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ORIGINAL PAPER

On structures of normal forms of complex points of small $\mathcal{C}^2$-perturbations of real 4-manifolds embedded in a complex 3-manifold

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Abstract

We extend our previous result on the behaviour of the quadratic part of a complex points of a small $\mathcal{C}^2$-perturbation of a real 4-manifold embedded in a complex 3-manifold. We describe the change of the structure of the quadratic normal form of a complex point. It is an immediate consequence of a theorem clarifying how small perturbations can change the bundle of a pair of one arbitrary and one symmetric 2×2 matrix with respect to an action of a certain linear group.

Keywords CR manifolds · Closure graphs · Complex points · Normal forms · Perturbations

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1 Introduction

Let M be a smooth real $2n$-submanifold in $\mathbb{C}^{n+1}$. A point $p \in M$ is called *complex* when T_pM is a complex subspace in $T_p\mathbb{C}^{n+1}$; its complex dimension is equal to n. Locally, near a complex point $p \in M$ we can see M as a graph (see e.g. [23]):

$$w = \overline{z}^T Az + \mathrm{Re}(z^T Bz) + o(|z|^2), \quad (z(p), w(p)) = (0, 0), \quad A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}_S^{n\times n}, \tag{1.1}$$

in which $(z, w) = (z_1, z_2, \dots, z_n, w)$ are suitable local coordinates on $\mathbb{C}^{n+1}$, and $\mathbb{C}^{n\times n}$, $\mathbb{C}_S^{n\times n}$ are sets of all $n \times n$ matrices and $n \times n$ symmetric matrices, respectively. A complex point p is quadratically flat, if the quadratic part of (1.1) is real valued.

When $n = 1$ complex points are well understood; see papers of Bishop [3], Kenig and Webster [18], Moser and Webster [19], Huang and Yin [16], Huang and Krantz [17], Bedford and Klingenberg [2] and Forstnerič [13]. They are quadratically flat and given locally by $w = z\overline{z} + \frac{\gamma}{2}(z^2 + \overline{z}^2) + o(|z|^2)$, $0 \leq \gamma$, or $w = z^2 + \overline{z}^2 + o(|z|^2)$. For $n = 2$ a relatively

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simple description of complex points up to quadratic terms was obtained by Coffman [7]; it includes two generic normal forms given by equations

$$w = \tau z_1\overline{z}_2 + z_2\overline{z}_1 + \tfrac{a}{2}z_1^2 + \tfrac{\overline{a}}{2}\overline{z}_1^2 + b(z_1z_2 + \overline{z}_1\overline{z}_2) + \tfrac{d}{2}z_2^2 + \tfrac{\overline{d}}{2}\overline{z}_2^2 + o(|z|^2), \tag{1.2}$$

$$w = z_1\overline{z}_1 + e^{i\theta}z_2\overline{z}_2 + \tfrac{a'}{2}(z_1^2 + \overline{z}_1^2) + b'z_1z_2 + \overline{b'}\overline{z}_1\overline{z}_2 + \tfrac{d'}{2}(z_2^2 + \overline{z}_2^2) + o(|z|^2), \tag{1.3}$$

in which $|a| = 1$, $b > 0$, $d \in \mathbb{C}$, $\tau \in (0, 1)$ and $\theta \in (0, \pi)$, $a', d' > 0$, $b' \in \mathbb{C}^*$ (apply $(A, B) = \left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & d \end{bmatrix}\right)$ and $(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \begin{bmatrix} a' & b' \\ b' & d' \end{bmatrix}\right)$ to (1.1) for $n = 2$, respectively). If $n > 2$ quadratically flat complex points were studied by Slapar and the author [22]. We refer to the papers of Dolbeault, Tomassini and Zaitsev [9] and Fang and Huang [12] for results on holomorphic flattenability of CR-nonminimal real analytic submanifolds near complex points. Formal normal forms of CR-singularities were considered by Burcea [6] and Gong and Stolovitch [15], among others.

In this paper we continue a research started in the paper [23], in which we explained when the quadratic part of a complex point of a real 4-manifold embedded in a complex 3-manifold can be transformed under small $\mathcal{C}^2$-perturbations to the quadratic part of another complex point. For instance, [23, Corollary 3.8] implies that no sufficiently small $\mathcal{C}^2$-deformation of $w = z_1\overline{z}_1 + \frac{1}{2}z_2^2 + \frac{1}{2}\overline{z}_2^2 + o(|z|^2)$ near $(0, 0)$ (with $(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ in (1.1) for $n = 2$) can lead to $w = \frac{1}{2}\overline{z}_1z_2 + z_1\overline{z}_2 + z_1^2 + \overline{z}_1^2 + z_1z_2 + \overline{z}_1\overline{z}_2 + z_2^2 + \overline{z}_2^2 + o(|z|^2)$ (with $(A, B) = \left(\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ in (1.1) for $n = 2$).

We now focus on the change of the type of a complex point, i.e. on the structure of (A, B) in (1.1). In particular, we provide the following result describing possible arbitrarily small $\mathcal{C}^2$-deformations to generic normal forms.

Theorem 1.1 *Let M be a real 4-manifold in $\mathbb{C}^3$ and let $p \in M$ be a complex point given locally by (1.1) with $n = 2$ and $A \in \mathbb{C}^{2\times 2}$, $B \in \mathbb{C}_S^{2\times 2}$. It follows that $A \neq \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta_0} \end{bmatrix}$ with $\theta_0 \in [0, \pi)$ (or $A \neq \begin{bmatrix} 0 & 1 \\ \tau_0 & 0 \end{bmatrix}$ with $\tau_0 \in [0, 1)$) if and only if there exists an arbitrarily small $\mathcal{C}^2$-perturbation M' of M, and such that M' has a complex point p', arbitrarily close to p, and p' is locally given by the equation of the form (1.2) (of the form (1.3)).*

A more general situation is considered in Theorem 3.4 and Corollary 3.6. Due to technical reasons, these results are stated in Sec. 3. A substantial difference in comparison to [23] is that our problem now reduces to a system of nonlinear equations with larger set of parameters. It makes the analysis considerably more involved. However, the number of cases to be considered now is smaller. We add that Theorem 3.4 is of independent interest in matrix analysis since it clarifies how small perturbations can change the bundle of a pair $(A, B) \in \mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ with respect to transformations (cP^*AP, P^TBP) with nonsingular matrix P and $|c| = 1$.

2 Normal forms in dimension 2

Any holomorphic change of coordinates that preserves (1.1) for $n = 2$ transforms (1.1) into the equation that can by a slight abuse of notation be written as

$$w = \overline{z}^T \left(cP^*AP\right) z + \mathrm{Re}\left(z^T (P^T BP)z\right) + o(|z|^2), \qquad P \in GL_2(\mathbb{C}), \quad c \in S^1,$$

where S^1 and $GL_2(\mathbb{C})$ are a unit circle and the group of invertible 2×2 matrices, respectively. Studying the quadratic part of a complex point thus means examining the action of $S^1 \times GL_2(\mathbb{C})$ on $\mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ (see also [7] and [23, Sec. 3]):

$$\Psi\colon \big((c, P), (A, B)\big) \mapsto (cP^*AP, P^T BP), \qquad P \in GL_2(\mathbb{C}), \quad c \in S^1. \tag{2.1}$$

An orbit at $(A, B) \in \mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ with respect to (2.1) is denoted by $\mathrm{Orb}_\Psi(A, B)$.

For some applications it is useful to have a stratification into *bundles* of matrices, i.e. sets of all matrices having similar properties. This notion was first introduced by Arnold [1, Section 30] for the action of similarity; two matrices are in the same bundle under similarity precisely when their Jordan canonical forms have the same structure (with bijection between the sets of distinct eigenvalues). For instance, matrices with all distinct eigenvalues form the generic bundle.

Three bundles with respect to the action (2.1) can be formed according to the sign of $\det \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix}$ for $(A, B) \in \mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ (see [7, Sec. 4]). Slapar [20] (see also [21]) proved that the bundles with nonvanishing determinant are connected components of $\mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ and showed that up to smooth isotopy complex points of a real 4-submanifold in $\mathbb{C}^3$ are locally given either by $w = z_1\overline{z}_1 + z_2\overline{z}_2$ or $w = z_1\overline{z}_1 + \overline{z}_2^2$.

Our goal is to understand the change of normal forms of (2.1) under small perturbations, thus we use the list [7, Sec. 7,Table 1] (see also [23, Lemma 2.2]) of normal forms for orbits under (2.1), to form bundles so that they contain normal forms of similar structure. To be more precise, each such set of normal forms is parameterized by smooth maps $\Lambda \to \mathbb{C}^{2\times 2}$, $\lambda \mapsto A(\lambda)$ and $\Lambda \to \mathbb{C}_S^{2\times 2}$, $\lambda \mapsto B(\lambda)$, in which $\Lambda \subset \mathbb{R}^k$ is a parameter set, and we define the bundle of $(A_0, B_0) = (A(\lambda_0), B(\lambda_0))$ for $\lambda_0 \in \Lambda$ with respect to the action Ψ in (2.1) as:

$$\mathrm{Bun}_\Psi(A_0, B_0) := \bigcup_{\lambda\in\Lambda} \mathrm{Orb}_\Psi(A(\lambda), B(\lambda)). \tag{2.2}$$

Elements of a bundle must behave similarly under small perturbations (Sect. 3).

To simplify the notation, $a \oplus d$ denotes the diagonal matrix with a, d on the diagonal, while the 2×2 identity-matrix and the 2×2 zero-matrix are I_2 and 0_2.

Lemma 2.1 *Bundles of the action (2.1), represented by pairs (A, B) given in Table 1, are immersed submanifolds in $\mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ with dimensions noted in the first column.*

Note that we arranged orbits $\mathrm{Orb}_\Psi(1 \oplus \sigma, d_0 \oplus d)$ for $\sigma \in \{1, -1\}$, $d > 0$, $d_0 \in \{0, d\}$ into bundles $\mathrm{Bun}_\Psi(1 \oplus \sigma, 0 \oplus d) = \cup_{d>0}\mathrm{Orb}_\Psi(1 \oplus \sigma, 0 \oplus d)$ and $\mathrm{Bun}_\Psi(1 \oplus \sigma, dI_2) = \cup_{d>0}\mathrm{Orb}_\Psi(1 \oplus \sigma, dI_2)$, $\sigma \in \{1, -1\}$. Next, $\mathrm{Orb}_\Psi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \zeta & b \\ b & 1 \end{bmatrix}\right)$ for $\zeta \in \mathbb{C}$, $b > 0$ are split into bundles with representatives $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \zeta^* & b \\ b & 1 \end{bmatrix}\right)$ and $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}\right)$ for $\zeta^* \in \mathbb{C}^*$, $b > 0$.

Table 1 Bundles of the action (2.1). Here $0<\tau<1$, $0<\theta<\pi$, $a,b,d>0$, $\zeta\in\mathbb{C}$, $\varphi\in\mathbb{R}$, $\zeta^*\in\mathbb{C}^*$ are the parameters

dim	A	B	A	B	A	B	A	B
14	$1\oplus e^{i\theta}$	$\begin{bmatrix} a & \zeta^* \\ \zeta^* & d \end{bmatrix}$,	$\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$	$\begin{bmatrix} e^{i\varphi} & b \\ b & \zeta \end{bmatrix}$,	$\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$		$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	
		$-\zeta^*\sim\zeta^*$		$\varphi+\pi\sim\varphi$				
12		$\begin{bmatrix} 0 & b \\ b & d \end{bmatrix}$		$\begin{bmatrix} 0 & b \\ b & e^{i\varphi} \end{bmatrix}$,		$a\oplus\zeta$		$\begin{bmatrix} \zeta^* & b \\ b & 1 \end{bmatrix}$
		$\begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$		$\varphi+\pi\sim\varphi$				
		$a\oplus d$		$1\oplus\zeta$				
10		$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$		$0\oplus 1$		$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$		$\begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}$
		$a\oplus 0$		$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$				$a\oplus 1$
		$0\oplus d$						$\begin{bmatrix} 1 & b \\ b & 0 \end{bmatrix}$
9						$0\oplus d$		
8		0_2		0_2				$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$
								$1\oplus 0$
								$0\oplus 1$
7						0_2		
6								0_2
11	I_2	$a\oplus d, a<d$	$1\oplus -1$	$a\oplus d, a<d$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$1\oplus de^{i\theta}$	$1\oplus 0$	
10						$\begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}$		$a\oplus 1$
9		dI_2		dI_2				
		$0\oplus d$		$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$				
				$0\oplus d$				
8						$1\oplus 0$		$0\oplus 1$
								$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
6					0_2	I_2		$a\oplus 0$
5		0_2		0_2				
4						$1\oplus 0$		0_2
0						0_2		

Sketch of the proof of Lemma 2.1 Fix $(A_0,B_0)\in\mathbb{C}^{2\times 2}\times\mathbb{C}_S^{2\times 2}$ from Table 1 and define

$$\Psi_\Lambda\colon S^1\times GL_2(\mathbb{C})\times\Lambda\to\mathbb{C}^{2\times 2}\times\mathbb{C}_S^{2\times 2},\quad (c,P,\lambda)\mapsto\Psi\big(c,P,A(\lambda),B(\lambda)\big),\tag{2.3}$$

where $\Psi_\Lambda(1,I_2,\lambda_0)=(A_0,B_0)$. For every $g\in S^1\times GL_2(\mathbb{C})$ the maps $\Psi^g\colon(A,B)\mapsto\Psi(g,(A,B))$ and $R_g\colon h\mapsto hg$ are automorphisms of $\mathbb{C}^{2\times 2}\times\mathbb{C}_S^{2\times 2}$ and $S^1\times GL_2(\mathbb{C})$, respectively, and we have $\Psi^g\circ\Psi_\Lambda=\Psi_\Lambda\circ(R_g\times\mathrm{id}_\Lambda)$. Thus the rank of $d\Psi_\Lambda$ does not depend on $\lambda \in \Lambda$, $g \in S^1 \times GL_2(\mathbb{C})$ and by the constant rank theorem (e.g. [5, Theorem IV.5.8]) the bundle $\mathrm{Bun}_\Psi(A_0, B_0) \subset \mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ is an immersed manifold.

In a similar manner as tangent spaces of orbits in [23, Lemma 2.2] are computed, tangent spaces of bundles are obtained. We choose paths $(A(t), B(t))$ in $\mathrm{Bun}_\Psi(A_0, B_0)$ and

$$\gamma\colon (-\delta, \delta) \to S^1 \times GL_2(\mathbb{C}), \quad \gamma(t) = (e^{i\alpha t}, I + tX), \qquad \alpha \in \mathbb{R}, X \in \mathbb{C}^{2\times 2}, \delta > 0,$$

and calculate:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} e^{i\alpha t}\big((I+tX)^* A(t)(I+tX)\big) &= i\alpha A_0 + \tfrac{d}{dt}\big|_{t=0} A(t) + (X^* A_0 + A_0 X), \\ \frac{d}{dt}\Big|_{t=0} \big((I+tX)^T B(t)(I+tX)\big) &= \tfrac{d}{dt}\big|_{t=0} B(t) + (X^T B_0 + B_0 X). \end{aligned}$$

Writing $X = \sum_{j,k=1}^2 (x_{jk} + i y_{jk}) E_{jk}$, where E_{jk} is the elementary matrix with one in the j-th row and k-th column and zeros otherwise, we deduce that

$$\begin{aligned} X^* A_0 + A_0 X &= \sum_{j,k=1}^2 (x_{jk} - i y_{jk}) E_{kj} A_0 + \sum_{j,k=1}^2 (x_{jk} + i y_{jk}) A_0 E_{jk} \\ &= \sum_{j,k=1}^2 x_{jk}(E_{kj}A_0 + A_0 E_{jk}) + \sum_{j,k=1}^2 y_{jk} i(-E_{kj}A_0 + A_0 E_{jk}), \\ \tfrac{d}{dt}\big|_{t=0} A(t) &= \beta_{21}E_{21} + \beta_{22}E_{22}, \beta_{22} = \begin{cases} \beta i e^{i\theta}, & A = 1 \oplus e^{i\theta}, 0 < \theta < \pi \\ 0, & \text{otherwise} \end{cases}, \\ \beta_{21} &= \begin{cases} \beta, & A = \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, 0 < \tau < 1 \\ 0, & \text{otherwise} \end{cases}, \quad \beta \in \mathbb{R}. \end{aligned}$$

In a similar fashion we conclude that

$$\begin{aligned} X^T B_0 + B_0 X &= \sum_{j,k=1}^2 x_{jk}(E_{kj}B_0 + B_0 E_{jk}) + \sum_{j,k=1}^2 y_{jk} i(E_{kj}B_0 + B_0 E_{jk}), \\ \tfrac{d}{dt}\big|_{t=0} B(t) &= \sum_{j,k=1}^2 \gamma_{jk} E_{jk}, \qquad \gamma_{jk} = \begin{cases} z_{jk}, & B_{jk}(t) = (B_0)_{jk} + z_{jk}t, z_{jk} \in \mathbb{C} \\ i(B_0)_{jk}\omega_{jk}, & B_{jk}(t) = (B_0)_{jk} e^{i\omega_{jk}t}, \omega_{jk} \in \mathbb{R} \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Note that if $A_{jk}(t)$ (or $B_{jk}(t)$) is constant, then $\beta_{jk} = 0$ ($\gamma_{jk} = 0$).

In view of the identification $\mathbb{R}^8 \times \mathbb{R}^6 \approx \mathbb{C}^{2\times 2} \times \mathbb{C}_S^{2\times 2}$ we denote ($j, k \in \{1, 2\}$):

$$\begin{aligned} &\widetilde{u}_{jk} \approx (0, E_{jk}), \qquad \widetilde{v}_{jk} \approx (0, iE_{jk}), \qquad j \le k \\ &u_{jk} \approx (E_{kj}A_0 + A_0E_{jk}, E_{kj}B_0 + B_0E_{jk}), \quad v_{jk} \approx i(-E_{kj}A_0 + A_0E_{jk}, E_{kj}B_0 + B_0E_{jk}), \\ &w_1 \approx \begin{cases} (ie^{i\theta}E_{22}, 0), & A = 1 \oplus e^{i\theta}, 0 < \theta < \pi \\ (E_{21}, 0), & A = \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, 0 < \tau < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \begin{aligned} w_2 &\approx (iA, 0), \\ w_3 &\approx (0, i(B_0)_{11}E_{11}), \\ w_4 &\approx (0, i(B_0)_{22}E_{22}). \end{aligned} \end{aligned}$$

The tangent space of $\mathrm{Bun}_\Psi(A_0, B_0)$ can be seen as a linear space spanned by vectors $\{w_1, w_2\} \cup \{u_{jk}, v_{jk}\}_{j,k\in\{1,2\}}$ and a subset of vectors $\{w_3, w_4\} \cup \{\widetilde{u}_{jk}, \widetilde{v}_{jk}\}_{j,k\in\{1,2\}, j\le k}$. If $B_{jj}(t) = (B_0)_{jj}(\lambda_0)e^{i\omega_{jj}t}$ for $j \in \{1, 2\}$, then w_{j+2} is in the span, while for $B_{jk}(t) = (B_0)_{jk} + z_{jk}t$, $z_{jk} \neq 0$ vectors $\widetilde{u}_{jk}, \widetilde{v}_{jk}$ are in the span. It is straightforward to compute the dimensions; see [23, Lemma 2.2] for the details in the case of orbits. □

3 Change of the normal form under small perturbations

In this section we study how small deformations of a pair of one arbitrary and one symmetric matrix can change its bundle under the action (2.1). For the sake of clarity the notion *closure graph* for bundles for an action is introduced; compare it with the closure graph for orbits in [23]. Given an action Φ, *vertices* of its closure graph are pairwise disjoint bundles of orbits with respect to Φ, and there is a *path* from a vertex $\widetilde{\mathcal{V}}$ to a vertex $\mathcal{V}$ precisely when $\widetilde{\mathcal{V}}$ lies in the closure of $\mathcal{V}$. The path from $\widetilde{\mathcal{V}}$ to $\mathcal{V}$ is denoted by $\widetilde{\mathcal{V}} \to \mathcal{V}$. To simplify the notation we usually write $\widetilde{V} \to V$ for $\widetilde{V} \in \widetilde{\mathcal{V}}$, $V \in \mathcal{V}$ (instead of $\widetilde{\mathcal{V}} \to \mathcal{V}$). We also require that if $\widetilde{V} \in \widetilde{\mathcal{V}}$ (hence $\mathrm{Orb}_\Phi(\widetilde{V})$) is contained in the closure of $\mathcal{V}$, then whole bundle $\widetilde{\mathcal{V}}$ must lie in the closure of $\mathcal{V}$; it does not hold in general. Closure graphs are reflexive and transitive.

When $\widetilde{V} \not\to V$ it is useful to know the distance from $\widetilde{V}$ to the bundle $\mathcal{V} \ni V$. It suffices to consider the distance from the normal form of $\widetilde{V}$ (see e.g. [23, Remark 3.2]). We use the max norm $\|X\| = \max_{j,k\in\{1,2\}} |x_{j,k}|$, $X = [x_{j,k}]_{j,k=1}^2 \in \mathbb{C}^{2\times 2}$ to measure the distance between matrices.

To emphasize the difference between the closure graphs for orbits and bundles we take look at the action of similarity on $\mathbb{C}^{2\times 2}$. Given $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$ we have $\lambda \oplus \lambda \not\to \lambda \oplus \mu$ in the closure graph for orbits (eigenvalues depend continuously on the entries of the matrix), but $\lambda \oplus \lambda \to \lambda \oplus \mu$ in the closure graph for bundles (the bundle of $\lambda \oplus \mu$ is dense in $\mathbb{C}^{2\times 2}$). For a comprehensive theory on closure hierarchy of matrices under similarity we refer to [10] and [11].

The action (2.1) is closely related to the following two actions:

$$\Psi_1\colon (c, P, A)\big) \mapsto cP^*AP, \quad P \in GL_2(\mathbb{C}),\ c \in S^1,\ A \in \mathbb{C}^{2\times 2} \tag{3.1}$$

$$\Psi_2\colon (P, B) \mapsto P^T BP, \quad P \in GL_2(\mathbb{C}),\ B \in \mathbb{C}_S^{2\times 2}. \tag{3.2}$$

Bundles under these actions are defined the same way as bundles for Ψ in (2.2).

The closure graph for (3.2) with trivial bundles (hence orbits) is simple (see [23, Lemma 3.2]); we add a few necessary conditions on its paths and prove them in Sec. 4. For closure graphs of all 2×2 or 3×3 matrices see [8].

Lemma 3.1 *The closure graph for the action (3.2) is*

$$0_2 \to 1 \oplus 0 \to I_2, \tag{3.3}$$

in which $1 \oplus 0$ and I_2 correspond to bundles of symmetric matrices of rank 1 and 2. Furthermore, let $B = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in \mathbb{C}_S^{2\times 2}$, $\widetilde{B} = \begin{bmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{b} & \widetilde{d} \end{bmatrix} \in \mathbb{C}_S^{2\times 2}$, $P = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \in GL_2(\mathbb{C})$ and $F = \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_2 & \epsilon_4 \end{bmatrix} \in \mathbb{C}_S^{2\times 2}$ be such that $P^T BP = \widetilde{B} + F$. Then the following statements hold:

(1) If $\widetilde{B}$, B are normal forms in (3.3) and such that $\widetilde{B} \not\to B$, then $\|F\| \geq 1$.

(2) If $\widetilde{B} \to B$, then there exist $\epsilon_2', \epsilon_2'' \in \mathbb{C}$, $|\epsilon_2'|, |\epsilon_2''| \leq \begin{cases} \frac{\|F\|(4\|\widetilde{B}\|+2+\sqrt{|\det \widetilde{B}|})}{\sqrt{|\det \widetilde{B}|}}, & \det \widetilde{B} \neq 0 \\ \sqrt{\|F\|(4\|\widetilde{B}\|+3)}, & \det \widetilde{B} = 0 \end{cases}$, so that equations listed in the third column (and in the line corresponding to B) of Table 2 are valid.

By making a more detailed analysis than in [23, Lemma 3.4] (see also [14, Theorem 2.2]) we get the closure graph for bundles under the action (3.1) along with necessary conditions related to its paths; the proof is given in Sect. 4.

Table 2 Necessary conditions on B and P (given that $P^T BP = \widetilde{B} + F$)

	B	
D1	$\begin{bmatrix} 0 & b \\ b & d \end{bmatrix}$	$u(i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2') = v(\widetilde{a}+\epsilon_4)$, $v(-i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2'') = u(\widetilde{d}+\epsilon_4)$
D2	$\begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$	$y(i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2') = x(\widetilde{d}+\epsilon_4)$, $x(-i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2'') = y(\widetilde{a}+\epsilon_1)$
D3	$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$	$2bvx = i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2'$, $2buy = (-i(-1)^l\sqrt{\det\widetilde{B}}+\widetilde{b}+\epsilon_2''$
D4	$0 \oplus d$	$u(\widetilde{b}+\epsilon_2) = v(\widetilde{a}+\epsilon_4)$, $v(\widetilde{b}+\epsilon_2) = u(\widetilde{d}+\epsilon_4)$
D5	$a \oplus 0$	$y(\widetilde{b}+\epsilon_2) = x(\widetilde{d}+\epsilon_4)$, $x(\widetilde{b}+\epsilon_2) = y(\widetilde{a}+\epsilon_1)$

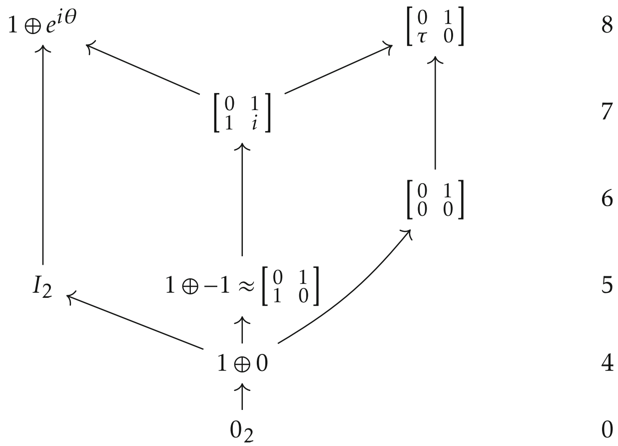

Fig. 1 The closure graph for the action (3.1)

Lemma 3.2 *The closure graph for bundles under the action (3.1) is drawn in Fig. 1. It contains six vertices corresponding to bundles (orbits) with normal forms* 0_2, $1 \oplus 0$, I_2, $1 \oplus -1$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$, *and two vertices for bundles with normal forms* $1 \oplus e^{i\theta}$ *for* $\theta \in (0, \pi)$ *and* $\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$ *for* $\tau \in (0, 1)$.

Furthermore, let $\widetilde{A}$, A *be normal forms in Fig. 1, and let* $E = cP^*AP - \widetilde{A}$ *for some* $c \in S^1$, $P = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \in GL_2(\mathbb{C})$, $E \in \mathbb{C}^{2\times 2}$ *with* $\|E\| < 1$. *Then the following statements hold:*

(1) If $\widetilde{A} \not\to A$, *then there exists a constant* $\mu > 0$ *which does not depend on* c, P *and such that* $\|E\| \geq \mu$.

(2) *If* $\widetilde{A} \to A$, *then there is a constant* $\nu > 0$ *such that the moduli of expressions listed in the fourth column (and in the line corresponding to* $\widetilde{A}$, A*) of Table 3 are bounded by* $\nu\sqrt{\|E\|}$. *(If* $\widetilde{A} \in GL_2(\mathbb{C})$ *then also* $\|E\| \leq \frac{|\det \widetilde{A}|}{8\|\widetilde{A}\|+4}$ *is assumed.)*

Remark 3.3 For calculations of μ, ν in Lemma 3.2 see the proof of the lemma.

We are ready to state the main results of the paper. The proof is given in Sect. 5.

