & H_n(z) \\ H_n^*(z) & 0 \end{bmatrix}. \tag{2.3}$$

The author [21, Theorem 1.1] showed that (2.1) is uniquely determined up to a permutation of its blocks. We add that *congruce canonical forms and dimensions of their orbits for general matrices are known as well ([12], [23]), and that the isotropy subgroups of invertible integer matrices under congruence at symmetric Gram matrices of edge-bipartite graphs were studied in [17], [20].

By applying results from [2, Sec. 2] on solutions of the equation $A\overline{Y} = YA$, we immediately conclude the following facts; check also Proposition 3.4 (1).

Proposition 2.1.

1. *Let $\rho_1, \ldots, \rho_n \in \mathbb{C}$ be all distinct and let $\mathcal{H}^\varepsilon = \bigoplus_{j=1}^n \mathcal{H}_j^\varepsilon$ be of the form (2.1), in which $\mathcal{H}_j^\varepsilon$ is a direct sum whose summands correspond to the eigenvalue ρ_j of $\mathcal{H}^\varepsilon\overline{\mathcal{H}^\varepsilon}$. Then $\Sigma_{\mathcal{H}^\varepsilon} = \bigoplus_{j=1}^n \Sigma_{\mathcal{H}_j^\varepsilon}$.*

2. If $\mathcal{H}^{\varepsilon} = \bigoplus_{j=1}^{n} \varepsilon_j \lambda_j \oplus \bigoplus_{l=1}^{m} \begin{bmatrix} 0 & \xi_l \\ \overline{\xi}_l & 0 \end{bmatrix}$ *(a generic canonical form), in which* $\lambda_j \geq 0$*,* $\xi_l \in \mathbb{C} \setminus \mathbb{R}$ *are all distinct constants and all* $\varepsilon_j \in \{1, -1\}$*, then* $\Sigma_{\mathcal{H}^{\varepsilon}}$ *is trivial.*

In Sec. 3, we describe nonsingular solutions of $A\overline{Y} = YA$ by the following matrices. Given $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_1 > \ldots > \alpha_N$ and $\mu = (m_1, \ldots, m_N)$ let $\mathbb{T}^{\alpha,\mu}$ and $\mathbb{T}_c^{\alpha,\mu}$ consist of N-by-N block matrices with α_r-by-α_s blocks of the form:

$$\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^{N}, \qquad \mathcal{X}_{rs} = \begin{cases} [0 \quad \mathcal{T}_{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} \mathcal{T}_{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ \mathcal{T}_{rs}, & \alpha_r = \alpha_s \end{cases}, \qquad b_{rs} := \min\{\alpha_s, \alpha_r\}, \tag{2.4}$$

in which $\mathcal{T}_{rs} = T(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$ and $\mathcal{T}_{rs} = T_c(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$, respectively; $A_n^{rs} \in \mathbb{C}^{m_r \times m_s}$ and all A_0^{rr} are nonsingular. We use the standard notation $\mathbb{C}^{m\times n}$ to denote the set of m-by-n matrices, and let a β-by-β *block upper triangular Toeplitz* and a β-by-β block *complex-alternating upper triangular Toeplitz* matrix be:

$$T(A_0, \ldots, A_{\beta-1}) := \begin{bmatrix} A_0 & A_1 & \ldots & A_{\beta-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_1 \\ 0 & \ldots & 0 & A_0 \end{bmatrix}, T_c(A_0, \ldots, A_{\beta-1}) := \begin{bmatrix} A_0 & A_1 & \ldots & \ldots & A_{\beta-1} \\ 0 & \overline{A}_0 & \overline{A}_1 & & \vdots \\ \vdots & \ddots & A_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & \ddots \end{bmatrix},$$

respectively, in which $A_0, \ldots, A_{\beta-1}$ are of the same size, and $T(A_0, \ldots, A_{\beta-1}) = [T_{jk}]_{j,k=1}^{\beta}$, $T_c(A_0, \ldots, A_{\beta-1}) = [T'_{jk}]_{j,k=1}^{\beta}$ with $T_{jk} = T'_{jk} = 0$ for $j > k$ and with $T_{(j+1)(k+1)} = T_{jk}$, $T'_{(j+1)(k+1)} = \overline{T}'_{jk}$. When in addition A_0 is the identity matrix, they are called block (complex-alternating) upper *unitriangular* Toeplitz.

Example 2.2. Examples of matrices of the form (2.4) are ($\alpha_1 = 3$, $\alpha_2 = 2$):

$$\mathcal{X} = \left[\begin{array}{ccc|cc} A_1 & B_1 & C_1 & G_1 & H_1 \\ 0 & A_1 & B_1 & 0 & G_1 \\ 0 & 0 & A_1 & 0 & 0 \\ \hline 0 & N_1 & P_1 & A_2 & B_2 \\ 0 & 0 & N_1 & 0 & A_2 \end{array}\right], \qquad \mathcal{X}_c = \left[\begin{array}{ccc|cc} A_1 & B_1 & C_1 & G_1 & H_1 \\ 0 & \overline{A}_1 & \overline{B}_1 & 0 & \overline{G}_1 \\ 0 & 0 & A_1 & 0 & 0 \\ \hline 0 & N_1 & P_1 & A_2 & B_2 \\ 0 & 0 & \overline{N}_1 & 0 & \overline{A}_2 \end{array}\right].$$

Let I_n be the n-by-n identity matrix. Given $g = I_p \oplus -I_q$ denote by $O_{p,q}(\mathbb{C})$ (by $O_{p,q}(\mathbb{R})$) and $U_{p,q}(\mathbb{C})$ the complex (real) pseudo-orthogonal and pseudo-unitary group, consisting of matrices of all complex (real) matrices Q such that $Q^{-1} = gQ^T g$ and $Q^{-1} = gQ^* g$, respectively.

We state our first main result; we prove it in Sec. 5.

Theorem 2.3. *For $\mu = (m_1, \ldots, m_N)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_1 > \ldots > \alpha_N$, and $\varepsilon = \{\varepsilon_{r,j}\}_{r=1,\ldots,N}^{j=1,\ldots,m_r}$ with all $\varepsilon_{r,j} \in \{1, -1\}$, let*

$$\mathcal{H}^\varepsilon = \bigoplus_{r=1}^{N} \Big(\bigoplus_{j=1}^{m_r} M_j^r \Big), \quad M_j^r = \begin{cases} \varepsilon_{r,j} H_{\alpha_r}(\sqrt{\rho}), & \rho \geq 0 \\ K_{\alpha_r}(\sqrt{-\rho}), & \rho < 0 \\ L_{\alpha_r}(\sqrt{\rho}), & \rho \in \mathbb{C} \setminus \mathbb{R} \end{cases},$$

$$B_r := \begin{cases} \oplus_{j=1}^{m_r} \varepsilon_{r,j}, & \rho \geq 0 \\ \rho I_{m_r} \oplus I_{m_r}, & \rho < 0 \\ I_{2m_r}, & \rho \in \mathbb{C} \setminus \mathbb{R} \end{cases},$$

i.e. $\mathcal{H}^\varepsilon \overline{\mathcal{H}^\varepsilon}$ has precisely one eigenvalue ρ. Then the isotropy group $\Sigma_{\mathcal{H}^\varepsilon}$ is conjugate (hence isomorphic) to the subgroup $\mathbb{X} \subset \begin{cases} \mathbb{T}^{\alpha,\mu}, & \rho > 0 \\ \mathbb{T}^{\alpha,\mu} \oplus \overline{\mathbb{T}}^{\alpha,\mu}, & \rho \in \mathbb{C} \setminus \mathbb{R} \\ \mathbb{T}_c^{\alpha,\mu}, & \rho = 0 \\ \mathbb{T}^{\alpha,2\mu}, & \rho < 0 \end{cases}$, where $\mathbb{T}^{\alpha,\mu}$, $\mathbb{T}_c^{\alpha,\mu}$ and $\mathbb{T}^{\alpha,2\mu}$ are defined by (2.4). Furthermore, each $\mathcal{X} \in \mathbb{X}$ for $\rho \in \mathbb{R}$ and each $\mathcal{X} \oplus \overline{\mathcal{X}} \in \mathbb{X}$ for $\rho \in \mathbb{C} \setminus \mathbb{R}$, with $\mathcal{X}$ of the form (2.4), satisfy the following properties:

(a) If $\rho > 0$ then all $\mathcal{X}_{rs}$ are real, while for $\rho < 0$ the upper triangular parts $\mathcal{T}_{rs} = T(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$ of $\mathcal{X}_{rs}$ consist of 2-by-2 block matrices of the form:

$$A_n = \begin{bmatrix} V_n^{rs} & W_n^{rs} \\ \rho \overline{W}_n^{rs} + \overline{W}_{n-1}^{rs} & \overline{V}_n^{rs} \end{bmatrix}, \quad V_n^{rs}, W_n^{rs} \in \mathbb{C}^{m_r \times m_s}, \; n \in \{1, \ldots, b_{rs} - 1\};$$
$$V_{-1}^{rs} = W_{-1}^{rs} = 0. \tag{2.5}$$

(b) The nonzero entries of $\mathcal{X}_{rs}$ for $r, s \in \{1, \ldots, N\}$ with $r > s$ can be chosen freely in accordance with (a). If either $\rho \in \mathbb{C} \setminus \{0\}$ or $\rho = 0$ with α_r odd, then $(\mathcal{X}_{rr})_{11} = A_0^{rr}$ is a solution of the equation $B_r = (A_0^{rr})^T B_r A_0^{rr}$ and such that (a) remains valid, while for $\rho = 0$ with α_r even, the matrix A_0^{rr} is any solution of the equation $B_r = (A_0^{rr})^* B_r A_0^{rr}$.

(c) For $r \in \{1, \ldots, N\}$ with $\alpha_r \geq 2$, $j \in \{1, \ldots, \alpha_r - 1\}$ we have $(\mathcal{X}_{rr})_{1(1+j)} = A_j^{rr} = A_0^{rr} B_r Z_j^r + D_j^r$ for $Z_j^r = \begin{cases} -(Z_j^r)^*, & \alpha_r - j \text{ even}, \rho = 0 \\ -(Z_j^r)^T, & \text{otherwise} \end{cases}$ chosen arbitrarily in accordance with (a), and for some D_j^r depending polynomially on $A_{j'}^{r'r'}$ with $j' \in \{0, \ldots, j-1\}$, $r' \in \{1, \ldots, r\}$ and on the nonzero entries of $\mathcal{X}_{rs}$ for $r > s$ (described in (b)).

The nonzero entries of $\mathcal{X}_{rs}$ for $r, s \in \{1, \ldots, N\}$ with $r < s$ are uniquely determined (polynomially) by the entries of $\mathcal{X}_{rs}$ with $r \geq s$ (described above).

In particular,

$$\dim_{\mathbb{R}}(\Sigma_{\mathcal{H}^{\varepsilon}}) = \begin{cases} \sum_{r=1}^{N} m_r\big(\tfrac{1}{2}\alpha_r(m_r-1) + \sum_{s=1}^{r-1}\alpha_s m_s\big), & \rho > 0 \\ \sum_{r=1}^{N}\big(\alpha_r m_r^2 + 2\sum_{s=1}^{r-1}\alpha_s m_r m_s\big) - \sum_{\alpha_r \ even}\tfrac{\alpha_r}{2}m_r - \sum_{\alpha_r \ odd}\tfrac{\alpha_r+1}{2}m_r, & \rho = 0 \\ \sum_{r=1}^{N} m_r\big(\alpha_r(2m_r-1) + 2\sum_{s=1}^{r-1}\alpha_s m_s\big), & \rho < 0 \\ \sum_{r=1}^{N} 2m_r\big(\tfrac{1}{2}\alpha_r(m_r-1) + \sum_{s=1}^{r-1}\alpha_s m_s\big), & \rho \in \mathbb{C}\setminus\mathbb{R} \end{cases}.$$

Remark 1. An algorithm to compute matrices in Theorem 2.3 (c) is provided as part of its proof, more precisely, by Lemma 4.1 and Lemma 4.2.

The following significant examples of matrices satisfy Theorem 2.3 (b), (c).

Example 2.4. ([22, Example 3.1]) Fix B_r nonsingular symmetric and let Z_n^r be any skew-symmetric matrix (i.e. $Z_n^r = -(Z_n^r)^T$); all of size $m_r \times m_r$. We set $W_0^r := 0$ and

$$\mathcal{W} = \bigoplus_{r=1}^{N} T(I_{m_r}, W_1^r, \ldots, W_{\alpha_r-1}^r), \qquad W_n^r := \frac{1}{2}B_r^{-1}\big(Z_n^r - \sum_{j=1}^{n-1}(W_j^r)^T B_r W_{n-j}^r\big). \tag{2.6}$$

Example 2.5. For $r \in \{1,\ldots,N\}$, $n \in \{1,\ldots,\alpha_r-1\}$, we are given B_r nonsingular real symmetric with $\mathcal{B}_n^r := \oplus_{j=1}^n B_r$ and let Z_n^r be any skew-symmetric matrix for $\alpha_r - n$ odd (skew-Hermitian for $\alpha_r - n$ even); all of size $m_r \times m_r$. Set:

$$\mathcal{W} = \bigoplus_{r=1}^{N} T_c(I_{m_r}, W_1^r, \ldots, W_{\alpha_r-1}^r), \quad W_n^r := \frac{1}{2}B_r^{-1}\Big(Z_n^r - \begin{cases} \mathcal{A}_{n-1}^r\mathcal{B}_{n-1}^r\mathcal{P}_{n-1}^r, & \alpha_r \text{ even} \\ \overline{\mathcal{A}}_{n-1}^r\mathcal{B}_{n-1}^r\mathcal{P}_{n-1}^r & \alpha_r \text{ odd} \end{cases}\Big), \tag{2.7}$$

$$\mathcal{A}_n^r := \begin{cases} \big[\,(W_1^r)^T, (\overline{W}_2^r)^T, \ldots, (\overline{W}_{n-1}^r)^T, (W_n^r)^T\,\big], & n \text{ odd} \\ \big[\,(W_1^r)^T, (\overline{W}_2^r)^T, \ldots, (W_{n-1}^r)^T, (\overline{W}_n^r)^T\,\big], & n \text{ even} \end{cases}, \quad \mathcal{P}_{2n-1}^r := \begin{bmatrix} \overline{W}_{2n-1}^r \\ W_{2n-2}^r \\ \vdots \\ W_2^r \\ \overline{W}_1^r \end{bmatrix},$$

$$\mathcal{P}_{2n}^r := \begin{bmatrix} \overline{W}_{2n}^r \\ W_{2n-1}^r \\ \vdots \\ \overline{W}_2^r \\ W_1^r \end{bmatrix};$$

the entry in the j-th column of $\mathcal{A}_n^r$ is $(W_j^{kr})^T$ (and $(\overline{W}_j^{kr})^T$) for j odd (even), and the entry in the j-th row of $\mathcal{P}_n^{ks}$ is $(W_{n-j+1}^{ks})^T$ (and $(\overline{W}_{n-j+1}^{ks})^T$) for j even (odd), $\mathcal{P}_0^r := 0$.

Example 2.6. Let a matrix of the form (2.4) have the identity as principal submatrix, formed by all blocks except those at the p-th, the t-th columns and rows, i.e.

$$\mathcal{T}_{rs} = \begin{cases} \oplus_{j=1}^{\alpha_r} I_{m_r}, & r = s, \\ 0, & r \neq s \end{cases}, \{r, s\} \not\subset \{p, t\}. \tag{2.8}$$

In particular, given B_r nonsingular symmetric of size m_r-by-m_r, $F \in \mathbb{C}^{m_p \times m_t}$ and $0 \leq k \leq \alpha_t - 1$, $r \in \{1, \ldots, N\}$, in [22, Example 3.2] we set:

$$\begin{aligned} &\mathcal{T}_{rr} = T(I_{m_r}, A_1^{rr}, \ldots, A_{\alpha_r-1}^{rr}), \quad r \in \{p, t\}, \qquad p < t, \\ &A_j^{pp} = \begin{cases} a_{n-1} B_p^{-1} (F^T B_t F B_p^{-1})^n B_0^r, & j = n(2k + \alpha - \beta) \\ 0, & \text{otherwise} \end{cases}, \\ &A_j^{tt} = \begin{cases} a_{n-1} B_t^{-1} (B_t F B_p^{-1} F^T)^n B_t, & j = n(2k + \alpha - \beta) \\ 0, & \text{otherwise} \end{cases}, \\ &a_n := -\frac{1}{2^{2n+1}(n+1)} \binom{2n}{n}, \qquad \mathcal{T}_{tp} = N_{\alpha_t}^k(F), \qquad \mathcal{T}_{pt} = N_{\alpha_t}^k\big(-B_p^{-1} F^T B_p\big), \end{aligned} \tag{2.9}$$

where $N_\beta^k(X)$ is a β-by-β block matrix with X on the k-th upper diagonal (the main diagonal for $k = 0$) and zeros otherwise. For example, if $N = 2$, $\alpha_1 = 4$, $\alpha_2 = 2$, $m_1 = 2$, $m_2 = 3$, $B_1 = I_2$, $B_2 = I_3$, then $F \in \mathbb{C}^{2\times 3}$ and we obtain

$$\left[\begin{array}{cccc|cc} I_2 & 0 & -\frac{1}{2}F^T F & 0 & -F^T & 0 \\ 0 & I_2 & 0 & -\frac{1}{2}F^T F & 0 & -F^T \\ 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 \\ \hline 0 & 0 & F & 0 & I_3 & 0 \\ 0 & 0 & 0 & F & 0 & I_3 \end{array}\right]. \tag{2.10}$$

The other intrieguing choice, with $G := \begin{cases} F, & k + \alpha_t \text{ odd} \\ \overline{F}, & k + \alpha_t \text{ even} \end{cases}$, is

$$\begin{aligned} &\mathcal{T}_{rr} = T_c(I_{m_r}, A_1^{rr}, \ldots, A_{\alpha_r-1}^{rr}), \quad r \in \{p, t\}, \qquad p < t \\ &A_{n(2k+\alpha_p-\alpha_t)}^{pp} = a_{n-1} B_p^{-1} \begin{cases} (G^T B_t G B_p^{-1})^n B_p, & \alpha_p, \alpha_t \text{ odd} \\ (G^T B_t \overline{G} B_p^{-1})^n B_p, & \alpha_p, \alpha_t \text{ even} \\ (G^T B_t G B_p^{-1})^{\overline{n}} B_p, & \alpha_p \text{ even}, \alpha_t \text{ odd} \\ (G^T B_t \overline{G} B_p^{-1})^{\overline{n}} B_p, & \alpha_p \text{ odd}, \alpha_t \text{ even} \end{cases}, \end{aligned}$$

$$A^{tt}_{n(2k+\alpha_p-\alpha_t)} = a_{n-1}B_t^{-1}\begin{cases}(B_tFB_p^{-1}F^T)^nB_t, & \alpha_p,\alpha_t \text{ odd}\\ (B_tFB_p^{-1}\overline{F}^T)^nB_t, & \alpha_p,\alpha_t \text{ even}\\ (B_tFB_p^{-1}F^T)^{\overline{n}}B_t, & \alpha_p \text{ even}, \alpha_t \text{ odd}\\ (B_tFB_p^{-1}\overline{F}^T)^{\overline{n}}B_t, & \alpha_p \text{ odd}, \alpha_t \text{ even}\end{cases}, \tag{2.11}$$

$$A^{tt}_j = 0, \quad A^{pp}_j = 0, \qquad j \neq n(2k+\alpha_p-\alpha_t), \quad a_n = -\frac{1}{2^{2n+1}(n+1)}\binom{2n}{n}$$

$$\mathcal{T}_{tp} = Nc^k_{\alpha_t}(F), \qquad \mathcal{U}_{pt} = Nc^k_{\alpha_t}\big(-B_p^{-1}G^TB_t\big),$$

in which $X^{\overline{n}} := \begin{cases}X\overline{X}X\cdots\overline{X}X, & n \text{ odd}\\ X\overline{X}\cdots X\overline{X}, & n \text{ even}\end{cases}$ is the complex-alternating product of n factors with X as odd (with $\overline{X}$ as even) factor, and $Nc^k_\beta(X)$ is a complex-alternating β-by-β Toeplitz with $X, \overline{X}, X, \ldots$ on the k-th upper diagonal (the main diagonal for $k=0$) and zeros otherwise. If $N=2$, $\alpha_1=4$, $\alpha_2=2$, $m_1=2$, $m_2=3$, $B_1=I_2$, $B_2=I_3$, we have (cf. (2.10)):

$$\left[\begin{array}{cccc|cc} I_2 & 0 & -\frac{1}{2}F^*F & 0 & -F^* & 0\\ 0 & I_2 & 0 & -\frac{1}{2}F^T\overline{F} & 0 & -F^T\\ 0 & 0 & I_2 & 0 & 0 & 0\\ 0 & 0 & 0 & I_2 & 0 & 0\\ \hline 0 & 0 & F & 0 & I_3 & 0\\ 0 & 0 & 0 & \overline{F} & 0 & I_3\end{array}\right]$$

We now exihibit the structure of isotropy groups; the proof is given in Sec. 5.

Theorem 2.7. *Let $\mathcal{H}^\varepsilon$, $\mathbb{T}^{\alpha,\mu}$, $\mathbb{T}^{\alpha,2\mu}$ and $\mathbb{T}^{\alpha,\mu}_c$ be as in Theorem 2.3. Then $\Sigma_{\mathcal{H}^\varepsilon}$ is isomorphic to a semidirect product:*

$$\Sigma_{\mathcal{H}^\varepsilon} \cong \mathbb{O} \ltimes \mathbb{V},$$

in which $\mathbb{O}$ and $\mathbb{V}$ are described as follows:

(I) *Suppose $\mathcal{H}^\varepsilon = \bigoplus_{r=1}^N \Big(\bigoplus_{j=1}^{p_r} H_{\alpha_r}(\lambda) \oplus \bigoplus_{j=1}^{q_r} -H_{\alpha_r}(\lambda)\Big)$ for $\lambda \geq 0$, $m_r := p_r + q_r$.*

- *i. If $\lambda > 0$, then $\mathbb{O} \subset \mathbb{T}^{\alpha,\mu}$ consists of all matrices $\mathcal{Q} = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{\alpha_r} Q_r\big)$ with $Q_r \in O_{p_r,q_r}(\mathbb{R})$, while $\mathbb{V} \subset \mathbb{T}^{\alpha,\mu}$ is generated by all real matrices of the form (2.6) and of the form (2.4) with (2.8), (2.9) for $B_r = I_{p_r} \oplus -I_{q_r}$.*
- *ii. If $\lambda = 0$ and for α_r odd $m_r = p_r$, then $\mathbb{O} \subset \mathbb{T}^{\alpha,\mu}_c$ consists of all matrices $\mathcal{Q} = \bigoplus_{r=1}^N (Q_r \oplus \overline{Q}_r \oplus Q_r \oplus \cdots)$ with $Q_r \in O_{m_r}(\mathbb{C})$ for α_r odd and $Q_r \in U_{p_r,q_r}(\mathbb{C})$ for α_r even, while $\mathbb{V} \subset \mathbb{T}^{\alpha,\mu}_c$ is generated by matrices of the form (2.7) and of the form (2.4) with (2.8), (2.11) for $B_r = I_{p_r} \oplus -I_{q_r}$.*

(The possible summands $\bigoplus_{j=1}^0 \pm H_{\alpha_r}(\lambda)$ and $\pm I_0$ are left out.)

(II) *If* $\mathcal{H}^{\varepsilon} = \bigoplus_{r=1}^{N} (\bigoplus_{k=1}^{m_r} K_{\alpha_r}(\mu))$ *for* $\mu > 0$*, then* $\mathbb{O} \subset \mathbb{T}^{\alpha,2\mu}$ *consists of all matrices* $\mathcal{Q} = \bigoplus_{r=1}^{N} \big(\bigoplus_{j=1}^{\alpha_r} Q_r\big)$ *such that* $(\mu I_{m_r} \oplus I_{m_r})Q_r(\mu I_{m_r} \oplus I_{m_r}) \in O_{m_r,m_r}(\mathbb{C})$*, and such that* $Q_r = \begin{bmatrix} V_0^{rr} & W_0^{rr} \\ -\mu^2\overline{W}_0^{rr} & \overline{V}_0^{rr} \end{bmatrix}$ *for some* $V_0^{rr}, W_0^{rr} \in \mathbb{C}^{m_r \times m_r}$*, while each* $\mathcal{V} \in \mathbb{V} \subset \mathbb{T}^{\alpha,2\mu}$ *can be written as* $\mathcal{V} = \mathcal{V}_0 \prod_{j=1}^{n} \mathcal{V}_j$*, where* $\mathcal{V}_0 = \bigoplus_{r=1}^{N} \mathcal{W}_r$ *with* $\mathcal{W}_r$ *upper unitriangular Toeplitz and* $\mathcal{V}_1, \ldots, \mathcal{V}_n$ *of the form (2.8); all* $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n$ *are of the form (2.4) with (2.5) and satisfying (b), (c) for* $\rho < 0$ *in Theorem 2.3.*

(III) *If* $\mathcal{H}^{\varepsilon} = \bigoplus_{r=1}^{N} (\bigoplus_{l=1}^{m_r} L_{\alpha_r}(\xi))$*,* $\xi^2 \in \mathbb{C} \setminus \mathbb{R}$*, then* $\mathbb{O} \subset \mathbb{T}^{\alpha,\mu}$ *consists of all matrices* $\mathcal{Q} = \bigoplus_{r=1}^{N} \big(\bigoplus_{j=1}^{\alpha_r} Q_r\big)$ *with* $Q_r \in O_{m_r}(\mathbb{C})$*, and* $\mathbb{V} \subset \mathbb{T}^{\alpha,\mu}$ *is generated by matrices of the form (2.6) and of the form (2.4) with (2.8), (2.9) for* $B_r = I_{m_r}$*.*

In particular, $\mathbb{V}$ *is unipotent of order at most* $\alpha_1 - 1$ *(nilpotent of class* $\leq \alpha_1$*).*

Remark 2. Isotropy groups for A and iA under orthogonal *conjugation coincide, thus analogues of Theorem 2.7 and Theorem 2.7 for skew-Hermitian matrices are valid.

3. The matrix equation $A\overline{Y} = YA$

Given a square matrix A we consider the matrix equation

$$A\overline{Y} = YA. \tag{3.1}$$

For $Y = PXP^{-1}$ with P nonsingular, (3.1) transforms to $B\overline{X} = XB$ for $B = P^{-1}A\overline{P}$; such A and B are said to be *consimilar*. Bevis, Hall and Hartwig [2] used the canonical form under consimilarity, given by Hong and Horn [10, Theorem 3.1], to reduce (3.1) to Sylvester equations. In a similar fashion we shall solve (3.1) by using Hong's Hermitian consimilarity canonical form (2.1) for $\varepsilon = (1, 1, \ldots)$ [9, p. 3-4]. Consimilarity canonical forms were first developed by Haantjes [7], Asano and Nakayama [1], but these are not suitable to solve (3.1).

Recall the classical result [5, Ch. VIII] on solutions of a Sylvester equation.

Theorem 3.1. *Given* $\lambda_1, \lambda_2 \in \mathbb{C}$*, an* m*-by-*n *matrix* Y *satisfies the matrix equation*

$$J_m(\lambda_1)X = XJ_n(\lambda_2), \qquad J_\alpha(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{C} \quad (\alpha\text{-by-}\alpha),$$

if and only if either $\lambda_1 \neq \lambda_2$ *and* $X = 0$*, or* $\lambda_1 = \lambda_2$ *and*

$$X = \begin{cases} \begin{bmatrix} 0 & T \end{bmatrix}, & m < n \\ \begin{bmatrix} T \\ 0 \end{bmatrix}, & m > n \\ T, & n = m \end{cases}, \tag{3.2}$$

in which T is an β-by-β upper triangular Toeplitz matrix ($\beta = \min\{m, n\}$).

Lemma 3.2. *Given matrices M and N, let us consider the following equation*

$$M\overline{Y} = YN. \tag{3.3}$$

Denote the n-by-n backward identity matrix by E_n (with ones on the anti-diagonal).

1. *If M and N are of the form (2.2) or (2.3) and such that $M\overline{M}$ and $N\overline{N}$ correspond to different eigenvalues, it then follows that $Y = 0$.*
2. *If $M = H_m(\lambda)$ and $N = H_n(\lambda)$ with λ positive (zero), then Y satisfies (3.3) if and only if $Y = P_m^{-1} X P_n$, in which X is an m-by-n matrix of the form (3.2) for T an β-by-β real (complex-alternating) upper triangular Toeplitz matrix with $\beta = \min\{m, n\}$, and $P_\alpha := \frac{1}{\sqrt{2}} e^{-\frac{i\pi}{4}} (I_\alpha + iE_\alpha)$ for $\alpha \in \{m, n\}$.*
3. *If $M = K_m(\mu)$ and $N = K_n(\mu)$ with $\mu > 0$, then Y satisfies (3.3) if and only if $Y = Q_m^{-1} V_m^{-1} S_m(\mu) X S_n^{-1}(\mu) V_n Q_n$, in which $Q_\alpha := e^{\frac{i\pi}{4}}(P_\alpha \oplus P_\alpha)$, $V_\alpha := e^{i\frac{\pi}{4}}(W_\alpha \oplus \overline{W}_\alpha)$ with $W_\alpha := \oplus_{j=0}^{\alpha-1} i^j$ for $\alpha \in \{m, n\}$, and*

$$X = \begin{bmatrix} X_1 & X_2 \\ J_m(-\mu^2)\overline{X}_2 & \overline{X}_1 \end{bmatrix}, \tag{3.4}$$

where X_1, X_2 are m-by-n matrices of the form (3.2) for an β-by-β upper triangular Toeplitz T with $\beta = \min\{m, n\}$, and $S_\alpha(\eta) := \left[\begin{smallmatrix} 0 & U_\alpha(\eta) \\ J_\alpha(-i\eta)\overline{U}_\alpha(\eta) & 0 \end{smallmatrix}\right]$ with $U_\alpha(\eta)$ as any solution of $U_\alpha(\eta) J_\alpha(-\eta^2) = (J_\alpha(i\eta))^2 U_\alpha(\eta)$ for $\alpha \in \{m, n\}$.
4. *If $M = L_m(\xi)$ and $N = L_n(\xi)$ with $\mathrm{Im}(\xi) > 0$ and ξ^2 nonreal, then Y satisfies (3.3) if and only if $Y = R_m^{-1} X R_n$, in which $X = X_1 \oplus \overline{X}_1$ and X_1 is an m-by-n matrix of the form (3.2) for T an β-by-β complex upper triangular Toeplitz matrix with $\beta = \min\{m, n\}$, and $R_\alpha := P_\alpha \oplus P_\alpha$ for $\alpha \in \{m, n\}$.*

The proof of the lemma relies very much on the ideas in [2].

Proof of Lemma 3.2. The following is a part of Hong's construction of the canonical form under consimilarity [9, p. 9-10]:

$$H_\alpha(\lambda) = P_\alpha^{-1} J_\alpha(\lambda) \overline{P}_\alpha, \quad K_\alpha(\mu) = Q_\alpha^{-1} \left[\begin{smallmatrix} 0 & J_\alpha(\mu) \\ -J_\alpha(\mu) & 0 \end{smallmatrix}\right] \overline{Q}_\alpha, \quad L_\alpha(\xi) = R_\alpha^{-1} \left[\begin{smallmatrix} 0 & J_\alpha(\xi) \\ J_\alpha(\overline{\xi}) & 0 \end{smallmatrix}\right] \overline{R}_\alpha,$$

in which $\lambda \geq 0$, $\mu > 0$, $\xi^2 \in \mathbb{C} \setminus \mathbb{R}$, and P_α, Q_α, R_α are as defined in the lemma.

The equation $H_m(\lambda)\overline{Y} = YH_n(\kappa)$ for $\lambda, \kappa \geq 0$ transforms to $J_m(\lambda)\overline{X} = XJ_n(\lambda)$ with $X = P_m Y P_n^{-1}$. By setting $X = U + iV$ with real m-by-n matrices U, V, we get $J_m(\lambda)U = UJ_n(\kappa)$ and $-J_m(\lambda)V = VJ_n(\kappa)$. The first equation for $\lambda \neq \kappa$ implies $U = 0$, while for $\lambda = \kappa$ we get U upper triangular Toeplitz (see Theorem 3.1). We write the second equation as $J_m(-\lambda)FV = FVJ_n(\kappa)$ with $F = -1 \oplus 1 \oplus -1 \oplus \cdots$. If either

$\lambda \neq \kappa$ or $\lambda = \kappa > 0$, then $V = 0$. When $\lambda = \kappa = 0$, then FV is real upper triangular Toeplitz, hence X is complex-alternating upper triangular Toeplitz. This proves (1) for $M = H_m(\lambda)$, $N = H_n(\mu)$ with $\lambda \neq \nu$ and (2).

If V_α and $S_\alpha(\mu)$ are defined as (3), it is not difficult to check that

$$\begin{bmatrix} 0 & J_\alpha(\eta) \\ -J_\alpha(\eta) & 0 \end{bmatrix} = V_\alpha^{-1} \begin{bmatrix} 0 & J_\alpha(i\eta) \\ J_\alpha(-i\eta) & 0 \end{bmatrix} \overline{V}_\alpha,$$
$$S_\alpha^{-1}(\eta) \begin{bmatrix} 0 & J_\alpha(i\eta) \\ J_\alpha(-i\eta) & 0 \end{bmatrix} \overline{S}_\alpha(\eta) = \begin{bmatrix} 0 & I_\alpha \\ J_\alpha(-\eta^2) & 0 \end{bmatrix}.$$

Thus $K_m(\mu)\overline{Y} = YK_n(\nu)$ for $\mu, \nu > 0$ transforms to

$$J'_m(\mu)\overline{X} = XJ'_n(\nu), \qquad X = S_m^{-1}(\mu)V_m Q_m Y Q_n^{-1} V_n^{-1} S_n(\nu), \quad J'_\alpha(\mu) := \begin{bmatrix} 0 & I_\alpha \\ J_\alpha(-\eta^2) & 0 \end{bmatrix}.$$

Set $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$: $\overline{X}_3 = X_2 J_n(-\nu^2)$, $J_m(-\mu^2)\overline{X}_1 = X_4 J_n(-\nu^2)$, $J_m(-\mu^2)\overline{X}_2 = X_3$, $\overline{X}_4 = X_1$. If $\mu = \nu$ we get (3), while $\mu \neq \nu$ gives (1) for $M = K_m(\mu)$, $N = K_n(\nu)$.

We transform $L_m(\xi)\overline{Y} = YL_n(\zeta)$ for $\mathrm{Im}(\xi), \mathrm{Im}(\zeta) > 0$ to

$$\begin{bmatrix} 0 & J_m(\xi) \\ J_m(\overline{\xi}) & 0 \end{bmatrix} \overline{X} = X \begin{bmatrix} 0 & J_n(\zeta) \\ J_n(\overline{\zeta}) & 0 \end{bmatrix}, \qquad R_m Y R_n^{-1} = X := \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right],$$

where X_1, X_2, X_3, X_4 are m-by-n matrices. We have

$$\begin{aligned} X_2 J_n(\overline{\zeta}) &= J_m(\xi)\overline{X}_3, \quad & X_3 J_n(\zeta) &= J_m(\overline{\xi})\overline{X}_2, \\ X_1 J_n(\zeta) &= J_m(\xi)\overline{X}_4, \quad & X_4 J_n(\overline{\zeta}) &= J_m(\overline{\xi})\overline{X}_1. \end{aligned} \tag{3.5}$$

By combining the first and the last pair of equations we deduce $\overline{X}_3(J_{\gamma_l}(\overline{\zeta}))^2 = (J_m(\xi))^2\overline{X}_3$, $\overline{X}_2(J_n(\zeta))^2 = (J_m(\overline{\xi}))^2\overline{X}_2$ and $\overline{X}_4(J_{\gamma_l}(\zeta))^2 = (J_m(\xi))^2\overline{X}_4$, $\overline{X}_1(J_n(\overline{\zeta}))^2 = (J_m(\overline{\xi}))^2\overline{X}_1$, respectively. Since $\mathrm{Im}(\xi), \mathrm{Im}(\zeta) > 0$, the first two equations imply $X_3 = X_2 = 0$, while the last two for $\xi \neq \zeta$ yield $X_1 = X_4 = 0$ (thus (1) for $M = L_m(\xi)$, $N = L_n(\zeta)$). Subtracting the conjugation of the last equation from the third equation of (3.5) for $\xi = \zeta$ gives $(X_1 - \overline{X}_4)J_n(\xi) = -J_m(\xi)(X_1 - \overline{X}_4)$. Hence $F(X_1 - \overline{X}_4)J_n(\xi) = J_m(-\xi)F(X_1 - \overline{X}_4)$, $F = -1 \oplus 1 \oplus -1 \oplus \cdots$, thus we obtain $X_4 = \overline{X}_1$. Using (3.5) then yields that X_1 is complex upper triangular Toeplitz, which shows (4).