Theorem 3.4 *Let bundles with normal forms of types from Lemma 2.1 be vertices in the closure graph for the action* Ψ *in (2.1). The graph contains precisely the paths described by the following statements:*

(1) *There is a path from* $(0_2, 0_2)$ *to any bundle. There exist paths from* $\mathrm{Bun}_\Psi(1 \oplus 0, 0_2)$ *to all bundles, except to* $\mathrm{Bun}_\Psi(0_2, B)$ *for* $B \in \mathbb{C}_S^{2\times 2}$. *Furthermore, there are paths from* $(1 \oplus 0, \widetilde{a} \oplus 0)$ *with* $\widetilde{a} > 0$ *to all bundles, except to* $(0_2, B)$ *for* $B \in \mathbb{C}_S^{2\times 2}$ *and* $(A, 0_2)$ *for* $A \in \mathbb{C}^{2\times 2}$.

(2) *There exist paths from* $\mathrm{Bun}_\Psi(0_2, 1 \oplus 0)$ *to all bundles, except to* $\mathrm{Bun}_\Psi(A, 0_2)$ *for* $A \in \mathbb{C}^{2\times 2}$.

(3) *From every bundle, except* $\mathrm{Bun}_\Psi(1 \oplus e^{i\theta}, B)$ *for* $0 \leq \theta < \pi$, $B \in \mathbb{C}_S^{2\times 2}$, *there exists a path to the bundle* $\mathrm{Bun}_\Psi\left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, \begin{bmatrix} e^{i\varphi} & b \\ b & \zeta \end{bmatrix}\right)$ *with* $0 \leq \varphi < \pi$, $0 < b$, $\zeta \in \mathbb{C}$.

(4) *From every bundle, except* $\mathrm{Bun}_\Psi\left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, B\right)$ *for* $0 \leq \tau < 1$, $B \in \mathbb{C}_S^{2\times 2}$, *there exists a path to the bundle* $\mathrm{Bun}_\psi\left(1 \oplus e^{i\theta}, \begin{bmatrix} a & \zeta^* \\ \zeta^* & d \end{bmatrix}\right)$ *with* $0 \leq \theta < \pi$, $\zeta^* \in \mathbb{C}^*$ *and* $a, d > 0$.

(5) *All other paths that are not mentioned in (1), (2), (3), (4) are noted in Figs. 2 and 3. (Dimensions of bundles are indicated on the right.)*

Remark 3.5 We prove $(\widetilde{A}, \widetilde{B}) \to (A, B)$ by finding $(A(s), B(s)) \in \mathrm{Bun}(A, B)$, $c(s) \in S^1$, $P(s) \in GL_2(\mathbb{C})$ such that $c(s)(P(s))^*A(s)P(s) \to \widetilde{A}$ and $(P(s))^T B(s)P(s) \to \widetilde{B}$ as $s \to 0$. It often includes tedious calculations and intriguing estimates; but since these do not seem to be of any special interest we omit them and thus shorten the proof significantly. When $(\widetilde{A}, \widetilde{B}) \not\to (A, B)$, then a lower bound for the distance from $(\widetilde{A}, \widetilde{B})$ to $\mathrm{Bun}_\Psi(A, B)$ will be provided as part of the proof of Theorem 3.4. Note that if $\dim\mathrm{Bun}_\Psi(A, B) \leq \dim\mathrm{Orb}_\Psi(\widetilde{A}, \widetilde{B})$, then it implies $(\widetilde{A}, \widetilde{B}) \not\to (A, B)$ ([4, Propositions 2.8.13,2.8.14]), but it gives no estimate on the distance from $(\widetilde{A}, \widetilde{B})$ to $\mathrm{Bun}_\Psi(A, B)$.

The following result is an immediate consequence of Theorem 3.4 (see [23, Corollary 3.8] for an analogous result in the case of the closure graph for orbits).

Corollary 3.6 *Let* M *be a compact real* 4*-manifold embedded* $\mathcal{C}^2$*-smoothly in a complex* 3*-manifold* X *and let* $p_1, \ldots, p_k \in M$ *be precisely (all) its complex points with the corresponding normal forms up to quadratic terms* $(A_1, B_1), \ldots, (A_k, B_k) \in \mathbb{C}^{n\times n} \times \mathbb{C}_S^{n\times n}$. *Assume that* M' *is a deformation of* M *obtained by a smooth isotopy of* M, *and let* $p \in M'$ *be a complex point with the corresponding quadratic normal form* (A, B). *If the isotopy is sufficiently* $\mathcal{C}^2$*-small then* p *is arbitrarily close to some* p_{j_0}, $j_0 \in \{1, \ldots, k\}$, *and* $(A_{j_0}, B_{j_0}) \to (A, B)$ *is a path in the closure graph for bundles for the action (2.1).*

Remark 3.7 The lower bounds for the distances from normal forms to other bundles give the estimate how small the isotopy in the corollary needs to be.

Table 3 Necessary conditions on A, P, c (given that $cP^*AP = \widetilde{A} + E$)

	$\widetilde{A}$	A		
C1	$\alpha \oplus 0$	$1 \oplus e^{i\theta}$	$\|x\|^2 + e^{i\theta}\|u\|^2 - c^{-1}\alpha$, $\|y\|^2 + e^{i\theta}\|v\|^2$	$\alpha \in \{0, 1\}$, $0 < \theta < \pi$
			$(\sin\theta)\overline{u}v$, $(\sin\theta)\overline{x}y$, $\overline{x}y + (\cos\theta)\overline{u}v$	
C2	$\begin{bmatrix} 0 & 1 \\ 1 & \omega \end{bmatrix}$	$1 \oplus e^{i\theta}$	$\|x\|^2 - \|u\|^2$, $\|y\|^2 - \|v\|^2$, $\overline{x}y - \overline{u}v - (-1)^k$, $\sin\theta$	$k \in \mathbb{Z}$; $0 < \theta < \pi$, $\omega \in \{0, i\}$ or $\theta = \pi$, $\omega = 0$
			$(\sin\theta)\|v\|^2 - 1$, $(\sin\theta)\|u\|^2$, $k = 0$	$\omega = i$
C3	$\alpha \oplus 0$	$\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$	$(1+\tau)\mathrm{Re}(\overline{x}u) + i(1-\tau)\mathrm{Im}(\overline{x}u) - \frac{\alpha}{c}$, $(1-\tau)\overline{x}v$	$0 \leq \tau \leq 1$, $\alpha \in \{0, 1\}$,
			$\mathrm{Re}(\overline{y}v)$, $(1-\tau)\mathrm{Im}(\overline{y}v)$, $(1-\tau)\overline{u}y$, $\overline{x}v + \overline{u}y$	
			$c^{-1} - (-1)^k$, $2\mathrm{Re}(\overline{x}u) - (-1)^k\alpha$	$\tau = \alpha = 1$, $\\|E\\| \leq \frac{1}{2}$
C4	$\begin{bmatrix} 0 & 1 \\ 1 & \omega \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$	$\mathrm{Re}(\overline{x}u)$, $(1-\tau)\mathrm{Im}(\overline{x}u)$, $1-\tau$,	$0 < \tau \leq 1$, $\omega \in \{0, i\}$
			$(1+\tau)\mathrm{Re}(\overline{y}v) + i(1-\tau)\mathrm{Im}(\overline{y}v) - (-1)^k\omega$	$k \in \mathbb{Z}$
			$\overline{x}v + \overline{u}y - (-1)^k$	
C5	$1 \oplus -1$	$\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$	$2\mathrm{Re}(\overline{y}v) - (-1)^k$, $2\mathrm{Re}(\overline{x}u) + (-1)^k$, $1-\tau$	$0 < \tau \leq 1$, $k \in \mathbb{Z}$
			$(1-\tau)\mathrm{Im}(\overline{y}v)$, $(1-\tau)\mathrm{Im}(\overline{x}u)$, $\overline{x}v + \overline{u}y$	
C6	$\alpha \oplus 0$	$\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$	$\mathrm{Re}(\overline{y}u)$, $2\mathrm{Re}(\overline{x}u) + i\|u\|^2 - \frac{\alpha}{c}$,	$\alpha \in \{0, 1\}$
			$\overline{x}v + \overline{u}y$, $\overline{u}v$, v^2	

Table 3 continued

	$\widetilde{A}$	A		
C7	$1 \oplus e^{i\widetilde{\theta}}$	$1 \oplus e^{i\theta}$	$u^2, y^2, \lvert x\rvert^2 - 1, \lvert v\rvert^2 - 1$	$0 < \theta < \pi, 0 < \widetilde{\theta} < \pi$
C8	$\begin{bmatrix} \alpha & \beta \\ \beta & \omega \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$	$2\mathrm{Re}(\overline{x}u) - (-1)^k\alpha, 2\mathrm{Re}(\overline{y}v) - (-1)^k\mathrm{Re}(\omega)$	$k \in \mathbb{Z}; \beta = 0, -\omega = \alpha \in \{0, 1\}$
			$\overline{x}v + \overline{u}y - (-1)^k\beta, u^2, \lvert v\rvert^2 - (-1)^k\mathrm{Im}(\omega)$	or $\beta = 1, \alpha = 0, \omega \in \{0, i\}$
C9	$\begin{bmatrix} 0 & 1 \\ \widetilde{\tau} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$	$\overline{x}u, \overline{y}v, \overline{y}u, \overline{x}v - c^{-1}$	$0 \leq \widetilde{\tau} < 1, 0 < \tau < 1$
				or $\tau = \widetilde{\tau} = 0$
			$c^{-1} - (-1)^k$	$0 < \widetilde{\tau}, \tau < 1, k \in \mathbb{Z}$
C10	$\alpha \oplus 0$	$1 \oplus \sigma$	$\lvert x\rvert^2 + \sigma\lvert u\rvert^2 - c^{-1}\alpha, \overline{x}y + \sigma\overline{u}v, \lvert y\rvert^2 + \sigma\lvert v\rvert^2$	$\sigma \in \{1, -1\}, \alpha \in \{1, 0\}$
C11	$1 \oplus \sigma$	$1 \oplus e^{\theta}$	$\lvert x\rvert^2 + \sigma\lvert u\rvert^2 - (-1)^k, \overline{x}y + \sigma\overline{u}v$	$\sigma \in \{1, -1\}, k \in \mathbb{Z}; \sigma = e^{i\theta}$
			$\lvert y\rvert^2 + \sigma\lvert v\rvert^2 - (-1)^k$	or $0 < \theta < \pi, \lVert E\rVert \leq \frac{1}{392}$
C12	$\alpha \oplus 0$	$1 \oplus 0$	$y^2, \lvert x\rvert^2 - \alpha$	$\alpha \in \{0, 1\}$

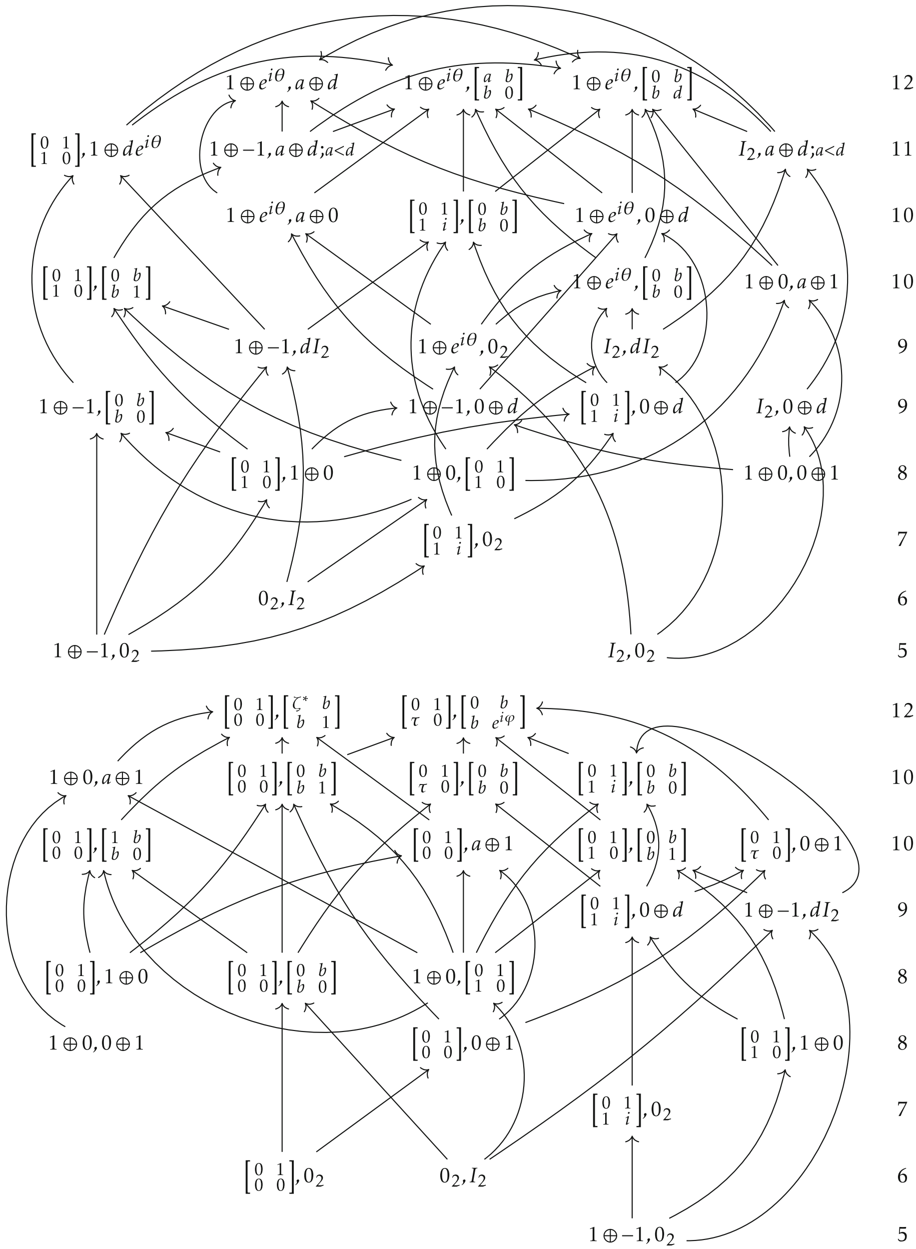

Fig. 2 Paths not mentioned in Theorem 3.4 (1), (2), (3), (4); $a, b, d > 0$, $\zeta^* \in \mathbb{C}^*$, $\tau \in (0,1)$, $\theta \in (0,\pi)$, $\varphi \in [0,\pi)$

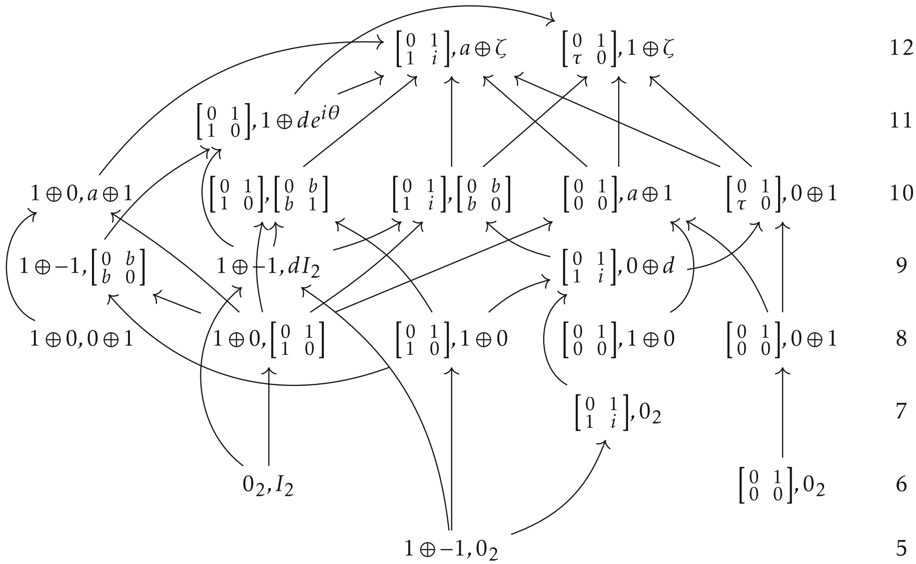

Fig. 3 Paths not mentioned in Theorem 3.4 (1), (2), (3), (4); $a, b, d > 0$, $\zeta \in \mathbb{C}$, $\tau \in (0, 1)$, $\theta \in (0, \pi)$

4 Proof of Lemmas 3.1 and 3.2

In this section we prove Lemma 3.1 and Lemma 3.2. We start with a technical lemma which is an adaptation of [23, Lemma 4.1] to the case of bundles.

Lemma 4.1 *Suppose* $P \in GL_2(\mathbb{C})$, $\widetilde{A}, A, E, \widetilde{B}, B, F \in \mathbb{C}^{2\times 2}$, $c \in S^1$.

(1) If $cP^*AP = \widetilde{A} + E$, $\|E\| \leq \begin{cases} \min\{\frac{|\det \widetilde{A}|}{8\|\widetilde{A}\|+4}, 1\}, & \det \widetilde{A} \neq 0 \\ 1 & \det \widetilde{A} = 0 \end{cases}$, *it then follows that*

$$\left|\sqrt{\det A}\right| |\det P| = \left|\sqrt{\det \widetilde{A}}\right| + r, \qquad |r| \leq \begin{cases} \frac{\|E\|(4\|\widetilde{A}\|+2)}{\sqrt{|\det \widetilde{A}|}}, & \det \widetilde{A} \neq 0 \\ \sqrt{\|E\|(4\|\widetilde{A}\| + 2)}, & \det \widetilde{A} = 0 \end{cases}. \tag{4.1}$$

Moreover, if $A, \widetilde{A} \in GL_2(\mathbb{C})$ *and* $\Delta := \arg\left(\frac{\det \widetilde{A}}{\det A}\right)$ *we have*

$$c = (-1)^k e^{\frac{i\Delta}{2}} + g, \quad c^{-1} = (-1)^k e^{-\frac{i\Delta}{2}} + \overline{g}, \qquad k \in \mathbb{Z}, \quad |g| \leq \frac{\|E\|(8\|\widetilde{A}\|+4)}{|\det \widetilde{A}|}. \tag{4.2}$$

(2) If $P^T BP = \widetilde{B} + F$, $\|F\| \leq \min\{\frac{|\det \widetilde{B}|}{4\|\widetilde{B}\|+2}, 1\}$, *then*

$$\sqrt{\det B}\, \det P = \sqrt{\det \widetilde{B}} + r, \qquad |r| \leq \begin{cases} \frac{\|F\|(4\|\widetilde{B}\|+2)}{\sqrt{|\det \widetilde{B}|}}, & \det \widetilde{B} \neq 0 \\ \sqrt{\|F\|(4\|\widetilde{B}\| + 2)}, & \det \widetilde{B} = 0 \end{cases}.$$

(3) Let further $A, \widetilde{A} \in GL_2(\mathbb{C})$, $\|E\| \leq \min\{1, ||\widetilde{A}^{-1}||^{-1}, \frac{|\det \widetilde{A}|}{8\|\widetilde{A}\|+4}\}$ *and* $cP^*AP = \widetilde{A} + E$, $P^TBP = \widetilde{B} + F$. *It then implies that*

$$\begin{aligned} &|\det \widetilde{A} \det B| = |\det \widetilde{B} \det A| + r, \\ &|r| \leq \max\{\|E\|, \|F\|\} \tfrac{|\det A|}{|\det \widetilde{A}|} \big(4\max\{\|\widetilde{A}\|, \|\widetilde{B}\|, |\det \widetilde{A}|, |\det \widetilde{B}|\} + 2\big)^2. \end{aligned}$$

Moreover, if in addition $B, \widetilde{B}$ *are nonsingular and* $|\det A| = |\det \widetilde{A}| = \|\widetilde{A}\| = 1$, $\|E\|, \|F\| \leq \frac{|\det \widetilde{B}|}{4(4\max\{1, \|\widetilde{B}\|, \det \widetilde{B}\}+2)^2}$, $\Gamma := \arg\big(\frac{\det \widetilde{B}}{\det B}\big)$, *then we have*

$$\det P = (-1)^l e^{i\frac{\Gamma}{2}} + p, \qquad l \in \mathbb{Z}, \quad |p| \leq \|F\| \tfrac{8\|\widetilde{B}\|+4}{\sqrt{3}|\det \widetilde{B}|}.$$

Proof For $\xi, h \in \mathbb{C}$, $\zeta \in \mathbb{C}^*$ we have $\xi\zeta^{-1} = 1 + \frac{h}{\zeta} = |1 + \frac{h}{\zeta}|e^{i\psi}$ with $|\frac{h}{\zeta}| \leq \frac{1}{2}$, hence $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $|\sin\psi| = \big|\mathrm{Im}\big(\frac{1+\frac{h}{\zeta}}{|1+\frac{h}{\zeta}|}\big)\big| \leq \frac{|\mathrm{Im}\frac{h}{\zeta}|}{|1+\frac{h}{\zeta}|} \leq \frac{|\frac{h}{\zeta}|}{1-|\frac{h}{\zeta}|} \leq \frac{2|h|}{|\zeta|}$. Thus

$$\xi = \zeta + h,\ |h| \leq \tfrac{|\zeta|}{2} \neq 0 \quad \text{implies} \quad \arg(\xi) - \arg(\zeta) = \psi \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big),\ \ |\sin\psi| \leq 2\left|\tfrac{h}{\zeta}\right|. \tag{4.3}$$

Estimating the absolute values of the entries of the matrices by the max norm of the matrices, and by slightly simplifying, we obtain that for any $X, D \in \mathbb{C}^{2\times 2}$:

$$\big||\det(X + D)| - |\det X|\big| \leq \big|\det(X + D) - \det X\big| \leq \|D\|\big(4\|X\| + 2\|D\|\big). \tag{4.4}$$

Furthermore, we apply the determinant to $cP^*AP = \widetilde{A} + E$, $Q^TBQ = \widetilde{B} + F$ to get

$$c^2|\det P|^2 \det A = \det(\widetilde{A} + E), \qquad (\det Q)^2 \det B = \det(\widetilde{B} + F). \tag{4.5}$$

Assuming $\|E\|, \|F\| \leq 1$ and using (4.4) for $X = \widetilde{A}$, $D = E$ and $X = \widetilde{B}$, $D = F$ gives

$$\begin{aligned} |\det A||\det P|^2 &= |\det \widetilde{A}| + p, \qquad |p| \leq \|E\|(4\|\widetilde{A}\| + 2), \\ \det B(\det Q)^2 &= \det \widetilde{B} + q, \qquad |q| \leq \|E\|(4\|\widetilde{B}\| + 2), \end{aligned} \tag{4.6}$$

respectively. Equations (4.6) immediately imply statements (4.1) for $\det \widetilde{A} = 0$ and (2) for $\det \widetilde{B} = 0$. Next, we observe another simple fact. If $|s| \leq 1$ then there exists s' so that

$$\sqrt{1 + s} = (-1)^l(1 + s'), \qquad l \in \mathbb{Z},\ \mathrm{Re}(s') \geq -1,\ |s'| \leq |s|. \tag{4.7}$$

We apply (4.7) to (4.6) with $\|E\| \leq \frac{|\det \widetilde{A}|}{4\|\widetilde{A}\|+2}$ and $\|F\| \leq \frac{|\det \widetilde{B}|}{4\|\widetilde{B}\|+2}$ to obtain (4.1) for $\det \widetilde{A} = 0$ and (2) for $\det \widetilde{B} \neq 0$.

The right-hand side of (4.4) for X nonsingular and D with $\|D\| \leq 1$ leads to

$$\left|\tfrac{\det(X+D)}{\det(X)} - 1\right| \leq \tfrac{\|D\|(4\|X\|+2)}{|\det X|}. \tag{4.8}$$

By assuming $\|E\| \leq \frac{|\det \widetilde{A}|}{8\|\widetilde{A}\|+4}$ and applying (4.3) to (4.8) for $X = \widetilde{A}$, $D = E$ we obtain

$$\psi = \arg\big(\tfrac{\det(\widetilde{A}+E)}{\det \widetilde{A}}\big) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big), \qquad |\sin\psi| \leq \tfrac{\|E\|(8\|\widetilde{A}\|+4)}{|\det \widetilde{A}|}. \tag{4.9}$$

From (4.5) we get

$$c^2|\det P|^2 = \tfrac{\det(\widetilde{A}+E)}{\det A} = \tfrac{\det(\widetilde{A}+E)}{\det \widetilde{A}} \tfrac{\det \widetilde{A}}{\det A}, \tag{4.10}$$

and it follows that $c = (-1)^k e^{i(\frac{\Delta}{2}+\frac{\psi}{2})}$ with $k \in \mathbb{Z}$, $\Delta = \arg\big(\frac{\det \widetilde{A}}{\det A}\big)$. Using the identity $e^{i\frac{\psi}{2}} = 1 + 2i(\sin\frac{\psi}{4})e^{i\frac{\psi}{4}}$ and the inequality $2|\sin\frac{\psi}{4}| \le |\frac{\psi}{2}| \le |\sin\psi|$ for $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we deduce (4.2).

We multiply (4.5) for $P = Q$ by $\det\widetilde{B}$ and $\det\widetilde{A}$. By comparing the moduli of the expressions, and assuming $\|E\| \le ||\widetilde{A}^{-1}||^{-1}$ (hence $\det(\widetilde{A}+E) \ne 0$), we get

$$|\det B||\det\widetilde{A}| = |\det A|\frac{|\det\widetilde{A}||\det(\widetilde{B}+F)|}{|\det(\widetilde{A}+E)|}. \tag{4.11}$$

Set $d_{X,D} := |\det(X+D)| - |\det(X)|$ and apply (4.4) for $X = \widetilde{A}$, $D = E$ and $X = \widetilde{B}$, $D = F$:

$$\Big|\frac{|\det\widetilde{A}||\det(\widetilde{B}+F)|}{|\det(\widetilde{A}+E)|} - |\det\widetilde{B}|\Big| = \Big|\frac{d_{\widetilde{B},F}|\det\widetilde{A}| - d_{\widetilde{A},E}|\det\widetilde{B}|}{d_{\widetilde{A},E} + |\det(\widetilde{A})|}\Big| \le \frac{|\det\widetilde{B}|\|E\|\big(4\|\widetilde{A}\|+2\big) + |\det\widetilde{A}|\|F\|\big(4\|\widetilde{B}\|+2\big)}{|\det\widetilde{A}| - \|E\|\big(4\|\widetilde{A}\|+2\big)},$$

provided that $\|E\| \le \min\big\{||\widetilde{A}^{-1}||^{-1}, \frac{|\det\widetilde{A}|}{8\|\widetilde{A}\|+4}\big\}$. We combine it with (4.11):

$$\begin{aligned}&\big||\det\widetilde{A}\det B| - |\det\widetilde{B}\det A|\big| = |\det A|\Big|\frac{|\det\widetilde{A}||\det(\widetilde{B}+F)|}{|\det(\widetilde{A}+E)|} - |\det\widetilde{B}|\Big|\\ &\le \frac{|\det A|}{|\det\widetilde{A}|}\max\{\|E\|, \|F\|\}4\max\big\{|\det\widetilde{A}|, |\det\widetilde{B}|\big\}\big(4\max\{\|\widetilde{A}\|, \|\widetilde{B}\|\}+2\big).\end{aligned} \tag{4.12}$$

Further, let B, $\widetilde{B}$ be nonsingular and $|\det A| = |\det\widetilde{A}| = \|\widetilde{A}\| = 1$, $\|F\| \le \{\frac{|\det\widetilde{B}|}{4\|\widetilde{B}\|+2}, 1\}$, $r := |\det B| - |\det\widetilde{B}|$. Applying (4.8) for $X = \widetilde{B}$, $D = F$ and (4.5) for $Q = P$ yields

$$\begin{aligned}(\det P)^2 &= \frac{\det(\widetilde{B}+F)}{\det\widetilde{B}}\frac{\det\widetilde{B}}{\det B} = e^{i\Gamma}\big(1 - \tfrac{r}{|\det\widetilde{B}|+r}\big)(1+\epsilon'),\\ \Gamma &= \arg\big(\tfrac{\det\widetilde{B}}{\det B}\big),\ |\epsilon'| \le \|F\|\tfrac{4\|\widetilde{B}\|+2}{|\det\widetilde{B}|}.\end{aligned}$$

Provided that $\|E\|, \|F\| \le \frac{|\det\widetilde{B}|}{4(4\max\{1,\|\widetilde{B}\|,\det\widetilde{B}\}+2)^2}$ we use (4.12) to assure $|r| \le \frac{|\det\widetilde{B}|}{4}$ (hence $|1 - \frac{r}{|\det\widetilde{B}|+r}| \le \frac{4}{3}$). By applying (4.7) we complete the proof of (3). □

We proceed with a simple proof of Lemma 3.1.