Similarly, $K_m(\mu)\overline{Y} = YL_n(\xi)$ for $\mu > 0$, $\xi^2 \in \mathbb{C} \setminus \mathbb{R}$ reduces to $\begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu) & 0 \end{bmatrix}\overline{Y} = Y\begin{bmatrix} 0 & J_n(\xi) \\ J_n(\overline{\xi}) & 0 \end{bmatrix}$ with $Q_m X R_n^{-1} = Y := \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$ and X_1, X_2, X_3, X_4 of size $m \times n$. Thus

$$\begin{aligned} X_2 J_n(\overline{\xi}) &= J_m(\mu)\overline{X}_3, \quad & X_3 J_n(\xi) &= -J_m(\mu)\overline{X}_2, \\ X_1 J_n(\xi) &= J_m(\mu)\overline{X}_4, \quad & X_4 J_n(\overline{\xi}) &= -J_m(\mu)\overline{X}_1. \end{aligned}$$

By combining these equations we get $\overline{X}_3(J_n(\overline{\xi}))^2 = -(J_m(\mu))^2\overline{X}_3$ and $\overline{X}_4(J_n(\xi))^2 = -(J_m(\mu))^2\overline{X}_4$, which implies $X_1 = X_2 = X_3 = X_4 = 0$, hence $X = 0$.

Next, $H_m(\lambda)\overline{Y} = YK_n(\mu)$ for $\lambda \geq 0$, $\mu > 0$ reduces to $J_m(\lambda)\overline{X} = X\left[\begin{smallmatrix} 0 & J_n(\mu) \\ -J_n(\mu) & 0 \end{smallmatrix}\right]$, where $P_m Y Q_n^{-1} = X := \left[\begin{smallmatrix} X_1 & X_2 \end{smallmatrix}\right]$ with m-by-n matrices X_1, X_2. We get $J_m(\lambda)\overline{X}_1 = -X_2 J_n(\mu)$, $J_m(\lambda)\overline{X}_2 = X_1 J_n(\mu)$, thus $(J_m(\lambda))^2 X_1 = -J_m(\lambda)\overline{X}_2 J_n(\mu) = -X_1(J_n(\mu))^2$. It yields $S^{-1}J_m(\lambda^2)SX_1 = -X_1 T^{-1} F J_n(-\mu^2) F^{-1} T$ and for some nonsingular S, T and $F = -1 \oplus 1 \oplus -1 \oplus \cdots$. Since $\lambda^2 \geq 0 > -\mu^2$, Theorem 3.1 implies $SX_1 T^{-1} F = 0$ with $X_1 = 0$ (hence $X_2 = 0$), and therefore $X = 0$.

Further, $H_m(\lambda)\overline{Y} = YL_n(\xi)$ yields $J_m(\lambda)\overline{X} = X\left[\begin{smallmatrix} 0 & J_n(\xi) \\ J_n(\overline{\xi}) & 0 \end{smallmatrix}\right]$ with $P_m Y R_n^{-1} = X := \left[\begin{smallmatrix} X_1 & X_2 \end{smallmatrix}\right]$ for some m-by-n matrices X_1, X_2. We obtain $J_m(\lambda)\overline{X}_1 = X_2 J_n(\overline{\xi})$ and $J_m(\lambda)\overline{X}_2 = X_1 J_n(\xi)$, therefore $(J_m(\lambda))^2 X_1 = J_m(\lambda)\overline{X}_2 J_n(\xi) = X_1(J_n(\xi))^2$. If $\lambda \geq 0$ and ξ^2 is nonreal, Theorem 3.1 yields $X_1 = X_2 = 0$), thus $X = 0$.

Since $H_m(\lambda)$, $K_n(\mu)$, $L_n(\xi)$ are Hermitian, by conjugating and transposing $K_n(\mu)\overline{Y} = YH_m(\lambda)$, $L_n(\xi)\overline{Y} = YH_m(\lambda)$, $L_n(\xi)\overline{Y} = YK_m(\mu)$ we obtain $Y^T K_n(\mu) = H_m(\lambda)\overline{Y}^T$, $L_n(\xi)Y^T = H_m(\lambda)\overline{Y}^T$, $Y^T L_m(\xi) = K_n(\mu)\overline{Y}^T$, respectively. These equations have already been solved with solution $Y = 0$. This concludes (1). □

Remark 3. The form of a solution of (3.3) for $M = L_m(\xi)$, $N = L_n(\xi)$ with $\xi^2 \in \mathbb{C} \setminus \mathbb{R}$ in [2] is not suited for our application in the proof of Theorem 2.7; the usage of $\left[\begin{smallmatrix} 0 & J_m(\xi) \\ J_m(\overline{\xi}) & 0 \end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} 0 & I_m \\ J_m(\xi^2) & 0 \end{smallmatrix}\right]$ in the proof of Lemma 3.2 is essential.

We proceed with a technical lemma based on the idea from the paper by Lin, Mehrmann and Xu [16, Sec. 3.1] (see also [22, Sec. 2]). It enables us to transform a block matrix with (complex-alternating) upper triangular Toeplitz blocks to a block (complex-alternating) upper triangular Toeplitz matrix. Set

$$\Omega_{\alpha,m} := \left[e_1 \; e_{\alpha+1} \; \ldots \; e_{(m-1)\alpha+1} \; e_2 \; e_{\alpha+2} \; \ldots \; e_{(m-1)\alpha+2} \; \ldots \; e_\alpha \; e_{2\alpha} \; \ldots \; e_{\alpha m}\right], \tag{3.6}$$

where $e_1, e_2, \ldots, e_{\alpha m}$ is the standard orthonormal basis in $\mathbb{C}^{\alpha m}$. Multiplication with $\Omega_{\alpha,m}$ from the right (with $\Omega_{\alpha,m}^T$ from the left) puts the k-th, the $(\alpha+k)$-th, …, the $((m-1)\alpha+k)$-th column (row) together for all $k \in \{1, \ldots, \alpha\}$. For example,

$$\Omega_{3,2}^T \left[\begin{array}{cc|cc|cc} a_1 & b_1 & a_2 & b_2 & a_3 & b_3 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a_4 & b_4 & a_5 & b_5 & a_6 & b_6 \\ 0 & a_4 & 0 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Omega_{2,3} = \left[\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \\ \hline 0 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_4 & a_5 & a_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Similarly, multiplication with the following matrix from the right puts the k-th, the $(2\alpha+k)$-th, …, the $((2m-1)\alpha+k)$-th column (row) together:

$$\Omega'_{\alpha,m} := \Big[e_1 \; e_{2\alpha+1} \; \ldots \; e_{2(m-1)\alpha+1} \; e_{\alpha+1} \; e_{3\alpha+1} \; \ldots \; e_{(2m-1)\alpha+1} \; e_2 \; e_{2\alpha+2} \; \ldots \; e_{2(m-1)\alpha+2}$$

$$e_{\alpha+2}\, e_{3\alpha+2}\, \dots\, e_{(2m-1)\alpha+2}\, \dots \dots\, e_{\alpha}\, e_{3\alpha}\, \dots\, e_{\alpha(2m-1)}\, e_{2\alpha}\, e_{4\alpha}\, \dots\, e_{\alpha(2m)}\Big]. \tag{3.7}$$

It is then immediate:

Lemma 3.3. *Suppose $X = [X_{rs}]_{r,s=1}^{N}$ such that each block $X_{rs} = [(X_{rs})_{jk}]_{j,k=1}^{m_r,m_s}$ is an m_r-by-m_s block matrix with blocks of the same size, and let $\alpha_1 > \ldots > \alpha_N$ with $b_{rs} := \{\alpha_r, \alpha_s\}$. Also, set $\Omega := \bigoplus_{r=1}^{N} \Omega_{\alpha_r,m_r}$ and $\Omega' := \bigoplus_{r=1}^{N} \Omega'_{\alpha_r,m_r}$.*

1. *Assume that each X_{rs} consists of blocks of size $\alpha_r \times \alpha_s$ and such that*

$$(X_{rs})_{jk} = \begin{cases} [0 \quad T_{jk}^{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} T_{jk}^{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ T_{jk}^{rs}, & \alpha_r = \alpha_s \end{cases}, \qquad \begin{array}{l} T_{jk}^{rs} = T(a_{0,jk}^{rs}, a_{1,jk}^{rs}, \ldots, a_{b_{rs}-1,jk}^{rs}) \\ \\ (\text{or } T_{jk}^{rs} = T_c(a_{0,jk}^{rs}, a_{1,jk}^{rs}, \ldots, a_{b_{rs}-1,jk}^{rs})) \end{array}$$

for $j \in \{1, \ldots, m_r\}$, $k \in \{1, \ldots, m_s\}$, $a_{n,jk}^{rs} \in \mathbb{C}$, and set $A_n^{rs} := [a_{n,jk}^{rs}]_{j,k=1}^{m_r,m_s}$. Then

$$\mathcal{X} := \Omega^T X \Omega, \quad \mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^{N}, \qquad \mathcal{X}_{rs} = \begin{cases} [0 \quad \mathcal{T}_{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} \mathcal{T}_{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ \mathcal{T}_{rs}, & \alpha_r = \alpha_s \end{cases}, \tag{3.8}$$

with $\mathcal{X}_{rs}$ of size $\alpha_r \times \alpha_s$ and $\mathcal{T}_{rs} = T(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$ (or $\mathcal{T}_{rs} = T_c(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$).

2. *Let each X_{rs} consist of four blocks of size $\alpha_r \times \alpha_s$, and such that:*

$$(X_{rs})_{jk} = \begin{bmatrix} \tau_{jk}^{rs} & \sigma_{jk}^{rs} \\ J_{\alpha_r}(\eta)\overline{\sigma}_{jk}^{rs} & \overline{\tau}_{jk}^{rs} \end{bmatrix}, \qquad j \in \{1, \ldots m_r\}, \quad k \in \{1, \ldots m_s\}, \quad \eta \in \mathbb{C},$$

$$\tau_{jk}^{rs} = \begin{cases} [0 \ T_{jk}^{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} T_{jk}^{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ T_{jk}^{rs}, & \alpha_r = \alpha_s \end{cases}, \quad \sigma_{jk}^{rs} = \begin{cases} [0 \ S_{jk}^{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} S_{jk}^{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ S_{jk}^{rs}, & \alpha_r = \alpha_s \end{cases},$$

$$T_{jk}^{rs} = T(v_{0,jk}^{rs}, \ldots, v_{b_{rs}-1,jk}^{rs}), \;\; S_{jk}^{rs} = T(w_{0,jk}^{rs}, \ldots, w_{b_{rs}-1,jk}^{rs}), \quad \text{all } v_{n,jk}^{rs}, w_{n,jk}^{rs} \in \mathbb{C}.$$

Set $W_{-1}^{rs} := 0$ and further $V_n^{rs} := [v_{n,jk}^{rs}]_{j,k=1}^{m_r,m_s}$, $W_n^{rs} := [w_{n,jk}^{rs}]_{j,k=1}^{m_r,m_s}$ with $A_n^{rs} := \begin{bmatrix} V_n^{rs} & W_n^{rs} \\ \eta\overline{W}_n^{rs} + \overline{W}_{n-1}^{rs} & \overline{V}_n^{rs} \end{bmatrix}$ for $n \in \{0, \ldots, b_{rs}-1\}$. Then

$$\mathcal{X}' := (\Omega')^T X \Omega'$$

is of the form (3.8) with $\mathcal{T}_{rs} = T(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs})$.

Furthermore, if all $W_n^{rs} = 0$, then there exists a permutation matrix Ω_0 such that $\Omega_0^T X \Omega_0 = \mathcal{V} \oplus \overline{\mathcal{V}}$ with $\mathcal{V}$ of the form (3.8) for $\mathcal{T}_{rs} = T(V_0^{rs}, \ldots, V_{b_{rs}-1}^{rs})$.

The following proposition describes the (nonsingular) solutions of (3.1).

Proposition 3.4.

1. *Let $\rho_1, \ldots, \rho_n \in \mathbb{C}$ be all distinct and let $\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j$, in which $\mathcal{H}_j$ is a direct sum whose summands are either of the form (2.2) or (2.3), and such that they correspond to the eigenvalue ρ_j of $\mathcal{H}\overline{\mathcal{H}}$. Then the solution of $\mathcal{H}\overline{Y} = Y\mathcal{H}$ is of the form $Y = \bigoplus_{j=1}^{n} Y_j$ with Y_j as a solution of $\mathcal{H}_j\overline{Y}_j = Y_j\mathcal{H}_j$.*
2. *For $\mu = (m_1, \ldots, m_N)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$ let $\mathbb{T}^{\alpha,\mu}$, $\mathbb{T}^{\alpha,\mu}$ and $\mathbb{T}_c^{\alpha,\mu}$ consist of matrices as described in (2.4), and let $\mathcal{H} = \mathcal{H}^{\varepsilon}$ be as in Theorem 2.3 for all $\varepsilon_{r,j} = 1$ ($\mathcal{H}\overline{\mathcal{H}}$ has precisely one eigenvalue ρ). The nonsingular solutions of $\mathcal{H}\overline{Y} = Y\mathcal{H}$ form a group conjugate to $\mathbb{T}^{\alpha,\mu} \oplus \overline{\mathbb{T}}^{\alpha,\mu}$ for $\rho \in \mathbb{C} \setminus \mathbb{R}$, conjugate to $\mathbb{T}_c^{\alpha,\mu}$ for $\rho = 0$, conjugate to the subgroup of all real matrices in $\mathbb{T}^{\alpha,\mu}$ for $\rho > 0$, and conjugate to the subgroup of all matrices in $\mathbb{T}^{\alpha,2\mu}$ of the form (2.4) with (2.5) for $\rho < 0$.*

Proof. Suppose $\mathcal{H} = \bigoplus_j M_j$ with all M_j either of the form (2.2) or of the form (2.3). The equation $\mathcal{H}\overline{Y} = Y\mathcal{H}$ is then equivalent to a system of equations:

$$M_j\overline{Y}_{jk} = Y_{jk}M_k, \quad j, k = 1, 2, \ldots, \qquad Y := [Y_{jk}]_{j,k}, \tag{3.9}$$

in which Y is partitioned conformally to $\mathcal{H}$. Lemma 3.2 (1) implies (1).

Next, let all $M_j\overline{M}_j$ have the same eigenvalue ρ. In view of Lemma 3.2 there exist nonsingular matrices U_j so that any solution Y of (3.9) is of the form

$$Y = U^{-1}XU \quad (Y_{jk} = U_j^{-1}X_{jk}U_k^{-1}); \qquad X := [X_{jk}]_{j,k}, U := \oplus_j U_j,$$

where all X_{jk} are of the form (3.2) with real (complex-alternating) upper triangular Toeplitz T for $\rho > 0$ (for $\rho = 0$), or of the form (3.4) with upper triangular Toeplitz X_1, X_2 (and $X_2 = 0$) for $\rho < 0$ (for $\rho \in \mathbb{C} \setminus \mathbb{R}$). Lemma 3.3 gives (2). □

We observe the group structures of $\mathbb{T}^{\alpha,\mu}$, $\mathbb{T}_c^{\alpha,\mu}$. The claim for $\mathbb{T}^{\alpha,\mu}$ coincides with [22, Lemma 2.2] and its proof is based on ideas from [18, Example 6.49] describing upper unitriangular matrices; it works mutatis mutandis for $\mathbb{T}_c^{\alpha,\mu}$.

Lemma 3.5. *Let $\mathbb{T}^{\alpha,\mu}$ and $\mathbb{T}_c^{\alpha,\mu}$ consist of matrices defined in (2.4). Then $\mathbb{T}^{\alpha,\mu} = \mathbb{D} \ltimes \mathbb{U}$ and $\mathbb{T}_c^{\alpha,\mu} = \mathbb{D}_c \ltimes \mathbb{U}_c$ are semidirect products of subgroups, where $\mathbb{D} \subset \mathbb{T}^{\alpha,\mu}$, $\mathbb{D}_c \subset \mathbb{T}_c^{\alpha,\mu}$ contain nonsingular block diagonal matrices, and $\mathbb{U} \subset \mathbb{T}^{\alpha,\mu}$, $\mathbb{U}_c \subset \mathbb{T}_c^{\alpha,\mu}$ are normal subgroups consisting of upper (complex-alternating) unitriangular Toeplitz diagonal blocks. Moreover, $\mathbb{U}$ and $\mathbb{U}_c$ are unipotent of order $\leq \alpha_1 - 1$.*

4. Certain block matrix equation

Let $\alpha_1 > \alpha_2 > \ldots > \alpha_N$ and suppose that we are given nonsingular matrices

$$\mathcal{B} = \bigoplus_{r=1}^{N} T\big(B_0^r, B_1^r, \ldots, B_{\alpha_r-1}^r\big), \quad \mathcal{C} = \bigoplus_{r=1}^{N} T\big(C_0^r, C_1^r, \ldots, C_{\alpha_r-1}^r\big), \quad \mathcal{F} = \bigoplus_{r=1}^{N} E_{\alpha_r}(I_{m_r}), \tag{4.1}$$

with symmetric $B_n^r, C_n^r \in \mathbb{C}^{m_r \times m_r}$ and $E_\beta(I_m) := \begin{bmatrix} 0 & & I_m \\ & \cdot^{\cdot^{\cdot}} & \\ I_m & & 0 \end{bmatrix}$ is an β-by-β block matrix with I_m on the anti-diagonal and zero matrices otherwise. We find all $\mathcal{X}$ in $\mathbb{T}^{\alpha,\mu}$ or $\mathbb{T}_c^{\alpha,\mu}$ for $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\mu = (m_1, \ldots, m_N)$ (see (2.4)) that solve

$$\mathcal{C} = \mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X}; \tag{4.2}$$

this is essential to prove Theorem 2.3 and Theorem 2.7. The observation

$$(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})^T = \mathcal{X}^T\mathcal{B}^T\mathcal{F}\mathcal{X}\mathcal{F} = \mathcal{F}\mathcal{F}\mathcal{X}^T\mathcal{F}(\mathcal{F}\mathcal{B}^T\mathcal{F})\mathcal{X}\mathcal{F} = \mathcal{F}(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})\mathcal{F}$$

shows that for $r \neq s$ we have $(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})_{rs} = 0$ if and only if $(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})_{sr} = 0$. When comparing the left-hand side with the right-hand side of (4.2) blockwise, it thus suffices to observe the upper triangular parts of $\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X}$ and $\mathcal{C}$. Since $(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})_{rs}$ and $\mathcal{C}_{rs}$ are rectangular upper triangular Toeplitz of the same form, it is enough to compare their first rows. By simplifying the notation with $\mathcal{Y} := \mathcal{B}\mathcal{X}$ and $\widetilde{\mathcal{X}} := \mathcal{F}\mathcal{X}^T\mathcal{F}$, we obtain the entry in the j-th column and in the first row of $(\mathcal{F}\mathcal{X}^T\mathcal{F}\mathcal{B}\mathcal{X})_{rs} = (\widetilde{\mathcal{X}}\mathcal{Y})_{rs}$ by multiplying the first rows of blocks $\widetilde{\mathcal{X}}_{r1}, \ldots, \widetilde{\mathcal{X}}_{rN}$ with the j-th columns of blocks $\mathcal{Y}_{1s}, \ldots, \mathcal{Y}_{Ns}$, respectively, and then adding them. Hence (4.2) reduces to:

$$\begin{aligned}(\mathcal{C}_{r(r+p)})_{1j} =& (\widetilde{\mathcal{X}}_{rr})_{(1)}(\mathcal{Y}_{r(r+p)})^{(j)} + \sum_{k=r+1}^{N} (\widetilde{\mathcal{X}}_{rk})_{(1)}(\mathcal{Y}_{k(r+p)})^{(j)} \\ &+ \sum_{k=1}^{r-1} (\widetilde{\mathcal{X}}_{rk})_{(1)}(\mathcal{Y}_{k(r+p)})^{(j)}, \quad 1 \leq j \leq \alpha_{r+p}, \quad 0 \leq p \leq N - r.\end{aligned} \tag{4.3}$$

It turns out to be important to consider the equations (4.3) in an appropriate order. The following lemmas provide this computation in detail.