Proof of Lemma 3.1 The closure graph for 2×2 symmetric matrices is obtained by an easy and straightforward calculation.

We write the matrix equation $P^TBP = F + \widetilde{B}$ for $B = \begin{bmatrix}0 & b\\ b & d\end{bmatrix}$ componentwise:

$$\begin{aligned}&2bux + du^2 = \widetilde{a} + \epsilon_1\\ &bvx + buy + duv = \widetilde{b} + \epsilon_2\\ &2byv + dv^2 = \widetilde{d} + \epsilon_4.\end{aligned} \tag{4.13}$$

By adding and subtracting $b\det P = b(vx - uy)$ from the second equation yields

$$2bvx + duv = b\det P + \widetilde{b} + \epsilon_2, \qquad 2buy + duv = \widetilde{b} + \epsilon_2 - b\det P. \tag{4.14}$$

We multiply the first (the second) equation of (4.14) by u (by v) and compare it with the first (the last) equation of (4.13), multiplied by v (by u):

$$u(b\det P + \widetilde{b} + \epsilon_2) = v(\widetilde{a} + \epsilon_4), \qquad v(-b\det P + \widetilde{b} + \epsilon_2) = u(\widetilde{d} + \epsilon_4). \tag{4.15}$$

For $b = 0$ we obtain (D4). Since $\det B = -b^2$ we deduce from Lemma 4.1 (2) that

$$b \det P = i(-1)^l \sqrt{\det \widetilde{B}} + r, \qquad l \in \mathbb{Z}, \ |r| \leq \begin{cases} \frac{\|F\|(4\|\widetilde{B}\|+2)}{\sqrt{|\det \widetilde{B}|}}, & \det \widetilde{B} \neq 0 \\ \sqrt{\|F\|(4\|\widetilde{B}\|+2)}, & \det \widetilde{B} = 0 \end{cases}. \tag{4.16}$$

Together with (4.14) for $d = 0$ and (4.15) this gives (D3) and (D1).

Next, the equation $P^T AP = F + \widetilde{B}$ for $B = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$ yields

$$\begin{aligned} ax^2 + 2bux &= \widetilde{a} + \epsilon_1 \\ axy + bvx + buy &= \widetilde{b} + \epsilon_2 \\ ay^2 + 2byv &= \widetilde{d} + \epsilon_4. \end{aligned} \tag{4.17}$$

We add and subtract $b \det P = b(vx - uy)$ from the second equation of (4.17):

$$2bvx + axy = b \det P + \widetilde{b} + \epsilon_2, \qquad 2buy + axy = \widetilde{b} + \epsilon_2 - b \det P.$$

By multiplying the first (the second) equation by y (by x) and comparing it with the last (the first) equation of (4.17), multiplied by x (by y), gives

$$y(b \det P + \widetilde{b} + \epsilon_2) = x(\widetilde{d} + \epsilon_4), \qquad x(-b \det P + \widetilde{b} + \epsilon_2) = y(\widetilde{a} + \epsilon_1). \tag{4.18}$$

For $b = 0$ we get (D5), while using (4.16) and (4.18) we obtain (D2). □

Proof of Lemma 3.2 For actions Ψ, Ψ_1 (see (2.1) and (3.1)), it follows that $(A', B') \in \mathrm{Orb}_\Psi(A, 0)$ if and only if $B' = 0$ and $A' \in \mathrm{Orb}_{\Psi_1}(A)$. Hence $\dim\big(\mathrm{Orb}_{\Psi_1}(A)\big) = \dim\big(\mathrm{Orb}_\Psi(A, 0)\big)$, where dimensions of orbits of Ψ are obtained from Lemma 2.1.

To prove $\widetilde{A} \to A$ it suffices to find $c(s) \in S^1$, $P(s) \in GL_2(\mathbb{C})$, $A(s) \in \mathrm{Bun}(A)$ so that

$$c(s)(P(s))^* A(s) P(s) - \widetilde{A} \to 0 \text{ as } s \to 0. \tag{4.19}$$

Trivially $0_2 \to 1 \oplus 0$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$ for $0 < \tau < 1$ and $1 \oplus e^{i\widetilde{\theta}} \to 1 \oplus e^{i\theta}$ for $\widetilde{\theta} \in \{0, \pi\}$, $0 < \theta < \pi$. It is not too difficult to show $1 \oplus 0 \to 1 \oplus \lambda$, $1 \oplus 0 \to \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$ for $0 \leq \tau \leq 1$, $1 \oplus -1 \to \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$ for $0 < \tau < 1$, we take $P(s) = 1 \oplus s$, $P(s) = \frac{1}{\sqrt{1+\tau}} \begin{bmatrix} 1 & 0 \\ 1 & s \end{bmatrix}$, $P(s) = \frac{1}{\sqrt{2}} \begin{bmatrix} s^{-1} & s^{-1} \\ s & -s \end{bmatrix}$ and $P(s) = \frac{1}{2\sqrt{s}} \begin{bmatrix} s & -2i \\ -is & 2 \end{bmatrix}$ with $\tau(s) = 1 - s$ in (4.19), respectively; in all cases $c(s) = 1$. Finally, $A(s) = 1 \oplus e^{i\theta(s)}$ with $\cos(\frac{\theta(s)}{2}) = \frac{s}{2}$, $c(s) = 1$, $P(s) = \sqrt{s} \begin{bmatrix} i & is^{-1} \\ 0 & -is^{-1} \end{bmatrix}$ proves $\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix} \to 1 \oplus e^{i\theta}$ for $0 < \theta < 1$.

It is left to find necessary conditions for the existence of these paths, i.e. given $\widetilde{A}$, E, we must find out how c, P, A depend on E, $\widetilde{A}$, if the following is satisfied:

$$cP^* AP = \widetilde{A} + E, \qquad c \in S^1, P \in GL_2(\mathbb{C}). \tag{4.20}$$

On the other hand, if (4.20) fails for every sufficiently small E, it gives $\widetilde{A} \not\to A$. In such cases the lower estimates for $\|E\|$ will be provided. These easily follow for $\widetilde{A} \neq 0$, $A = 0$ and $\det \widetilde{A} \neq 0$, $\det A = 0$ (Lemma 4.1 (1)).

Throughout the rest of the proof we denote

$$\widetilde{A}=\begin{bmatrix}\alpha & \beta\\ \gamma & \omega\end{bmatrix},\qquad E=\begin{bmatrix}\epsilon_1 & \epsilon_2\\ \epsilon_3 & \epsilon_4\end{bmatrix},\qquad P=\begin{bmatrix}x & y\\ u & v\end{bmatrix}. \tag{4.21}$$

Case I. $A=\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix}$ $(\mathrm{Bun}_{\Psi_1}(A)=\mathrm{Orb}_{\Psi_1}(A))$

This case coincides with [23, Lemma 3.4. Case I]; see (C6), (C8).

Case II. $A=1\oplus\lambda$, $|\lambda|\in\{1,0\}$

The equation (4.20) multiplied by c^{-1}, written componentwise and rearranged is:

$$\begin{aligned}&|x|^2+\lambda|u|^2-c^{-1}\alpha=c^{-1}\epsilon_1,\quad \overline{x}y+\lambda\overline{u}v-c^{-1}\beta=c^{-1}\epsilon_2,\\ &\overline{y}x+\lambda\overline{v}u-c^{-1}\gamma=c^{-1}\epsilon_3,\quad |y|^2+\lambda|v|^2-c^{-1}\omega=c^{-1}\epsilon_4.\end{aligned} \tag{4.22}$$

Subtracting the second complex-conjugated equation (and multiplied by λ) from the third equation (and multiplied by $\overline{\lambda}$) for $\beta,\gamma\in\mathbb{R}$ gives

$$\begin{aligned}&2\mathrm{Im}(\lambda)\overline{v}u-c^{-1}\gamma+\overline{c}^{-1}\beta=c^{-1}\epsilon_3-\overline{c}^{-1}\overline{\epsilon}_2,\\ &-2\mathrm{Im}(\lambda)\overline{y}x-c^{-1}\overline{\lambda}\gamma+\overline{c}^{-1}\lambda\beta=c^{-1}\overline{\lambda}\epsilon_3-\overline{c}^{-1}\lambda\overline{\epsilon}_2.\end{aligned} \tag{4.23}$$

(a) $\lambda=e^{i\theta},\quad 0\le\theta\le\pi$

From (4.23) for $\beta=\gamma=0$, $\mathrm{Im}(\lambda)=\sin\theta$ we get

$$\big|(\sin\theta)\overline{v}u\big|\le\|E\|,\qquad \big|(\sin\theta)\overline{x}y\big|\le\|E\|. \tag{4.24}$$

We take the (real) imaginary parts of the (last) first equation of (4.22) for $\lambda=e^{i\theta}$:

$$\begin{aligned}&(\sin\theta)|u|^2=\mathrm{Im}(c^{-1}\alpha+c^{-1}\epsilon_1),\qquad |x|^2+(\cos\theta)|u|^2=\mathrm{Re}(c^{-1}\alpha+c^{-1}\epsilon_1),\\ &(\sin\theta)|v|^2=\mathrm{Im}(c^{-1}\omega+c^{-1}\epsilon_4),\qquad |y|^2+(\cos\theta)|v|^2=\mathrm{Re}(c^{-1}\omega+c^{-1}\epsilon_4).\end{aligned} \tag{4.25}$$

If $\alpha=0$ we further have:

$$\begin{aligned}&(\sin\theta)|u|^2\le\|E\|,\quad (\sin\theta)|x|^2\le\|E\|(\sin\theta+|\cos\theta|),\\ &\big|(\sin\theta)|v|^2-\mathrm{Im}(c^{-1}\omega)\big|\le\|E\|,\quad \big|(\sin\theta)|y|^2-\mathrm{Re}(c^{-1}\omega)\big|\le\|E\|(\sin\theta+|\cos\theta|).\end{aligned} \tag{4.26}$$

(i) $\widetilde{A}=\begin{bmatrix}0 & 1\\ \widetilde{\tau} & 0\end{bmatrix}, 0\le\widetilde{\tau}\le 1$

If $1\le\widetilde{\tau}<1$, then by applying the triangle inequality to the first equation of (4.23) for $\beta=1$, $\gamma=\widetilde{\tau}$, $\mathrm{Im}(\lambda)=\sin\theta$ and using the first estimates of (4.26) for $\omega=0$ we obtain $2\|E\|\ge 2(\sin\theta)|uv|\ge 1-\widetilde{\tau}-2\|E\|$, which fails for $\|E\|<\frac{1-\widetilde{\tau}}{4}$.

(ii) $\widetilde{A}=\begin{bmatrix}0 & 1\\ 1 & \omega\end{bmatrix},\quad \omega\in\{0,i\}$

By applying the triangle inequality to the second equation of (4.22), and using (4.26) with $|c^{-1}\omega|\le 1$ leads to the inequality:

$$(\sin\theta)(1-\|E\|)\le\sin\theta|\overline{x}y+\lambda\overline{u}v|\le\sqrt{\|E\|(1+\|E\|)}+\sqrt{2\|E\|(1+2\|E\|)}.$$

If $\|E\|\le\frac{1}{12}$ then we deduce $\sin\theta\le 3\sqrt{\|E\|}$ and $\cos^2\theta\ge 1-9\|E\|$. If θ is close to 0 then the second and the last equation of (4.25) for $\alpha=0$, $|c^{-1}\omega|\le 1$ imply that

$|x|^2, |u|^2 \le \frac{\|E\|}{\sqrt{1-9\|E\|}}$ and $|y|^2, |v|^2 \le \frac{1+\|E\|}{\sqrt{1-9\|E\|}}$, respectively. For $\|E\|$ so small that $1 > 2\frac{\sqrt{\|E\|(1+\|E\|)}}{\sqrt{1-9\|E\|}} + \|E\|$, the second equation of (4.22) for $\beta = 1$ fails. Next, when θ is close to π, we deduce that $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$ is close to 0 and $\pi - \theta \in (0, \frac{\pi}{2})$, hence

$$\left|\cos\tfrac{\theta}{2}\right| = \left|\sin\left(\tfrac{\pi-\theta}{2}\right)\right| \le \sin(\pi-\theta) = \sin\theta, \quad \left|\cos\left(\tfrac{\theta+\pi}{4}\right)\right| = \left|\sin\left(\tfrac{\pi-\theta}{4}\right)\right| \le \sin\theta,$$
$$1+\cos\theta = \frac{\cos\frac{\theta}{2}\sin\theta}{\sin\frac{\theta}{2}} \le \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\sin\theta, \quad 1-\sin\tfrac{\theta}{2} = \frac{\cos^2\frac{\theta}{2}}{1+\sin\frac{\theta}{2}} \le \sin\theta. \tag{4.27}$$

We have $c^{-1} = -i(-1)^k e^{i\frac{\theta}{2}} + \overline{g}$, $|g| \le 12\|E\|$ with $\|E\| \le \frac{1}{12}$ (Lemma 4.1 (4.2)), thus $\left|\mathrm{Re}(c^{-1}i)\right| = |\cos\frac{\theta}{2} + i\overline{g}| \le 3\sqrt{\|E\|} + 12\|E\|$ (since $\sin\theta \le 3\sqrt{\|E\|}$). Using the second (fourth) equation of (4.25) and (4.26) for $\alpha = 0$, $\omega \in \{0, i\}$ with (4.27) further implies

$$\begin{aligned}
\left||x|^2 - |u|^2\right| - \|E\| &\le \left||x|^2 - |u|^2 + (1+\cos\theta)|u|^2\right| \\
&= \left||x|^2 + (\cos\theta)|u|^2\right| \le \|E\|, \\
\left||y|^2 - |v|^2\right| - \frac{3\sqrt{\|E\|(1+\|E\|)}}{\sqrt{1-9\|E\|}} &\le \left||y|^2 - |v|^2 + (1+\cos\theta)|v|^2\right| = \left||y|^2 + (\cos\theta)|v|^2\right| \\
&\le 3\sqrt{\|E\|} + 13\|E\|.
\end{aligned} \tag{4.28}$$

Using the second equation of (4.22) and (4.26), (4.27) (for $\alpha = 0$, $\omega \in \{0, i\}$) we get:

$$\begin{aligned}
14\|E\| &\ge |\overline{x}y + e^{i\theta}\overline{u}v + i(-1)^k e^{i\frac{\theta}{2}}| = \\
&= \left|(\overline{x}y - \overline{u}v - (-1)^k) + 2\left(\cos\tfrac{\theta}{2}\right)e^{i\frac{\theta}{2}}\overline{u}v + 2(-1)^k\left(\cos\tfrac{\theta+\pi}{4}\right)e^{i\frac{\theta+\pi}{4}}\right| \\
&\ge \left|\overline{x}y - \overline{u}v - (-1)^k\right| - 2\sqrt{\|E\|(1+\|E\|)} + 6\sqrt{\|E\|}.
\end{aligned} \tag{4.29}$$

For $\omega = i$ we have $\mathrm{Im}(c^{-1}i) = \sin\frac{\theta}{2} + \mathrm{Im}(i\overline{g})$, $|g| \le 12\|E\|$, therefore (4.26) yields

$$13\|E\| \ge \left|(\sin\theta)|v|^2 - (-1)^k - (-1)^k\left(\sin\tfrac{\theta}{2} - 1\right)\right| \ge \left|(\sin\theta)|v|^2 - (-1)^k\right| - 3\sqrt{\|E\|}.$$

Together with (4.28) and (4.29) it proves (C2). Note that the third equation of (4.25) for $\theta = \pi$, $\omega = i$ fails for $\|E\| < \frac{1}{13}$.

(iii) $\widetilde{A} = \alpha \oplus 0$, $\alpha \in \{0, 1\}$
If $\theta \in \{0, \pi\}$, then (4.22) for $e^{i\theta} = \sigma$ yields (C10).
By (4.24) and the second equation of (4.22) we have

$$\left|\overline{x}y + (\cos\theta)\overline{u}v\right| \le 2\|E\|. \tag{4.30}$$

If $0 < \theta \le \pi$, then (4.22), (4.24), (4.30) for $\omega = \beta = \gamma = 0$, $\lambda = e^{i\theta}$ give (C1).

(iv) $\widetilde{A} = 1 \oplus e^{i\widetilde{\theta}}$, $0 \le \widetilde{\theta} \le \pi$.
By Lemma 4.1 (4.2) we have $c^{-1} = (-1)^k e^{i\frac{\theta-\widetilde{\theta}}{2}} + \overline{g}$, $|\overline{g}| \le 12\|E\|$, assuming that $\|E\| \le \frac{1}{12}$. Thus the first and the last equation of (4.22) for $\alpha = 1$, $\lambda = e^{i\widetilde{\theta}}$ are of the form:

$$\begin{aligned}
|x|^2 + e^{i\theta}|u|^2 &= (-1)^k e^{i\frac{\theta-\widetilde{\theta}}{2}} + (\overline{g} + c^{-1}\epsilon_1), \\
|y|^2 + e^{i\theta}|v|^2 &= (-1)^k e^{i\frac{\widetilde{\theta}+\theta}{2}} + (\overline{g}e^{i\widetilde{\theta}} + c^{-1}\epsilon_4).
\end{aligned} \tag{4.31}$$

We take the imaginary parts of (4.31) and apply the triangle inequality:

$$\begin{aligned}\big||u|^2\sin\theta-(-1)^k\sin\left(\tfrac{\theta-\widetilde{\theta}}{2}\right)\big| &\le \big|\mathrm{Im}(\overline{g})+\mathrm{Im}(\epsilon_1)\big|\le 13\|E\|,\\ \big||v|^2\sin\theta-(-1)^k\sin\left(\tfrac{\widetilde{\theta}+\theta}{2}\right)\big| &\le \big|\mathrm{Im}(\overline{g}e^{i\widetilde{\theta}}+\overline{c}^{-1}\epsilon_4)\big|\le 13\|E\|.\end{aligned}\tag{4.32}$$

In particular we have

$$|u|^2\sin\theta\ge\left|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)\right|-13\|E\|,\quad |v|^2\sin\theta\ge|\sin\left(\tfrac{\widetilde{\theta}+\theta}{2}\right)|-13\|E\|.$$

By multiplying these inequalities and using the triangle inequality we deduce

$$\begin{aligned}(\sin^2\theta)|uv|^2\ge&\;|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)\sin\left(\tfrac{\widetilde{\theta}+\theta}{2}\right)|\\ &-13\|E\|\big(|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)|+|\sin\left(\tfrac{\widetilde{\theta}+\theta}{2}\right)|\big)-169\|E\|^2.\end{aligned}$$

By combining it with (4.24) and rearranging the terms we obtain

$$\tfrac{1}{2}|\cos\widetilde{\theta}-\cos\theta|=|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)\sin\left(\tfrac{\widetilde{\theta}+\theta}{2}\right)|\le 170\|E\|^2+26\|E\|\le 196\|E\|.\tag{4.33}$$

If $\theta\in\{0,\pi\}$ with $\widetilde{\theta}\ne\theta$ then (4.33) fails for $\|E\|<\frac{1-|\cos\widetilde{\theta}|}{392}$.
We take the real parts in the first equation of (4.31), multiply them by $\sin\theta$, then rearrange the terms and apply (4.32):

$$\begin{aligned}&(\sin\theta)\big||x|^2-(-1)^k\cos\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)|\\ &\quad=|-\sin\theta\cos\theta|u|^2+(\sin\theta)\mathrm{Re}(\overline{g}+c^{-1}\epsilon_1)\big|,\\ &(\sin\theta)\big||x|^2-(-1)^k\big|-(\sin\theta)\big|\cos\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)-1\big|\le|\sin\left(\tfrac{\theta-\widetilde{\theta}}{2}\right)|+13\|E\|+13\|E\|.\end{aligned}\tag{4.34}$$

Next, let $0<\widetilde{\theta},\theta<\pi$. Thus $\frac{\theta-\widetilde{\theta}}{2}\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\frac{\theta+\widetilde{\theta}}{2}\in(\frac{\widetilde{\theta}}{2},\frac{\widetilde{\theta}+\pi}{2})\subset(0,\pi)$ with $\sin(\frac{\widetilde{\theta}+\theta}{2})\ge\min\{\sin\frac{\widetilde{\theta}}{2},\cos\frac{\widetilde{\theta}}{2}\}$. We apply (4.33) and make a trivial estimate:

$$\frac{196\|E\|}{\min\{\sin\widetilde{\theta},\cos\widetilde{\theta}\}}\ge|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)|\ge|\sin\left(\tfrac{\widetilde{\theta}-\theta}{4}\right)|=\tfrac{1}{\sqrt{2}}\big|\cos\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)-1\big|^{\frac{1}{2}}.\tag{4.35}$$

By combining (4.34) and (4.35) it is straightforward to get a constant $C>0$ so that

$$(\sin\theta)\big||x|^2-(-1)^k\big|\le\frac{196\|E\|}{\min\{\sin\widetilde{\theta},\cos\widetilde{\theta}\}}+2\Big(\frac{196\|E\|}{\min\{\sin\widetilde{\theta},\cos\widetilde{\theta}\}}\Big)^2+26\|E\|\le C\|E\|.\tag{4.36}$$

We multiply the second equation of (4.31) by $e^{-i\theta}$. Then we take the imaginary parts or only rearrange the terms; in both cases we also use (4.35):

$$\begin{aligned}&(\sin\theta)|y|^2\le|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)|+14\|E\|\le C'\|E\|,\quad C':=\frac{196}{\min\{\sin\widetilde{\theta},\cos\widetilde{\theta}\}}+14,\\ &\big||v|^2-(-1)^k\big|\le|e^{i\frac{\widetilde{\theta}-\theta}{2}}-1|+|y|^2+|\overline{g}e^{i\widetilde{\theta}}+c^{-1}\epsilon_4|\\ &\qquad\le 2\Big(\frac{196\|E\|}{\min\{\sin\widetilde{\theta},\cos\widetilde{\theta}\}}\Big)^2+13\|E\|+|y|^2.\end{aligned}\tag{4.37}$$

From the first estimate in (4.32) we similarly obtain $(\sin\theta)|u|^2\le C'\|E\|$. If $\sin\theta\le\max\{\sqrt{C},\sqrt{C'}\}\sqrt{\|E\|}$, then (4.33) yields a contradiction for sufficiently small $\|E\|$. Otherwise $|u|^2\le\sqrt{C'\|E\|}$ and (4.36), (4.37) imply $\big||x|^2-(-1)^k\big|\le\sqrt{C\|E\|}$,

$|y|^2 \leq \sqrt{C'\|E\|}$, respectively. The last estimate in (4.37) concludes the proof of (C7).

Finally, suppose $0 < \theta < \pi$ and $\widetilde{\theta} \in \{0, \pi\}$; hence $\frac{\widetilde{\theta}-\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We apply (4.33) and use (4.32) for $\widetilde{\theta} = 0$ or $\widetilde{\theta} = \pi$ to deduce

$$14\sqrt{\|E\|} \geq \left|\sin\left(\tfrac{\widetilde{\theta}-\theta}{2}\right)\right| \geq \left|\sin\left(\tfrac{\widetilde{\theta}-\theta}{4}\right)\right|, \quad |u|^2 \sin\theta, |v|^2 \sin\theta \leq 13\|E\| + 14\sqrt{\|E\|}. \tag{4.38}$$

Assume now that $\sqrt{\|E\|} \leq \frac{\sqrt{2}}{28}$. If $\widetilde{\theta} = 0$, then $|\cos\frac{\theta}{2}| \geq \frac{\sqrt{2}}{2}$, therefore $1 - \cos\theta = (\sin\theta)|\tan\frac{\theta}{2}| \leq \sqrt{2}\sin\theta$. Similarly, for $\widetilde{\theta} = \pi$ we have $|\sin\frac{\theta}{2}| \geq \frac{\sqrt{2}}{2}$ and so $1 + \cos\theta = (\sin\theta)|\cot\frac{\theta}{2}| \leq \sqrt{2}\sin\theta$. We take the real parts of the first equation (4.31) for $\sigma = e^{i\widetilde{\theta}}$ with $\widetilde{\theta} \in \{0, \pi\}$, rearrange the terms, and apply the triangle inequality:

$$\begin{aligned} 13\|E\| &\geq \left||x|^2 + \sigma|u|^2 - (-1)^k + (-1)^k\left(1 - \cos\left(\tfrac{\theta-\widetilde{\theta}}{2}\right)\right) - |u|^2(\sigma - \cos\theta)\right| \\ &\geq \left||x|^2 + \sigma|u|^2 - (-1)^k\right| - 392\|E\| - \sqrt{2}(13\|E\| + 14\sqrt{\|E\|}). \end{aligned} \tag{4.39}$$

The same proof applies if we replace $x, u, (-1)^k$ by $y, v, \sigma(-1)^k$, respectively. The second equation (4.22) for $\beta = 0$, $\lambda = e^{i\theta}$ and (4.24) finally yield

$$\begin{aligned} \|E\| &\geq \left|\overline{x}y + e^{i\theta}\overline{u}v\right| = \left|\overline{x}y + \sigma\overline{u}v - (\sigma - \cos\theta)\overline{u}v + i(\sin\theta)\overline{u}v\right| \\ &\geq \left|\overline{x}y + \sigma\overline{u}v\right| - (1 + \sqrt{2})\big(13\|E\| + 14\sqrt{\|E\|}\big). \end{aligned}$$

Thus (C11) follows.

(b) $\lambda = 0$ (hence $\det\widetilde{A} = 0$.)
If $\widetilde{A} = \alpha \oplus 0$ for $\alpha \in \{0, 1\}$, then (C12) follows from (4.22) for $\omega = \lambda = 0$. Applying (4.3) for $\|E\| \leq \frac{1}{2}$ to the first equation of (4.22) for $\alpha = 1$, $\lambda = 0$ (multiplied by c), yields $c = e^{i\psi} = 1 + 2i(\sin\frac{\psi}{2})e^{i\frac{\psi}{2}}$ with $|\sin\frac{\psi}{2}| \leq 2\|E\|$. If $\widetilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then (4.22) for $\lambda = \alpha = \omega = 0$ yields $|x|^2, |y|^2 \leq \|E\|$, thus (4.22) fails for $\lambda = \gamma = 0$, $\|E\| < \frac{1}{2}$.

Case III. $A = \begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}$, $0 \leq \tau \leq 1$

From (4.20) multiplied by c^{-1} we obtain

$$\begin{aligned} &\overline{x}u + \tau\overline{u}x - c^{-1}\alpha = c^{-1}\epsilon_1, && \overline{x}v + \tau\overline{u}y - c^{-1}\beta = c^{-1}\epsilon_2, \\ &\tau\overline{v}x + \overline{y}u - c^{-1}\gamma = c^{-1}\epsilon_3, && \overline{y}v + \tau\overline{v}y - c^{-1}\omega = c^{-1}\epsilon_4. \end{aligned} \tag{4.40}$$

Rearranging the terms of the first and the last equation immediately yields

$$\begin{aligned} (1+\tau)\mathrm{Re}(\overline{x}u) + i(1-\tau)\mathrm{Im}(\overline{x}u) &= c^{-1}\alpha + c^{-1}\epsilon_1, \\ (1+\tau)\mathrm{Re}(\overline{y}v) + i(1-\tau)\mathrm{Im}(\overline{y}v) &= c^{-1}\omega + c^{-1}\epsilon_4, \end{aligned} \tag{4.41}$$

while multiplying the third (second) complex-conjugated equation with τ, subtracting it from the second (third) equation, and rearranging the terms, give

$$\begin{aligned} (1-\tau^2)\overline{x}v &= c^{-1}(\beta + \epsilon_2) - \tau\overline{c}^{-1}(\overline{\gamma} + \overline{\epsilon_3}) = (c^{-1}\beta - \tau\overline{c}^{-1}\overline{\gamma}) + (c^{-1}\epsilon_2 - \tau\overline{c}^{-1}\overline{\epsilon_3}) \\ (1-\tau^2)\overline{y}u &= c^{-1}(\gamma + \epsilon_3) - \tau\overline{c}^{-1}(\overline{\beta} + \overline{\epsilon_2}) = (c^{-1}\gamma - \tau\overline{c}^{-1}\overline{\beta}) + (c^{-1}\epsilon_3 - \tau\overline{c}^{-1}\overline{\epsilon_2}). \end{aligned} \tag{4.42}$$

For the existence of paths to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ($*$-congruent to $1 \oplus -1$) see Case II

Using (4.40) we obtain that

$$(1+\tau)|\overline{x}u| \geq |\alpha + \epsilon_1| \geq (1-\tau)|\overline{x}u|, \qquad (1+\tau)|\overline{y}v| \geq |\omega + \epsilon_4| \geq (1-\tau)|\overline{y}v|. \quad (4.43)$$

By multiplying the left-hand and the right-hand sides of these inequalities we get

$$(1+\tau)^2|\overline{x}u\overline{y}v| \geq |\alpha\omega| - \big(|\alpha| + |\omega|\big)\|E\| - \|E\|^2, \quad (4.44)$$
$$|\alpha\omega| + \big(|\alpha| + |\omega|\big)\|E\| + \|E\|^2 \geq (1-\tau)^2|\overline{x}u\overline{y}v|. \quad (4.45)$$

(a) $\widetilde{A} = \begin{bmatrix} 0 & 1 \\ \gamma & \omega \end{bmatrix}$, either $0 \leq \gamma \leq 1, \omega = 0$ or $\gamma = 1, \omega = i$

Equations (4.42) for $\beta = 1, 0 \leq \gamma \leq 1$ imply

$$(1-\tau^2)|\overline{x}v| \geq |\tau\gamma - 1| - (\tau+1)\|E\|, \quad (1-\tau^2)|\overline{u}y| \geq |\gamma - \tau| - (1+\tau)\|E\|.$$

By combining these inequalities and making some trivial estimates we deduce

$$(1-\tau^2)^2|\overline{y}u\overline{x}v| \geq |\tau\gamma - 1|\,|\gamma - \tau| - (1+\tau)\big(\tau\gamma + 1 + \gamma + \tau\big)\|E\| - (1+\tau)^2\|E\|^2.$$

Together with (4.45) for $\alpha = 0$ and using $\|E\| \geq \|E\|^2$ we get

$$(1+\tau)^2(1+|\omega|)\|E\| \geq |\tau\gamma - 1|\,|\gamma - \tau| - (1+\tau)^2(\gamma+1)\|E\| - (1+\tau)^2\|E\|,$$
$$(1+\tau)^2\big(3 + |\omega| + \gamma\big)\|E\| \geq |\tau\gamma - 1|\,|\gamma - \tau| \geq |1-\gamma|\,|\gamma - \tau|. \quad (4.46)$$

If $0 \leq \gamma < 1$ (if $\gamma = 1$) then the right-hand (the left-hand) side of (4.46) implies

$$|\gamma - \tau| \leq \begin{cases} \frac{(1+\tau)^2}{1-\gamma}\big(4 + |\omega|\big)\|E\|, & 0 \leq \gamma < 1 \\ (1+\tau)\sqrt{(4+|\omega|)\|E\|}, & \gamma = 1 \end{cases}. \quad (4.47)$$

When either $\tau = 0, \gamma > 0$ or $\tau = 1, \gamma < 1$ (and $\|E\|$ is small enough), then (4.47) fails. If $0 \leq \gamma < 1$ and $\|E\| \leq \frac{(1-\gamma)^2}{2(1+\tau)^2(4+|\omega|)}$ (hence $1 - \tau \geq |1-\gamma| - |\gamma - \tau| \geq \frac{1-\gamma}{2}$), then (4.43) for $\alpha = 0$ (for $\omega = 0$) yields $|\overline{x}u| \leq \frac{2}{1-\gamma}\|E\|$ (and $|\overline{y}v| \leq \frac{2}{1-\gamma}\|E\|$). Next, (4.42), (4.47) for $\beta = 1, \gamma = 0$, imply $|\overline{y}u| \leq C\|E\|$ and $|\overline{x}v - c^{-1}| \leq C\|E\|$ for some constant $C > 0$ (see (C9) for $\widetilde{\tau} = 0, 0 \leq \tau < 1$).

By Lemma 4.1 (4.2) for $1 \geq \tau > 0$, $\widetilde{A} = \begin{bmatrix} 0 & 1 \\ \gamma & \omega \end{bmatrix}$ with $1 \geq \gamma > 0$ and $\|E\| \leq \frac{\gamma}{12} \leq \frac{1}{12}$, we have $c^{-1} = (-1)^k + \overline{g}$, $k \in \mathbb{Z}$, $|g| \leq \frac{12}{\gamma}\|E\|$, thus (4.42) for $\beta = 1$ (and $\gamma \in \mathbb{R}$) gives

$$(1-\tau^2)\overline{x}v = \big((-1)^k(1-\tau\gamma) - g\tau\gamma + \overline{g}\big) + (c^{-1}\epsilon_2 - \tau\overline{c}^{-1}\overline{\epsilon_3})$$
$$(1-\tau^2)\overline{y}u = (-1)^k(\gamma - \tau) + \gamma\overline{g} - \tau g + (c^{-1}\epsilon_3 - \tau\overline{c}^{-1}\overline{\epsilon_2}).$$

We further obtain

$$(1-\tau^2)|\overline{y}u| \leq (\gamma - \tau) + (\tau\gamma + 1)\tfrac{12}{\gamma}\|E\| + (1+\tau)\|E\|,$$
$$(1-\tau^2)\big|\overline{x}v - (-1)^k\big| \leq \tau(\gamma - \tau) + \tfrac{12(\tau\gamma+1)}{\gamma}\|E\| + (\tau+1)\|E\|. \quad (4.48)$$

Using (4.47) for $0 < \gamma < 1$ we deduce that the left-hand sides of (4.48) are bounded by $D\|E\|$, where $D := \frac{4(4+|\omega|)}{1-\gamma} + \frac{12(\gamma+1)}{\gamma} + 2$. Thus either $1 - \tau^2 \leq \sqrt{D}\sqrt{\|E\|}$ and

$$|1-\gamma| \leq |\tau - \gamma| + |1 - \tau| \leq \tfrac{(1+\tau)^2}{1-\gamma}(2 + |\omega|)\|E\| + \tfrac{\sqrt{D}\sqrt{\|E\|}}{2}$$

fails for small $\|E\|$, or we have $|\overline{y}u|, \left|\overline{x}v-(-1)^k\right| \leq \sqrt{D}\sqrt{\|E\|}$ (see (C9) for $0 < \tau_0, \tau < 1$). The second equation of (4.40) with $\beta = 1$, $c^{-1} = (-1)^k + \overline{g}$, $k \in \mathbb{Z}$, $|g| \leq 12\|E\|$ gives

$$\left|\overline{x}v+\overline{u}y-(-1)^k\right|-(1-\tau)|\overline{u}y| \leq \left|\overline{x}v+\tau\overline{u}y-(-1)^k\right| \leq 12\|E\|+\|E\|. \quad (4.49)$$

From (4.40), (4.41), (4.47), (4.48), (4.49) for $\alpha = 0$, $\omega \in \{0, i\}$, $\gamma = 1$ we deduce (C4). If $\omega = i$, $\tau = 1$ and $\|E\| < \frac{1}{13}$, then the second equality of (4.41) fails.

(b) $\widetilde{A} = \alpha \oplus \omega$

From (4.42) for $\beta = \gamma = 0$ it follows that

$$(1-\tau^2)|\overline{x}v| \leq (1+\tau)\|E\|, \quad (1-\tau^2)|\overline{u}y| \leq (1+\tau)\|E\|, \quad (1-\tau)^2|\overline{x}v\overline{u}y| \leq \|E\|^2. \quad (4.50)$$

Next, (4.50) yields either $(1-\tau)|\overline{x}u| \leq \|E\|$ or $(1-\tau)|\overline{y}v| \leq \|E\|$.

By Lemma 4.1 (4.2) for $0 < \tau \leq 1$, $\widetilde{A} = 1 \oplus e^{i\widetilde{\theta}}$, we have $c^{-1} = (-1)^k e^{-i\frac{\widetilde{\theta}+\pi}{2}} + \overline{g}$, $k \in \mathbb{Z}$, $|g| \leq 12\|E\|$. We take the imaginary parts of (4.41) with $\alpha = 1$, $\omega = e^{i\widetilde{\theta}}$, $0 < \tau < 1$ to deduce $|\cos\frac{\widetilde{\theta}}{2}| \leq 14\|E\|$, which fails for $0 \leq \widetilde{\theta} < \pi$ and small $\|E\|$.

By combining (4.50) with (4.44) for $|\alpha| = |\omega| = 1$ and using $\|E\| \leq \frac{1}{4}$, we get

$$\tfrac{1}{4}(1-\tau)^2 \leq (1-\tau)^2(1-2\|E\|-\|E\|^2) \leq (1-\tau^2)^2|\overline{x}v\overline{u}y| \leq (1+\tau)^2\|E\|^2.$$

Thus $1-\tau \leq 4\|E\|$. (In particular, we obtain a contradiction for $\tau = 0$, $|\alpha| = |\omega| = 1$.) When $\widetilde{\theta} = \pi$ (i.e. $\widetilde{A} = 1 \oplus -1$, $c^{-1} = (-1)^{k+1} + \overline{g}$, $k \in \mathbb{Z}$, $|g| \leq 12\|E\|$), we use (4.40), (4.41) for $\beta = 0$, $\alpha = -\omega = 1$ to get $(1-\tau)\mathrm{Im}(\overline{x}u), (1-\tau)\mathrm{Im}(\overline{x}u) \leq 13\|E\|$ and

$$\begin{aligned}
&|\overline{x}v+\overline{u}y|-2\|E\| \leq |\overline{x}v+\overline{u}y|-(1-\tau)|\overline{u}y| \leq |\overline{x}v+\tau\overline{u}y| \leq \|E\|,\\
&\left|2\mathrm{Re}(\overline{x}u)-(-1)^{k+1}\right| = \tfrac{2}{1+\tau}\left|(1+\tau)\mathrm{Re}(\overline{x}u)-(-1)^{k+1}+(-1)^{k+1}\tfrac{1-\tau}{2}\right| \leq 30\|E\|,\\
&\left|2\mathrm{Re}(\overline{y}v)-(-1)^{k}\right| = \tfrac{2}{1+\tau}\left|(1+\tau)\mathrm{Re}(\overline{y}v)-(-1)^{k}+(-1)^{k}\tfrac{1-\tau}{2}\right| \leq 30\|E\|. \quad (4.51)
\end{aligned}$$

It gives (C5). The first line of (4.51) is valid also for $\alpha \in \{0, 1\}$, $\beta = \omega = 0$ (see (4.50)). If $\alpha = 1$, then (4.41) for $\tau = 1$ yields $2c\mathrm{Re}(\overline{x}u) = 1 + \epsilon_1$. By applying (4.3) for $\|E\| \leq \frac{1}{2}$ we get $c = (-1)^k e^{i\psi}$, $k \in \mathbb{Z}$, $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $|\sin\psi| \leq 2\|E\|$. Moreover, $\left|c-(-1)^k\right| = 2|\sin\frac{\psi}{2}| \leq 4\|E\|$. To conclude, (4.41), (4.43), (4.50) provides (C3).

This completes the proof of the lemma. □

5 Proof of Theorem 3.4

To prove the nonexistence of some paths in the closure graph for bundles under (2.1), the proof of [23, Theorem 3.6] (the closure graph for orbits) applies mutatis mutandis; we shall not rewrite the proof in these cases, instead we refer to [23] for the proof. However, we reprove the existence of paths for bundles consisting of one orbit, since short and plausible arguments can be given (see e.g. (5.2)).

Proof of Theorem 3.4 Given normal forms $(\widetilde{A}, \widetilde{B})$, (A, B) from Lemma 2.1 the existence of a path $(\widetilde{A}, \widetilde{B}) \to (A, B)$ in the closure graph for bundles for the action (2.1) immediately implies $\widetilde{A} \to A$, $\widetilde{B} \to B$. When this is not fulfilled, then $(\widetilde{A}, \widetilde{B}) \not\to (A, B)$ and we already have a lower estimate on the distance from $(\widetilde{A}, \widetilde{B})$ to the bundle of (A, B) (see Lemmas 3.1, 3.2). Further, $(\widetilde{A}, 0_2) \to (A, 0_2)$ (or $(0_2, \widetilde{B}) \to (0_2, B)$) if and only if $\widetilde{A} \to A$ (or $\widetilde{B} \to B$), and trivially $(A, B) \to (A, B)$ for any A, B.

From now on suppose $\widetilde{A} \to A$, $\widetilde{B} \to B \neq 0$ with $(\widetilde{A}, \widetilde{B}) \notin \mathrm{Bun}_\Psi(A, B)$ and let

$$cP^*AP = \widetilde{A} + E, \quad P^TBP = \widetilde{B} + F, \qquad c \in S^1,\ P \in GL_2(\mathbb{C}), \quad E, F \in \mathbb{C}^{2\times 2}. \tag{5.1}$$

Due to Lemmas 3.2 and 4.1 (1) the first equation of (5.1) yields restrictions on P, c, A imposed by $\|E\|$, $\widetilde{A}$. The trick of the proof is to use these to analyse the second equation of (5.1). We now work with equations with larger set of parameters than in [23, Theorem 3.6], and it usually makes the analysis more involved. If it eventually leads to an inequality that fails for any sufficiently small E and F, it will prove $(\widetilde{A}, \widetilde{B}) \not\to (A, B)$; it is then straightforward to estimate how small E, F should be, thus we omit this calculation. Otherwise, to prove $(\widetilde{A}, \widetilde{B}) \to (A, B)$, we find $c(s) \in S^1$, $P(s) \in GL_2(\mathbb{C})$, $(A(s), B(s)) \in \mathrm{Bun}(A, B)$ such that

$$c(s)\big(P(s)\big)^*A(s)P(s) - \widetilde{A} =: E(s) \overset{s\to 0}{\longrightarrow} 0, \quad \big(P(s)\big)^T B(s)P(s) - \widetilde{B} =: F(s) \overset{s\to 0}{\longrightarrow} 0. \tag{5.2}$$

When we can arrange the parameter s so that $A(s) \to \widetilde{A}$ and $B(s) \to \widetilde{B}$, this is trivial.

Throughout the proof we denote $\delta = \nu\sqrt{\|E\|}$ for $\nu > 0$ (Lemma 3.2 (2)), $\epsilon = \|F\|$,

$$B = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{b} & \widetilde{d} \end{bmatrix}, \qquad F = \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_2 & \epsilon_4 \end{bmatrix}, \qquad P = \begin{bmatrix} x & y \\ u & v \end{bmatrix},$$

where sometimes polar coordinates for x, y, u, v in P might be preferred:

$$x = |x|e^{i\phi}, \quad y = |y|e^{i\varphi}, \quad u = |u|e^{i\eta}, \quad v = |v|e^{i\kappa}, \qquad \phi, \varphi, \eta, \kappa \in \mathbb{R}. \tag{5.3}$$

The second matrix equation of (5.1) can thus be written componentwise as:

$$\begin{aligned} &ax^2 + 2bux + du^2 = \widetilde{a} + \epsilon_1, \\ &axy + buy + bvx + duv = \widetilde{b} + \epsilon_2, \\ &ay^2 + 2bvy + dv^2 = \widetilde{d} + \epsilon_4. \end{aligned} \tag{5.4}$$

For the sake of simplicity some estimates in the proof are crude, and it is always assumed $\epsilon, \delta \leq \frac{1}{2}$. Since we shall often apply Lemma 4.1, we take for granted that $(\frac{\delta}{\nu})^2 = \|E\| \leq \begin{cases} \min\{\frac{|\det \widetilde{A}|}{8\|\widetilde{A}\|+4}, 1\}, & \det \widetilde{A} \neq 0 \\ 1 & \det \widetilde{A} = 0 \end{cases}$, $\epsilon = \|F\| \leq \begin{cases} \min\{\frac{|\det \widetilde{B}|}{8\|\widetilde{B}\|+4}, 1\}, & \det \widetilde{B} \neq 0 \\ 1 & \det \widetilde{B} = 0 \end{cases}$. If $A, \widetilde{A}$ are nonsingular we also assume $\|E\| \leq \|\widetilde{A}^{-1}\|^{-1}$, while for $B, \widetilde{B}$ nonsingular with $1 = |\det A| = |\det \widetilde{A}| = \|\widetilde{A}\|$ it is assumed $\|E\|, \|F\| \leq \frac{|\det \widetilde{B}|}{4(4\max\{1, \|\widetilde{B}\|, |\det \widetilde{B}|\}+2)^2}$.

We split our analysis into several cases (see Lemma 2.1 for normal forms). The notation $(\widetilde{A}, \widetilde{B}) \dashrightarrow (A, B)$ is used when the existence of a path is yet to be considered.

Case I. $(1 \oplus e^{i\widetilde{\theta}}, \widetilde{B}) \dashrightarrow (1 \oplus e^{i\theta}, B)$, $0 \leq \widetilde{\theta} \leq \pi$, $0 \leq \theta \leq \pi$

(a) $0 < \widetilde{\theta}, \theta < \pi$

From Lemma 3.2 (2) for (C7) we get

$$|y|^2, |u|^2 \leq \delta, \qquad \big||v|^2 - 1\big|,\ \big||x|^2 - 1\big| \leq \delta. \tag{5.5}$$

(i) $B=\begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$, $b, a \geq 0$
Using (5.5) and Lemma 3.1 (D2) we get a contradiction for small ϵ, δ and $\widetilde{d} \neq 0$. Next, Lemma 4.1 (3) for $\widetilde{d}=0$ gives $b^2=|\widetilde{b}|^2+\delta_5$, $|\delta_5| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}\big(4\max\{1, |\widetilde{b}|^2, |\widetilde{b}|\}+2\big)^2$. It fails for $b=0$, $\widetilde{b} \neq 0$ and $\epsilon, \frac{\delta^2}{\nu^2} < \widetilde{b}^2\big(4\max\{1, |\widetilde{b}|^2, |\widetilde{b}|\}+2\big)^{-2}$, while the case $b=\widetilde{b}=0$ is trivial. For $a=0$, $\widetilde{a} \neq 0$ then the first equation of (5.4) for $a=d=0$ and (5.5) yields an inequality that fails for ϵ, δ small enough:

$$|\widetilde{a}|=|\epsilon_1-2bux| \leq \epsilon+2(\widetilde{b}+\delta_5)\sqrt{\delta(1+\delta)}.$$

(ii) $B=\begin{bmatrix} 0 & b \\ b & d \end{bmatrix}$, $b \geq 0, d \neq 0$
Due to a symmetry we deal with this case similarly as with Case I (a) (i).

(iii) $B=a \oplus d, \quad a, d>0$
From (5.4) for $b=0$ we obtain

$$\begin{aligned} ax^2+du^2 &= \widetilde{a}+\epsilon_1, \\ axy+duv &= \widetilde{b}+\epsilon_2, \\ ay^2+dv^2 &= \widetilde{d}+\epsilon_4. \end{aligned} \tag{5.6}$$

By multiplying the first and the last equation of (5.6) by $\delta_6=\frac{y}{x}$ and $\delta_5=\frac{u}{v}$, respectively, and by slightly simplifying them, we get

$$axy+duv\delta_5\delta_6=\delta_6(\widetilde{a}+\epsilon_1), \qquad axy\delta_5\delta_6+duv=\delta_5(\widetilde{d}+\epsilon_4).$$

Adding these two equations and using the second equation of (5.6) we deduce

$$(\widetilde{b}+\epsilon_2)(1+\delta_5\delta_6)=\delta_5(\widetilde{d}+\epsilon_4)+\delta_6(\widetilde{a}+\epsilon_1),$$

which fails for $\widetilde{b} \neq 0$ and sufficiently small ϵ, δ (by (5.5) we have $|\delta_5|, |\delta_6| \leq \frac{\delta}{1-\delta}$).

(b) $\widetilde{\theta} \in \{0, \pi\}$
Set $\sigma=e^{i\widetilde{\theta}} \in \{1,-1\}$. Lemma 3.2 (C11) yields

$$|x|^2+\sigma|u|^2=(-1)^k+\delta_1, \quad \overline{x}y+\sigma\overline{u}v=\delta_2, \quad |y|^2+\sigma|v|^2=\sigma(-1)^k+\delta_4, \tag{5.7}$$

where $|\delta_1|, |\delta_2|, |\delta_4| \leq \delta$. Next, for $v \neq 0$, $\big(|x|-|u|\big)^2 \leq \big||x|^2-|u|^2\big| =: 1+\delta_1'$ we deduce

$$\begin{aligned} |\overline{x}y+\sigma\overline{u}v| &\geq \big||\overline{x}y|-|\overline{x}v|+|\overline{x}v|-|\overline{u}v|\big| \geq |v|\big|\overline{x}|-|\overline{u}|\big|-|\overline{x}|\big||y|-|v|\big| \\ &\geq \frac{\big|\overline{x}|^2-|\overline{u}|^2\big|}{\frac{1}{|v|}(|\overline{x}|+|\overline{u}|)}-\frac{|x|\big|\overline{y}|^2-|\overline{v}|^2\big|}{|\overline{y}|+|\overline{v}|} \geq \frac{1-|\delta_1'|}{2\frac{|u|}{|v|}+\frac{\sqrt{1+|\delta_1'|}}{|v|}}-\left(\frac{|u|}{|v|}+\frac{\sqrt{1+|\delta_1'|}}{|v|}\right)\big||\overline{y}|^2-|\overline{v}|^2\big|. \end{aligned} \tag{5.8}$$

(i) $B=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $a, d, b \geq 0, a+d \neq 0$
Let first $B=a \oplus d$. Using the notation (5.3) the following calculation is trivial:

$$\begin{aligned} ax^2+du^2 &= ae^{2i\phi}\big(|x|^2+\sigma|u|^2\big)-u^2(\sigma ae^{2i(\phi-\eta)}-d), \qquad \sigma \in \{-1,1\}, \\ ay^2+dv^2 &= ae^{2i\varphi}\big(|y|^2+\sigma|v|^2\big)-v^2(\sigma ae^{2i(\varphi-\kappa)}-d), \end{aligned}$$

$$ay^2 + dv^2 = d\sigma e^{2i\kappa}\big(|y|^2 + \sigma|v|^2\big) - y^2(\sigma d e^{2i(\kappa-\varphi)} - a). \tag{5.9}$$

Furthermore, one easily writes:

$$\begin{aligned} axy + duv &= ae^{2i\phi}(\overline{x}y + \sigma\overline{u}v) - uv(\sigma a e^{2i(\phi-\eta)} - d), \qquad \sigma \in \{-1, 1\}, \\ axy + duv &= ae^{2i\varphi}(x\overline{y} + \sigma u\overline{v}) - uv(\sigma a e^{2i(\varphi-\kappa)} - d), \\ axy + duv &= d\sigma e^{2i\kappa}(x\overline{y} + \sigma u\overline{v}) - xy(d\sigma e^{2i(\kappa-\varphi)} - a). \end{aligned} \tag{5.10}$$

Rearranging the terms in (5.9), (5.10) and using (5.6), (5.7) yields for $\sigma \in \{-1, 1\}$:

$$\begin{aligned} u^2(\sigma a e^{2i(\phi-\eta)} - d) &= ae^{2i\phi}((-1)^k + \delta_1) - \widetilde{a} - \epsilon_1, \\ uv(\sigma a e^{2i(\phi-\eta)} - d) &= ae^{2i\phi}\delta_2 - \widetilde{b} - \epsilon_2, \\ v^2(\sigma a e^{2i(\varphi-\kappa)} - d) &= ae^{2i\varphi}(\sigma(-1)^k + \delta_4) - \widetilde{d} - \epsilon_4, \\ uv(\sigma a e^{2i(\varphi-\kappa)} - d) &= ae^{2i\varphi}\overline{\delta}_2 - \widetilde{b} - \epsilon_2, \\ y^2(\sigma d e^{2i(\kappa-\varphi)} - a) &= d\sigma e^{2i\kappa}(\sigma(-1)^k + \delta_4) - \widetilde{d} - \epsilon_4, \\ xy(d\sigma e^{2i(\kappa-\varphi)} - a) &= d\sigma\overline{\delta}_2 - \widetilde{b} - \epsilon_2. \end{aligned}$$

By dividing the equations in each line we get

$$\frac{u}{v} = \frac{ae^{2i\phi}((-1)^k+\delta_1)-\widetilde{a}-\epsilon_1}{ae^{2i\phi}\delta_2-\widetilde{b}-\epsilon_2} = \frac{ae^{2i\varphi}\overline{\delta}_2-\widetilde{b}-\epsilon_2}{ae^{2i\varphi}(\sigma(-1)^k+\delta_4)-\widetilde{d}-\epsilon_4}, \qquad \frac{x}{y} = \frac{d\sigma\overline{\delta}_2-\widetilde{b}-\epsilon_2}{d\sigma e^{2i\kappa}(\sigma(-1)^k+\delta_4)-\widetilde{d}-\epsilon_4}. \tag{5.11}$$

If $\widetilde{B} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 0 \end{bmatrix}$ for $\widetilde{b} > 0$ (hence $\sigma = -1$, $d \geq a > 0$), then Lemma 4.1 (3) implies $a^2 \leq ad = \widetilde{b}^2 + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}\big(4\max\{1, |\widetilde{b}|, |\widetilde{b}|^2\} + 2\big)^2$. From the first equation of (5.11) for $\widetilde{d} = \widetilde{a} = 0$, $\sigma = 1$ we now obtain a contradiction for small ϵ, δ. Similarly, it follows from Lemma 4.1 (3) for $\widetilde{B} = \widetilde{a} \oplus \widetilde{d}$ and $B = aI_2$ that $a^2 = \widetilde{a}\widetilde{d} + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{1, |\widetilde{d}|, |\widetilde{d}\widetilde{a}|\} + 2)^2$. If $\widetilde{d} > \widetilde{a} > 0$, then the first equation of (5.11) (with $\sigma \in \{-1, 1\}$, $\widetilde{b} = 0$) fails as well. Next, when $\widetilde{a} = 0$ we have $a^2 = \epsilon'$. Hence (5.11) for $\sigma \in \{-1, 1\}$, $\widetilde{a} = \widetilde{b} = 0$ yields $|\frac{u}{v}|, |\frac{x}{y}| \leq \frac{\epsilon+\sqrt{\epsilon'}\delta}{\widetilde{d}-\epsilon-\sqrt{\epsilon'}(1+\delta)}$. Further, the third equation of (5.9) with (5.6), (5.7) for $a = d = \sqrt{\epsilon'}$, $\widetilde{b} = 0$ gives $\frac{1}{|v|^2} \leq \frac{2\epsilon'}{\widetilde{d}-\epsilon-\epsilon'(1+\delta)}$. We apply this and (5.7) to (5.8) to deduce a contradiction for small ϵ, δ and $\widetilde{d} \neq 0$.