Lemma 4.1. *Let $\mathcal{B}, \mathcal{C}$ as in (4.1) be given. Then the dimension of the space of solutions of (4.2) that are of the form $\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^N$ (partitioned conformally to $\mathcal{B}, \mathcal{C}$) with*

$$\mathcal{X}_{rs} = \begin{cases} [0 \quad \mathcal{T}_{rs}], & \alpha_r < \alpha_s \\ \begin{bmatrix} \mathcal{T}_{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ \mathcal{T}_{rs}, & \alpha_r = \alpha_s \end{cases}, \quad \begin{array}{l} (\alpha_1 > \alpha_2 > \ldots > \alpha_N), \\ b_{rs} := \min\{\alpha_s, \alpha_r\}, \\ \mathcal{T}_{rs} = T\big(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs}\big), \ A_j^{rs} \in \mathbb{C}^{m_r \times m_s} \end{array} \tag{4.4}$$

is $\sum_{r=1}^{N} m_r(\frac{m_r-1}{2}\alpha_r + \sum_{s=1}^{r-1}\alpha_s m_s)$*, and each solution satisfies the following properties:*

(a) Each A_0^{rr} *is a solution of the equation* $C_0^r = (A_0^{rr})^T B_0^r A_0^{rr}$*. If* $N \geq 2$ *the matrices* A_j^{rs} *for* $j \in \{0,\dots,\alpha_r - 1\}$*,* $r,s \in \{1,\dots,N\}$ *with* $r > s$ *can be taken freely.*

(b) Assuming (a) and choosing matrices $Z_j^r = -Z_j^r \in \mathbb{C}^{m_r\times m_r}$ *for* $r \in \{1,\dots,N\}$*,* $j \in \{1,\dots,\alpha_r - 1\}$ *freely, the remaining entries of* $\mathcal{X}$ *are computed as follows:*

$\Psi_n^{krs} := \sum_{i=0}^{n}\sum_{l=0}^{n-i}(A_i^{kr})^T B_{n-i-l}^k A_l^{ks}$
$\widetilde{\Psi}_n^{krs} := \sum_{i=1}^{n}\sum_{l=0}^{n-i}(A_i^{kr})^T B_{n-i-l}^k A_l^{ks} + \sum_{l=0}^{n-1}(A_0^{kr})^T B_{n-l}^k A_l^{ks}$

for $j = 0 : \alpha_1 - 1$ ***do***
  if $r \in \{1,\dots,N\}$, $j \in \{1,\dots,\alpha_r - 1\}$ ***then***
  $A_j^{rr} = \frac{1}{2}A_0^{rr} - \frac{1}{2}A_0^{rr}(C_0^r)^{-1}(Z_j^r + \widetilde{\Psi}_j^{rrr} + \sum_{k=1}^{r-1}\Psi_{j-\alpha_k+\alpha_r}^{krr} + \sum_{k=r+1}^{N}\Psi_{j-\alpha_r+\alpha_k}^{krr})$
  end if
  for $p = 1 : N - 1$ ***do***
    if $r \in \{1,\dots,N\}$, $j \leq \alpha_{r+p} - 1$, $r + p \leq N$ ***then***

$$A_j^{r(r+p)} = -A_0^{r(r+p)}(C_0^r)^{-1}\big((A_j^{rr})^T B_0^r A_0^{r(r+p)} + \widetilde{\Psi}_j^{rr(r+p)} + \sum_{k=1}^{r-1}\Psi_{j-\alpha_k+\alpha_r}^{kr(r+p)} + \sum_{k=r+1}^{r+p}\Psi_j^{kr(r+p)} + \sum_{k=r+p+1}^{N}\Psi_{j-\alpha_{r+p}+\alpha_k}^{kr(r+p)}\big)$$

    end if
  end for
end for

For simplicity, we define $\sum_{j=l}^{n} a_j = 0$ *if* $l > n$*, and it is understood that the inner loop (i.e. for p =1: N-1) is not performed for* $N = 1$*.*

(c) (i) If $\mathcal{B},\mathcal{C}$ *are real, then* $\mathcal{X}$ *is real if and only if the following statements hold*

- *Matrices* B_0^r *and* C_0^r *in (4.1) have the same inertia for all* $r \in \{1,\dots,N\}$*.*
- *All matrices* A_0^{rr}*, matrices* A_j^{rs} *with* $r > s$*,* $j \in \{0,\dots,\alpha_r - 1\}$*, and* Z_j^r *for* $j \in \{1,\dots,\alpha_r - 1\}$ *in (a) and (b) are chosen to be real.*

(ii) For any $r \in \{1,\dots,N\}$*,* $n \in \{1,\dots,b_{rs} - 1\}$ *assume in (4.1) that* $m_r = 2m_r'$ *and*

$$B_n^r = u_n^r K_r + u_{n-1}^r L_r, \quad K_r := -\mu^2 I_{m_r'} \oplus I_{m_r'}, \quad L_r := I_{m_r'} \oplus 0, \quad \mu > 0, \tag{4.5}$$

$$C_n^r = v_n^r K_r + v_{n-1}^r L_r, \quad u_0, v_0, \dots, u_{b_{rs}-1}, v_{b_{rs}-1} \in \mathbb{R},\ u_0, v_0 \neq 0,$$
$$u_{-1} = v_{-1} = 0.$$

Then there are $V_j^{rs}, W_j^{rs} \in \mathbb{C}^{m_r'\times m_r'}$ *for* $j \in \{0,\dots,b_{rs} - 1\}$ *and such that*

$$A_0^{rs} = \begin{bmatrix} V_0^{rs} & W_0^{rs} \\ -\mu^2\overline{W}_0^{rs} & \overline{V}_0^{rs}\end{bmatrix}, \quad A_n^{rs} = \begin{bmatrix} V_n^{rs} & W_n^{rs} \\ -\mu^2\overline{W}_n^{rs} + \overline{W}_{n-1}^{rs} & \overline{V}_n^{rs}\end{bmatrix}, \quad n \in \{1,\dots,b_{rs} - 1\}, \tag{4.6}$$

precisely when A_0^{rs}, Z_j^r *in (a), (b) are of the form* $\begin{bmatrix} V & W \\ -\mu^2\overline{W} & \overline{V}\end{bmatrix}$*,* $V, W \in \mathbb{C}^{m_r'\times m_r'}$*.*

Lemma 4.1 (a), (b), (c) (i) coincides with [22, Lemma 3.1]; we apologize for minor errors in formulas providing A_j^{rr} and $A_j^{r(r+p)}$ in [22, Lemma 3.1 (b)]. Thus we only prove (c) (ii), in which solutions are of a special form, which makes the analysis considerably more involved.

Lemma 4.2. *Let $\mathcal{B}$, $\mathcal{C}$ as in (4.1) and real be given. Then the solution of (4.2) that is of the form $\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^N$ (partitioned conformally to $\mathcal{B}, \mathcal{C}$) with*

$$\mathcal{X}_{rs} = \begin{cases} \begin{bmatrix} 0 & \mathcal{T}_{rs} \end{bmatrix}, & \alpha_r < \alpha_s \\ \begin{bmatrix} \mathcal{T}_{rs} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_s \\ \mathcal{T}_{rs}, & \alpha_r = \alpha_s \end{cases}, \qquad \begin{array}{l} (\alpha_1 > \alpha_2 > \ldots > \alpha_N), \\ b_{rs} := \min\{\alpha_s, \alpha_r\} \\ \mathcal{T}_{rs} = T_c\big(A_0^{rs}, \ldots, A_{b_{rs}-1}^{rs}\big), \ A_j^{rs} \in \mathbb{C}^{m_r \times m_s}, \end{array} \tag{4.7}$$

exists if and only if the following condition holds:

$$B_0^r \text{ and } C_0^r \text{ have the same inertia for all } r \in \{1, \ldots, N\} \text{ such that } \alpha_r \text{ is even.} \tag{4.8}$$

If (4.8) is fulfilled, then the real dimension of the space of solutions is

$$\sum_{r=1}^{N} \Big(\alpha_r m_r^2 + 2\sum_{s=1}^{r-1} \alpha_s m_r m_s\Big) - \sum_{\alpha_r \ \text{even}} \tfrac{\alpha_r}{2} m_r - \sum_{\alpha_r \ \text{odd}} \tfrac{\alpha_r+1}{2} m_r.$$

Furthermore, such solutions satisfy the following properties:

(a) Each A_0^{rr} with α_r odd is a solution of $C_0^r = (A_0^{rr})^T B_0^r A_0^{rr}$, while A_0^{rr} for α_r even is a solution of $C_0^r = (A_0^{rr})^* B_0^r A_0^{rr}$. If $N \geq 2$ the entries of A_j^{rs} for $j \in \{0, \ldots, \alpha_r - 1\}$ and $r, s \in \{1, \ldots, N\}$ with $r > s$ can be taken as free variables.

(b) Assuming (a) and choosing all m_r-by-m_r matrices $Z_j^r = \begin{cases} -(Z_j^r)^T, & j - \alpha_r \ \text{odd} \\ -(Z_j^r)^*, & j - \alpha_r \ \text{even} \end{cases}$ for $j \in \{1, \ldots, \alpha_r - 1\}$ freely, the remaining entries of $\mathcal{X}$ are computed as follows:

$$\mathcal{A}_n^{kr} := \begin{cases} \big[(A_0^{kr})^T \ (\overline{A}_1^{kr})^T \ \ldots \ (\overline{A}_{n-1}^{kr})^T \ (A_n^{kr})^T \big], & n \ \text{even} \\ \big[(A_0^{kr})^T \ (\overline{A}_1^{kr})^T \ \ldots \ (A_{n-1}^{kr})^T \ (\overline{A}_n^{kr})^T \big], & n \ \text{odd} \end{cases}; \quad \mathcal{R}_n^k := \begin{cases} \big[B_n^k \ B_{n-1}^k \ \ldots \ B_1^k \big], & n \neq 0 \\ 0, & n = 0 \end{cases},$$

$$\phi_n^{ks} := \begin{cases} \mathcal{R}_n^k (\mathcal{A}_n^{ks})^T, & n \ \text{even} \\ \mathcal{R}_n^k (\overline{\mathcal{A}}_n^{ks})^T, & n \ \text{odd} \end{cases}, \qquad \Phi_n^{ks} := \phi_n^{ks} + B_0^k A_n^{ks},$$

$$\begin{array}{l} \mathcal{Q}_0^{ks} := 0 \\ \mathcal{Q}_1^{ks} := \phi_1^{ks} \end{array}, \quad \mathcal{Q}_{2n}^{ks} := \begin{bmatrix} \phi_{2n}^{ks} \\ \overline{\Phi}_{2n-1}^{ks} \\ \Phi_{2n-2}^{ks} \\ \vdots \\ \overline{\Phi}_1^{ks} \end{bmatrix}, \quad \mathcal{Q}_{2n+1}^{ks} := \begin{bmatrix} \phi_{2n+1}^{ks} \\ \overline{\Phi}_{2n}^{ks} \\ \overline{\Phi}_{2n-1}^{ks} \\ \vdots \\ \Phi_1^{ks} \end{bmatrix}, \quad \psi_n^{krs} := \begin{cases} \mathcal{A}_n^{kr} \mathcal{Q}_n^{ks}, & b_{kr} \ \text{odd} \\ \overline{\mathcal{A}}_n^{kr} \mathcal{Q}_n^{ks}, & b_{kr} \ \text{even} \end{cases},$$

$$\xi_n^{krs} := \psi_n^{krs} + \begin{cases} (\overline{A}_n^{kr})^T B_0^k \overline{A}_0^{ks}, & b_{kr}, n \ \text{odd} \\ (A_n^{kr})^T B_0^k A_0^{ks}, & b_{kr} \ \text{odd}, n \geq 2 \ \text{even} \\ (\overline{A}_n^{kr})^T B_0^k A_0^{ks}, & b_{kr}, n \geq 2 \ \text{even} \\ (A_n^{kr})^T B_0^k \overline{A}_0^{ks}, & b_{kr} \ \text{even}, n \ \text{odd} \\ 0, & n = 0 \end{cases} \qquad \Psi_n^{krs} := \xi_n^{krs} + \begin{cases} (A_0^{kr})^T B_0^k A_n^{ks}, & b_{kr} \ \text{odd} \\ (\overline{A}_0^{kr})^T B_0^k A_n^{ks}, & b_{kr} \ \text{even} \end{cases}$$

for $j = 0 : \alpha_1 - 1$ ***do***
  if $r \in \{1, \dots, N\}$, $j \in \{1, \dots, \alpha_r - 1\}$ ***then***
    $A_j^{rr} = \frac{1}{2} A_0^{rr} - \frac{1}{2} A_0^{rr} (C_0^r)^{-1} \big(Z_j^r + \psi_j^{rrr} + \sum_{k=1}^{r-1} \Psi_{j-\alpha_k+\alpha_r}^{krr} + \sum_{k=r+1}^{N} \Psi_{j-\alpha_r+\alpha_k}^{krr} \big)$
  end if
  for $p = 1 : N - 1$ ***do***
    if $r \in \{1, \dots, N\}$, $j \leq \alpha_{r+p} - 1$, $r + p \leq N$ ***then***
      $A_j^{r(r+p)} = -A_0^{r(r+p)} (C_0^r)^{-1} \big(\xi_j^{rr(r+p)} + \sum_{k=1}^{r-1} \Psi_{j-\alpha_k+\alpha_r}^{kr(r+p)} + \sum_{k=r+1}^{r+p} \Psi_j^{kr(r+p)} + \sum_{k=r+p+1}^{N} \Psi_{j-\alpha_{r+p}+\alpha_k}^{kr(r+p)} \big)$
    end if
  end for
end for

For simplicity, in this algorithm we define $\sum_{j=l}^{n} a_j = 0$ *if* $l > n$*, and it is understood that the inner loop (i.e. for p =1: N-1) is not performed for* $N = 1$*.*

To prove Lemma 4.2 we follow the same general approach as in [22, Lemma 4.1], however, some additional intrigueging technical problems arise.

For the sake of clarity we point out the correct order of calculating the entries of $\mathcal{X}$ in the lemmas. First, all nonzero entries of the blocks below the main diagonal of $\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^N$ (i.e. A_j^{rs} for $r > s$) can be chosen freely. We proceed by computing the upper triangular part of $\mathcal{X}$. We begin with the diagonal entries A_0^{rr} of the main diagonal blocks $\mathcal{X}_{rr}$. Next, the step $j = 0$, $p = 1$ (if $N \geq 2$) of the algorithm yields the diagonal entries of the first upper off-diagonal blocks of $\mathcal{X}$ (i.e. $(\mathcal{X}_{r(r+1)})_{11} = A_0^{r(r+1)}$). Further, the step $j = 0$, $p = 2$ gives the diagonal entries of the second upper off-diagonal blocks of $\mathcal{X}$ (i.e. $(\mathcal{X}_{r(r+2)})_{11} = A_0^{r(r+2)}$), and so forth. In the same fashion the step for fixed $j \in \{1, \dots, \alpha_1 - 1\}$, $p \in \{0, \dots, N\}$ yields the entries on the j-th upper off-diagonals of the p-th upper off-diagonal blocks of $\mathcal{X}$, i.e. $(\mathcal{X}_{r(r+p)})_{1(j+1)} = A_{j+1}^{r(r+p)}$ with $r + p \leq N$, $j \leq \alpha_{r+p} - 1$.

Proof of Lemma 4.1 (c) (ii). We analyze (4.3) for $\mathcal{Y} = \mathcal{B}\mathcal{X}$, $\widetilde{\mathcal{X}} = \mathcal{F}\mathcal{X}^T\mathcal{F}$ (see (4.1)) and $\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^N$ with $\mathcal{X}_{rs}$ as in (4.4). Observe that the fact

$$E_\alpha(I_n) \big(T(A_0, \dots, A_{\alpha-1}) \big)^T E_\alpha(I_m) = T(A_0^T, \dots, A_{\alpha-1}^T), \quad A_0, \dots, A_{\alpha-1} \in \mathbb{C}^{m \times n}, \tag{4.9}$$

implies

$$\widetilde{\mathcal{X}}_{rk} = E_{\alpha_r}(I_{m_r}) \mathcal{X}_{kr}^T E_{\alpha_k}(I_{m_k}) = \begin{cases} \begin{bmatrix} \widetilde{\mathcal{T}}_{rk} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_k \\ \begin{bmatrix} 0 & \widetilde{\mathcal{T}}_{rk} \end{bmatrix}, & \alpha_r < \alpha_k \\ \widetilde{\mathcal{T}}_{rk}, & \alpha_r = \alpha_k \end{cases}, \quad \widetilde{\mathcal{T}}_{rk} = T\big((A_0^{kr})^T, \dots, (A_{b_{kr}-1}^{kr})^T \big).$$

For simplicity we set $\Phi_n^{ks} := \sum_{i=0}^{n} B_{n-i}^k A_i^{ks}$, $n \in \{0, \ldots, b_{rs}-1\}$, and we have

$$\mathcal{Y}_{ks} = \begin{cases} \begin{bmatrix} \mathcal{S}_{ks} \\ 0 \end{bmatrix}, & \alpha_k > \alpha_s \\ \begin{bmatrix} 0 & \mathcal{S}_{ks} \end{bmatrix}, & \alpha_k < \alpha_s \\ \mathcal{S}_{ks}, & \alpha_k = \alpha_s \end{cases}, \qquad \begin{aligned} \mathcal{S}_{ks} &= T\big(B_0^k, \ldots, B_{b_{ks}-1}^k\big) T\big(A_0^{ks}, \ldots, A_{b_{ks}-1}^{ks}\big) \\ &= T\big(\Phi_0^{ks}, \ldots, \Phi_{b_{ks}-1}^{ks}\big) \end{aligned}. \tag{4.10}$$

Next, for $k, r, s \in \{1, \ldots, N\}$, $n \in \{0, \ldots, b_{rs}-1\}$ we set:

$$\Psi_n^{krs} := \begin{cases} \begin{bmatrix} (A_0^{kr})^T & (A_1^{kr})^T & \ldots & (A_n^{rr})^T \end{bmatrix} \begin{bmatrix} \Phi_n^{ks} \\ \vdots \\ \Phi_0^{ks} \end{bmatrix}, & n \geq 0 \\ 0, & n < 0 \end{cases} = \begin{cases} \sum_{i=0}^{n} (A_i^{kr})^T \Phi_{n-i}^{ks}, & n \geq 0 \\ 0, & n < 0 \end{cases}, \tag{4.11}$$

$$\begin{aligned} (\Psi_n^{krs})^T &= \sum_{i=0}^{n} (\Phi_i^{ks})^T A_{n-i}^{kr} = \sum_{i=0}^{n} \sum_{l=0}^{i} (A_l^{ks})^T (B_{i-l}^k)^T A_{n-i}^{kr} = \sum_{l=0}^{n} \sum_{i=l}^{n} (A_l^{ks})^T B_{i-l}^k A_{n-i}^{kr} \\ &= \sum_{l=0}^{n} (A_l^{ks})^T \sum_{i'=0}^{n-l} B_{i'}^k A_{n-l-i'}^{kr} = \sum_{l=0}^{n} (A_l^{ks})^T \Phi_{n-l}^{kr} = \Psi_n^{ksr}, \qquad n \geq 0. \end{aligned} \tag{4.12}$$

Furthermore,

$$(\widetilde{\mathcal{X}}_{rk})_{(1)} (\mathcal{Y}_{k(r+p)})^{(n+1)} = \begin{cases} \Psi_{n-\alpha_{r+p}+\alpha_k}^{kr(r+p)}, & k \geq r+p+1 \\ \Psi_n^{kr(r+p)}, & r+p \geq k \geq r+1, p \geq 1 \\ \Psi_{n-\alpha_k+\alpha_r}^{kr(r+p)}, & k \leq r \end{cases}. \tag{4.13}$$

We now calculate matrices A_0^{rr} for $r \in \{1, \ldots, N\}$. Since

$$(\widetilde{\mathcal{X}}_{rk})_{(1)} = \begin{cases} \begin{bmatrix} (A_0^{kr})^T & * & \ldots & * \end{bmatrix}, & k \geq r \\ \begin{bmatrix} 0 & * & \ldots & * \end{bmatrix}, & k < r \end{cases}, \qquad (\mathcal{Y}_{kr})^{(1)} = \begin{cases} \begin{bmatrix} B_0^k A_0^{kr} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, & k \leq r \\ 0, & k > r \end{cases},$$

we deduce $\sum_{k=1}^{N} (\widetilde{\mathcal{X}}_{rk})_{(1)} ((\mathcal{Y})_{kr})^{(1)} = (A_0^{rr})^T B_0^r A_0^{rr}$, thus (4.3) for $r = s$, $j = 1$ yields:

$$C_0^r = (A_0^{rr})^T B_0^r A_0^{rr}, \qquad r \in \{1, \ldots, N\}. \tag{4.14}$$

If B_0^r, C_0^r are as in (c) (ii) then $\sqrt{\frac{v_0}{u_0}} I_{m_r}$ is one solution of (4.14) of the form (4.6).

Proceed to the key step: an inductive computation of the remaining entries. Fix $p \in \{0, \ldots, N-1\}$, $j \leq \alpha_r - 1$, but not $p = j = 0$. For r, s, n satisfying

$$\begin{aligned} &j \geq 1, n \in \{0, \ldots, j-1\}, s \geq r \quad \text{or} \quad p \geq 1, n = j, r \leq s \leq r+p-1 \\ &\text{or} \quad s \leq r, n \in \{0, \ldots, b_{rs}-1\}, N \geq 2, \end{aligned} \tag{4.15}$$

we assume that there exist $V_n^{rs}, W_n^{rs} \in \mathbb{C}^{m_r \times m_s}$ ($W_{-1}^{rs} := 0$, hence $F_0^{rs} = 0$) so that

$$A_n^{rs} = \widetilde{A}_n^{rs} + F_n^{rs}, \qquad \widetilde{A}_n^{rs} := \begin{bmatrix} V_n^{rs} & W_n^{rs} \\ -\mu^2 \overline{W}_n^{rs} & \overline{V}_n^{rs} \end{bmatrix}, \quad F_n^{rs} := \begin{bmatrix} 0 & 0 \\ \overline{W}_{n-1}^{rs} & 0 \end{bmatrix}.$$

We need to prove that $\widetilde{A}_j^{r(r+p)} = A_j^{r(r+p)} - F_j^{r(r+p)}$ is of the form $\begin{bmatrix} V & W \\ -\mu^2 \overline{W} & \overline{V} \end{bmatrix}$ as well.

The trick of the proof is to reduce $(\mathcal{C}_{r(r+p)})_{1j} = ((\widetilde{\mathcal{X}}\mathcal{Y})_{r(r+p)})_{1j}$ to a certain linear matrix equation in $\widetilde{A}_j^{r(r+p)}$ (and possibly $(\widetilde{A}_j^{r(r+p)})^T$) with coefficients of the appropriate form and depending only on A_n^{rs} for r, s, n satisfying (4.15).

If n, r, s satisfy (4.15) or if $n = j$, $s = r + p$, we have ($L_r F_j^{rs} = 0$, $(F_{j-1}^{kr})^T L_r = 0$):

$$\begin{aligned} \Phi_n^{rs} &= \sum_{i=0}^{n} B_{n-i}^r A_i^{rs} = K_r \sum_{i=0}^{n} u_{n-i}^r (\widetilde{A}_i^{rs} + F_i^{rs}) + L_r \sum_{i=0}^{n-1} u_{n-1-i}^r (\widetilde{A}_i^{rs} + F_i^{rs}) \\ &= K_r D_n^{rs} + K_r E_n^{rs} + L_r D_{n-1}^{rs}, \qquad (4.16) \\ D_{-1}^{rs} &:= 0, \quad D_n^{rs} := \sum_{i=0}^{n} u_{n-i}^r \widetilde{A}_i^{rs}, \quad E_n^{rs} := \sum_{i=0}^{n} u_{n-i}^r F_i^{rs}. \end{aligned}$$

Further, we set

$$U_n^{rs} := \sum_{i=0}^{n} u_{n-i}^r V_i^{rs}, \;\; Z_n^{rs} := \sum_{i=0}^{n} u_{n-i}^r W_i^{rs},$$

$$\left(D_n^{rs} = \begin{bmatrix} U_n^{rs} & Z_n^{rs} \\ -\mu^2 \overline{Z}_n^{rs} & \overline{U}_n^{rs} \end{bmatrix}, E_n^{rs} = \begin{bmatrix} 0 & 0 \\ \overline{Z}_{n-1}^{rs} & 0 \end{bmatrix}\right).$$

Using this and (4.16) it is straightforward to compute

$$\begin{aligned} \Psi_n^{krs} = \sum_{i=0}^{n} (A_i^{kr})^T \Phi_{n-i}^{ks} &= \sum_{i=0}^{n} (\widetilde{A}_i^{kr})^T K_r D_{n-i}^{ks} + \sum_{i=0}^{n-1} (\widetilde{A}_i^{kr})^T L_r D_{n-i-1}^{ks} \\ &\quad + \sum_{i=0}^{n-1} (\widetilde{A}_i^{kr})^T K_r E_{n-i}^{ks} \\ &\quad + \sum_{i=1}^{n} (F_i^{kr})^T K_r D_{n-i}^{ks} + \sum_{i=1}^{n} (F_{i-1}^{kr})^T K_r E_{n-i}^{ks} = \end{aligned}$$

$$= \sum_{i=0}^{n} \begin{bmatrix} -\mu^2((V_i^{rs})^T U_{n-i} - \mu^2 (\overline{W}_i^{rs})^T \overline{Z}_{n-i}) & -\mu^2((V_i^{rs})^T Z_{n-i} + (\overline{W}_i^{rs})^T \overline{U}_{n-i}) \\ -\mu^2((\overline{V}_i^{rs})^T \overline{Z}_{n-i} + (W_i^{rs})^T U_{n-i}) & (\overline{V}_i^{rs})^T \overline{U}_{n-i} - \mu^2 (W_i^{rs})^T Z_{n-i} \end{bmatrix}$$

$$+ \sum_{i=0}^{n-1} \begin{bmatrix} -\mu^2 (\overline{W}_i^{rs})^T \overline{Z}_{n-1-i}^{ks} & (\overline{W}_i^{rs})^T \overline{U}_{n-1-i}^{ks} + (V_i^{rs})^T Z_{n-1-i}^{ks} \\ (\overline{V}_i^{rs})^T \overline{Z}_{n-1-i} + (W_i^{rs})^T U_{n-1-i} & (W_i^{rs})^T Z_{n-1-i} \end{bmatrix}$$

$$+\sum_{i=0}^{n-1}\begin{bmatrix} (V_i^{rs})^T U_{n-1-i}-\mu^2(\overline{W}_i^{rs})^T\overline{Z}_{n-1-i}^{ks} & 0 \\ 0 & 0\end{bmatrix}+\sum_{i=0}^{n-2}\begin{bmatrix} (\overline{W}_i^{rs})^T\overline{Z}_{n-2-i}^{ks} & 0 \\ 0 & 0\end{bmatrix}.$$

Finally, we define

$$\Gamma_{-1}^{krs}:=0,\quad \Gamma_n^{krs}:=\sum_{i=0}^{n}\begin{bmatrix} -\mu^2((V_i^{rs})^T U_{n-i}-\mu^2(\overline{W}_i^{rs})^T\overline{Z}_{n-i}) & -\mu^2((V_i^{rs})^T Z_{n-i}+(\overline{W}_i^{rs})^T\overline{U}_{n-i}) \\ -\mu^2((\overline{V}_i^{rs})^T\overline{Z}_{n-i}+(W_i^{rs})^T U_{n-i}) & (\overline{V}_i^{rs})^T\overline{U}_{n-i}-\mu^2(W_i^{rs})^T Z_{n-i}\end{bmatrix} \tag{4.17}$$

$$+\sum_{i=0}^{n-1}\begin{bmatrix} -\mu^2(\overline{W}_i^{rs})^T\overline{Z}_{n-1-i}^{ks} & (\overline{W}_i^{rs})^T\overline{U}_{n-1-i}^{ks}+(V_i^{rs})^T Z_{n-1-i}^{ks} \\ (\overline{V}_i^{rs})^T\overline{Z}_{n-1-i}+(W_i^{rs})^T U_{n-1-i} & (W_i^{rs})^T Z_{n-1-i}\end{bmatrix}.$$

Therefore, for r,s,n satisfying (4.15) or for $n=j$, $s=r+p$ we can write

$$\Psi_n^{krs}=\Gamma_n^{krs}+\begin{bmatrix} -\frac{1}{\mu^2}[\Gamma_{n-1}^{krs}]_{11} & 0 \\ 0 & 0\end{bmatrix}. \tag{4.18}$$

Next, by applying (4.18) and (4.13) we further write; $\Gamma_n^{kr(r+p)}:=0$ for $n<0$:

$$\begin{aligned}\sum_{k=1}^{N}(\widetilde{\mathcal{X}}_{rk})_{(1)}(\mathcal{Y}_{k(r+p)})^{(n+1)}&=\sum_{k=1}^{r}\Psi_{n-\alpha_k+\alpha_r}^{kr(r+p)}+\sum_{k=r+1}^{r+p}\Psi_n^{kr(r+p)}+\sum_{k=r+p+1}^{N}\Psi_{n-\alpha_{r+p}+\alpha_k}^{kr(r+p)}\\ &=\gamma(n,r,p)+\begin{bmatrix} -\frac{1}{\mu^2}[\gamma(n-1,r,p)]_{11} & 0 \\ 0 & 0\end{bmatrix},\end{aligned} \tag{4.19}$$

$$\gamma(n,r,p):=\Gamma_n^{kr(r+p)}+\left(\sum_{k=1}^{r-1}\Gamma_{n-\alpha_k+\alpha_r}^{kr(r+p)}+\sum_{k=r+1}^{r+p}\Gamma_j^{kr(r+p)}+\sum_{k=r+p+1}^{N}\Gamma_{n-\alpha_{r+p}+\alpha_k}^{kr(r+p)}\right). \tag{4.20}$$

Using (4.19), the equation $((\widetilde{\mathcal{X}}\mathcal{Y})_{r(r+p)})_{1(j+1)}=(\mathcal{C}_{r(r+p)})_{1(j+1)}$ can be seen as

$$\gamma(j,r,p)+\begin{bmatrix} -\frac{1}{\mu^2}[\gamma(j-1,r,p)]_{11} & 0 \\ 0 & 0\end{bmatrix}=\begin{cases} v_jK_r+v_{j-1}L_r, & p=0\\ 0, & p\neq 0\end{cases}.$$

We show by induction that it is actually reduces to

$$\gamma(j,r,p)=\begin{cases} v_jK_r, & p=0\\ 0, & p\neq 0\end{cases}. \tag{4.21}$$

Indeed, it is clear for $j=0$ (since $v_{-1}=\gamma(-1,r,p)=0$), while assuming (4.21) for some $n<j$ we easily conclude the following fact yielding the claim for $n+1$:

$$\begin{bmatrix} [-\frac{1}{\mu^2}\gamma(n,r,p)]_{11} & 0 \\ 0 & 0\end{bmatrix}=\begin{cases} \begin{bmatrix} v_n[-\frac{1}{\mu^2}K_r]_{11} & 0 \\ 0 & 0\end{bmatrix}, & p=0\\ 0, & p\neq 0\end{cases}=\begin{cases} v_nL_r, & p=0\\ 0, & p\neq 0\end{cases}.$$

Observe that $\Gamma_j^{rr(r+p)}$ (see (4.17)), $u_0^r(\widetilde{A}_0^{rr})^T K_r \widetilde{A}_j^{r(r+p)}$, $u_0^r(\widetilde{A}_j^{rr})^T K_r \widetilde{A}_0^{r(r+p)}$, and hence the expressions below are both of the form $\left[\begin{smallmatrix} -\mu^2 V & W \\ \overline{W} & \overline{V} \end{smallmatrix}\right]$:

$$\Gamma_j^{rrr} - u_0^r(\widetilde{A}_0^{kr})^T K_r \widetilde{A}_j^{ks} + u_0^r(\widetilde{A}_j^{kr})^T K_r \widetilde{A}_0^{ks}, \qquad \Gamma_j^{rr(r+p)} - u_0^r(\widetilde{A}_0^{kr})^T K_r \widetilde{A}_j^{ks}.$$

Moreover, the equation of (4.21) can be seen as:

$$\begin{aligned} \big(u_0^r(A_0^{rr})^T K_r\big)\widetilde{A}_j^{r(r+p)} &= \kappa(j,r,p), \qquad p \geq 1, \\ \big(u_0^r(A_0^{rr})^T K_r\big)\widetilde{A}_j^{rr} + (\widetilde{A}_j^{rr})^T\big(u_0^r K_r A_0^{rr}\big) &= \kappa(j,r,0), \qquad p = 0, \end{aligned} \tag{4.22}$$

with $\kappa(j,r,p)$ of the form $\left[\begin{smallmatrix} -\mu^2 V & W \\ \overline{W} & \overline{V} \end{smallmatrix}\right]$ and depending on A_n^{rs} for n,r,s satisfying (4.15). Similarly, as we proved (4.12), we see that $\kappa(j,r,0)$ is symmetric.