Take $P(s) = \frac{1}{\sqrt{\widetilde{d}+\sigma\widetilde{a}}}\begin{bmatrix} -i\sqrt{\widetilde{d}} & \sqrt{\widetilde{a}} \\ i\sqrt{\widetilde{a}} & \sigma\sqrt{\widetilde{d}} \end{bmatrix}$, $B(s) = \begin{bmatrix} 0 & \sqrt{\widetilde{a}\widetilde{d}} + s \\ \sqrt{\widetilde{a}\widetilde{d}} + s & \widetilde{d} - \sigma\widetilde{a} + s \end{bmatrix}$, $c(s) = 1$, $e^{i\theta} \to \sigma$ in (5.2) to see $\big(1 \oplus \sigma, \widetilde{a} \oplus \widetilde{d}\big) \to \big(1 \oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & d \end{bmatrix}\big)$, and $P(s) = \frac{1}{\sqrt{\widetilde{d}+\sigma\widetilde{a}}}\begin{bmatrix} i\sqrt{\widetilde{a}} & \sigma\sqrt{\widetilde{d}} \\ -i\sqrt{\widetilde{d}} & \sqrt{\widetilde{a}} \end{bmatrix}$, $B(s) = \begin{bmatrix} \widetilde{d} - \sigma\widetilde{a} + s & \sqrt{\widetilde{a}\widetilde{d}} + s \\ \sqrt{\widetilde{a}\widetilde{d}} + s & 0 \end{bmatrix}$, $c(s) = \sigma$, $e^{i\theta} \to \sigma$ to show $\big(1 \oplus \sigma, \widetilde{a} \oplus \widetilde{d}\big) \to \big(1 \oplus e^{i\theta}, \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}\big)$, $0 < \theta < \pi$.

(ii) $B = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$, $b > 0$

From (5.4) for $a = d = 0$ we obtain

$$2bux = \widetilde{a} + \epsilon_1,$$

$$\begin{aligned} buy + bvx &= \widetilde{b} + \epsilon_2, \\ 2bvy &= \widetilde{d} + \epsilon_4. \end{aligned} \tag{5.12}$$

It suffices to take $0 \le \widetilde{a} \le \widetilde{d}$, $\widetilde{d} > 0$, $\widetilde{b} = 0$. By Lemma 4.1 (3) and (4.7) we have $b = \sqrt{\widetilde{a}\widetilde{d}} + \delta_5 > 0$, $|\delta_5| \le \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{1, \widetilde{d}, \widetilde{a}\widetilde{d}\} + 2)^2$. Thus (5.12) and (5.7) give:

$$|v|^2, |y|^2 \le \frac{\widetilde{d}+\epsilon}{2(\sqrt{\widetilde{a}\widetilde{d}}+\delta_5)} + 1 + \delta, \quad |u|^2, |x|^2 \le \frac{\widetilde{a}+\epsilon}{2(\sqrt{\widetilde{a}\widetilde{d}}+\delta_5)} + 1 + \delta. \tag{5.13}$$

Using Lemma 3.1 (D1), (D2) for $\det \widetilde{B} = \widetilde{a}\widetilde{d}$ we get

$$\begin{aligned} u(\widetilde{d} + \epsilon_4) &= v\big(-i(-1)^l\sqrt{\widetilde{a}\widetilde{d}} + \epsilon_2''\big) \\ x(\widetilde{d} + \epsilon_4) &= y\big(i(-1)^l\sqrt{\widetilde{a}\widetilde{d}} + \epsilon_2'\big) \end{aligned}, \ |\epsilon_2'|, |\epsilon_2''| \le \begin{cases} \frac{\epsilon(4\max\{\widetilde{d},\widetilde{a}\}+2+\widetilde{d}\widetilde{a})}{\widetilde{d}\widetilde{a}}, & \widetilde{a}\widetilde{d} \ne 0 \\ \sqrt{\epsilon}(4\max\{\widetilde{d},\widetilde{a}\}+3)^{\frac{1}{2}}, & \widetilde{a}\widetilde{d} = 0 \end{cases}. \tag{5.14}$$

By further applying the first and the third equality of (5.7) we deduce

$$\begin{aligned} (-1)^k + \delta_1 = |x|^2 + \sigma|u|^2 &= \frac{|(-1)^l\sqrt{\widetilde{a}\widetilde{d}}+\epsilon_2'|^2}{|\widetilde{d}+\epsilon_4|^2}|y|^2 + \sigma\frac{|-(-1)^l\sqrt{\widetilde{a}\widetilde{d}}+\epsilon_2''|^2}{|\widetilde{d}+\epsilon_4|^2}|v|^2 \\ &= \frac{|-(-1)^l\sqrt{\widetilde{a}\widetilde{d}}+\epsilon_2'|^2}{|\widetilde{d}+\epsilon_4|^2}\big(\sigma(-1)^k + \delta_4\big) + \delta'|v|^2 \end{aligned} \tag{5.15}$$

with $|\delta'| \le C\max\{\epsilon, \delta\}$, where $C > 0$ is a constant that can be computed easily. By combining (5.15) and (5.13) we obtain a contradiction for $0 < \widetilde{a} < \widetilde{d}$ and sufficiently small ϵ, δ. Next, let $\widetilde{a} = 0$, $\widetilde{d} > 0$. From (5.14) it follows $|\frac{u}{v}| \le \frac{|\epsilon_2'|}{|\widetilde{d}|-\epsilon}$, $|\frac{x}{y}| \le \frac{|\epsilon_2''|}{|\widetilde{d}|-\epsilon}$ ($y = 0$ or $v = 0$ would contradict (5.12) for $|\widetilde{d}| > \epsilon$). By applying this with (5.7) and (5.15) (hence $|v|$ is large) to (5.8), we obtain a contradiction for small ϵ, δ.

We take $P(s) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$, $B(s) = (\widetilde{d} + s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $c(s) = 1$, $e^{i\theta} \to 1$ in (5.2) to prove $\big(I_2, \widetilde{d}I_2\big) \to \big(1 \oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}\big)$, $b > 0$, $0 \le \widetilde{d}$, $0 < \theta < \pi$. Using (5.7) for $\sigma = -1$ leads to

$$\delta \ge |\overline{x}y - \overline{u}v| \ge \big||x|^2|\tfrac{y}{x}| - |u|^2|\tfrac{v}{u}|\big| \ge \big||x|^2 - |u|^2\big| - |x|^2\big(1 - |\tfrac{y}{x}|\big) - |u|^2\big(1 - |\tfrac{v}{u}|\big). \tag{5.16}$$

From (5.14) we get that $|\frac{y}{x}|$, $|\frac{v}{u}|$ are close to 1, and (5.13) implies that $|u|^2, |x|^2$ are bounded. Thus the last two terms on the right-hand side of (5.16) are small, while the first one is close to 1 (see (5.7) for $\sigma = -1$). For small ϵ, δ we get a contradiction.

Case II. $\left(\begin{bmatrix} 0 & 1 \\ \widetilde{\tau} & 0 \end{bmatrix}, \begin{bmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{b} & \widetilde{d} \end{bmatrix}\right) \dashrightarrow \left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & d \end{bmatrix}\right)$, $\widetilde{b}, b \ge 0$, $(\widetilde{\tau}, \tau) \in \big([0,1) \times (0,1)\big) \cup \{(0,0)\}$

By Lemma 3.2 (2) for (C9) we have

$$|xu|, |yu|, |vy| \le \delta, \qquad \big||vx| - 1\big| \le \delta. \tag{5.17}$$

It yields $\delta_6 = \frac{y}{x} = \frac{yv}{xv}$ with $|\delta_6| \le \frac{\delta}{1-\delta} \le 2\delta$, $\delta_5 = \frac{u}{v} = \frac{ux}{xv}$ with $|\delta_5| \le \frac{\delta}{1-\delta} \le 2\delta$ and $\delta_7 = \frac{uy}{vx}$ with $|\delta_7| \le 2\delta$ (note $\delta \le \frac{1}{2}$).

(a) $B=\begin{bmatrix}0 & b\\ b & d\end{bmatrix}$, $b\geq 0$, $|d|\in\{0,1\}$, $|b|+|d|\neq 0$

By multiplying the last two equations of (4.13) by $\delta_5=\frac{u}{v}$ and using $\delta_7=\frac{uy}{vx}$ we get

$$du^2+(1+\delta_7)bux=(\widetilde{b}+\epsilon_2)\delta_5,\qquad 2\delta_7 bvx+dvu=(\widetilde{d}+\epsilon_4)\delta_5. \tag{5.18}$$

Subtracting the first and the second equation of (5.18) from the first and the second equation of (4.13) (in the form $duv+b(1+\delta_7)vx=\widetilde{b}+\epsilon_2$), we deduce

$$(1-\delta_7)bux=\widetilde{a}+\epsilon_1-(\widetilde{b}+\epsilon_2)\delta_5,\quad (1-\delta_7)bvx=\widetilde{b}+\epsilon_2-(\widetilde{d}+\epsilon_4)\delta_5. \tag{5.19}$$

It is clear that the first (the second) equality in (5.19) fails for $\widetilde{a}\neq 0$ (for $\widetilde{b}\neq 0$) and $b=0$, provided that ϵ, δ are sufficiently small. Next, from the second equation of (5.19) and using $vx=e^{i\vartheta}-\delta_0$ with $|\delta_0|\leq\delta$, $\vartheta\in\mathbb{R}$ (see (5.17)) we obtain

$$b=\frac{\widetilde{b}+\epsilon_2-(\widetilde{d}+\epsilon_4)\delta_5}{(1-\delta_7)(e^{i\vartheta}-\delta_0)}=e^{-i\vartheta}\widetilde{b}+\frac{e^{-i\vartheta}\widetilde{b}(\delta_7+e^{i\vartheta}\delta_0-\delta_0\delta_7)+\epsilon_2-(\widetilde{d}+\epsilon_4)\delta_5}{(1-\delta_7)(e^{i\vartheta}-\delta_0)} \tag{5.20}$$

From (5.20) and $|ux|\leq\delta$ (and $|yv|\leq\delta$) we get that the first equation of (5.19) fails for $\widetilde{a}\neq 0$ (the last equation of (4.13) fails for $\widetilde{d}\neq 0$, $d=0$), and ϵ, δ small enough. Finally, it is easy to check that $P(s)=\begin{bmatrix}s^{-1} & 0\\ s^2 & s\end{bmatrix}$, $B(s)=\begin{bmatrix}0 & b(s)\\ b(s) & d\end{bmatrix}$ with $b(s)\to\widetilde{b}$, $A(s)=\begin{bmatrix}0 & 1\\ \widetilde{\tau}+s & 0\end{bmatrix}$, $c(s)=1$ in (5.2) proves $\left(\begin{bmatrix}0 & 1\\ \widetilde{\tau} & 0\end{bmatrix},\begin{bmatrix}0 & \widetilde{b}\\ \widetilde{b} & 0\end{bmatrix}\right)\to\left(\begin{bmatrix}0 & 1\\ \tau & 0\end{bmatrix},\begin{bmatrix}0 & b\\ b & d\end{bmatrix}\right)$, $b\geq\widetilde{b}\geq 0$.

(b) $B=\begin{bmatrix}1 & b\\ b & 0\end{bmatrix}$, $b\geq 0$, $\tau=0$

We argue similarly as in Case II (a). We have equations (4.17); by multiplying the first two equations by $\delta_6=\frac{y}{x}$ and using $\delta_7=\frac{uy}{vx}$ we obtain

$$ay^2+(1+\delta_7)bvy=(\widetilde{b}+\epsilon_2)\delta_6,\qquad 2\delta_7 bvx+axy=(\widetilde{a}+\epsilon_1)\delta_6. \tag{5.21}$$

Subtracting the first and the second equation of (5.21) from the last and the second equation of (4.13) (written as $axy+b(1+\delta_7)vx=\widetilde{b}+\epsilon_2$), respectively, we get

$$(1-\delta_7)bvy=\widetilde{d}+\epsilon_4-(\widetilde{b}+\epsilon_2)\delta_5,\quad (1-\delta_7)bvx=\widetilde{b}+\epsilon_2-(\widetilde{a}+\epsilon_1)\delta_6. \tag{5.22}$$

The first (the second) equality in (5.22) fails for $\widetilde{d}\neq 0$ (for $\widetilde{b}\neq 0$) and $b=0$, provided that ϵ, δ are sufficiently small. We obtain a similar expression for b as in (5.20). It yields a contradiction for $b=0$, $\widetilde{b}\neq 0$ and δ,ϵ small enough, while by combining it with $|yv|\leq\delta$ (and $|ux|\leq\delta$) we contradict the first equation of (5.22) for $\widetilde{d}\neq 0$ (or (4.17) for $\widetilde{a}\neq 0$, $a=0$), provided that ϵ, δ are small. Take $P(s)=\begin{bmatrix}s & s^2\\ 0 & s^{-1}\end{bmatrix}$, $B(s)=\begin{bmatrix}1 & b(s)\\ b(s) & 0\end{bmatrix}$, $b(s)\to\widetilde{b}$, $c(s)=1$ in (5.2) to prove $\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & \widetilde{b}\\ \widetilde{b} & 0\end{bmatrix}\right)\to\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}1 & b\\ b & 0\end{bmatrix}\right)$, $b\geq\widetilde{b}$.

(c) $B=1\oplus d$, $d\in\mathbb{C}$ $(0<\tau<1)$ or $B=a\oplus 1$, $a>0$ $(\tau=0)$

Since $|\frac{y}{x}|\leq\frac{\delta}{1-\delta}$ and $|\frac{u}{v}|\leq\frac{\delta}{1-\delta}$ the same proof as in Case I (a) (iii) applies. From (5.2) for $P(s)=\begin{bmatrix}s & s^2\\ 0 & s^{-1}\end{bmatrix}$, $B(s)=1\oplus s^2\widetilde{d}$ and $P(s)=\sqrt{\widetilde{a}}\oplus\frac{1}{\sqrt{\widetilde{a}}}$, $B(s)=1\oplus\widetilde{a}\widetilde{d}$ with $\tau\to\widetilde{\tau}$, $c(s)=1$, in (5.2) we obtain $\left(\begin{bmatrix}0 & 1\\ \widetilde{\tau} & 0\end{bmatrix},\widetilde{a}\oplus\widetilde{d}\right)\to\left(\begin{bmatrix}0 & 1\\ \tau & 0\end{bmatrix},1\oplus d\right)$ (with $0<\tau<1$) for $\widetilde{a}=0$ and $\widetilde{a}>0$, respectively. Finally, $P(s)=\begin{bmatrix}s^{-1} & 1\\ s^2 & s\end{bmatrix}$, $B(s)=$

$(\widetilde{d}s^2+s^3)\oplus 1$ with $c(s)=1$ gives $\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},\widetilde{d}\oplus 0\right)\to\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},a\oplus 1\right)$, $a>0$, $\widetilde{d}\in\{0,1\}$.

(d) $B=\begin{bmatrix}e^{i\varphi}&b\\b&\zeta\end{bmatrix}$, $\zeta\in\mathbb{C}$, $\varphi\in[0,\pi)$, $\tau\in(0,1)$ or $B=\begin{bmatrix}\zeta^*&b\\b&1\end{bmatrix}$, $\zeta^*\in\mathbb{C}^*$, $\tau=0$; $b>0$

Let $B=\begin{bmatrix}e^{i\varphi}&b\\b&\zeta\end{bmatrix}$, $\zeta\in\mathbb{C}$, $0\le\varphi<\pi$. If $\widetilde{B}$ is either $\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&\widetilde{d}\end{bmatrix}$ or $\begin{bmatrix}\widetilde{\zeta}&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}$ with $\widetilde{\zeta}\neq 0$ we take $P(s)=\begin{bmatrix}s&s^2\\1&s^{-1}\end{bmatrix}$, $B(s)=\begin{bmatrix}e^{i\varphi}&\widetilde{b}+s\\\widetilde{b}+s&\widetilde{d}s^2\end{bmatrix}$ or $P(s)=|\widetilde{\zeta}|e^{i\frac{k\pi}{2}}\oplus\frac{1}{|\widetilde{\zeta}|}e^{i\frac{k\pi}{2}}$, $c(s)=(-1)^k$, $B(s)=\begin{bmatrix}e^{i\varphi}&\widetilde{b}+s\\\widetilde{b}+s&(-1)^k|\widetilde{\zeta}|^2\end{bmatrix}$ with $\arg\widetilde{\zeta}=\arg(\varphi+k\pi)$ in (5.2) to get a path. Next, $B(s)=\begin{bmatrix}\widetilde{a}s^2+s^3&\widetilde{b}+s\\\widetilde{b}+s&1\end{bmatrix}$, $c(s)=1$, $P(s)=\begin{bmatrix}s^{-1}&1\\s^2&s\end{bmatrix}$ shows $\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}\widetilde{a}&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}a&b\\b&1\end{bmatrix}\right)$, $\widetilde{b}\ge 0$, $\widetilde{a}\in\{0,1\}$.

Case III. $(1\oplus -1,\widetilde{B})\dashrightarrow\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},B\right)$, $0<\tau\le 1$

Lemma 3.2 (2) with (C5) for $\alpha=-\omega=1$, $\beta=0$ gives ($|\delta_1|,|\delta_2|,|\delta_4|\le\delta$):

$$2\mathrm{Re}(\overline{x}u)=(-1)^k+\delta_1,\quad 2\mathrm{Re}(\overline{y}v)=-(-1)^k+\delta_2,\quad \overline{x}v+\overline{u}y=\delta_4,\quad 1-\tau,\quad k\in\mathbb{Z},\tag{5.23}$$

Observe that $u,v\neq 0$, otherwise (5.23) fails. We compute

$$\overline{x}v+\overline{u}y=e^{-2i\phi}(xv-yu)+2\cos(\phi-\eta)e^{-i(\phi+\eta)}uy=e^{-2i\phi}\det P+\mathrm{Re}(\overline{x}u)\tfrac{y}{x},\tag{5.24}$$

$$\overline{x}v+\overline{u}y=-e^{-2i\eta}\det P+2\cos(\phi-\eta)e^{-i(\phi+\eta)}vx=-e^{-2i\eta}\det P+\mathrm{Re}(\overline{x}u)\tfrac{v}{u}.\tag{5.25}$$

Therefore, by combining (5.24) and (5.25) with (5.23) we obtain

$$\tfrac{y}{x}=\tfrac{\delta_4-e^{-2i\phi}\det P}{(-1)^k+\delta_1},\qquad \tfrac{v}{u}=\tfrac{\delta_4+e^{-2i\eta}\det P}{(-1)^k+\delta_1}.\tag{5.26}$$

(a) $B=a\oplus d$, $\quad a\ge 0$

Equations (5.6) and (5.26) yield

$$\begin{aligned}\widetilde{b}+\epsilon_2=axy+duv&=\tfrac{1}{(-1)^k+\delta_1}\big(ax^2(\delta_4-e^{-2i\phi}\det P)+du^2(\delta_4+e^{-2i\eta}\det P)\big)\\&=\tfrac{1}{(-1)^k+\delta_1}\big(\delta_4(ax^2+du^2)+\det P(-ax^2e^{-2i\phi}+du^2e^{-2i\eta})\big)\\&=\tfrac{1}{(-1)^k+\delta_1}\big(\delta_4(\widetilde{a}+\epsilon_1)+\det P(-a|x|^2+d|u|^2)\big),\end{aligned}\tag{5.27}$$

and further for $a,\widetilde{a}\in\mathbb{R}$:

$$\begin{aligned}\widetilde{d}+\epsilon_4=ay^2+dv^2&=\tfrac{1}{((-1)^k+\delta_1)^2}\big(ax^2(\delta_4-e^{-2i\phi}\det P)^2+du^2(\delta_4+e^{-2i\eta}\det P)^2\big)\\&=\tfrac{1}{((-1)^k+\delta_1)^2}\big(\delta_4^2(ax^2+du^2)+2\delta_4\det P(-a|x|^2+d|u|^2)+(\det P)^2(a\overline{x}^2+d\overline{u}^2)\big)\\&=\tfrac{1}{((-1)^k+\delta_1)^2}\Big(\delta_4^2(\widetilde{a}+\epsilon_1)+2\delta_4\big((\widetilde{b}+\epsilon_2)((-1)^k+\delta_1)-\delta_4(\widetilde{a}+\epsilon_1)\big)\end{aligned}$$

$$+\,(\det P)^2(\widetilde{a}+\overline{\epsilon}_1+2i\mathrm{Im}(d)\overline{u}^2)\Big). \tag{5.28}$$

The equation (5.27) gives ($a\in\mathbb{R}$):

$$\mathrm{Im}(d)|u|^2=\mathrm{Im}\left(\tfrac{1}{\det P}\big((\widetilde{b}+\epsilon_2)((-1)^k+\delta_1)-\delta_4(\widetilde{a}+\epsilon_1)\big)\right). \tag{5.29}$$

Lemma 4.1 (1) yields $|\det P|\geq\frac{1-\frac{6\delta^2}{\nu^2}}{\sqrt{1-\delta}}$ (note $1-\tau\leq\delta$ by (5.23). It follows for $\widetilde{b}=0$ that $|\mathrm{Im}(d)u^2|\leq\nu^2\frac{\epsilon(1+\delta)^{\frac{3}{2}}+\delta(\widetilde{a}+\epsilon)\sqrt{1+\delta}}{\nu^2-6\delta^2}$. It contradicts (5.28) for $\widetilde{a}<\widetilde{d}$, $\widetilde{b}=0$ and ϵ,δ small enough. Next, $c(s)=1$, $P(s)=\sqrt{\frac{\widetilde{b}}{2}}\begin{bmatrix}1&1\\\widetilde{b}^{-1}&-\widetilde{b}^{-1}\end{bmatrix}$, $B(s)=1\oplus\widetilde{b}^2e^{is}$ (or $B(s)=1\oplus-\widetilde{b}^2e^{-is}$) yields a path from $(1\oplus-1,\widetilde{b}I_2)$, $\widetilde{b}>0$ (from $\left(1\oplus-1,\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}\right)$, $\widetilde{b}>0$) to $\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},1\oplus d\right)$, $\mathrm{Im}(d)>0$. For $P(s)=\frac{1}{2}\begin{bmatrix}2s&s^{-1}\\2s&-s^{-1}\end{bmatrix}$ we get $(1\oplus-1,0_2)\to\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},1\oplus 0\right)$.

(b) $B=\begin{bmatrix}a&b\\b&d\end{bmatrix}$, $b>0$

Let $\widetilde{B}=\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}$, $\widetilde{b}>0$. For $a=0$ we have $b^2=\widetilde{b}^2-(1-\tau)\widetilde{b}^2+\epsilon'$ with $1-\tau\leq\delta$, $|\epsilon'|\leq\max\{\epsilon,\frac{\delta^2}{\nu^2}\}(4\max\{\widetilde{b},\widetilde{b}^2,1\}+2)^2$ (Lemma 4.1 (3)). If $d=e^{i\varphi}$ with $\varphi<\pi$, the proof in [22, Theorem 3.6, Case VII. (b) (i)] applies, while for $d=0$ the first equation of (5.12) for $\widetilde{a}=0$ and (5.23) yield $b(1-\delta)\leq 2b|ux|\leq\epsilon$, which fails for small ϵ,δ. Suppose $\widetilde{B}=\widetilde{a}\oplus\widetilde{d}$ for $0\leq\widetilde{a}\leq\widetilde{d}$. If $d=e^{i\varphi}$ the proof in [22, Theorem 3.6, Case VII. (b) (ii)] for $\widetilde{a}\neq\widetilde{d}$ applies almost mutatis mutandis, we only replace $\frac{|\det\widetilde{A}|}{|\det A|}=\frac{|\det\widetilde{B}|}{|\det B|}=|\frac{\widetilde{a}\widetilde{d}}{b^2}|=1$ with $b^2=\widetilde{a}\widetilde{d}-(1-\tau)\widetilde{a}\widetilde{d}+\epsilon'$, $1-\tau\leq\delta$, $|\epsilon'|\leq\max\{\epsilon,\frac{\delta^2}{\nu^2}\}(4\max\{\widetilde{d},\widetilde{a}\widetilde{d},1\}+2)^2$ (Lemma 4.1 (3)). If $d=0$, the first equation of (5.12) for $\widetilde{a}=0$ and (5.23) give $(\widetilde{a}\widetilde{d}-\delta\widetilde{a}\widetilde{d}-|\epsilon'|)(1-\delta)^2\leq 4b^2|ux|^2\leq|\widetilde{a}+\epsilon|^2$, which fails for small ϵ,δ. Note, $c(s)=1$, $P(s)=\begin{bmatrix}\frac{1}{2s}&-\frac{i}{2s}\\s&is\end{bmatrix}$, $B(s)=\begin{bmatrix}0&\widetilde{d}+s\\\widetilde{d}+s&1\end{bmatrix}$ in (5.2) implies $(1\oplus-1,\widetilde{d}I_2)\to\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&b\\b&1\end{bmatrix}\right)$ for $\widetilde{d}\geq 0$. By conjugating with $\frac{1}{2}\begin{bmatrix}1&-2\\-1&-2\end{bmatrix}$ and $r\oplus\frac{1}{r}$ for $r>0$, we get a path

$$(1\oplus-1,\widetilde{a}\oplus\widetilde{d})\approx\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\frac{1}{4}\begin{bmatrix}\widetilde{a}+\widetilde{d}&2(\widetilde{d}-\widetilde{a})\\2(\widetilde{d}-\widetilde{a})&4(\widetilde{a}+\widetilde{d})\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}r^2e^{i\varphi}&b\\b&r^{-2}\zeta\end{bmatrix}\right)$$
$$\approx\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}e^{i\varphi}&b\\b&\zeta\end{bmatrix}\right).$$

Case IV. $\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},\widetilde{B}\right)\dashrightarrow\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},B\right)$

Lemma 3.2 (2) with (C8) for, $\beta=1,\omega=i,\alpha=k=0$ (since $||v|^2-(-1)^k|<\delta$) gives

$$\left|\overline{x}v+\overline{u}y-1\right|\leq\delta,\quad |u|^2\leq\delta,\quad \left||v|^2-1\right|\leq\delta,\quad |\mathrm{Re}(\overline{x}u)|,|\mathrm{Re}(\overline{y}v)|\leq\delta. \tag{5.30}$$

(a) $B=a\oplus d$, $a\geq 0$, $d\in\mathbb{C}$

It is not difficult to check that $B(s)=s\oplus\frac{\widetilde{b}^2}{s}$, $c(s)=1$, $P(s)=e^{-i\frac{\pi}{4}}\begin{bmatrix}1&i\widetilde{b}s^{-1}\\s^2e^{i\frac{\pi}{4}}&1\end{bmatrix}$

in (5.2) proves $\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},a\oplus d\right)$, $d\in\mathbb{C}$, $a>0$, $\widetilde{b}\geq 0$.

Next, let $B=0\oplus d$, $d>0$, $\widetilde{B}=\widetilde{a}\oplus\widetilde{d}$, $\widetilde{a}>0$. Using (5.6) for $a=0$ and $|u|^2\leq\delta$ we get

$$\widetilde{a}+\epsilon\leq|du^2|\leq d\delta,\qquad d(1-\delta)\leq|dv^2|\leq\epsilon+|\widetilde{d}|.$$

Hence $\frac{|\widetilde{d}|+\epsilon}{1-\delta}\delta\geq\widetilde{a}+\epsilon$, which fails for sufficiently small ϵ,δ.

(b) $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$, (hence $\widetilde{B}=\widetilde{a}\oplus\widetilde{d}$ by Lemma 2.1)

The proof in [22, Theorem 3.6, Case V. (b)] applies mutatis mutandis. Note, $B(s)=\frac{\widetilde{d}s}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $P(s)=e^{-i\frac{\pi}{4}}\begin{bmatrix}e^{i\frac{\pi}{4}}s&s^{-1}\\s&i\end{bmatrix}$ in (5.2) implies $\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},0\oplus\widetilde{d}\right)\to\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},\begin{bmatrix}0&b\\b&0\end{bmatrix}\right)$, $\widetilde{d}>0$.

Case V. $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\widetilde{B}\right)\dashrightarrow\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},B\right)$

Lemma 3.2 (2) with (C8) for $\alpha=\omega=0$, $\beta=0$ yields

$$|u|^2,|v|^2\leq\delta,\quad\left|2\mathrm{Re}(\overline{y}v)\right|\leq\delta,\quad\left|2\mathrm{Re}(\overline{x}u)\right|\leq\delta,\quad\left|\overline{x}v+\overline{u}y-(-1)^k\right|\leq\delta,\ k\in\mathbb{Z}.\tag{5.31}$$

(a) $B=a\oplus d$, $a\geq 0$

Taking $c(s)=1$, $P(s)=\begin{bmatrix}1&s^{-1}\\s&0\end{bmatrix}$, $B(s)=0\oplus\frac{1}{s}$ in (5.2) proves $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\widetilde{a}\oplus 0\right)\to\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},0\oplus d\right)$, $\widetilde{a}\in\{0,1\}$, $d>0$. Next, $c(s)=1$, $P(s)=e^{i\frac{1}{2}\widetilde{\vartheta}}\begin{bmatrix}1&s^{-1}\\s&0\end{bmatrix}$, $B(s)=(|\widetilde{d}|+s)s^2\oplus\frac{1}{s^2}e^{-i\widetilde{\vartheta}}$ yields $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},1\oplus\widetilde{d}\right)\to\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},a\oplus d\right)$, $\widetilde{d}=|\widetilde{d}|e^{i\widetilde{\vartheta}}$, $a>0$, $d\in\mathbb{C}$.

Proceed with $\widetilde{b}=\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}$, $\widetilde{b}>0$; we conjugate the first pair with $\frac{1}{2}\begin{bmatrix}2&-2\\1&1\end{bmatrix}$:

$$\begin{aligned}&\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}\right)\approx\\&\qquad\left(1\oplus-1,\tfrac{1}{4}\begin{bmatrix}4\widetilde{b}+1&1\\1&-4\widetilde{b}+1\end{bmatrix}\right)\dashrightarrow\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},a\oplus d\right),\quad d\in\mathbb{C},a>0.\end{aligned}\tag{5.32}$$

Using ideas from Case III (a) we find $c(s)=-1$, $P(s)=\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\begin{bmatrix}\frac{i}{s^2}e^{i\alpha(s)}&\frac{i}{s^2}e^{-i\alpha(s)}\\se^{-i\alpha(s)}&se^{i\alpha(s)}\end{bmatrix}$ with $\sin(2\alpha(s))=s$, $B(s)=\widetilde{b}s^3\oplus(\frac{\widetilde{b}}{s^3}-\frac{i}{2s^2})$ (see (5.2)), which proves the existence of (5.32).

(b) $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$

If $\widetilde{B}=\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}$, $\widetilde{b}>0$, the proof in [22, Theorem 3.6, Case VI. (b) (i)] applies mutatis mutandis, we only use $b^2=\widetilde{b}^2+\epsilon'$, $|\epsilon'|\leq\max\{\epsilon,\frac{\delta^2}{\nu^2}\}(4\max\{\widetilde{b},\widetilde{b}^2,1\}+2)^2$ (see Lemma 4.1 (3)) instead of $1=\frac{|\det\widetilde{A}|}{|\det A|}=\frac{|\det\widetilde{B}|}{|\det B|}=\frac{\widetilde{b}^2}{b^2}$. For $\widetilde{B}=1\oplus\widetilde{d}$, $\widetilde{d}\neq 0$ we apply

[22, Theorem 3.6, Case VI. (b) (ii)], we only replace $\frac{|\det \widetilde{A}|}{|\det A|} = \frac{|\det \widetilde{B}|}{|\det B|} = |\frac{\widetilde{d}}{b^2}| = 1$ with $b^2 = |\widetilde{d}| + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{|\widetilde{d}|, 1\} + 2)^2$ (Lemma 4.1 (3)).

Case VI. $(1 \oplus -1, \widetilde{B}) \dashrightarrow \left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, B\right)$

Lemma 3.2 (2) with (C8) for $-\omega = \alpha = 1$, $\beta = 0$ yields ($|\delta_1|, |\delta_2|, |\delta_4| < \delta$, $k \in \mathbb{Z}$):

$$2\mathrm{Re}(\overline{x}u) = (-1)^k + \delta_1, \quad 2\mathrm{Re}(\overline{y}v) = -(-1)^k + \delta_2, \ |u|^2, |v|^2 \leq \delta, \quad \overline{x}v + \overline{u}y = \delta_4. \tag{5.33}$$

(a) $B = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}, b > 0$

The proof in [22, Theorem 3.6, Case V. (b) (i)] applies mutatis mutandis for $\widetilde{B} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 0 \end{bmatrix}$, $\widetilde{b} > 0$; recall $b^2 = \widetilde{b}^2 + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(2\max\{1, \widetilde{b}, \widetilde{b}^2\} + 1)^2$ (Lemma 4.1 (3)). Let $\widetilde{B} = \widetilde{a} \oplus \widetilde{d}$, $\widetilde{d} \geq \widetilde{a} \geq 0$. If $\widetilde{d} > \widetilde{a} > 0$ the proof in [22, Theorem 3.6, Case V. (b) (ii)] applies for $b^2 = \widetilde{a}\widetilde{d} + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{1, \widetilde{d}, \widetilde{a}\widetilde{d}\} + 2)^2$ (Lemma 4.1 (3)). For $c(s) = -1$, $P(s) = \frac{1}{\sqrt{2}}\begin{bmatrix} is^{-1} & s^{-1} \\ -is & s \end{bmatrix}$, $B(s) = (\widetilde{d} + s)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in (5.2) we get $(1 \oplus -1, \widetilde{d}I_2) \to \left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}\right)$, $\widetilde{d} \geq 0$. If $\widetilde{a} = 0$, $\widetilde{d} > 0$ then Lemma 3.1 (D1) yields $\widetilde{d} - \epsilon \leq |\frac{v}{u}|\sqrt{\epsilon}(4\widetilde{d} + 2)^{\frac{1}{2}}$, and Lemma 4.1 (1) gives $|\det P| \leq 1 + \frac{6\delta^2}{\nu^2}$. By applying this and (5.33) to (5.25) implies $(1 - \delta)(\widetilde{d} - \epsilon) \leq \sqrt{\epsilon}(4\widetilde{d} + 2)^{\frac{1}{2}}(\delta + 1 + \frac{6\delta^2}{\nu^2})$, which fails for small ϵ, δ.

(b) $B = a \oplus d$, $\quad a \geq 0, d \in \mathbb{C}$

If $\widetilde{b} = 0$, $0 \leq \widetilde{a} < \widetilde{d}$ the same proof as in Case III (a) applies (see (5.23) and (5.33)).

Case VII. $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \widetilde{B}\right) \dashrightarrow (1 \oplus -1, B)$

Lemma 3.2 (2) with (C2) for $\omega = 0$, $\theta = \pi$ gives

$$|x|^2 - |u|^2 = \delta_1, \ \ \overline{x}y - \overline{u}v - (-1)^k = \delta_2, \ \ |y|^2 - |v|^2 = \delta_4, \ \ |\delta_1|, |\delta_2|, |\delta_4| \leq \delta, \ k \in \mathbb{Z}. \tag{5.34}$$

(a) $B = a \oplus d$, $\quad 0 \leq a \leq d, d > 0$

(i) $\widetilde{B} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 1 \end{bmatrix}, \widetilde{b} > 0$

First, $c(s) = -1$, $P(s) = \begin{bmatrix} \frac{-i}{2}s & is^{-1} \\ \frac{s}{2} & s^{-1} \end{bmatrix}$, $B(s) = \widetilde{b} \oplus (\widetilde{b} + s^2)$ in (5.2) gives $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 1 \end{bmatrix}\right) \to (1 \oplus -1, a \oplus d)$, $a < d$. For $a = d$ we apply the proof of [22, Theorem 3.6, Case VIII (a) (ii)], but replace $\begin{bmatrix} 0 & d \\ d & 1 \end{bmatrix}$ with $\begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 1 \end{bmatrix}$; and use $d^2 = |\widetilde{b}|^2 + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}\big(4\max\{|\widetilde{b}|, |\widetilde{b}|^2, 1\} + 2\big)^2$ (Lemma 4.1 (3)) at the end of the proof.

(ii) $\widetilde{B} = 1 \oplus \widetilde{d}, \widetilde{d} \in \mathbb{C}$
We prove $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1 \oplus 0\right) \to (1 \oplus -1, 0 \oplus d)$ with $P(s) = \begin{bmatrix} s^{-1} & \frac{1}{2}s \\ s^{-1} & -\frac{1}{2}s \end{bmatrix}$, $c(s) = 1$, $B(s) = 0 \oplus s^2$.
Proceed with $B = a \oplus d$, $0 < a \leq d$. We have equations (5.6) for $\widetilde{a} = 1$, $\mathrm{Im}\widetilde{d} > 0$, $\widetilde{b} = 0$. By combining them with (5.9), (5.10) for $\sigma = -1$ and with (5.34) we get

$$\begin{aligned}
&\epsilon_1 + 1 - ae^{2i\phi}\delta_1 = u^2(ae^{2i(\phi-\eta)} + d),\\
&a\big((-1)^k + \delta_2\big) - e^{-2i\phi}\epsilon_2 = -e^{-2i\phi}\big(uv(ae^{2i(\phi-\eta)} + d)\big),\\
&a\big((-1)^k + \delta_2\big) - e^{-2i\varphi}\epsilon_2 = -e^{-2i\varphi}\big(uv(ae^{2i(\varphi-\kappa)} + d)\big),\\
&\epsilon_4 + \widetilde{d} - ae^{2i\varphi}\delta_4 = v^2(ae^{2i(\varphi-\kappa)} + d),\\
&d\big((-1)^k + \delta_2\big) + e^{-2i\kappa}\epsilon_2 = e^{-2i\kappa}\big(xy(de^{2i(\kappa-\varphi)} + a)\big),
\end{aligned} \tag{5.35}$$

We have $ad = |\widetilde{d}| + \delta'$, $|\delta'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}\big(4\max\{1, |\widetilde{d}|\} + 2\big)^2$ (Lemma 4.1), hence $a \leq \sqrt{|\widetilde{d}|} + 1$, provided that $\max\{\epsilon, \frac{\delta^2}{\nu^2}\} \leq \frac{1}{(4\max\{1,|\widetilde{d}|\}+2)^2}$. Next, we divide the first and the second (the third and the fourth) two equations of (5.35) to get

$$\frac{u}{v} = \frac{\epsilon_1 + 1 - ae^{2i\phi}\delta_1}{a\big((-1)^k + \delta_2\big) - e^{-2i\phi}\epsilon_2}(-e^{-2i\phi}) = \frac{a\big((-1)^k + \delta_2\big) - e^{-2i\varphi}\epsilon_2}{\epsilon_4 + \widetilde{d} - ae^{2i\varphi}\delta_4}(-e^{2i\varphi}).$$

The second equality yields that there is a (computable) constant $D > 0$ so that

$$a^2 = \widetilde{d}e^{-2i(\phi+\varphi)} + \delta_5, \quad d^2 = \frac{(|\widetilde{d}| + \delta')^2}{\widetilde{d}e^{2i(\phi-\varphi)} + \delta_5}, \qquad |\delta_5| \leq D\max\{\epsilon, \delta\}, \tag{5.36}$$

Furthermore, we divide the third and the fifth equation of (5.35) to conclude:

$$\frac{xy}{uv} = \frac{(d((-1)^k + \delta_2) + e^{-2i\kappa}\epsilon_2)}{(a((-1)^k + \delta_2)e^{2i\varphi}\epsilon_2)} = 1 + \delta_6, \qquad |\delta_6| \leq C\max\{\epsilon, \delta\}, \tag{5.37}$$

while the first four equations of (5.35) yield

$$\begin{aligned}
\frac{1}{\widetilde{d}} + \delta_0 &= \frac{\big(1 + \epsilon_1 - ae^{2i\phi}\delta_1\big)\big(a(-1)^k + a\delta_2 - e^{-2i\phi}\epsilon_2\big)}{\big(\widetilde{d} + \epsilon_1 - ae^{2i\varphi}\delta_4\big)\big(a(-1)^k + a\delta_2 - e^{-2i\varphi}\epsilon_2\big)}\\
&= e^{i(2\eta - 2\kappa - 2\phi + 2\varphi)}\frac{|u|^2}{|v|^2}, \quad |\delta_0| \leq K\max\{\epsilon, \delta\},
\end{aligned}$$

where constants $C, K > 0$ can be computed. By applying (4.7) for $\widetilde{d} = |\widetilde{d}|e^{i\widetilde{\vartheta}}$ we get $2\eta - 2\kappa - 2\phi + 2\varphi + \widetilde{\vartheta} = \psi$ with $|e^{i\frac{\psi}{2}} - 1| = |\sin\frac{\psi}{4}| \leq |\sin\psi| \leq \delta_0$. Using (5.37) we get

$$\begin{aligned}
\Big|\frac{\overline{x}y}{\overline{u}v} - 1\Big| &= \Big|\frac{|xy|}{|uv|}e^{i(\phi-\varphi-\kappa+\eta)} - 1\Big| = \Big||1 + \delta_6|e^{i\left(-\frac{\widetilde{\vartheta}}{2} + \frac{\psi}{2}\right)} - 1\Big|\\
&= \Big|e^{i\frac{\psi}{2}}\Big(e^{-i\frac{\widetilde{\vartheta}}{2}} + 1\Big) - \Big(e^{i\frac{\psi}{2}} - 1\Big) + (|1 - \delta_6| - 1)e^{i\left(-\frac{\widetilde{\vartheta}}{2} + \frac{\psi}{2}\right)}\Big|\\
&\geq |e^{-i\frac{\widetilde{\vartheta}}{2}} + 1| - |\delta_0| - |\delta_6| \geq \cos\frac{\widetilde{\vartheta}}{4},
\end{aligned}$$

provided that ϵ, δ are such that $\frac{1}{4}|e^{-i\frac{\widetilde{\vartheta}}{2}} + 1| = \frac{1}{2}\cos\frac{\widetilde{\vartheta}}{4} \geq |\delta_0|, |\delta_6|$ with $0 < \widetilde{\vartheta} < \pi$. Thus:

$$2 \geq 1 + \delta \geq |\overline{x}y - \overline{u}v| = |\overline{u}v|\Big|\frac{\overline{x}y}{\overline{u}v} - 1\Big| \geq \frac{1}{2}|uv|\cos\frac{\widetilde{\vartheta}}{4}, \quad |u|^2 = \frac{|u|}{|v|}|uv| \leq 4\frac{|\widetilde{d}|^{-1} + |\delta_0|}{\cos\frac{\widetilde{\vartheta}}{4}}.$$

We simplify the first and the third equation of (5.35) and rearrange the terms:

$$\begin{aligned}&2au^2\cos(\phi-\eta)e^{i(\phi-\eta)}=1+\epsilon_1-ae^{2i\phi}\delta_1-(d-a)u^2,\\&-2auv\cos(\varphi-\kappa)e^{-i(\varphi+\kappa)}=a(-1)^k+a\delta_2-e^{-2i\varphi}\epsilon_2+(d-a)uve^{-2i\varphi}.\end{aligned}\tag{5.38}$$

By applying (4.3) with $\widetilde{d}=|\widetilde{d}|e^{i\widetilde{\vartheta}}$ to (5.36) and (5.38) we deduce ($L>0$ is a constant):

$$\begin{aligned}&\psi_0=\widetilde{\theta}-2(\phi+\varphi), && |\sin\psi|\le\tfrac{2|\delta_5|}{|\widetilde{d}|},\\&\psi_1=(\phi+\eta)-\pi l_1, && |\sin\psi_1|\le L\max\{\epsilon,\delta\},\quad l_1\in\mathbb{Z},\\&\psi_2=(\eta-\varphi)-\pi(k+l_2), && |\sin\psi_2|\le L\max\{\epsilon,\delta\},\quad l_2\in\mathbb{Z}.\end{aligned}$$

Thus $\big|\sin(\psi_0+2\psi_1-2\psi_2)\big|=|\sin\widetilde{\vartheta}|\le\frac{2|\delta_5|}{|\widetilde{d}|}+4L\max\{\epsilon,\delta\}$ and it fails for small ϵ,δ.

(b) $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$

If $\widetilde{B}=\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}$ for $\widetilde{b}>0$ we can apply the proof of [22, Theorem 3.6, Case VIII (b) (i)], recall $b^2=\widetilde{b}^2+\epsilon'$, $|\epsilon'|\le\max\{\epsilon,\frac{\delta^2}{\nu^2}\}(4\max\{|\widetilde{b}|,|\widetilde{b}|^2,1\}+2)^2$ (Lemma 4.1 (3)).

Let $\widetilde{B}=1\oplus\widetilde{d}$, $\widetilde{d}\in\mathbb{C}$. To get $\Big(\begin{bmatrix}0&1\\1&0\end{bmatrix},1\oplus0\Big)\to\Big(1\oplus-1,\begin{bmatrix}0&b\\b&0\end{bmatrix}\Big)$, we take $P(s)=\frac{1}{2}\begin{bmatrix}2s^{-1}&s\\2s^{-1}&-s\end{bmatrix}$, $c(s)=1$, $B(s)=\frac{s^2}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$ in (5.2). If $\widetilde{d}=|\widetilde{d}|e^{i\widetilde{\vartheta}}$, $0<\widetilde{\vartheta}<\pi$ Lemma 3.1 (D3) implies

$$bvx=\tfrac{1}{2}\Big(\epsilon_2'+(-1)^l i\sqrt{|\widetilde{d}|}e^{i\frac{\widetilde{\vartheta}}{2}}\Big),\qquad buy=\tfrac{1}{2}\Big(\epsilon_2''-(-1)^l i\sqrt{|\widetilde{d}|}e^{i\frac{\widetilde{\vartheta}}{2}}\Big),$$

where $|\epsilon_2'|,|\epsilon_2''|\le\frac{\epsilon(4\max\{1,|\widetilde{d}|\}+2+|\widetilde{d}|)}{|\widetilde{d}|}$. By applying (4.3) to these two equations and to the first equality of (5.12) we get $\psi_1,\psi_2,\psi_3\in(-\frac{\pi}{2},\frac{\pi}{2})$ such that:

$$\begin{aligned}&\psi_1=\phi+\kappa-\tfrac{\pi}{2}-\tfrac{\widetilde{\vartheta}}{2}-l\pi+2\pi l_3, && |\sin\psi_1|\le\tfrac{2|\epsilon_2'|}{\sqrt{|\widetilde{d}|}},\\&\psi_2=\varphi+\eta-\tfrac{\pi}{2}-\tfrac{\widetilde{\vartheta}}{2}-(l+1)\pi+2\pi l_4, && |\sin\psi_2|\le\tfrac{2|\epsilon_2''|}{\sqrt{|\widetilde{d}|}},\\&\psi_3=\phi+\eta+2\pi l_1, && |\sin\psi_3|\le\epsilon.\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^k+\delta_2\\&=\overline{x}y-\overline{u}v=|xy|e^{i(\varphi-\phi)}-|uv|e^{i(\kappa-\eta)}=e^{-i(\phi+\eta)}\big(|xy|e^{i(\varphi+\eta)}-|uv|e^{i(\kappa+\phi)}\big)\\&=e^{i(-\psi_3+2\pi l_1)}\big(|xy|e^{i(\psi_2-2\pi l_4+\frac{\pi}{2}+\frac{\widetilde{\vartheta}}{2}+(l+1)\pi)}-|uv|e^{i(\frac{\pi}{2}+\frac{\widetilde{\vartheta}}{2}+l\pi-2\pi l_3+\psi_1)}\big)\\&=e^{i(\psi_2-\psi_3+\frac{\widetilde{\vartheta}}{2}+(l+1)\pi+\frac{\pi}{2})}\big(|xy|+|uv|e^{i(\psi_1-\psi_2)}\big).\end{aligned}$$

Since $\psi_1, \psi_2, \psi_3 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ are close to 0, the argument of the second factor is close to 0, too. Using (4.3) again we obtain a contradiction for ϵ, δ small enough:

$$\psi = k\pi - \left(\psi_2 - \psi_3 + \tfrac{\widetilde{\vartheta}}{2} + (l+1)\pi + \tfrac{\pi}{2}\right) - (\psi_1 - \psi_2), \qquad |\sin\psi| \le 2\delta,$$
$$0 \neq |\cos\tfrac{\widetilde{\vartheta}}{2}| = \left|\sin\left(\tfrac{\widetilde{\vartheta}}{2} + \tfrac{\pi}{2}\right)\right| \le \left|\sin(\psi_3 + \psi_1)\right| \le 2\epsilon_2' + 2\delta.$$

Case VIII. $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \widetilde{B}\right) \dashrightarrow \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B\right)$

(a) $B = 1 \oplus d$, $\mathrm{Im} d > 0$, $\widetilde{B} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 1 \end{bmatrix}$, $\widetilde{b} > 0$

We can apply the proof of [22, Theorem 3.6, Case IX (b)], and use $|d| = |\widetilde{b}|^2 + \epsilon'$, $|\epsilon'| \le \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{|\widetilde{b}|, |\widetilde{b}|^2, 1\} + 2)^2$ (Lemma 4.1 (3)).

(b) $B = \begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}$, $b > 0$

For $P(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & s \\ s & 1 \end{bmatrix}$, $c(s) = 1$ in (5.2) we get $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1 \oplus 0\right) \to \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ b & 1 \end{bmatrix}\right)$, $b > 0$. For $\widetilde{B} = 1 \oplus \widetilde{d}$, $\widetilde{d} \neq 0$ we use the proof of [22, Theorem 3.6, Case IX (c)], but replace $\frac{|\det \widetilde{B}|}{|\det B|} = \frac{\widetilde{a}\widetilde{d}}{b^2}$ with $b^2 = |\widetilde{d}| + \epsilon'$, $|\epsilon'| \le \max\{\epsilon, \frac{\delta^2}{\nu^2}\}(4\max\{|\widetilde{b}|, |\widetilde{b}|^2, 1\} + 2)^2$ (Lemma 4.1 (3)).

Case IX. $\left(\begin{bmatrix} 0 & 1 \\ 1 & \omega \end{bmatrix}, \widetilde{B}\right) \dashrightarrow (1 \oplus e^{i\theta}, B)$, $0 < \theta < \pi$, $\omega \in \{0, i\}$

From Lemma 3.2 (C2) we get

$$\left||u|^2 - |x|^2\right| \le \delta, \quad \left||v|^2 - |y|^2\right| \le \delta, \quad \left|\overline{x}y - \overline{u}v - (-1)^k\right| \le \delta, \quad k \in \mathbb{Z}, \ \sin\theta \le \delta;$$
$$\text{if} \quad \omega = i, \quad \text{then} \quad (\sin\theta)|v|^2 = 1 + \delta_2, \ (\sin\theta)|u|^2 = \delta_3, \ |\delta_2|, |\delta_3| \le \delta. \tag{5.39}$$

For $\omega = i$ we further deduce

$$\left|(\sin\theta)|y|^2 - 1\right| \le \delta + \delta^2, \quad (\sin\theta)|x|^2 \le \delta + \delta^2. \tag{5.40}$$

(a) $B = \begin{bmatrix} 0 & b \\ b & d \end{bmatrix}$, $b \ge 0, d > 0$

Lemma 3.1 (D1) for $\widetilde{B} = \widetilde{a} \oplus \widetilde{d}$, $\widetilde{a} \neq 0$ and (5.39) for $\omega = i$ (hence $(1+\delta_2)|u|^2 = \delta_3|v|^2$) yield a contradiction for small ϵ, δ. Next, $c(s) = 1$, $P(s) = i\sqrt{\widetilde{d} + s}\begin{bmatrix} \frac{s}{\widetilde{d}+s} & s^{-1} \\ 0 & -s^{-1} \end{bmatrix}$, $\cos(\frac{\theta}{2}) = \begin{cases} \frac{s^2}{2(\widetilde{d}+s)}, & \omega = i \\ s^3, & \omega = 0 \end{cases}$, $B(s) = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 2\widetilde{b} - s^2 \end{bmatrix}$ in (5.2) proves $\left(\begin{bmatrix} 0 & 1 \\ 1 & \omega \end{bmatrix}, \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & \widetilde{d} \end{bmatrix}\right) \to \left(1 \oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & d \end{bmatrix}\right)$, $\widetilde{b} > 0$, either $\omega = 0$, $\widetilde{d} = 1$ or $\omega = i$, $\widetilde{d} = 0$. Taking $c(s) = 1$, $P(s) = \sqrt{\widetilde{d} + s}\begin{bmatrix} \frac{s}{\widetilde{d}+s} & s^{-1} \\ 0 & -s^{-1} \end{bmatrix}$, $\cos(\frac{\theta}{2}) = \frac{s^2}{2(\widetilde{d}+s)}$, $B(s) = 0 \oplus s^2$ shows $\left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, 0 \oplus \widetilde{d}\right) \to (1 \oplus e^{i\theta}, 0 \oplus d)$. Finally, $c(s) = -ie^{i\frac{\widetilde{\vartheta}}{2}}$, $\cos(\frac{\theta(s)}{2}) = s^3$, $P(s) =$

$\frac{1}{s}e^{-i\frac{\pi}{4}}\begin{bmatrix} e^{i\alpha(s)} & ie^{-i\alpha(s)}\sqrt{\widetilde{d}+s} \\ -e^{-i\alpha(s)} & -ie^{i\alpha(s)}\sqrt{\widetilde{d}+s}\end{bmatrix}$, $\sin(2\alpha(s)) = \frac{s^2}{2|\sqrt{\widetilde{d}+s}|}$, $B(s) = |\sqrt{\widetilde{d}+s}|\begin{bmatrix} 0 & 1 \\ 1 & 2\end{bmatrix}$ in (5.2) proves $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, 1\oplus\widetilde{d}\right) \to \left(1\oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & d\end{bmatrix}\right)$, $b>0$, $\mathrm{Im}(\widetilde{d})>0$.

(b) $B = \begin{bmatrix} 0 & b \\ b & 0\end{bmatrix}$, $b>0$

Let $\widetilde{B} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 0\end{bmatrix}$, $\widetilde{b}>0$ and $\omega = i$. It follows from Lemma 4.1 (3) that $b^2 = \widetilde{b}^2 + \epsilon'$, $|\epsilon'| \leq \max\{\epsilon, \frac{\delta^2}{\nu^2}\}\big(4\max\{1, |\widetilde{b}|, |\widetilde{b}|^2\}+2\big)^2$, so the third equation of (5.12) for $\widetilde{d}=0$ yields $(y v)^2 = \frac{\epsilon_4^2}{4(\widetilde{b}^2+\epsilon')}$. By combining it with (5.39) and (5.40) we deduce

$$\big(1-\delta(1+\delta)\big)(1-\delta) \leq (\sin\theta)^2|yv|^2 = \frac{\delta^2\epsilon^2}{4|\widetilde{b}^2+\epsilon'|},$$

which fails for ϵ,δ small enough. Next, $c(s)=1$, $\cos(\frac{\theta(s)}{2}) = s^2$, $P(s) = \frac{i}{\sqrt{2}}\begin{bmatrix} s & s^{-1} \\ s & -s^{-1}\end{bmatrix}$, $B(s) = (\widetilde{d}+s)s^2\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$ in (5.2) gives $\left(\begin{bmatrix} 0 & 1 \\ 1 & i\end{bmatrix}, 0\oplus\widetilde{d}\right) \to \left(1\oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & 0\end{bmatrix}\right)$, $\widetilde{d}\geq 0$. We apply the same proof as in Case VII (compare (5.34) and (5.39)) to show $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & 1\end{bmatrix}\right) \not\to \left(1\oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & 0\end{bmatrix}\right)$, $\widetilde{b}>0$ and $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, 1\oplus\widetilde{d}\right) \not\to \left(1\oplus e^{i\theta}, \begin{bmatrix} 0 & b \\ b & 0\end{bmatrix}\right)$, $\mathrm{Im}\widetilde{d}>0$.