Thus $\widetilde{A}_j^{r(r+p)}$ for $p \geq 1$ and $p = 0$ is a solution of equations $A^T Y = B$ and $A^T X + X^T A = B$ for $A = u_0^r(A_0^{rr})^T K_r$, $B = \kappa(j,r,p)$, respectively (i.e. $Y = (A^T)^{-1}B$ and $X = (A^T)^{-1}(\frac{1}{2}B + Z)$ with Z skew-symmetric). Since $(A^T)^{-1} = ((A_0^{rr})^T B_0^r)^{-1} = A_0^r(C_0^r)^{-1} = \frac{1}{v_0^r} A_0^r (K_0^r)^{-1} = \frac{1}{v_0^r}\left[\begin{smallmatrix} -\frac{1}{\mu^2}V_0^{rr} & W_0^{rr} \\ \overline{W}_0^{rr} & \overline{V}_0^{rr} \end{smallmatrix}\right]$ (see (4.14)) and $B = \kappa(j,r,p)$ is of the form $\left[\begin{smallmatrix} -\mu^2 V & W \\ \overline{W} & \overline{V} \end{smallmatrix}\right]$, it follows that Y is of the form $\left[\begin{smallmatrix} V & W \\ -\mu^2\overline{W} & \overline{V} \end{smallmatrix}\right]$, while X is of this form precisely when Z is of this form. This completes the inductive step. □

Proof of Lemma 4.2. Let $\mathcal{X} = [\mathcal{X}_{rs}]_{r,s=1}^N$ with $\mathcal{X}_{rs}$ as in (4.7) and $\mathcal{Y} = \mathcal{B}\mathcal{X}$, $\widetilde{\mathcal{X}} = \mathcal{F}\mathcal{X}^T\mathcal{F}$ (see (4.1)). Next, for $A_0, A_1, \ldots, A_{\alpha-1} \in \mathbb{C}^{m\times n}$ we have

$$E_\alpha(I_n)\big(T_c(A_0, A_1, \ldots, A_{\alpha-1})\big)^T E_\alpha(I_m) = \begin{cases} T_c(\overline{A}_0^T, A_1^T, \ldots, \overline{A}_{\alpha-2}^T, A_{\alpha-1}^T), & \alpha \text{ even} \\ T_c(A_0^T, \overline{A}_1^T, \ldots, \overline{A}_{\alpha-2}^T, A_{\alpha-1}^T), & \alpha \text{ odd} \end{cases};$$

the entry in the first row and in the j-th column of the matrix $T_c(\overline{A}_0^T, A_1^T, \overline{A}_2^T, \ldots)$ (or $T_c(A_0^T, \overline{A}_1^T, \overline{A}_2^T, \ldots)$) is A_{j-1} for j odd (even) and $\overline{A}_{j-1}$ for j even (odd). Thus

$$\widetilde{\mathcal{X}}_{rk} := E_{\alpha_r}(I_{m_r})\mathcal{X}_{sr}^T E_{\alpha_s}(I_{m_s}) = \begin{cases} \begin{bmatrix} \widetilde{\mathcal{T}}_{rk} \\ 0 \end{bmatrix}, & \alpha_r > \alpha_k \\ \begin{bmatrix} 0 & \widetilde{\mathcal{T}}_{rk} \end{bmatrix}, & \alpha_r < \alpha_k \\ \widetilde{\mathcal{T}}_{rk}, & \alpha_r = \alpha_k \end{cases},$$

$$\widetilde{\mathcal{T}}_{rk} = \begin{cases} T_c\big((\overline{A}_0^{kr})^T, (A_1^{kr})^T, \ldots, (\overline{A}_{b_{kr}-2}^{kr})^T, (A_{b_{kr}-1}^{kr})^T\big), & b_{kr} \text{ even} \\ T_c\big((A_0^{kr})^T, (\overline{A}_1^{kr})^T, \ldots, (\overline{A}_{b_{kr}-2}^{kr})^T, (A_{b_{kr}-1}^{kr})^T\big), & b_{kr} \text{ odd} \end{cases}.$$

We also have

$$\mathcal{Y}_{ks} = \begin{cases} \begin{bmatrix} S_{ks} \\ 0 \end{bmatrix}, & \alpha_k > \alpha_s \\ \begin{bmatrix} 0 & S_{ks} \end{bmatrix}, & \alpha_k < \alpha_s \\ S_{ks}, & \alpha_k = \alpha_s \end{cases}, \quad \begin{aligned} S_{ks} &= T\big(B_0^k, B_1^k, \ldots, B_{b_{ks}-1}^k\big) T_c\big(A_0^{ks}, A_1^{ks}, \ldots, A_{b_{ks}-1}^{ks}\big) \\ &= T_c\big(\Phi_0^{ks}, \Phi_1^{ks}, \ldots, \Phi_{b_{ks}-1}^{rs}\big) \end{aligned},$$

$$\Phi_{2n}^{ks} := \sum_{j=0}^{n} B_{2n-2j}^k A_{2j}^{ks} + \sum_{j=0}^{n-1} B_{2n-2j-1}^k \overline{A}_{2j+1}^{ks},$$

$$\Phi_{2n+1}^{ks} := \sum_{j=0}^{n} (B_{2n-2j}^k A_{2j+1}^{ks} + B_{2n-2j+1}^k \overline{A}_{2j}^{ks}).$$

Let us now compute A_0^{rr} for $r \in \{1, \ldots, N\}$. Since

$$(\widetilde{X}_{rk})_{(1)} = \begin{cases} \big[\, (A_0^{rr})^T \ * \ \ldots * \big], & k \geq r, \alpha_r \text{ odd} \\ \big[\, (A_0^{rr})^* \ * \ \ldots * \big], & k \geq r, \alpha_r \text{ even} \\ \big[\, 0 \ * \ \ldots \ * \big], & k < r \end{cases}, \quad ((\mathcal{Y})_{kr})^{(1)} = \begin{cases} \begin{bmatrix} B_0^k A_0^{kr} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, & k \leq r \\ 0, & k > r \end{cases},$$

it follows from (4.3) for $r = s$, $j = 1$ that

$$C_0^r = \begin{cases} (A_0^{rr})^T B_0^r A_0^{rr}, & \alpha_r \text{ odd} \\ (A_0^{rr})^* B_0^r A_0^{rr}, & \alpha_r \text{ even} \end{cases}. \tag{4.23}$$

Since B_0^r, C_0^r are real symmetric, then by Sylvester's inertia theorem this equation for α_r even has a solution A_0^{rr} precisely when B_0^r, C_0^r are of the same inertia.

Next, if $N \geq 2$, we fix arbitrarily the blocks below the main diagonal of $[\mathcal{X}_{rs}]_{r,s=1}^N$, and then inductively compute the remaining entries (as in the proof of Lemma 4.1). We fix $p \in \{0, \ldots, N-1\}$ and $j \leq \alpha_r - 1$, but not $p = j = 0$. To get $A_j^{r(r+p)}$ (step j, p of the algorithm in (b)), we solve $(\mathcal{C}_{r(r+p)})_{1j} = ((\widetilde{\mathcal{X}}\mathcal{Y})_{r(r+p)})_{1j}$, while assuming that we have already determined matrices A_n^{rs} for

$$\begin{aligned} &j \geq 1, n \in \{0, \ldots, j-1\}, s \geq r \quad \text{or} \quad p \geq 1, n = j, r \leq s \leq r+p-1 \\ &\text{or} \quad s \leq r, n \in \{0, \ldots, b_{rs}-1\}, N \geq 2, \qquad (1 \leq r, s \leq N). \end{aligned} \tag{4.24}$$

To simplify calculations we use $\mathcal{A}_0^{kr}$, Φ_n^{krs}, Ψ_n^{krs} defined in the algorithm in (b), and in addition we introduce the matrix vectors $\mathcal{P}_n^{ks}$ with Φ_{n-j+1}^{ks} (and $\overline{\Phi}_{n-j+1}^{ks}$) in the j-th row for $j \geq 2$ odd (even):

$$\Psi_n^{krs} = \begin{cases} \mathcal{A}_n^{rk} \mathcal{P}_n^{ks} & b_{kr} \text{ odd} \\ \overline{\mathcal{A}}_n^{rk} \mathcal{P}_n^{ks} & b_{kr} \text{ even} \end{cases}, \qquad \mathcal{P}_{2n}^{ks} := \begin{bmatrix} \Phi_{2n}^{ks} \\ \overline{\Phi}_{2n-1}^{ks} \\ \vdots \\ \overline{\Phi}_1^{ks} \\ \Phi_0^{ks} \end{bmatrix}, \quad \mathcal{P}_{2n+1}^{ks} := \begin{bmatrix} \Phi_{2n+1}^{ks} \\ \overline{\Phi}_{2n}^{ks} \\ \vdots \\ \Phi_1^{ks} \\ \overline{\Phi}_0^{ks} \end{bmatrix}, \quad n \geq 0.$$

Further, for $n \geq 0$ we obtain:

$$
\begin{aligned}
(\mathcal{A}^{kr}_{2n+1}\overline{\mathcal{P}}^{ks}_{2n+1})^T &= \sum_{j=0}^{n} (\overline{\Phi}^{kr}_{2j+1})^T A^{ks}_{2n-2j} + \sum_{j=0}^{n} (\Phi^{kr}_{2j})^T \overline{A}^{ks}_{2n+1-2j} \\
&= \sum_{j=0}^{n}\sum_{l=0}^{j} \big((\overline{A}^{kr}_{2l+1})^T B^k_{2j-2l} + (A^{kr}_{2l})^T B^k_{2j+1-2l}\big) A^{ks}_{2n-2j} \\
&\quad + \big(\sum_{j=0}^{n}\sum_{l=0}^{j}(A^{kr}_{2l})^T B^k_{2j-2l} + \sum_{j=1}^{n}\sum_{l=0}^{j-1}(\overline{A}^{kr}_{2l+1})^T B^k_{2j-1-2l}\big)\overline{A}^{ks}_{2n+1-2j} \\
&= \sum_{l=0}^{n}\sum_{j=l}^{n}(\overline{A}^{kr}_{2l+1})^T B^k_{2j-2l} A^{ks}_{2n-2j} + \sum_{l=0}^{n}\sum_{j=l}^{n}(A^{kr}_{2l})^T B^k_{2j+1-2l} A^{ks}_{2n-2j} \\
&\quad + \sum_{l=0}^{n}\sum_{j=l}^{n}(A^{kr}_{2l})^T B^k_{2j-2l}\overline{A}^{ks}_{2n+1-2j} \\
&\quad + \sum_{l=0}^{n-1}\sum_{j=l+1}^{n}(\overline{A}^{kr}_{2l+1})^T B^k_{2j-1-2l}\overline{A}^{ks}_{2n+1-2j} \\
&= (\overline{A}^{kr}_{2n+1})^T B^k_0 A^{ks}_0 + \sum_{l=0}^{n}(A^{kr}_{2l})^T \sum_{j'=0}^{n-l}\big(B^k_{2j'+1}A^{ks}_{2n-2l-2j'} \\
&\quad + B^k_{2j'}\overline{A}^{ks}_{2n+1-2l-2j'}\big) \\
&\quad + \sum_{l=0}^{n-1}(\overline{A}^{kr}_{2l+1})^T\big(B^k_0 A^{ks}_{2n-2l} + \sum_{j'=0}^{n-1-l}(B^k_{2j'+2}A^{ks}_{2n-2l-2j'-2} \\
&\quad + B^k_{2j'+1}\overline{A}^{ks}_{2n-1-2l-2j})\big) \\
&= \sum_{l=0}^{n}(\overline{A}^{kr}_{2l+1})^T \Phi^{ks}_{2n-2l} + \sum_{l=0}^{n}(A^{kr}_{2l})^T\overline{\Phi}^{ks}_{2n+1-2l} = \mathcal{A}^{kr}_{2n+1}\overline{\mathcal{P}}^{ks}_{2n+1},
\end{aligned}
$$

$$
\begin{aligned}
(\overline{\mathcal{A}}^{kr}_{2n}\mathcal{P}^{ks}_{2n})^T &= \sum_{j=1}^{n}(\overline{\Phi}^{kr}_{2j-1})^T A^{ks}_{2n+1-2j} + \sum_{j=0}^{n}(\Phi^{kr}_{2j})^T\overline{A}^{ks}_{2n-2j} \\
&= \sum_{j=1}^{n}\sum_{l=0}^{j-1}\big((A^{kr}_{2l})^T B^k_{2j-1-2l} + (\overline{A}^{kr}_{2l+1})^T B^k_{2j-2-2l}\big)A^{ks}_{2n+1-2j} \\
&\quad + \big(\sum_{j=0}^{n}\sum_{l=0}^{j}(A^{kr}_{2l})^T B^k_{2j-2l} + \sum_{j=1}^{n}\sum_{l=0}^{j-1}(\overline{A}^{kr}_{2l+1})^T B^k_{2j-1-2l}\big)\overline{A}^{ks}_{2n-2j} \\
&= \sum_{l=0}^{n-1}\sum_{j=l+1}^{n}\big((A^{kr}_{2l})^T B^k_{2j-1-2l}A^{ks}_{2n+1-2j}\big) + \sum_{l=0}^{n}\sum_{j=l}^{n}\big((A^{kr}_{2l})^T B^k_{2j-2l}\overline{A}^{ks}_{2n-2j}\big)
\end{aligned}
$$

$$+\sum_{l=0}^{n-1}\sum_{j=l+1}^{n}\left((\overline{A}_{2l+1}^{kr})^T B_{2j-2l}^k A_{2n+1-2j}^{ks}+(\overline{A}_{2l+1}^{kr})^T B_{2j-1-2l}^k \overline{A}_{2n-2j}^{ks}\right)$$

$$=(A_{2n}^{kr})^T B_0^k \overline{A}_0^{ks}+\sum_{l=0}^{n-1}(\overline{A}_{2l+1}^{kr})^T\sum_{j'=0}^{n-l-1}\big(B_{2j'-2}^k A_{2n-1-2j-2l}^{ks}$$
$$+B_{2j'+1}^k \overline{A}_{2n-2j'-2l-2}^{ks}\big)$$
$$+\sum_{l=0}^{n-1}(A_{2l}^{kr})^T\left(B_0^k \overline{A}_{2n-2l}^{ks}+\sum_{j'=0}^{n-l-1}\big(B_{2j'+1}^k A_{2n-1-2j'-2l}^{ks}\right.$$
$$\left.+B_{2j'+2}^k \overline{A}_{2n-2j'-2l-2}^{ks}\big)\right)$$
$$=\sum_{l=0}^{n}(A_{2l}^{kr})^T\overline{\Phi}_{2n-2l}^{ks}+\sum_{l=0}^{n-1}(\overline{A}_{2l+1}^{kr})^T\Phi_{2n-1-2l}^{ks}=\mathcal{A}_{2n}^{kr}\overline{\mathcal{P}}_{2n}^{ks}.$$

In a similar manner we prove

$$(\mathcal{A}_{2n+1}^{kr}\mathcal{P}_{2n+1}^{ks})^T=\overline{\mathcal{A}}_{2n+1}^{kr}\overline{\mathcal{P}}_{2n+1}^{ks},\qquad (\mathcal{A}_{2n}^{kr}\mathcal{P}_{2n}^{ks})^T=\mathcal{A}_{2n}^{kr}\mathcal{P}_{2n}^{ks}.$$

The above computations thus yield

$$(\Psi_j^{krs})^T=\begin{cases}\Psi_j^{ksr} & j-b_{kr}\ \text{odd}\\ \overline{\Psi}_j^{ksr} & j-b_{kr}\ \text{even}\end{cases}. \tag{4.25}$$

Since

$$(\widetilde{X}_{rr})_{(1)}=\begin{cases}\mathcal{A}_{\alpha_r-1}^{rr}, & \alpha_r\ \text{odd}\\ \overline{\mathcal{A}}_{\alpha_r-1}^{rr}, & \alpha_r\ \text{even}\end{cases},\quad (\mathcal{Y}_{r(r+p)})^{(j+1)}=\begin{cases}\mathcal{P}_{\alpha_r-1}^{rr}, & p=0, j=\alpha_r-1\\ \begin{bmatrix}\mathcal{P}_j^{r(r+p)}\\ 0\end{bmatrix}, & j<\alpha_r-1 \text{ or } p\geq 1\end{cases},$$

we have $(\widetilde{\mathcal{X}}_{rr})_{(1)}(\mathcal{Y}_{r(r+p)})^{(j+1)}=\Psi_j^{rr(r+p)}$. In particular, for $\xi(j,r,p)$ as defined in the algorithm in (b), we deduce for $j\geq 1$, $p\geq 1$ and for $p=0$, respectively:

$$(\widetilde{\mathcal{X}}_{rr})_{(1)}(\mathcal{Y}_{r(r+p)})^{(j+1)}=\Psi_j^{rr(r+p)}=\xi(j,r,p)+\begin{cases}(A_0^{rr})^T B_0^r A_j^{r(r+p)}, & \alpha_r\ \text{odd}\\ (A_0^{rr})^* B_0^r A_j^{r(r+p)}, & \alpha_r\ \text{even,}\end{cases} \tag{4.26}$$

$$(\widetilde{\mathcal{X}}_{rr})_{(1)}(\mathcal{Y})_{rr}^{(j+1)}=\Psi_j^{rrr}=\xi(j,r,0)+\begin{cases}(\overline{A}_0^{rr})^* B_0^r A_j^{rr}+(A_j^{rr})^* B_0^r \overline{A}_0^{rr}, & \alpha_r, j\ \text{odd}\\ (A_0^{rr})^T B_0^r A_j^{rr}+(A_j^{rr})^T B_0^r A_0^{rr}, & \alpha_r\ \text{odd}, j\ \text{even}\\ (A_0^{rr})^* B_0^r A_j^{rr}+(A_j^{rr})^* B_0^r A_0^{rr}, & \alpha_r, j\ \text{even}\\ (\overline{A}_0^{rr})^T B_0^r A_j^{rr}+(A_j^{rr})^T B_0^r \overline{A}_0^{rr}, & \alpha_r\ \text{even}, j\ \text{odd}\end{cases}$$

Summands of the second term in (4.3) for $j+1$ instead of j consist of

$$(\widetilde{X}_{rk})_{(1)}=\begin{cases}\mathcal{A}^{kr}_{\alpha_r-1}, & \alpha_r \text{ odd}\\ \overline{\mathcal{A}}^{kr}_{\alpha_r-1}, & \alpha_r \text{ even}\end{cases},\quad (\mathcal{Y}_{k(r+p)})^{(j+1)}=\begin{cases}\mathcal{P}^{r(r+p)}_j, & k=r+p\\ \begin{bmatrix}\mathcal{P}^{r(r+p)}_j\\ 0\end{bmatrix}, & k<r+p,\\ \begin{bmatrix}\mathcal{P}^{r(r+p)}_{j-\alpha_{r+p}-\alpha_k}\\ 0\end{bmatrix}, & k>r+p,\end{cases},$$

hence (for $N\geq r+1\geq 2$):

$$\begin{aligned}\Theta(j,r,p) &:= \sum_{k=r+1}^{N}(\widetilde{\mathcal{X}}_{rk})_{(1)}(\mathcal{Y}_{k(r+p)})^{(j+1)} \qquad (4.27)\\ &=\begin{cases}\sum_{k=r+1}^{N}\Psi^{krr}_{j-\alpha_r+\alpha_k}, & j\geq 1, p=0\\ \sum_{k=r+1}^{r+p}\Psi^{kr(r+p)}_j+\sum_{k=r+p+1}^{N}\Psi^{kr(r+p)}_{j-\alpha_{r+p}+\alpha_k}, & j\geq 0, p\geq 1.\end{cases}\end{aligned}$$

For simplicity, we defined $\sum_{k=r+p+1}^{N}\Psi^{rr(r+p)}_{j-\alpha_{r+p}-\alpha_k}=0$ for $r+p+1>N$.