(c) $\begin{bmatrix} a & b \\ b & 0\end{bmatrix}$, $a>0$, $b\geq 0$

We multiply the squared equation in Lemma 3.1 (D2) for $\widetilde{B} = \widetilde{a}\oplus\widetilde{d}$ with $(\sin\theta)^2$:

$$(\widetilde{a}+\epsilon_1)^2y^2\sin^2\theta = \big(-i(-1)^l\sqrt{\widetilde{a}\widetilde{d}}+\epsilon_2'\big)x^2\sin^2\theta,\ |\epsilon_2'| \leq \begin{cases} \frac{\epsilon(4\max\{\widetilde{a},\widetilde{d}\}+2+|\widetilde{a}\widetilde{d}|)}{|\widetilde{a}\widetilde{d}|}, & \widetilde{a}\widetilde{d}\neq 0 \\ \sqrt{\epsilon(4\max\{\widetilde{a},\widetilde{d}\}+3)}, & \widetilde{a}\widetilde{d}=0\end{cases}.$$

By applying (5.39) and (5.40) (for $\omega = i$) we get $|\widetilde{a}+\epsilon_1|^2(1-\delta) \leq \big(\sqrt{|\widetilde{a}\widetilde{d}|}+|\epsilon_2'|\big)(\delta+\delta^2)$, which fails for $\widetilde{a}\neq 0$ and small ϵ,δ. For $c(s) = e^{i\frac{\widetilde{\vartheta}}{2}}$, $P(s) = \frac{1}{s}e^{-i\frac{\pi}{4}}\begin{bmatrix} -e^{-i\alpha} & -ie^{i\alpha}\sqrt{\widetilde{d}+s} \\ e^{i\alpha} & ie^{-i\alpha}\sqrt{\widetilde{d}+s}\end{bmatrix}$ with $\sin(2\alpha(s)) = \frac{s^2}{2|\sqrt{\widetilde{d}+s}|}$, $\cos\frac{\theta}{2} = s^3$, $B(s) = |\sqrt{\widetilde{d}+s}|\begin{bmatrix} 2 & 1 \\ 1 & 0\end{bmatrix}$ in (5.2), it follows $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, 1\oplus\widetilde{d}\right) \to \left(1\oplus e^{i\theta}, \begin{bmatrix} a & b \\ b & 0\end{bmatrix}\right)$, $\pi > \widetilde{\vartheta} = \arg\widetilde{d} > 0$ or $\widetilde{d}=0$. Taking $c(s)=1$, $P(s) = \frac{1}{\sqrt{\widetilde{d}+s}}\begin{bmatrix} 0 & \frac{1}{s}(\widetilde{d}+s) \\ s & -\frac{1}{s}(\widetilde{d}+s)\end{bmatrix}$, $B(s) = \begin{bmatrix} 2b(s)+s^2 & b(s) \\ b(s) & 0\end{bmatrix}$, $b(s)\to\widetilde{b}$, $\cos(\frac{\theta}{2}) = \begin{cases} \frac{s^2}{2(\widetilde{d}+s)}, & \omega = i \\ s^3, & \omega = 0\end{cases}$ proves $\left(\begin{bmatrix} 0 & 1 \\ 1 & \omega\end{bmatrix}, \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{b} & \widetilde{d}\end{bmatrix}\right) \to \left(1\oplus e^{i\theta}, \begin{bmatrix} a & b \\ b & 0\end{bmatrix}\right)$, $b\geq\widetilde{b}\geq 0$, either $\omega=0$, $\widetilde{b}>0$, $\widetilde{d}=1$ or $\omega = i$, $\widetilde{d},\widetilde{b}\geq 0$.

(d) $B = \begin{bmatrix} a & b \\ b & d\end{bmatrix}$, $a, d>0$, $b\in\mathbb{C}$

First let $b=0$. We deal with the case $\omega=0$ in the same manner as in Case VII (a) (ii) (compare also (5.34) and (5.39); observe that the proof works in the case $a>d$, too).

If $\omega = i$ we have $|v|^2 \geq 1$, $\frac{|u|^2}{|v|^2} \leq \delta \leq \frac{1}{2}$ and using (5.39) we easily verify

$$\tfrac{|x|^2}{|y|^2} = \tfrac{|u|^2+\delta}{|v|^2-\delta} = \tfrac{|u|^2}{|v|^2} + \tfrac{|v|^2\delta+|u|^2\delta}{(|v|^2-\delta)|v|^2} \leq \delta + \tfrac{|v|^2\delta+\frac{1}{2}|v|^2\delta}{\frac{1}{2}|v|^2} \leq 4\delta.$$

Multiplying the second equation of (5.6) with $\delta_5 = \frac{x}{y}$ and $\delta_6 = \frac{u}{v}$ yields

$$ax^2 + dv^2\delta_6\delta_5 = (\widetilde{b}+\epsilon_2)\delta_5, \qquad ay^2\delta_6\delta_5 + du^2 = (\widetilde{b}+\epsilon_2)\delta_6.$$

By adding them and using (5.6) yields a contradiction for $\widetilde{a} \neq 0$ and small ϵ, δ:

$$(\widetilde{a}+\epsilon_2) + (\widetilde{d}+\epsilon_2)\delta_6\delta_5 = (\widetilde{b}+\epsilon_2)(\delta_5+\delta_6).$$

It is tedious to find $c(s) = 1$, $\cos(\frac{\theta(s)}{2}) = s^2$, $B(s) = \frac{1}{2}\begin{bmatrix} \widetilde{a}s^{-2} & \widetilde{a}s^{-2} - 2ds^2 \\ \widetilde{a}s^{-2} - 2ds^2 & \widetilde{a}s^{-2} \end{bmatrix}$, $P(s) = \frac{1}{\sqrt{2}}\begin{bmatrix} s & s^{-1} \\ s & -s^{-1} \end{bmatrix}$ in (5.2) to prove $\left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, \widetilde{a}\oplus\widetilde{d}\right) \to \left(1\oplus e^{i\theta}, \begin{bmatrix} a & b \\ b & d \end{bmatrix}\right)$, $a, d, \widetilde{a} > 0$, $b \in \mathbb{C}^*$, $\widetilde{d} \in \mathbb{C}$.

Case X. $\left(\begin{bmatrix} 0 & 1 \\ 1 & \omega \end{bmatrix}, \widetilde{B}\right) \dashrightarrow \left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & d \end{bmatrix}\right)$, $0 < \tau < 1$, $\omega \in \{0, i\}$

The following expressions are bounded by δ (Lemma 3.2 (2) (C4) for $\alpha = 1$):

$$\mathrm{Re}(\overline{x}u),\ (1-\tau)\mathrm{Im}(\overline{x}u),\ \mathrm{Re}(\overline{y}v),\ (1-\tau)\mathrm{Im}(\overline{y}v) - (-1)^k|\omega|,\ 1-\tau,\ \overline{x}v + \overline{u}y - (-1)^k, \tag{5.41}$$

where $k \in \mathbb{Z}$. If in addition $\omega = i$, it then follows that

$$\begin{aligned} &\delta_5 = \tfrac{|xu|}{|yv|} = \tfrac{(1-\tau)|xu|}{(1-\tau)|yv|} \leq \tfrac{\left|(1-\tau)\mathrm{Re}(\overline{x}u)\right| + \left|(1-\tau)\mathrm{Im}(\overline{x}u)\right|}{\left|(1-\tau)\mathrm{Im}(\overline{y}v)\right| - \left|(1-\tau)\mathrm{Re}(\overline{y}v)\right|} \leq \tfrac{\delta+\delta^2}{1-\delta-\delta^2}, \\ &\delta|yv| \geq \left|(1-\tau)\mathrm{Im}(\overline{y}v)\right| \geq 1-\delta, \end{aligned} \tag{5.42}$$

$$\begin{aligned} &(1+\delta)\tfrac{|v|}{|u|} \geq |\overline{u}y + \overline{x}v|\tfrac{|v|}{|u|} \geq |vy| - \tfrac{|xv|}{|uy|}|vy| = |vy|\big(1 - \tfrac{|xu|}{|vy|}\,|\tfrac{v}{u}|^2\big), \\ &(1+\delta)\tfrac{|y|}{|x|} \geq |\overline{u}y + \overline{x}v|\tfrac{|y|}{|x|} \geq |vy| - \tfrac{|uy|}{|xv|}|vy| = |vy|\big(1 - \tfrac{|xu|}{|vy|}\,|\tfrac{y}{x}|^2\big). \end{aligned} \tag{5.43}$$

(a) $B = a \oplus d$

Let $B = 0 \oplus 1$. If $\widetilde{a} \neq 0$ (hence $\widetilde{b} = \widetilde{d} = 0, \omega = i$) then (5.4) for $a = b = \widetilde{d} = 0, d = 1$ yields $(\frac{v}{u})^2 = \frac{\epsilon_2}{\widetilde{a}+\epsilon_1}$, thus (5.42), (5.43) give a contradiction for small ϵ, δ. Taking $c(s) = 1$, $\tau(s) = 1 - s$, $P(s) = \frac{1}{\sqrt{\widetilde{d}+s}}\begin{bmatrix} 1 & -is^{-1} \\ 0 & \widetilde{d}+s \end{bmatrix}$ proves $\left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, 0\oplus\widetilde{d}\right) \to \left(\begin{bmatrix} 0 & 1 \\ \tau & 0 \end{bmatrix}, 0\oplus 1\right)$, $\widetilde{d} \geq 0$.

Next, $B = 1 \oplus d$, $d \in \mathbb{C}$. If either $|\frac{x}{y}| \geq 1$ (or $|\frac{u}{v}| \geq 1$), then in case $\omega = i$ the second (the first) inequality of (5.43) yields a contradiction. When $|\frac{x}{y}|, |\frac{u}{v}| \leq 1$ we multiply the second equation of (5.4) for $b = 0$, $a = 1$ with $\frac{u}{v}$ and $\frac{x}{y}$, and simplify them:

$$\delta_5 y^2 + du^2 = (\widetilde{b}+\epsilon_2)\tfrac{u}{v}, \quad x^2 + \delta_5 dv^2 = (\widetilde{b}+\epsilon_2)\tfrac{x}{y} \qquad (\delta_5 \leq \tfrac{\delta+\delta^2}{1-\delta-\delta^2}).$$

We add these equations and use (5.4) for $b = 0$, $a = 1$ to get $\delta_5(\widetilde{d}+\epsilon_4) + (\widetilde{a}+\epsilon_1) = (\widetilde{b}+\epsilon_2)\frac{u}{v} + (\widetilde{b}+\epsilon_2)\frac{x}{y}$. Since $|\frac{x}{y}|, |\frac{u}{v}| \leq 1$, it fails for $\widetilde{a} \neq 0$, $\widetilde{b} = 0$ and small ϵ, δ. Finally, $c(s) = 1$, $\tau(s) = 1 - s^2$, $P(s) = \frac{1}{\sqrt{\widetilde{b}}}e^{-i\frac{\pi}{4}}\begin{bmatrix} s^2 e^{i\frac{\pi}{4}} & \widetilde{b}s^{-1} \\ s & s^{-1} \end{bmatrix}$,

$B(s)=1\oplus\widetilde{b}^2\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},1\oplus d\right)$, while, $c(s)=1$, $\tau(s)=1-s^3$, $P(s)=\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\begin{bmatrix}s\widetilde{b}e^{-i\alpha(s)}&-is^{-1}e^{i\alpha(s)}\\-ise^{i\alpha(s)}&(\widetilde{b}s)^{-1}e^{-i\alpha(s)}\end{bmatrix}$, $B(s)=1\oplus\widetilde{b}^2e^{4\alpha(s)+\beta(s)}$, $\sin(\alpha(s))=s^3$, $\sin(\frac{\beta(s)}{2})=-s^2$ gives $\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&1\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},1\oplus d\right)$, $\widetilde{b}>0$, $d\in\mathbb{C}$.

(b) $B=\begin{bmatrix}0&b\\b&e^{i\varphi}\end{bmatrix}$, $\quad 0\leq\varphi<\pi, b>0$

Let $a=0$ and $\widetilde{B}=\widetilde{a}\oplus\widetilde{d}$. Lemma 3.1 (D1) for $\widetilde{a}\neq 0$ implies $\frac{v}{u}=\frac{i(-1)^l\sqrt{\widetilde{a}\widetilde{d}}+\epsilon_2'}{\widetilde{a}+\epsilon_1}=i(-1)^l\sqrt{\frac{\widetilde{d}}{\widetilde{a}}}+\epsilon_2''$, $|\epsilon_2'|\leq\begin{cases}\frac{\epsilon(4|\max\{\widetilde{d},\widetilde{a}\}|+2+|\widetilde{a}\widetilde{d}|)}{|\widetilde{a}\widetilde{d}|}, & \det\widetilde{B}\neq 0\\ \sqrt{\epsilon(4|\max\{\widetilde{d},\widetilde{a}\}|+3)}, & \det\widetilde{B}=0\end{cases}$, $|\epsilon_2''|\leq\frac{2}{\widetilde{a}}(|\epsilon_2'|+\epsilon\sqrt{\frac{\widetilde{d}}{\widetilde{a}}})$, $l\in\mathbb{Z}$, provided that $\epsilon\leq\frac{|\widetilde{a}|}{2}$. It contradicts (5.42), (5.43) for $\omega=i$. If $\widetilde{a}=1$, $\widetilde{d}=|\widetilde{d}|e^{i\widetilde{\vartheta}}\neq 0$, $0<\widetilde{\vartheta}<\pi$ we apply (4.3) to deduce $\psi=\kappa-\eta-\frac{\widetilde{\vartheta}}{2}-\frac{\pi}{2}-l\pi$ with $|\sin\psi|\leq\frac{|\epsilon_2''|}{|\sqrt{\widetilde{d}}|}$. Hence

$$\begin{aligned}\overline{x}v+\overline{u}y=\overline{x}u\tfrac{v}{u}+y\overline{v}\tfrac{\overline{u}}{\overline{v}}&=-(-1)^le^{i(\frac{\widetilde{\vartheta}}{2}+\psi)}\left(\mathrm{Im}(\overline{x}u)|\tfrac{v}{u}|+\mathrm{Im}(y\overline{v})|\tfrac{\overline{u}}{\overline{v}}|\right)\\&\quad+\mathrm{Re}(\overline{x}u)\tfrac{v}{u}+\mathrm{Re}(y\overline{v})\tfrac{\overline{u}}{\overline{v}}.\end{aligned}$$

Using (5.41)) and $\left||\frac{v}{u}|-|\sqrt{\frac{\widetilde{d}}{\widetilde{a}}}|\right|\leq|\epsilon_2''|$, the above calculation and (4.3) gives

$$\psi'=k\pi-\left(\tfrac{\widetilde{\vartheta}}{2}+\psi+(l+1)\pi\right),\quad |\sin\psi'|\leq 2\delta\left(1+|\sqrt{\tfrac{\widetilde{d}}{\widetilde{a}}}|+|\epsilon_2''|+(|\sqrt{\tfrac{\widetilde{d}}{\widetilde{a}}}|-|\epsilon_2''|)^{-1}\right),$$

which fails for small ϵ,δ (recall $|\sin\psi|\leq\frac{|\epsilon_2''|}{|\sqrt{\widetilde{d}}|}$, $0<\widetilde{\vartheta}<\pi$). Next, $c(s)=-1$, $P(s)=\begin{bmatrix}-\frac{2is}{3\widetilde{b}}&\frac{1}{s}\\\frac{s}{3}&\frac{2i\widetilde{b}}{s}\end{bmatrix}$, $B(s)=\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&i\end{bmatrix}$, $\tau(s)=1-\frac{s^2}{2\widetilde{b}}$ implies $\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},\begin{bmatrix}0&\widetilde{b}\\\widetilde{b}&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}0&b\\b&e^{i\varphi}\end{bmatrix}\right)$, $\widetilde{b}>0$.

For $\tau(s)=\begin{cases}1-s\sqrt{\widetilde{a}+s}, & \omega=i\\1-s^2, & \omega=0\end{cases}$, $P(s)=e^{i\frac{\pi}{4}}\begin{bmatrix}\sqrt{\widetilde{a}+s}&\frac{-i}{s}\\s^3e^{-i\frac{\pi}{4}}&\frac{1}{\sqrt{\widetilde{a}+s}}\end{bmatrix}$, $B(s)=\begin{bmatrix}-i&\frac{\sqrt{\widetilde{a}+s}}{s}\\\frac{\sqrt{\widetilde{a}+s}}{s}&-i(\widetilde{a}+s)(\widetilde{d}-\frac{1}{s^2})\end{bmatrix}$, $c(s)=-1$ we get $\left(\begin{bmatrix}0&1\\1&\omega\end{bmatrix},\widetilde{a}\oplus\widetilde{d},\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}e^{i\varphi}&b\\b&d\end{bmatrix}\right)$, $d\in\mathbb{C}$, $\widetilde{a}\geq 0$.

(c) $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$

We multiply the first and the second equality of (D1) and (D2) of Lemma 3.1 for $\widetilde{a}\neq 0$ to get a contradiction with (5.42) for $\omega=i$ and small ϵ,δ. Taking $c(s)=1$, $\tau(s)=1-s$, $P(s)=e^{-i\frac{\pi}{4}}\begin{bmatrix}se^{i\frac{\pi}{4}}&s^{-1}\\s&i\end{bmatrix}$, $B(s)=\frac{\widetilde{d}+s}{2}s\begin{bmatrix}0&1\\1&0\end{bmatrix}$ shows $\left(\begin{bmatrix}0&1\\1&i\end{bmatrix},0\oplus\widetilde{d}\right)\to\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}0&b\\b&0\end{bmatrix}\right)$.

For $\widetilde{a}=0,\widetilde{b}>0$ we have $b=\widetilde{b}+\delta'$, with $|\delta'|\leq\delta\widetilde{b}^2+\max\{\epsilon,\frac{\delta^2}{\nu^2}\}(4\max\{\widetilde{b},\widetilde{b}^2,1\}+2)^2$

(see Lemma 4.1 (3 and (4.7)); recall $1-\tau \leq \delta$). If $\omega = i$ (hence $\widetilde{d} = 0$) the last equation of (5.12) for $\widetilde{d} = 0$ contradicts the second estimate of (5.42). Next, let $\omega = 0$ (hence $\widetilde{d} = 1$). Using $2bvy = 1 + \epsilon_4$ (see (5.12)) and $|\mathrm{Re}(\overline{y}v)| \leq \delta$ (see (5.41)), we have $|\mathrm{Im}(\overline{y}v)| \geq |yv| - |\mathrm{Re}(\overline{y}v)| \geq \frac{1-\epsilon}{\widetilde{b}+|\delta'|} - \delta$. Further Lemma 3.1 gives $2bvx = ((-1)^{l+1} + 1)\widetilde{b} + \epsilon_2'$, $2buy = ((-1)^l + 1)\widetilde{b} + \epsilon_2''$, $l \in \mathbb{Z}$, where $|\epsilon_2'|, |\epsilon_2''| \leq \frac{\epsilon(4\max\{1,\widetilde{b}\}+2+\widetilde{b}^2)}{\widetilde{b}^2}$. So either $2bvx = 2\widetilde{b} + \epsilon_2'$, $2buy = \epsilon_2''$ or $2buy = 2\widetilde{b} + \epsilon_2''$, $2bvx = \epsilon_2'$. In the first case we also have $\overline{x}v = (-1)^k + \delta_2'$ with $|\delta_2'| \leq \delta + \frac{|\epsilon_2''|}{2(\widetilde{b}-|\delta'|)}$ (see (5.41)). We combine all facts:

$$|\tfrac{y}{x}|^2 = \frac{2bvy\,\overline{y}v}{2bvx\,\overline{x}v} = \frac{(1+\epsilon_4)(i\mathrm{Im}(\overline{y}v)+\delta_0)}{(2\widetilde{b}+\epsilon_2')((-1)^k+\delta_2')}$$

For sufficiently small ϵ, δ the right-hand (the left-hand) side is (not) real, a contradiction. The other case is treated similarly and yields a contradiction as well.

Case XI. $(1 \oplus 0, \widetilde{B}) \dashrightarrow (1 \oplus 0, B)$

If $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\widetilde{B} = \widetilde{a} \oplus 1$, $\widetilde{a} \geq 0$, then [22, Theorem 3.6, Case XI (a)] applies. (Taking $c(s) = 1$, $P(s) = \begin{bmatrix} 1 & s \\ \frac{\widetilde{a}}{2} & 0 \end{bmatrix}$ in (5.2) proves $(1 \oplus 0, \widetilde{a} \oplus 0) \to \big(1 \oplus 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\big)$.)

Next, Lemma 3.2 (2) with (C12) for $\alpha = 1$ gives $\big||x|^2 - 1\big| \leq \delta$ and $|y|^2 \leq \delta$, hence $|\frac{y}{x}|^2 \leq \frac{\delta}{1-\delta}$. When $B = a \oplus 0$ for $a \geq 0$, then dividing the last two equalities of (5.4) for $b = d = \widetilde{b} = 0$, $\widetilde{d} = 1$ yields $\frac{x}{y} = \frac{\epsilon_2}{1+\epsilon_4}$, which contradicts $|\frac{y}{x}|^2 \leq \frac{\delta}{1-\delta}$ for small ϵ, δ.

Finally, $c(s) = 1$, $P(s) = \begin{bmatrix} 1 & 0 \\ \sqrt{\widetilde{a} - a} & s \end{bmatrix}$ in (5.2) proves $(1 \oplus 0, \widetilde{a} \oplus 0) \to (1 \oplus 0, a \oplus 1)$, $a \geq 0$, and $c(s) = 1$, $P(s) = \begin{bmatrix} i & s^3 \\ s^{-1} & s \end{bmatrix}$, $B(s) = \frac{1}{s^2} \oplus 1$ implies $\big(1 \oplus 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\big) \to \big(1 \oplus 0, a \oplus 1\big)$, $a > 0$.

Case XII. $(1 \oplus 0, \widetilde{B}) \to \big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B\big)$

Lemma 3.2 (2) for (C5) for $\alpha = 1$, $\beta = \omega = 0$ yields

$$2\mathrm{Re}(\overline{x}u) = (-1)^k + \delta_1,\ 2\mathrm{Re}(\overline{y}v) = \delta_2,\ \overline{x}v + \overline{u}y = \delta_4, \quad k \in \mathbb{Z}, |\delta_1|, |\delta_2|, |\delta_4| \leq \delta. \tag{5.44}$$

Next, (5.26) (compare (5.44) with (5.23)) is valid in this case as well. Since $|\det P| \leq \frac{\delta\sqrt{6}}{\nu}$ by Lemma 4.1 (1), it follows from (5.26) that

$$|\tfrac{v}{u}|, |\tfrac{y}{x}| \leq \delta \frac{\nu+\sqrt{6}}{\nu(1-\delta)}. \tag{5.45}$$

(a) $B = 1 \oplus 0$
The bundle consists of one orbit, hence [22, Theorem 3.6, Case XV (c)] applies. (We take $c(s) = 1$ and $P(s) = \begin{bmatrix} \sqrt{\widetilde{a} + s} & 0 \\ \frac{1}{2\sqrt{\widetilde{a}+s}} & s \end{bmatrix}$ to get $(1 \oplus 0, \widetilde{a} \oplus 0) \to \big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1 \oplus 0\big)$ for $\widetilde{a} \geq 0$.)

(b) $B = 1 \oplus d$, $\mathrm{Im}(d) > 0$
For $\widetilde{B} = \widetilde{a} \oplus 1$ we have (5.6) with $\widetilde{b} = 0$, $a = 1$. By multiplying the second equation of (5.6) for $\widetilde{b} = 0$ with $\delta_4 := \frac{v}{u}$, $\delta_5 := \frac{y}{x}$ and by simplifying it we obtain

$$ax^2\delta_4\delta_5 + dv^2 = \epsilon_2\delta_4, \qquad ay^2 + dv^2\delta_4\delta_5 = \epsilon_2\delta_5, \tag{5.46}$$

respectively. We add these equalities and using the first and the last equation of (5.6) we get the equality that fails for $\widetilde{d}\neq 0$ and ϵ, δ small enough (recall (5.45)):

$$\epsilon_2(\delta_4+\delta_5)=(ax^2+du^2)\delta_4\delta_5+(ay^2+dv^2)=(\widetilde{a}+\epsilon_1)\delta_4\delta_5+\widetilde{d}+\epsilon_4. \tag{5.47}$$

Note that $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},1\oplus d\right)$ will follow after we prove $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(1\oplus -1,\begin{bmatrix}0&b\\b&0\end{bmatrix}\right)$ (see Case XIV (a)).

(c) $B=\begin{bmatrix}0&b\\b&1\end{bmatrix}$, $b>0$

Let $\widetilde{B}=\widetilde{a}\oplus 1$, $\widetilde{a}\geq 0$. From Lemma 3.1 (D1) for $\widetilde{b}=0$, $\widetilde{d}=1$ we get:

$$|u|\leq \tfrac{\sqrt{\widetilde{a}}+|\epsilon_2''|}{1-\epsilon}|v|, \tag{5.48}$$

which clearly contradicts (5.45) for sufficiently small ϵ, δ.

For $P(s)=\begin{bmatrix}-\frac{1}{2}s&s^4\\s^{-1}&2s\end{bmatrix}$, $c(s)=1$, $B(s)=\begin{bmatrix}0&s^{-2}\\s^{-2}&1\end{bmatrix}$ we show $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&b\\b&1\end{bmatrix}\right)$.

Case XIII. $(1\oplus 0,\widetilde{B})\dashrightarrow\left(\begin{bmatrix}0&1\\\tau&0\end{bmatrix},\begin{bmatrix}a&b\\b&d\end{bmatrix}\right)$, $0\leq\tau<1$

From Lemma 3.2 (2) for (C3) with $\alpha=1$ we get

$$\begin{aligned}&\mathrm{Re}(\overline{y}v)\leq\delta,\quad (1-\tau)\mathrm{Im}(\overline{y}v)\leq\delta,\quad (1-\tau)|\overline{x}v|\leq\delta,\quad (1-\tau)|\overline{u}y|\leq\delta,\\ &\overline{x}v+\overline{u}y\leq\delta,\qquad \left|(1+\tau)\mathrm{Re}(\overline{x}u)+i(1-\tau)\mathrm{Im}(\overline{x}u)-\tfrac{1}{c}\right|\leq\delta.\end{aligned} \tag{5.49}$$

The last estimate yields either $\left|(1+\tau)\mathrm{Re}(\overline{x}u)\right|\geq\frac{1-\delta}{2}$ or $\left|(1-\tau)\mathrm{Im}(\overline{x}u)\right|\geq\frac{1-\delta}{2}$, thus

$$|\overline{x}u|\geq\tfrac{1-\delta}{4}. \tag{5.50}$$

(a) $B=\begin{bmatrix}a&b\\b&d\end{bmatrix}$, either $b>0$ or $b=0$ and $ad=0$

First, let $\widetilde{B}=\widetilde{a}\oplus 1$, $\widetilde{a}\geq 0$; we have (5.48). Using (5.49), (5.50) we thus get

$$\delta\tfrac{\sqrt{\widetilde{a}}+|\epsilon_2''|}{|1-\epsilon|}\geq(1-\tau)|\overline{x}v||\tfrac{u}{v}|=(1-\tau)|\overline{x}u|\geq\tfrac{1}{4}(1-\tau).$$

Similarly, when $d=0$ then Lemma 3.1 (D2) for $\widetilde{b}=0$, $\widetilde{d}=1$ and (5.49), (5.50) yield $|\frac{x}{y}|\leq\frac{\sqrt{\widetilde{a}}+|\epsilon_2'|}{1-\epsilon}$ and $\frac{4\delta(\sqrt{\widetilde{a}}+|\epsilon_2'|)}{1-\epsilon}\geq 1-\tau$. From Lemma 4.1 (1) we obtain $\sqrt{\tau}|\det P|\leq\frac{\delta\sqrt{6}}{v}$. By combining the above statements with (5.24), (5.25) we get $\mathrm{Re}(\overline{x}u)\leq C\delta$, where a constant $C>0$ can be computed. Hence $(1-\tau)\mathrm{Im}(\overline{x}u)\geq 1-\delta-C\delta$, and further

$$\frac{|yv|}{|xu|}=\frac{(1-\tau)|yv|}{(1-\tau)|xu|}\leq\frac{\left|(1-\tau)\mathrm{Im}(\overline{y}v)\right|+\left|(1-\tau)\mathrm{Re}(\overline{y}v)\right|}{\left|(1-\tau)\mathrm{Im}(\overline{x}u)\right|-\left|(1-\tau)\mathrm{Re}(\overline{x}u)\right|}\leq\frac{2\delta}{1-\delta-2C\delta}. \tag{5.51}$$

It is also easy to validate

$$|\overline{x}v+\overline{u}y||\tfrac{u}{v}|\geq|ux|\left|1-\tfrac{|yv|}{|xu|}|\tfrac{u}{v}|^2\right|,\qquad |\overline{x}v+\overline{u}y||\tfrac{x}{y}|\geq|ux|\left|1-\tfrac{|yv|}{|xu|}|\tfrac{x}{y}|^2\right|. \tag{5.52}$$

We apply (5.49) and the estimates on $|\frac{u}{v}|$, $|\frac{x}{y}|$, $|\frac{yv}{xu}|$ to (5.52) to get a contradiction for small ϵ,δ. Next, $P(s)=\begin{bmatrix}-se^{i(\alpha(s)+\frac{\pi}{4})} & s^3\\ s^{-1}e^{i\frac{\pi}{4}} & 1\end{bmatrix}$, $B(s)=\begin{bmatrix}s^{-4}e^{-i\alpha(s)} & s^{-2}\\ s^{-2} & 1\end{bmatrix}$, $c(s)=-1$, $\tau(s)\to 0$, $\sin(\frac{\alpha(s)}{2})=\frac{\widetilde{a}s^2}{2}$ implies $(1\oplus 0,\widetilde{a}\oplus 1)\to\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}\zeta^* & b\\ b & 1\end{bmatrix}\right)$, $\zeta^*\in\mathbb{C}^*$.