Finally, the third term in (4.3) for $j+1$ instead of j (with $N\geq 2$, $k\leq r-1$) is

$$\Lambda(j,r,p):=\sum_{k=1}^{r-1}(\widetilde{\mathcal{X}}_{rk})_{(1)}(\mathcal{Y}_{k(r+p)})^{(j+1)}=\sum_{k=1}^{r-1}\Psi^{kr(r+p)}_{j-\alpha_k+\alpha_r}, \qquad (4.28)$$

since we have

$$(\widetilde{\mathcal{X}}_{rk})_{(1)}=\begin{cases}\begin{bmatrix}0 & \mathcal{A}^{kr}_{\alpha_k}\end{bmatrix}, & \alpha_k \text{ odd}\\ \begin{bmatrix}0 & \overline{\mathcal{A}}^{kr}_{\alpha_k}\end{bmatrix}, & \alpha_k \text{ even}\end{cases},\qquad (\mathcal{Y}_{k(r+p)})^{(j+1)}=\begin{bmatrix}\mathcal{P}^{k(r+p)}_j\\ 0\end{bmatrix},\quad 1\leq k\leq r-1.$$

For $j,p\geq 0$ with $j+p\geq 1$ we define

$$D^{r(r+p)}_j:=\Xi(j,r,p)+\Theta(j,r,p)+\Lambda(j,r,p). \qquad (4.29)$$

We combine $(\mathcal{C}_{r(r+p)})_{1j}=((\widetilde{\mathcal{X}}\mathcal{Y})_{r(r+p)})_{1j}$ in (4.3) with (4.26), (4.27), (4.28), (4.29):

$$\begin{aligned}
&(A^{rr}_0)^*B^r_0A^{r(r+p)}_j=-D^{r(r+p)}_j, && \alpha_r \text{ even}, && p\geq 1 && (4.30)\\
&(A^{rr}_0)^TB^r_0A^{r(r+p)}_j=-D^{r(r+p)}_j, && \alpha_r \text{ odd}, && p\geq 1,\\
&(\overline{A}^{rr}_0)^*B^r_0A^{rr}_j+(A^{rr}_j)^*B^r_0\overline{A}^{rr}_0=C^r_j-D^{rr}_j, && \alpha_r, j \text{ odd}, (p=0), j\geq 1 && && (4.31)\\
&(A^{rr}_0)^TB^r_0A^{rr}_j+(A^{rr}_j)^TB^r_0A^{rr}_0=C^r_j-D^{rr}_j, && \alpha_r \text{ odd}, j \text{ even } (p=0), j\geq 1\\
&(A^{rr}_0)^*B^r_0A^{rr}_j+(A^{rr}_j)^*B^r_0A^{rr}_0=C^r_j-D^{rr}_j, && \alpha_r, j \text{ even }, (p=0), j\geq 1\\
&(\overline{A}^{rr}_0)^TB^r_0A^{rr}_j+(A^{rr}_j)^TB^r_0\overline{A}^{rr}_0=C^r_j-D^{rr}_j, && \alpha_r \text{ even }, j \text{ odd}(p=0), j\geq 1
\end{aligned}$$

Moreover, from (4.25) it follows that Ψ^{krs}_n for $r=s$, and thus $\xi(j,r,0)$, $\Theta(j,r,0)$, $\Lambda(j,r,0)$, $C^r_j-D^{rr}_j$ are all symmetric (Hermitian) if α_r-j is odd (even).

Since (4.23) is equivalent to $A_0^r(C_0^r)^{-1} = \begin{cases} ((A_0^{rr})^T B_0^r)^{-1}, & \alpha_r \text{ odd} \\ ((A_0^{rr})^* B_0^r)^{-1}, & \alpha_r \text{ even} \end{cases}$, (4.30) yields $A_j^{r(r+p)} = -A_0^r(C_0^r)^{-1}D_j^{r(r+p)}$ for $p \geq 1$. Next, we get A_j^{rr} by solving (4.31), i.e. an equation of the form $A^TX + X^TA = B$ for $\alpha_r - j$ odd and of the form $A^*X + X^*A = B$ for $\alpha_r - j$ even, with given A nonsingular and B symmetric or Hermitian; the solution in the first case is $X = \frac{1}{2}(A^T)^{-1}B + (A^T)^{-1}Z$ with Z skew-symmetric and in the second case $X = \frac{1}{2}(A^*)^{-1}B + (A^*)^{-1}Z$ with Z skew-Hermitian. If α_r is odd (even), then for j even (odd) we have $A = B_0^r A_0^{rr}$ ($A = B_0^r \overline{A}_0^{rr}$), hence $(A^T)^{-1} = A_0^r(C_0^r)^{-1}$, while a similar argument for $\alpha_r - j$ even gives $(A^*)^{-1} = A_0^r(C_0^r)^{-1}$. Furthermore, $B = C_j^r - D_j^{rr}$ and it depends only on A_n^{rs} with n, r, s satisfying (4.24). It is straightforward to conclude the algorithm in (b).

It is only left to sum up the dimensions:

$$\begin{aligned} &2\sum_{r=1}^{N}\sum_{s=1}^{r-1}\alpha_s m_r m_s + \sum_{\alpha_r \text{ even}} \left(m_r^2 + \tfrac{(\alpha_r-2)m_r^2}{2} + \tfrac{\alpha_r}{2}m_r(m_r-1)\right) \\ &+ \sum_{\alpha_r \text{ odd}} \left(\tfrac{1}{2}m_r(m_r-1) + \tfrac{\alpha_r-1}{2}m_r^2 + \tfrac{\alpha_r-1}{2}m_r(m_r-1)\right) \\ =&\sum_{r=1}^{N}\left(\alpha_r m_r^2 + 2\sum_{s=1}^{r-1}\alpha_s m_r m_s\right) - \sum_{\alpha_r \text{ even}} \tfrac{1}{2}m_r\alpha_r - \sum_{\alpha_r \text{ odd}} \tfrac{1}{2}m_r(\alpha_r+1). \end{aligned}$$

This completes the proof of the lemma. □

Remark 4.

1. It would be interesting to find a nice description of A_0^{rr} of the form (4.6) and such that $C_0^r = (A_0^{rr})^T B_0^r A_0^{rr}$ with B_0^r, C_0^r as in (4.5).
2. One could consider (4.2) even for nonsingular $\mathcal{B}$ and $\mathcal{C}$, since the solutions of $A^TX + X^TA = B$ and $A^*X + X^*A = B$ in this case are known (see [3], [15]).

Example 4.3. We solve (4.2) for $\mathcal{F} = E_3(I) \oplus E_2(I)$, $\mathcal{B} = \mathcal{B}' = I_6(I)$ with the identity matrix I, and where the solution $\mathcal{X}_c$ is of the form as in Example 2.2. We have:

$$\widetilde{\mathcal{X}}_c\mathcal{X}_c = \left[\begin{array}{ccc|cc} A_1^T & B_1^* & C_1^T & N_1^* & P_1^T \\ 0 & A_1^* & B_1^T & 0 & N_1^T \\ 0 & 0 & A_1^T & 0 & 0 \\ \hline 0 & H_1^* & F_1^T & A_2^* & B_2^T \\ 0 & 0 & H_1^T & 0 & A_2^T \end{array}\right] \left[\begin{array}{ccc|cc} A_1 & B_1 & C_1 & H_1 & F_1 \\ 0 & \overline{A}_1 & \overline{B}_1 & 0 & \overline{H}_1 \\ 0 & 0 & A_1 & 0 & 0 \\ \hline 0 & N_1 & P_1 & A_2 & B_2 \\ 0 & 0 & \overline{N}_1 & 0 & \overline{A}_2 \end{array}\right] =$$

$$
= \left[\begin{array}{ccc|cc}
A_1^T A_1 & A_1^T B_1 + B_1^* \overline{A}_1 & A_1^T C_1 + C_1^T A_1 & A_1^T H_1 + N_1^* A_2 & A_1^T F_1 + B_1^* \overline{H}_1 \\
 & +N_1^* N_1 & +B_1^* \overline{B}_1 + N_1^* P_1 + P_1^T \overline{N}_1 & & +N_1^* B_2 + P_1^T \overline{A}_2 \\
0 & A_1^* \overline{A}_1 & A_1^* \overline{B}_1 + B_1^T A_1 + N_1^T \overline{N}_1 & 0 & A_1^* \overline{H}_1 + N_1^T \overline{A}_2 \\
0 & 0 & A_1^T A_1 & 0 & 0 \\
\hline
 & & & A_2^* A_2 & A_2^* B_2 + B_2^T \overline{A}_2 \\
 & & & & +H_1^* \overline{H}_1 \\
 & & & 0 & A_2^T \overline{A}_2
\end{array}\right]
$$

By comparing diagonals of the main diagonal blocks in $\widetilde{\mathcal{X}}_c \mathcal{X}_c = \mathcal{I}$, we deduce that A_1 is orthogonal, while A_2 is unitary. Next, we choose N_1, P_1 arbitrarily. The diagonal element of the right upper block gives $A_1^* H_1 + N_1^* A_2 = 0$, thus $H_1 = -\overline{A}_1 N_1^* A_2$.

We observe the first upper diagonals of the blocks to get $A_1^T B_1 + B_1^* \overline{A}_1 + N_1^* N_1 = 0$, $A_2^* B_2 + B_2^T \overline{A}_2 + H_1^* \overline{H}_1 = 0$ and $A_1^T F_1 + B_1^* \overline{H}_1 + N_1^* B_2 + P_1^T \overline{A}_2 = 0$. Thus $B_1 = -\frac{1}{2} A_1 N_1^* N_1 + A_1 Z_1$, $B_2 = -\frac{1}{2} A_2 H_1^* \overline{H}_1 + A_2 Z_2 = -\frac{1}{2} N_1 N_1^T \overline{A}_2 + A_2 Z_2$ for any $Z_1 = -Z_1^*$, $Z_2 = -Z_2^T$, and $F_1 = -A_1(B_1^* \overline{H}_1 + N_1^* B_2 + P_1^T \overline{A}_2)$. Finally, the second upper diagonal of the left upper block yields $A_1^T C_1 + C_1^T A_1 + B_1^* B_1 + N_1^* P_1 + P_1^T \overline{N}_1 = 0$, therefore C_1 follows.

Solutions of (4.2) with $\mathcal{C} = \mathcal{B}$ form a group. Indeed, for any pair of solutions $\mathcal{X}_1, \mathcal{X}_2$ the product $\mathcal{X}_1 \mathcal{X}_2^{-1}$ is a solution, too:

$$
\begin{aligned}
\mathcal{F}(\mathcal{X}_1 \mathcal{X}_2^{-1})^T \mathcal{F} \mathcal{B} (\mathcal{X}_1 \mathcal{X}_2^{-1}) &= \mathcal{F}(\mathcal{X}_2^{-1})^T \mathcal{F} \mathcal{F} \mathcal{X}_1^T \mathcal{F} \mathcal{B} \mathcal{X}_1 \mathcal{X}_2^{-1} = \mathcal{F}(\mathcal{X}_2^{-1})^T \mathcal{F} \mathcal{B} \mathcal{X}_2^{-1} = \\
&= \mathcal{F}(\mathcal{X}_2^{-1})^T \mathcal{F} \mathcal{B} (\mathcal{B}^{-1} \mathcal{F} \mathcal{X}_2^T \mathcal{F} \mathcal{B}) = \mathcal{B}.
\end{aligned}
$$

Generators of this group are relatively simple as described below.

Lemma 4.4. *Assume $\mathbb{T}^{\alpha,\mu}$ and $\mathbb{T}_c^{\alpha,\mu}$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\mu = (m_1, \ldots, m_N)$ are as in (2.4). Let $\mathbb{X} \subset \mathbb{T}^{\alpha,\mu}$ and $\mathbb{X}_c \subset \mathbb{T}_c^{\alpha,\mu}$ be the sets of solutions $[\mathcal{X}_{rs}]_{r=1}^N$ with $\mathcal{X}_{rs}$ of the form (4.4) and of the form (4.7), respectively, of the equation (4.2) for $\mathcal{C} = \mathcal{B}$. Then*

$$
\mathbb{X} = \mathbb{O} \ltimes \mathbb{V} \subset \mathbb{T}^{\alpha,\mu}, \qquad \mathbb{X}_c = \mathbb{O}_c \ltimes \mathbb{V}_c \subset \mathbb{T}_c^{\alpha,\mu},
$$

in which the group $\mathbb{O}$ (the group $\mathbb{O}_c$) consists of all matrices $\mathcal{Q} = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{\alpha_r} Q_r\big)$ ($\mathcal{Q} = \bigoplus_{r=1}^N (Q_r \oplus \overline{Q}_r \oplus Q_r \oplus \cdots)$) for $Q_r \in \mathbb{C}^{m_r \times m_r}$ such that $B_0^r = Q_r^T B_0^r Q_r$ (such that $B_0^r = Q_r^T B_0^r Q_r$ for α_r odd and $B_0^r = Q_r^* B_0^r Q_r$ for α_r even), $B_0^r = [\mathcal{B}_{rr}]_{11}$, while any $\mathcal{V} \in \mathbb{V}$ (any $\mathcal{V} \in \mathbb{V}_c$) can be written as $\mathcal{V} = \prod_{j=0}^{n_\mathcal{V}} \mathcal{V}_j$, where $\mathcal{V}_0 = \bigoplus_{r=1}^N \mathcal{W}_r$ with $\mathcal{W}_r$ (complex-alternating) upper unitriangular Toeplitz and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of the form (2.4) with (2.8). Both, $\mathbb{V}$ and $\mathbb{V}_c$, are unipotent of order at most $\leq \alpha_1 - 1$. Furthermore:

1. *If $\mathcal{B}$ is of the form (4.5) and $\mathcal{V} \in \mathbb{V}$ is of the form (4.4) with (4.6), then $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_n$ can be choosen of the form (4.4) with (4.6) as well.*

2. *If $\mathcal{B}=\bigoplus_{r=1}^{N}\big(\bigoplus_{j=1}^{\alpha_r}B_0^r\big)$, then $\mathbb{V}$ is generated by matrices of the form (2.6) and of the form (2.4) with (2.8), (2.9), while $\mathbb{V}_c$ is generated by matrices of the form (2.7) and of the form (2.4) with (2.8), (2.11) for $B_r=B_0^r$.*

If solutions of (4.2) consist of rectangular upper triangular Toeplitz blocks, the lemma coincides with [22, Lemma 4.2]. Its proof works mutatis mutandis for solutions with rectangular complex-alternating upper triangular Toeplitz blocks, and also for $\mathcal{B}$ of the form (4.5) and $\mathcal{X}$ of the form (4.4) with (4.6).

5. Proofs of Theorem 2.3 and Theorem 2.7

To get the isotropy group at $\mathcal{H}^\varepsilon$ we shall find all orthogonal Q that solve

$$\mathcal{H}^\varepsilon\overline{Q}=Q\mathcal{H}^\varepsilon. \tag{5.1}$$

We shall first apply Lemma 3.2 to obtain a general solution of (5.1) (Proposition 3.4 (2)). It will then be written in a suitable form by using permutation matrices from Lemma 3.3. Finally, we take into account the orthogonality of solutions, which yields to the crux of the problem, i.e. the equation (4.2) considered in Sec. 4. Applying Lemma 4.1 and Lemma 4.2 will thus immediately imply Theorem 2.3, while further using Lemma 4.4 will furnish Theorem 2.7.

Case I. Suppose

$$\mathcal{H}^\varepsilon=\bigoplus_{r=1}^{N}\Big(\bigoplus_{j=1}^{m_r}\varepsilon_{r,j}H_{\alpha_r}(\lambda)\Big),\qquad \rho:=\lambda^2,\quad \lambda\geq 0,\quad \text{all } \varepsilon_{r,j}\in\{-1,1\},$$

where $H_{\alpha_r}(\lambda)$ is as in (2.2) for $z=\lambda$, $m=\alpha_r$. We have

$$\mathcal{H}:=\bigoplus_{r=1}^{N}\Big(\bigoplus_{j=1}^{m_r}H_{\alpha_r}(\lambda)\Big)=S_\varepsilon\mathcal{H}^\varepsilon\overline{S}_\varepsilon^{-1},\qquad S_\varepsilon=\bigoplus_{r=1}^{N}\Big(\bigoplus_{j=1}^{m_r}\sqrt{\varepsilon_{r,j}}I_{\alpha_j}\Big). \tag{5.2}$$

Using (5.2), the equation (5.1) further transforms to

$$\mathcal{H}\overline{Y}=Y\mathcal{H},\qquad Y=S_\varepsilon QS_\varepsilon^{-1}. \tag{5.3}$$

Lemma 3.2 (2) gives the solution $Y=P^{-1}XP$ of (5.3), so the solution of (5.1) is

$$Q=S_\varepsilon^{-1}P^{-1}XPS_\varepsilon,\qquad P=\bigoplus_{r=1}^{N}\Big(\bigoplus_{j=1}^{m_r}P_{\alpha_r}\Big),\quad P_\alpha:=\tfrac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}(I_\alpha+iE_\alpha),$$

in which $X = [X_{rs}]_{r,s=1}^N$ is such that X_{rs} is an m_r-by-m_s block matrix with blocks of the form (3.2) for $m = \alpha_r$, $n = \alpha_s$ and T is an b_{rs}-by-b_{rs} real (complex-alternating) upper triangular Toeplitz matrix for $\lambda > 0$ ($\lambda = 0$); $b_{rs} = \min\{\alpha_r, \alpha_s\}$.

Since $P_\alpha = P_\alpha^T$, $P_\alpha^2 = E_\alpha$, we get $P^2 = (P^{-1})^2 = E := \bigoplus_{r=1}^N \Big(\bigoplus_{j=1}^{m_r} E_{\alpha_r}\Big)$. Next, $S_\varepsilon = S_\varepsilon^T$, $S_\varepsilon^2 = (S_\varepsilon^2)^{-1} = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{m_r} \varepsilon_{r,j} I_{\alpha_j}\big)$, $PS_\varepsilon^2 = S_\varepsilon^2 P$. Thus $I = Q^T Q$ becomes

$$\begin{aligned} I =& \big(S_\varepsilon^T P^T X^T (P^{-1})^T (S_\varepsilon^{-1})^T\big)\big(S_\varepsilon^{-1} P^{-1} X P S_\varepsilon\big) \\ I =& PS_\varepsilon\big(S_\varepsilon^T P^T X^T (P^{-1})^T (S_\varepsilon^{-1})^T S_\varepsilon^{-1} P^{-1} X P S_\varepsilon\big) S_\varepsilon^{-1} P^{-1} \\ I =& S_\varepsilon^2 P^2 X^T (P^{-1})^2 S_\varepsilon^{-2} X \\ S_\varepsilon^2 =& E X^T E S_\varepsilon^2 X. \end{aligned} \tag{5.4}$$

We conjugate matrices of (5.4) by $\Omega = \bigoplus_{r=1}^N \Omega_{\alpha_r, m_r}$ from Lemma 3.3:

$$\begin{aligned} \Omega^T S_\varepsilon^2 \Omega =& (\Omega^T E \Omega)(\Omega^T Y^T \Omega)(\Omega^T E \Omega)(\Omega^T S_\varepsilon^2 \Omega)(\Omega^T Y \Omega) \\ \mathcal{B} =& \mathcal{F} \mathcal{X}^T \mathcal{F} \mathcal{B} \mathcal{X}, \end{aligned} \tag{5.5}$$

where $\mathcal{F} = \Omega^T E \Omega = \bigoplus_{r=1}^N E_{\alpha_r}(I_{m_r})$, $\mathcal{B} = \Omega^T S_\varepsilon^2 \Omega = \bigoplus_{r=1}^N \big(\bigoplus_{k=1}^{\alpha_r} (\bigoplus_{j=1}^{m_r} \varepsilon_{r,j})\big)$ and $\mathcal{X} = \Omega^T X \Omega$ for $\lambda > 0$ (for $\lambda = 0$) is of the form (3.8) with real (complex-alternating) upper triangular Toeplitz blocks. Lemma 4.1 (a), (b), (c) (i), Lemma 4.2 and Lemma 4.4 (2) give Theorem 2.3 for $\rho \geq 0$ and Theorem 2.7 (I).

Case II. Let

$$\mathcal{H}^\varepsilon = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{m_r} K_{\alpha_r}(\mu)\big), \qquad \rho := -\mu^2, \quad \mu > 0,$$

where $K_{\alpha_r}(\mu)$ is as in (2.3) for $z = \mu$, $m = \alpha_r$. Lemma 3.2 (3) now solves (5.1):

$$Q = P^{-1} V^{-1} S X S^{-1} V P, \tag{5.6}$$

in which $X = [X_{rs}]_{r,s=1}^N$ with an m_r-by-m_s block matrix X_{rs} whose blocks are of the form (3.4) for T_1, T_2 of the form (3.2) for $m = \alpha_r$, $n = \alpha_s$ and T upper triangular Toeplitz of size $b_{rs} \times b_{rs}$ with $b_{rs} = \min\{\alpha_r, \alpha_s\}$, and

$$P = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{m_r} e^{\frac{i\pi}{4}} (P_{\alpha_r} \oplus P_{\alpha_r})\big), \qquad V = \bigoplus_{r=1}^N \big(\bigoplus_{j=1}^{m_r} e^{i\frac{\pi}{4}} (W_{\alpha_r} \oplus \overline{W}_{\alpha_r})\big),$$

$$S=\bigoplus_{r=1}^{N}\Big(\bigoplus_{k=1}^{m_r}\begin{bmatrix} 0 & U_{\alpha_r} \\ J_{\alpha_r}(-i\mu)\overline{U}_{\alpha_r} & 0\end{bmatrix}\Big), \qquad P_\alpha:=\tfrac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}(I_\alpha+iE_\alpha), \quad W_\alpha:=\oplus_{j=0}^{\alpha-1} i^j,$$

where U_α is a solution of the equation $U_\alpha J_\alpha(-\mu^2)=(J_\alpha(i\mu))^2U_\alpha$. Observe that $P=P^T$, $P^2=-(P^{-1})^2=iE$ with $E:=\bigoplus_{r=1}^N\big(\bigoplus_{j=1}^{m_r}(E_{\alpha_r}\oplus E_{\alpha_r})\big)$ and $V=V^T$, $V^{-1}=\overline{V}$. If we define $B=-iES^T\overline{V}E\overline{V}S$, then $I=Q^TQ$ is equivalent to

$$\begin{aligned} I=&\big(P^TV^T(S^{-1})^TX^TS^T(V^{-1})^T(P^T)^{-1}\big)\\ &\times\big(P^{-1}V^{-1}SXS^{-1}VP\big)\\ (S^TV^{-1}P^{-1})(P^{-1}V^{-1}S)=&S^TV^{-1}P^{-1}\big(PV(S^{-1})^TX^TS^T\overline{V}(-iE)\\ &\times\overline{V}SXS^{-1}VP\big)P^{-1}V^{-1}S\\ -iES^T\overline{V}E\overline{V}S=&EX^TE(-iES^T\overline{V}E\overline{V}S)X\\ B=&EX^TEBX. \end{aligned}\tag{5.7}$$

Next, since $(J_{\alpha_r}(-i\mu))^TE_{\alpha_r}=E_{\alpha_r}J_{\alpha_r}(-i\mu)$, we have

$$\begin{aligned}\overline{U}_{\alpha_r}^T(J_{\alpha_r}(-i\mu))^TE_{\alpha_r}J_{\alpha_r}(-i\mu)\overline{U}_{\alpha_r}&=\overline{U}_{\alpha_r}^TE_{\alpha_r}(J_{\alpha_r}(-i\mu))^2\overline{U}_{\alpha_r}\\ &=\overline{U}_{\alpha_r}^TE_{\alpha_r}\overline{U}_{\alpha_r}J_{\alpha_r}(-\mu^2).\end{aligned}$$

We combine it with a calculation $\overline{V}E\overline{V}=\oplus_{r=1}^N\big(i^{\alpha_r}\oplus_{j=1}^{m_r}((-1)^{\alpha_r}E_{\alpha_r}\oplus -E_{\alpha_r})\big)$:

$$B=\bigoplus_{r=1}^{N}\Big(i^{\alpha_r-1}\bigoplus_{j=1}^{m_r}\begin{bmatrix}-E_{\alpha_r}\overline{U}_{\alpha_r}^TE_{\alpha_r}\overline{U}_{\alpha_r}J_{\alpha_r}(-\mu^2) & 0\\ 0 & (-1)^{\alpha_r}E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}U_{\alpha_r}\end{bmatrix}\Big). \tag{5.8}$$

Furthermore, we show that $E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}U_{\alpha_r}$ is upper triangular Toplitz:

$$\begin{aligned}\big(E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}U_{\alpha_r}\big)J_{\alpha_r}(-\mu^2)=&E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}(J_{\alpha_r}(i\mu))^2U_{\alpha_r}\\ =&E_{\alpha_r}U_{\alpha_r}^T((J_{\alpha_r}(i\mu))^2)^TE_{\alpha_r}U_{\alpha_r}\\ =&E_{\alpha_r}(U_{\alpha_r}J_{\alpha_r}(-\mu^2))^TE_{\alpha_r}U_{\alpha_r}\\ =&J_{\alpha_r}(-\mu^2)\big(E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}U_{\alpha_r}\big).\end{aligned}$$

We choose U_{α_r} so that the odd (even) rows have real (purely imaginary or zero) entries, e.g. $U_{\alpha_r}=\begin{bmatrix}v_1 & v_2 & \ldots & v_{\alpha_r}\end{bmatrix}$ is formed by taking real eigenvector v_0 of $(J_{\alpha_r}(i\mu))^2$ and then recursively solve equations $((J_{\alpha_r}(i\mu))^2+\mu^2)v_n=v_{n-1}$ for $n\in\{2,\ldots,\alpha_r\}$. All nonvanishing entries of $E_{\alpha_r}U_{\alpha_r}^TE_{\alpha_r}U_{\alpha_r}$ are hence purely imaginary for α_r even and real for α_r odd. Up to real scaling U_{α_r}, we deduce

$$B = \bigoplus_{r=1}^{N} \Big(\bigoplus_{j=1}^{m_r} \Big[\begin{smallmatrix} T(1,u_1^r,\ldots,u_{\alpha_r-1}^r) J_{\alpha_r}(-\mu^2) & 0 \\ 0 & T(1,u_1^r,\ldots,u_{\alpha_r-1}^r) \end{smallmatrix} \Big] \Big),$$
$$u_1^r, \ldots, u_{\alpha_r-1}^r \in \mathbb{R}.$$

Proceed by conjugating (5.7) with $\Omega' = \bigoplus_{r=1}^{N} \Omega'_{\alpha_r,m_r}$ as in Lemma 3.3 (2):

$$\begin{aligned}(\Omega')^T B\Omega' =&\big((\Omega')^T E\Omega'\big)\big((\Omega')^T X\Omega'\big)^T\big((\Omega')^T E\Omega'\big)\big((\Omega')^T B\Omega'\big)\big((\Omega')^T X\Omega'\big) \\ \mathcal{B} =&\mathcal{F}'\mathcal{X}^T\mathcal{F}'\mathcal{B}\mathcal{X};\end{aligned} \tag{5.9}$$

where $\mathcal{F}' = (\Omega')^T E\Omega' = \bigoplus_{r=1}^{N} E_{\alpha_r}(I_{2m_r})$, $\mathcal{B} = (\Omega')^T B\Omega' = \bigoplus_{r=1}^{N} T(B_0^r, \ldots, B_{\alpha_r-1}^r)$ with B_n^r as in (4.5), and $\mathcal{X} = (\Omega')^T X\Omega'$ of the form (2.4) with (2.5) for $\rho = -\mu^2$. To prove Theorem 2.7 (II), we apply Lemma 4.1 (a), (b), (c) (ii) and Lemma 4.4 (1) to (5.9), while to conclude the proof of Theorem 2.3 for $\rho < 0$, it remains to find $\dim(\Sigma_{\mathcal{H}})$, since Lemma 4.1 does not provide it in this case.

We directly compute the dimension of the tangent space of $\mathrm{Orb}(\mathcal{H}^\varepsilon)$ which is diffeomorphic to the quotient of the orthogonal group over $\Sigma_{\mathcal{H}^\varepsilon}$ ([6, Ch. II.1]). If $Q(t)$ is a complex-differentiable path of orthogonal matrices with $Q(0) = I$, then differentiation of $(Q(t))^T Q(t) = I$ at $t = 0$ yields $Z_0 := \frac{d}{dt}\big|_{t=0} Q(t) = -\frac{d}{dt}\big|_{t=0} Q^T(t) = -Z_0^T$, and the tangent vector of the orbit at $\mathcal{H}^\varepsilon$ is

$$\frac{d}{dt}\Big|_{t=0}\big(Q^*(t)\mathcal{H}^\varepsilon Q(t)\big) = \tfrac{d}{dt}\big|_{t=0} Q^*(t)\mathcal{H}^\varepsilon + \mathcal{H}^\varepsilon \tfrac{d}{dt}\big|_{t=0} Q(t) = -\overline{Z}_0\mathcal{H}^\varepsilon + \mathcal{H}^\varepsilon Z_0;$$

e^{tZ} is orthogonal for $Z = -Z^T$ with $\frac{d}{dt}\big|_{t=0} e^{tZ} = Z$. Hence the codimension of $\Sigma_{\mathcal{H}^\varepsilon}$ in the set of orthogonal matrices is equal to the codimension of $\{-\overline{Z}\mathcal{H}^\varepsilon + \mathcal{H}^\varepsilon Z = 0 \mid Z = -Z^T\}$ in the space of skew-symmetric matrices. We must thus find those Q in (5.6) (solving (5.1)) that satisfy $Q = -Q^T$. By recalling (5.7) with $P = P^T$, $P^{-2} = E$, $V = V^T$, $V^{-1} = \overline{V}$ and $B = -iES^T\overline{V}E\overline{V}S$, we deduce:

$$\begin{aligned} P^{-1}V^{-1}SXS^{-1}VP =& -P^T V^T (S^T)^{-1} X^T S^T (V^{-1})^T (P^{-1})^T \\ ES^T(V^T)^{-1}(P^T)^{-1}P^{-1}V^{-1}SX =& -EX^T S^T (V^{-1})^T (P^{-1})^T P^{-1} V^{-1} S \\ BX =& -EX^T EB. \end{aligned} \tag{5.10}$$

In the same manner as we transformed (5.7) to (5.9), we transform (5.10) to

$$\begin{aligned} \mathcal{B}\mathcal{X} &= -\mathcal{F}'\mathcal{X}^T\mathcal{F}'\mathcal{B} \\ \mathcal{B}_{rr}\mathcal{X}_{rs} &= -E_{\alpha_r}(I_{2m_r})\mathcal{X}_{sr}^T E_{\alpha_s}(I_{2m_s})\mathcal{B}_{ss}, \qquad r, s \in \{1, \ldots, N\}. \end{aligned} \tag{5.11}$$

Clearly, $\mathcal{X}_{sr}$ for $r \neq s$ is uniquely determined by $\mathcal{X}_{rs}$. We now examine the case $r = s$. We compare the entries in the first row of the $(j+1)$-th column in (5.11):

$$\sum_{n=0}^{j} B_n^r A_{j-n}^{rr} = -\sum_{n=0}^{j} (A_{j-n}^{rr})^T B_n^r, \qquad r \in \{1, \ldots, N\}. \tag{5.12}$$

Since $B_n^r = (-\mu^2 u_n^r + u_{n-1}^r) I_{m_r} \oplus u_n^r I_{m_r}$, $A_n^{rr} = \begin{bmatrix} V_n^{rr} & W_n^{rr} \\ -\mu^2 \overline{W}_n^{rr} + \overline{W}_{n-1}^{rr} & \overline{V}_n^{rr} \end{bmatrix}$ with $V_n^{rr}, W_n^{rr} \in \mathbb{C}^{m_r \times m_r}$ for $n \in \{0, \ldots, \alpha_r - 1\}$ and $u_{-1}^r = 0$, $W_{-1}^{rr} = 0$ (see (4.5), (4.6)), then (5.12) for $j = 0$ gives $\begin{bmatrix} -\mu^2 V_0^{rr} & -\mu^2 W_0^{rr} \\ -\mu^2 \overline{W}_0^{rr} & \overline{V}_0^{rr} \end{bmatrix} = -\begin{bmatrix} -\mu^2 (V_0^{rr})^T & -\mu^2 (\overline{W}_0^{rr})^T \\ -\mu^2 (W_0^{rr})^T & (\overline{V}_0^{rr})^T \end{bmatrix}$, while for $j \geq 1$ it yields:

$$\begin{aligned} &\sum_{n=0}^{j} u_n^r \begin{bmatrix} -\mu^2 V_{j-n}^{rr} & -\mu^2 W_{j-n}^{rr} \\ -\mu^2 \overline{W}_{j-n}^{rr} & \overline{V}_{j-n}^{rr} \end{bmatrix} + \sum_{n=0}^{j-1} u_n^r \begin{bmatrix} V_{j-1-n}^{rr} & W_{j-1-n}^{rr} \\ \overline{W}_{j-1-n}^{rr} & 0 \end{bmatrix} = \\ &= -\sum_{n=0}^{j} u_n^r \begin{bmatrix} -\mu^2 (V_{j-n}^{rr})^T & -\mu^2 (\overline{W}_{j-n}^{rr})^T \\ -\mu^2 (W_{j-n}^{rr})^T & (\overline{V}_{j-n}^{rr})^T \end{bmatrix} \\ &\quad -\sum_{n=0}^{j-1} u_n^r \begin{bmatrix} (V_{j-1-n}^{rr})^T & (W_{j-1-n}^{rr})^T \\ (\overline{W}_{j-1-n}^{rr})^T & 0 \end{bmatrix}. \end{aligned} \tag{5.13}$$

We prove by induction that $V_j^{rr} = -(V_j^{rr})^T$, $W_j^{rr} = -(W_j^{rr})^*$ for all j. Clearly, $V_0^{rr} = -(V_0^{rr})^T$, $W_0^{rr} = -(W_0^{rr})^*$. If we assume that the statement holds for $n < j$, it then follows from (5.13) that $\begin{bmatrix} -\mu^2 V_j^{rr} & -\mu^2 W_j^{rr} \\ -\mu^2 \overline{W}_j^{rr} & \overline{V}_j^{rr} \end{bmatrix} = -\begin{bmatrix} -\mu^2 (V_{n-1}^{rr})^T & -\mu^2 (\overline{W}_{n-1}^{rr})^T \\ -\mu^2 (W_{n-1}^{rr})^T & (\overline{V}_{n-1}^{rr})^T \end{bmatrix}$, thus $V_j^{rr} = -(V_j^{rr})^T$, $W_j^{rr} = -(W_j^{rr})^*$. It remains to count all free parameters.

Case III. Let

$$\mathcal{H}^\varepsilon = \bigoplus_{r=1}^{N} \Big(\bigoplus_{j=1}^{m_r} L_{\alpha_r}(\xi) \Big), \qquad \rho := \xi^2 \in \mathbb{C} \setminus \mathbb{R};$$

$L_{\alpha_r}(\xi)$ is as in (2.3) for $z = \xi$, $m = \alpha_r$. Lemma 3.2 (2) gives the solution of (5.1):

$$Q = P^{-1} X P, \qquad P = \bigoplus_{r=1}^{N} \big(\bigoplus_{j=1}^{m_r} P_{\alpha_r} \oplus P_{\alpha_r} \big), \quad P_\alpha := \tfrac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} (I_\alpha + i E_\alpha), \tag{5.14}$$

where $X = [X_{rs}]_{r,s=1}^N$ such that X_{rs} is an m_r-by-m_s block matrix whose blocks are of the form (3.4) for $m = \alpha_r$, $T_2 = 0$ and T_1 of the form (3.2) for $m = \alpha_r$, $n = \alpha_s$ with T an b_{rs}-by-b_{rs} complex upper triangular Toeplitz; $b_{rs} = \min\{\alpha_r, \alpha_s\}$.

Similarly, (5.4) was obtained, we now apply (5.14) to $I = Q^T Q$ to deduce

$$\begin{aligned} I =& P^T X^T (P^{-1})^T P^{-1} X P \\ I =& E X^T E X, \end{aligned}$$

in which $E := \bigoplus_{r=1}^{N} \left(\bigoplus_{j=1}^{m_r} (E_{\alpha_r} \oplus E_{\alpha_r}) \right)$. Using Ω_0 from Lemma 3.3 (2) we get

$$\begin{aligned} I =& (\Omega_0^T E \Omega_0)(\Omega_0^T X \Omega_0)^T (\Omega_0^T E \Omega_0)(\Omega_0^T X \Omega_0) \\ I =& (\mathcal{F} \oplus \mathcal{F}) \mathcal{X}^T (\mathcal{F} \oplus \mathcal{F}) \mathcal{X}, \\ I =& \mathcal{F} \mathcal{V}^T \mathcal{F} \mathcal{V}, \end{aligned} \tag{5.15}$$

in which $\mathcal{F} = \bigoplus_{r=1}^{N} E_{\alpha_r}(I_{m_r})$, $\mathcal{X} = \Omega_0^T X \Omega_0 = \mathcal{V} \oplus \overline{\mathcal{V}}$ for $\mathcal{V}$ of the form (2.4) with upper triangular Toeplitz blocks. Finally, we apply Lemma 4.1 (a), (b) and Lemma 4.4 (2) to prove Theorem 2.3 for $\rho \in \mathbb{C} \setminus \mathbb{R}$ and Theorem 2.7 (II).

This concludes the proof of the theorems.

Remark 5.

1. Solvability of (5.1) was first studied by the author [21, Eq. 2.12] to prove the uniqueness of Hong's normal form under orthogonal *congruence. The technique used there was developed in [22, Lemma 4.1] to the extent of solving (5.15), and finally in this paper we give a complete solution of (5.1).
2. By applying the general approach from this paper or [22], the isotropy groups under orthogonal similarity on skew-symmetric or orthogonal matrices are described by equations involving a significant difference in comparison to (4.2). However, this problem is expected to be addressed in a future study.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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