Let $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$, $\widetilde{B}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. The first (the second) equation of (5.12) for $\widetilde{a}=0$ (for $\widetilde{b}=1$) combined with (5.50) (with (5.49) for $0\leq\tau\leq\frac{1}{2}$) yields $\epsilon\geq b|ux|\geq b\frac{1-\delta}{4}$ (and $1+\epsilon\geq b|vx+uy|\geq 4b\delta$), thus a contradiction for sufficiently small ϵ,δ and $0\leq\tau\leq\frac{1}{2}$. If $1\geq\tau\geq\frac{1}{2}$ then Lemma 4.1 (1), (2) leads to $|\det P|\leq\frac{2\sqrt{3}\delta}{\nu}$ and $b|\det P|\geq 1-6\epsilon$, hence $\frac{8\sqrt{3}\epsilon\delta}{\nu(1-\delta)}\geq b\frac{2\sqrt{3}\delta}{\nu}\geq 1-6\epsilon$, which fails for small ϵ,δ. Taking $P(s)=\begin{bmatrix}-s & s^4\\ s^{-1} & 2s\end{bmatrix}$, $B(s)=\begin{bmatrix}\zeta & \frac{1}{2}s^{-2}\\ \frac{1}{2}s^{-2} & 1\end{bmatrix}$ with $\frac{\zeta}{s^2}\to 0$ and $P(s)=\begin{bmatrix}s^{-1} & 2s\\ -s & s^4\end{bmatrix}$, $B(s)=\begin{bmatrix}1 & \frac{1}{2}s^{-2}\\ \frac{1}{2}s^{-2} & 0\end{bmatrix}$ (both with $c(s)=-1$, $\tau(s)\to 0$) in (5.2) proves $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\ \tau&0\end{bmatrix},\begin{bmatrix}\zeta & b\\ b & e^{i\varphi}\end{bmatrix}\right)$, $\zeta\in\mathbb{C}$ and $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\ \tau&0\end{bmatrix},\begin{bmatrix}e^{i\varphi} & b\\ b & 0\end{bmatrix}\right)$ with $b>0$, $0\leq\varphi<\pi$, respectively.

Finally, to see $(1\oplus 0,\widetilde{a}\oplus 0)\to\left(\begin{bmatrix}0&1\\ \tau&0\end{bmatrix},B\right)$, $0\leq\tau<1$, $\widetilde{a}\geq 0$, where B is any of the matrices $\begin{bmatrix}a & b\\ b & e^{i\varphi}\end{bmatrix}$, $a,b\geq 0$ and $\begin{bmatrix}e^{i\varphi} & b\\ b & d\end{bmatrix}$, $d,b\geq 0$, we take $P(s)=\begin{bmatrix}\frac{1}{\sqrt{\widetilde{a}+s}} & s\\ \sqrt{\widetilde{a}+s} & s\end{bmatrix}$, $B(s)=\begin{bmatrix}a(s) & b(s)\\ b(s) & 1\end{bmatrix}$ with $b(s)\to 0$, $\frac{a(s)}{\sqrt{s}}\to 0$ or $P(s)=\begin{bmatrix}\sqrt{\widetilde{a}+s} & s\\ \frac{1}{\sqrt{\widetilde{a}+s}} & s\end{bmatrix}$, $B(s)=\begin{bmatrix}1 & b(s)\\ b(s) & d(s)\end{bmatrix}$ with $b(s)\to 0$, $\frac{d(s)}{\sqrt{s}}\to 0$ in (5.2) ($c(s)=1$, $\tau(s)\to 0$ in both cases). To prove $(1\oplus 0,\widetilde{a}\oplus 0)\to\left(\begin{bmatrix}0&1\\ \tau&0\end{bmatrix},\begin{bmatrix}0&b\\b&0\end{bmatrix}\right)$, $b>0$, we put $P(s)=\begin{bmatrix}1&s\\1&0\end{bmatrix}$, $B(s)=\frac{\widetilde{a}+s}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $c(s)=1$, $\tau(s)\to 0$ in (5.2).

(b) $B=a\oplus d,\quad a,d\neq 0$

For $c(s)=-i$, $P(s)=\begin{bmatrix}s & s^3\\ is^{-1} & s^2\end{bmatrix}$, $B(s)=\frac{1}{s^4}\oplus 1$ we get $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to\left(\begin{bmatrix}0&1\\0&0\end{bmatrix},a\oplus 1\right)$.

If $\tau\leq\frac{1}{2}$ then we have $|xv|,|uy|\leq 2\delta$, thus using (5.50) we get $|\frac{v}{u}|=|\frac{vx}{ux}|\leq 8\delta$ and $|\frac{y}{x}|=|\frac{uy}{ux}|\leq 8\delta$. On the other hand for $\tau\geq\frac{1}{2}$ we get $|\det P|\leq\frac{2\sqrt{3}\delta}{\nu}$ (Lemma 4.1 (1)), therefore (5.24), (5.25), (5.49) imply $|\mathrm{Re}(\overline{x}u)||\frac{v}{u}|$, $|\mathrm{Re}(\overline{x}u)||\frac{v}{u}|\leq 2\sqrt{3}\delta+\delta$. If $|\mathrm{Re}(\overline{x}u)|\leq\sqrt{2\sqrt{3}\delta+\delta}$, then $(1-\tau)|\mathrm{Im}(\overline{x}u)|\geq 1-\sqrt{(2\sqrt{3}+1)\delta}$ and similarly as in (5.51) we obtain $|\frac{v}{u}|,|\frac{y}{x}|\leq\frac{2\delta}{1-2\sqrt{(2\sqrt{3}+1)\delta}}$. If $\widetilde{B}=\widetilde{a}\oplus\widetilde{d}$ with $\widetilde{d}\neq 0$, then in any case we proceed mutatis mutandis as in Case XII (b) to get a contradiction for small ϵ,δ.

Case XIV. $(1\oplus 0,\widetilde{B})\dashrightarrow(1\oplus e^{i\theta},B)$, $0\leq\theta\leq\pi$

From Lemma 3.2 (2) with (C1) for $\alpha=1$ and $0<\theta<\pi$ we have

$$\big||x|^2+e^{i\theta}|u|^2-c^{-1}\big|\leq\delta,\ \big||y|^2+e^{i\theta}|v|^2\big|\leq\delta,\ \sin(\theta)|\overline{u}v|\leq\delta,\ |\overline{x}y+\cos(\theta)\overline{u}v|\leq\delta. \tag{5.53}$$

Furthermore, Lemma 3.2 (2) with (C10) for $\alpha=1$, $\sigma=-1$ yields

$$\big||x|^2-|u|^2\big|=1+\delta_1,\quad \big||y|^2-|v|^2\big|=\delta_4,\quad |\overline{x}y-\overline{u}v|=\delta_2,\quad |\delta_1|,|\delta_2|,|\delta_4|\le\delta, \tag{5.54}$$

while from (C10) for $\alpha=1$, $\sigma=1$ we deduce

$$|x|^2+|u|^2=1+\delta_1,\qquad |y|^2,|v|^2\le\delta,\quad |\delta_1|\le\delta. \tag{5.55}$$

(a) $\widetilde{B}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

Taking $c(s)=1$, $P(s)=\frac{1}{\sqrt{2}}\begin{bmatrix}1&s\sqrt{2}\\i&-is\sqrt{2}\end{bmatrix}$, $B(s)=\frac{\sqrt{2}}{2s}I_2$ gives $\left(1\oplus 0,\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\to(I_2,aI_2)$, $a>0$. If $B=dI_2$ and $\theta=\pi$, then Lemma 4.1 (1), (2) gives $|\det P|\le\frac{\delta\sqrt{6}}{\nu}$ and $d|\det P|\ge 1-6\epsilon$. The first equation of (5.4) for $a=d$,$b=\widetilde{a}=0$ yields $\epsilon\ge\big|d(x^2+u^2)\big|\ge|d|\big||x|^2-|u|^2\big|\ge|d|(1-\delta)$ (see (5.54)). Thus $\frac{\epsilon\delta\sqrt{6}}{\nu}\ge(1-\delta)(1-6\epsilon)$, which fails for $\epsilon,\delta\le\frac{1}{12}$.

(b) $\widetilde{B}=\widetilde{a}\oplus 0,\widetilde{a}\ge 0$

We take $c(s)=e^{-i\theta}$, $P(s)=\begin{bmatrix}s&s\\1&s\end{bmatrix}$, $B(s)=\begin{bmatrix}a(s)&b(s)\\b(s)&d(s)\end{bmatrix}$ with $d(s)\to\widetilde{a}$, $sa(s),b(s)\to 0$ to prove a path $(1\oplus 0,\widetilde{a}\oplus 0)\to\left(1\oplus e^{i\theta},\begin{bmatrix}a&b\\b&d\end{bmatrix}\right)$ for $d>0$, $b\ge 0$, $0\le\theta\le\pi$.

(c) $\widetilde{B}=\widetilde{a}\oplus 1,\widetilde{a}\ge 0$

(i) $B=\begin{bmatrix}a&b\\b&d\end{bmatrix}$, $|a|+|d|\ne 0$, $a\ne d$

For $c(s)=1$, $P(s)=\begin{bmatrix}1&s\\0&s\end{bmatrix}$, $B(s)=\begin{bmatrix}a(s)&b(s)\\b(s)&s^{-2}\end{bmatrix}$, $a(s)\to\widetilde{a}$, $sb(s)\to 0$ and $c(s)=e^{-i\theta}$, $P(s)=\begin{bmatrix}0&s\\1&s\end{bmatrix}$, $B(s)=\begin{bmatrix}s^{-2}&b(s)\\b(s)&d(s)\end{bmatrix}$ with $d(s)\to\widetilde{a}$, $sb(s)\to 1$, we get $(1\oplus 0,\widetilde{a}\oplus 1)\to\left(1\oplus e^{i\theta},\begin{bmatrix}a&b\\b&d\end{bmatrix}\right)$ for $b\ge 0$, $d>0$ and $b\ge 0$, $a>0$, respectively. Next, $c(s)=1$, $P(s)=\begin{bmatrix}-i&0\\is&\frac{s}{\sqrt{\widetilde{a}+s}}\end{bmatrix}$, $B(s)=(\widetilde{a}+s)\begin{bmatrix}0&s^{-1}\\s^{-1}&s^{-2}\end{bmatrix}$ and $c(s)=e^{-i\theta}$, $P(s)=\begin{bmatrix}is&\frac{s}{\sqrt{\widetilde{a}+s}}\\-i&0\end{bmatrix}$, $B(s)=(\widetilde{a}+s)\begin{bmatrix}s^{-2}&s^{-1}\\s^{-1}&0\end{bmatrix}$ in (5.2) imply $(1\oplus 0,\widetilde{a}\oplus 1)\to\left(1\oplus e^{i\theta},\begin{bmatrix}0&b\\b&d\end{bmatrix}\right)$ and $(1\oplus 0,\widetilde{a}\oplus 1)\to\left(1\oplus e^{i\theta},\begin{bmatrix}a&b\\b&0\end{bmatrix}\right)$ for $b,a,d>0$.

(ii) $B=\begin{bmatrix}0&b\\b&0\end{bmatrix}$, $b>0$ $\quad(0<\theta\le\pi)$

The second estimate of (5.53) gives $(\sin\theta)|v|^2\le\delta$, thus either $|v|^2\le\sqrt{\delta}$ or $\sin\theta\le\sqrt{\delta}$ (or both). If $|v|^2\le\sqrt{\delta}$, then the second estimate of (5.53) (or (5.54)) implies $|y|^2\le\delta+\sqrt{\delta}$. Since we have (5.14) for $\widetilde{d}=1$, we further get $|u|,|x|\le(\delta+\sqrt{\delta})\frac{|\sqrt{\widetilde{a}}|+\max\{|\epsilon_2'|,|\epsilon_2''|\}}{(1-\epsilon)}$, which contradicts the first estimate of (5.53) and (5.54).

Let now $v,y\ne 0$ and $\sin\theta\le\sqrt{\delta}$. If $\theta\in(0,\frac{\pi}{4})$, then $(1-\cos\theta)|v|^2=$

$2(\sin^2\frac{\theta}{2})|v|^2 \leq 2(\sin^2\theta)|v|^2$, hence the second estimate of (5.53) yields

$$\delta \geq \left||y|^2 + \cos\theta|v|^2\right| \geq \left||y|^2 + |v|^2\right| - (1-\cos\theta)|v|^2 \geq \left||y|^2 + |v|^2\right| - 2\delta.$$

Hence $|y|^2, |v|^2 \leq 3\delta$ and it gives a contradiction again. If $\theta \in (\frac{3\pi}{4}, \pi]$, then $|\cos\frac{\theta}{2}| = |\sin\frac{\pi-\theta}{2}| \leq |\sin(\pi-\theta)|$, and by combining it with the first equation in (5.14) for $\widetilde{d} = 1$ and the third estimate of (5.53) we get $(\cos\frac{\theta}{2})|u|^2 \leq (\sin\theta)|uv||\frac{u}{v}| \leq \delta\frac{\sqrt{\widetilde{a}}+|\epsilon_2'|}{1-\epsilon}$. Since $|x|^2 + e^{i\theta}|u|^2 = |x|^2 - |u|^2 + 2(\cos\frac{\theta}{2})|u|^2 e^{i\frac{\theta}{2}}$, the first estimate of (5.53) yields

$$|x|^2 - |u|^2 = c^{-1} + \delta_5, \qquad |\delta_5| \leq \delta + 2\delta\frac{\sqrt{\widetilde{a}}+|\epsilon_2'|}{1-\epsilon}. \tag{5.56}$$

Next, (5.14) for $\widetilde{d} = 1$ yields $|\frac{x}{y}|, |\frac{u}{v}| \leq \frac{|\sqrt{\widetilde{a}}|+\max\{|\epsilon_2'|,|\epsilon_2''|\}}{(1-\epsilon)}$. From the first estimate of (5.53) (or (5.54)) we deduce either $|x|^2 \geq \frac{1-\delta}{2}$ or $|u|^2 \geq \frac{1-\delta}{2}$, and the second estimate of (5.53) (or (5.54)) gives $|y|, |v| \geq \frac{(1-\epsilon)(1-\delta)}{2(|\sqrt{\widetilde{a}}|+\max\{|\epsilon_2'|,|\epsilon_2''|\})} - \sqrt{\delta}$. To conclude we use the (5.8) with (5.53), (5.54) and (5.56) to obtain an inequality that fails for small ϵ, δ:

$$\begin{aligned}\delta &\geq \left|\overline{x}y + (\cos\theta)\overline{u}v\right| \geq \left||\overline{x}y| - |\overline{u}v|\right| - |uv||1-\cos\theta| \\ &\geq \frac{1-|\delta_5|}{2\frac{|u|}{|v|} + \frac{\sqrt{1+|\delta_5|}}{|y|}} - \left(\frac{|u|}{|v|} + \frac{\sqrt{1+|\delta_5|}}{|v|}\right)\delta - 2\delta.\end{aligned}$$

(iii) $B = aI_2, a > 0$ (hence $A = 1\oplus\sigma, \sigma = e^{i\theta} \in \{1,-1\}$)
The first equation of (5.6) for $a = d$ and (5.54) yield

$$\frac{\epsilon+|\widetilde{a}|}{a} \geq |x^2 + u^2| \geq \left||x|^2 - |u|^2\right|. \tag{5.57}$$

If $\sigma = 1$, then the last equation of (5.6) for $a = d$ and the last estimates in (5.55) imply $\frac{1-\epsilon}{a} \leq y^2 + v^2 \leq 2\delta$. Hence (5.57) gives $\left||x|^2 - |u|^2\right| \leq \delta_0 := \frac{2\delta(|\widetilde{a}|+\epsilon)}{1-\epsilon}$. The first equation of (5.6) further yields that $|x|, |u| \geq \frac{1-\delta}{2} - \delta_0$ with $\frac{|v|}{|u|}, \frac{|y|}{|x|} \leq \frac{2\delta}{\frac{1-\delta}{2}-2\delta_0}$. If $\widetilde{B} = \widetilde{a}\oplus 1$, we proceed mutatis mutandis as in Case XII (b) to get a contradiction. Let $\sigma = -1$. By Lemma 4.1 (1), (2) we have $a\frac{\delta\sqrt{6}}{v} \geq a|\det P| = |\sqrt{\widetilde{a}} + \delta'|$ with $\delta' \leq \epsilon\frac{4\widetilde{a}+2}{\widetilde{a}}$ if $\widetilde{a} \neq 0$ (or $\delta' \leq \epsilon\sqrt{4\widetilde{a}+2}$ if $\widetilde{a} = 0$). If $\widetilde{a} \neq 0$, we combine it with the first equality of (5.54) and (5.57), to obtain $\frac{\delta\sqrt{6}(\epsilon+|\widetilde{a}|)}{v(|\sqrt{\widetilde{a}}|-|\delta'|)} \geq \left||x|^2 - |u|^2\right| \geq (1-\delta)$, which fails for small ϵ, δ. Next, if $\widetilde{a} = 0$ then (5.57) and (5.54) imply $a \leq \frac{\epsilon}{1-\delta}$. Using the second equation of (5.9) and (5.10) we deduce $\frac{u}{v} = \frac{ae^{2i\varphi}\delta_2 - \epsilon_4}{ae^{2i\varphi}\delta_4 - 1 - \epsilon_4}$, while the last equation of (5.6) for $a = d, \widetilde{d} = 1$ and the second inequality of (5.54) give $2|v^2| \geq \frac{1-\epsilon}{2a} - \delta$. Applying this and (5.54) to (5.8) leads to an inequality that fails for small ϵ, δ.

Case XV. $(1\oplus 0, \widetilde{B}) \dashrightarrow \left(\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, B\right)$,

From Lemma 3.2 (2) for (C6) with $\alpha = 1, c^{-1} = e^{i\Gamma}$ we deduce

$$|\overline{x}v + \overline{u}y| \leq \delta, \quad |v|^2, |\overline{u}v| \leq \delta, \quad \left|2\mathrm{Re}(\overline{y}v)\right| \leq \delta, \quad \left|2\mathrm{Re}(\overline{x}u) + i|u|^2 - e^{i\Gamma}\right| \leq \delta. \tag{5.58}$$

(a) $B=\begin{bmatrix}0 & b\\ b & 0\end{bmatrix}$, $b>0$
If $\widetilde{B}=\widetilde{a}\oplus 1$ we again have (5.14) for $\widetilde{d}=1$, and by combining it with $|v|^2\leq\delta$ (see (5.58)), we get $|u|\leq\frac{(\sqrt{\widetilde{a}}+|\epsilon_2''|)\sqrt{\delta}}{1-\epsilon}$ with ϵ_2'' is as in (5.14). The last estimate of (5.58) then yields $|2\mathrm{Re}(\overline{x}u)|\geq 1+\delta+\frac{\delta(\sqrt{\widetilde{a}}+|\epsilon_2'|)^2}{(1-\epsilon)^2}$. By applying this, $\frac{\delta\sqrt{6}}{\nu}\geq|\det P|$ (Lemma 4.1 (1)) and the first estimate of (5.58) to (5.25) we get $|\frac{v}{u}|(1-\delta-\frac{\delta(\sqrt{\widetilde{a}}+|\epsilon_2''|)^2}{(1-\epsilon)^2})\leq\delta-\frac{\delta\sqrt{6}}{\nu}$, which contradicts (5.14) for small ϵ, δ.
Taking $B(s)=\frac{1}{s}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $P(s)=\begin{bmatrix}s^2 & s\\ 1 & s\end{bmatrix}$, $c(s)=-i$ gives $\big(1\oplus 0,\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\big)\to\big(\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix},\begin{bmatrix}0 & b\\ b & 0\end{bmatrix}\big)$.

(b) $B=a\oplus d,\quad a\geq 0, d\in\mathbb{C}$
First, $B(s)=\frac{1}{s^2}\oplus\widetilde{a}$, $P(s)=\begin{bmatrix}s^2 & s\\ 1 & s^2\end{bmatrix}$, $c(s)=-i$ give $(1\oplus 0,\widetilde{a}\oplus 1)\to\big(\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix},a\oplus d\big)$, $a>0$.
If $\widetilde{B}=\widetilde{a}\oplus 1$, $\widetilde{a}\geq 0$ and $a=0$, then the last two equations of (5.6) for $\widetilde{d}=1$, $a=\widetilde{b}=0$ give $(1+\epsilon_4)u=\epsilon_2 v$ with $|\frac{u}{v}|\leq\frac{\epsilon}{1-\epsilon}$. Thus $|u|^2=|\frac{u}{v}||uv|\leq\frac{\epsilon^2\delta}{(1-\epsilon)^2}$ and $|2\mathrm{Re}(\overline{u}x)|\geq 1-\delta-\frac{\delta\epsilon^2}{(1-\epsilon)^2}$. By applying this, $\frac{\delta\sqrt{6}}{\nu}\geq|\det P|$ (Lemma 4.1 (1)) and (5.58) to (5.25) we get $|\frac{v}{u}|(1-\delta-\frac{\delta\epsilon^2}{(1-\epsilon)^2})\leq\delta-\frac{\delta\sqrt{6}}{\nu}$; it contradicts $|\frac{u}{v}|\leq\frac{\epsilon}{1-\epsilon}$ for small ϵ, δ.

So far we have proved Theorem 3.4 (1), (5). Furthermore, Theorem 3.4 (3), (4) can be concluded for all cases except maybe for $(0_2,1\oplus 0)$.

Case XVI. $(0_2,1\oplus\sigma)\dashrightarrow(A,B)$

(a) $\sigma=1\quad(\widetilde{B}=I_2)$
We prove $(0_2,I_2)\to\big(A,\begin{bmatrix}a & b\\ b & d\end{bmatrix}\big)$, $b>0$, $A\in\mathbb{C}^{2\times 2}$ by taking $P(s)=\frac{s}{\sqrt{2}}e^{i\frac{\pi}{4}}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}$, $c(s)=1$, $B(s)=\begin{bmatrix}a(s) & s^{-2}\\ s^{-2} & d(s)\end{bmatrix}$, $a(s),d(s)\leq\frac{1}{s}$. Next, $P(s)=\frac{1}{\sqrt{2}}\begin{bmatrix}s & s\\ 1 & -1\end{bmatrix}$, $B(s)=\frac{1}{s^2}\oplus 1$, $c(s)=1$ give $(0_2,I_2)\to(A,a\oplus 1)$ with $a>0$ and either $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ or $A=1\oplus 0$, while $P(s)=\frac{1}{\sqrt{2}}\begin{bmatrix}s & s\\ s & -s\end{bmatrix}$, $B(s)=\frac{1}{s^2}\oplus(\frac{1}{s^2}+\frac{d-a}{s})$, $c(s)=1$ yield $(0_2,I_2)\to(1\oplus\sigma,a\oplus d)$, $d\geq a>0$.

(b) $\sigma=0\quad(\widetilde{B}=1\oplus 0)$
To prove $(0_2,1\oplus 0)\to(1\oplus 0,a\oplus 0)$ for $a>0$ we take $B(s)=\frac{1}{s^2}\oplus 0$, $P(s)=sI_2$, $c(s)=1$ in (5.2). From what we proved so far this implies $(0_2,1\oplus 0)\to(A,B)$ for all $B\neq 0_2$.

This completes the proof of the theorem. □

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References

1. Arnold, V.I.: Geometric Methods in the Theory or Ordinary Differential Equations. Springer, New York (NY) (1988)
2. Bedford, E., Klingenberg, W.: On the envelope of holomorphy of a 2-sphere in $\mathbb{C}^2$. J. Amer. Math. Soc. **4**(3), 623–646 (1991)
3. Bishop, E.: Differential manifolds in complex Euclidean space. Duke Math. J. **32**, 1–22 (1965)
4. Bochnak, J., Coste, M., Roy, M.F.: Real Algebraic Geometry. Springer-Verlag, Berlin (1988)
5. Boothby, W.M.: An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic Press, New York (NY) (1975)
6. Burcea, V.: A normal form for a real 2-codimensional submanifold in $\mathbb{C}^{N+1}$ near a CR singularity. Adv. Math. **243**, 262–295 (2013)
7. Coffman, A.: CR singularities of real fourfolds in $\mathbb{C}^3$. Illinois J. Math. **53**(3), 939–981 (2009)
8. Dmytryshyn, A.R., Futorny, V., Kågström, B., Klimenko, L., Sergeichuk, V.V.: Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence. Linear Algebra Appl. **469**, 305–334 (2015)
9. Dolbeault, P., Tomassini, G., Zaitsev, D.: On boundaries of Levi-flat hypersurfaces in $\mathbb{C}^n$. C. R. Math. Acad. Sci. Paris **341**(6), 343–348 (2005)
10. Edelman, E., Elmroth, E., Kågström, B.: A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification enhanced staircase algorithm. SIAM J. Matrix Anal. Appl. **20**, 667–669 (1999)
11. Elmroth, E., Johansson, P., Kågström, B.: Computation and presentation of graph displaying closure hierarchies of Jordan and Kronecker structures. Numer. Linear Algebra Appl. **8**, 381–399 (2001)
12. Fang, H., Huang, X.: Flattening a non-degenerate CR singular point of real codimension two. Geom. Funct. Anal. **28**, 289–333 (2018)
13. Forstnerič, F.: Complex tangents of real surfaces in complex surfaces. Duke Math. J. **67**(2), 353–367 (1992)
14. Futorny, V., Klimenko, L., Sergeichuk, V.V.: Change of the $*$-congruence canonical form of 2-by-2 matrices under perturbations. Electron. J. Linear Algebra **27**, 146–154 (2014)
15. Gong, X., Stolovitch, L.: Real submanifolds of maximum complex tangent space at a CR singular point I. Invent. Math. **206**(1), 293–377 (2016)
16. Huang, X., Yin, W.: A Bishop surface with a vanishing Bishop invariant. Invent. Math. **176**(3), 461–520 (2009)
17. Huang, X., Krantz, S.: On a problem of Moser. Duke Math. J. **78**(1), 213–228 (1995)
18. Kenig, C.E., Webster, S.M.: The local hull of holomorphy of a surface in the space of two complex variables. Invent. Math. **67**(1), 1–21 (1982)
19. Moser, J.K., Webster, S.M.: Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations. Acta Math. **150**(3–4), 255–296 (1983)
20. Slapar, M.: Modeling complex points up to isotopy. J. Geom. Anal. **23**, 1932–1943 (2013)
21. Slapar, M.: On complex points of codimension 2 submanifolds. J. Geom. Anal. **26**(1), 206–219 (2016)
22. Slapar, M., Starčič, T.: On normal forms of complex points of codimension 2 submanifolds. J. Math. Anal. Appl. **461**(2), 1308–1326 (2018)
23. Starčič, T.: On normal forms of complex points of small $\mathcal{C}^2$-perturbations of real 4-manifolds embedded in a complex 3-manifold. Complex Var. Elliptic Equ. **66**(3), 376–436 (2021)