ANALYSIS & PDE

Volume 17 No. 3 2024

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ANALYSIS AND PDE
Vol. 17 (2024), No. 3, pp. 981–1003
DOI: 10.2140/apde.2024.17.981

SCHWARZ–PICK LEMMA FOR HARMONIC MAPS WHICH ARE CONFORMAL AT A POINT

FRANC FORSTNERIČ AND DAVID KALAJ

We obtain a sharp estimate on the norm of the differential of a harmonic map from the unit disc $\mathbb{D}$ in $\mathbb{C}$ into the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, $n \geq 2$, at any point where the map is conformal. For $n = 2$ this generalizes the classical Schwarz–Pick lemma, and for $n \geq 3$ it gives the optimal Schwarz–Pick lemma for conformal minimal discs $\mathbb{D} \to \mathbb{B}^n$. This implies that conformal harmonic maps $M \to \mathbb{B}^n$ from any hyperbolic conformal surface are distance decreasing in the Poincaré metric on M and the Cayley–Klein metric on the ball $\mathbb{B}^n$, and the extremal maps are the conformal embeddings of the disc $\mathbb{D}$ onto affine discs in $\mathbb{B}^n$. Motivated by these results, we introduce an intrinsic pseudometric on any Riemannian manifold of dimension at least three by using conformal minimal discs, and we lay foundations of the corresponding hyperbolicity theory.

1. Introduction

In this paper, we establish precise estimates of derivatives and the rate of growth of conformal harmonic maps from hyperbolic conformal surfaces into the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ for any $n \geq 3$; see Theorem 2.6. Such maps parametrize minimal surfaces, objects of high interest in geometry. To motivate the discussion, we begin with the following special case of one of our main results, Theorem 2.1. This generalizes the classical Schwarz–Pick lemma, due to H. A. Schwarz [1890, Band II, p. 108], H. Poincaré [1884], C. Carathéodory [1912], and G. A. Pick [1915], to a substantially larger class of maps.

Theorem 1.1. *Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc. If $f : \mathbb{D} \to \mathbb{D}$ is a harmonic map which is conformal at a point $z \in \mathbb{D}$, then at this point we have*

$$\|df_z\| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \tag{1-1}$$

with equality if and only if f is a conformal diffeomorphism of the disc $\mathbb{D}$.

The classical Schwarz–Pick lemma gives the same conclusion under the much stronger hypothesis that the map f is holomorphic or antiholomorphic, which means that it is conformal at every noncritical point; see, e.g., [Dineen 1989; Kobayashi 2005; Royden 1988]. This fundamental rigidity result in complex analysis leads to the notion of Kobayashi hyperbolic manifolds [1967; 1976; 2005] and provides a connection to complex differential geometry via the Ahlfors lemma (see [Ahlfors 1938; Kobayashi 2005, Theorem 2.1; Royden 1988]) and its generalizations by S.-T. Yau [1978] and others.

MSC2020: primary 53A10; secondary 30C80, 31A05, 32Q45.
Keywords: harmonic map, minimal surface, Schwarz–Pick lemma, Cayley–Klein metric.

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The conditions in Theorem 1.1 are invariant under precompositions by holomorphic automorphisms of $\mathbb{D}$, so the proof reduces to the case $z = 0$. On the other hand, postcompositions of harmonic maps into $\mathbb{D}$ by holomorphic automorphisms of $\mathbb{D}$ need not be harmonic, so we cannot exchange $f(0)$ and 0. Hence, the standard proof of the classical Schwarz–Pick lemma breaks down. The estimate (1-1) fails for some nonconformal harmonic diffeomorphisms of $\mathbb{D}$ (see Example 4.1), as well as for harmonic maps $\mathbb{D} \to D$ to more general domains which are conformal at a point (see Example 4.2 and Problem 4.3).

Our main results are precise estimates of the differential and the rate of growth of conformal harmonic maps $M \to \mathbb{B}^n$ from an open conformal surface M to the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ for any $n \geq 3$. It is classical that such maps parametrize minimal surfaces. Indeed, a smooth conformal map $f : M \to \mathbb{R}^n$ from an open conformal surface M into $\mathbb{R}^n$ with the Euclidean metric parametrizes a minimal surface in $\mathbb{R}^n$ if and only if f is a harmonic map; see [Alarcón et al. 2021, Chapter 2; Duren 2004; Osserman 1969], among many other sources. Note that an oriented conformal surface is a Riemann surface.

The focal point of the paper is Theorem 2.1, which gives a precise upper bound on the norm $\|df_z\|$ of the differential df_z of a harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ at any point $z \in \mathbb{D}$ where the map is conformal. The estimate is similar to the one in Theorem 1.1, except that, for $n \geq 3$, it also involves the angle θ between the position vector $f(z) \in \mathbb{B}^n$ and the 2-plane $df_z(\mathbb{R}^2) \subset \mathbb{R}^n$. A related result (see Theorem 2.2) shows that the worst case estimate, which occurs for $\theta = \frac{\pi}{2}$ (i.e., when the vector $f(z)$ is orthogonal to the plane $df_z(\mathbb{R}^2)$), holds for all harmonic maps $f : \mathbb{D} \to \mathbb{B}^n$ provided that $\|df_z\|$ is replaced by $\sqrt{2}^{-1}|\nabla f(z)|$; these quantities coincide if f is conformal at z.

We then give a differential geometric formulation and an extension of Theorem 2.1. Let $\mathcal{CK}$ denote the Cayley–Klein metric on the ball $\mathbb{B}^n$ ($n \geq 2$), also called the Beltrami–Klein metric; see (2-6) and the footnote on page 985. This metric is one of the classical models of hyperbolic geometry. It coincides with the restriction of the Kobayashi metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ (2-5) (which is the same as $1/\sqrt{n+1}$ times the Bergman metric on $\mathbb{B}^n_{\mathbb{C}}$) to points of the real ball $\mathbb{B}^n$ and real tangent vectors. Theorem 2.1 implies that any conformal harmonic map $f : M \to \mathbb{B}^n$, $n \geq 3$, from a hyperbolic conformal surface is metric and distance decreasing in the Poincaré metric on M and the Cayley–Klein metric on $\mathbb{B}^n$; see Theorem 2.6. Furthermore, if the differential df_p has the operator norm equal to 1 at some point $p \in M$ in this pair of metrics, or if f preserves the distance between a pair of distinct points in M, then M is necessarily the disc $\mathbb{D}$ and f is a conformal diffeomorphism of $\mathbb{D}$ onto a proper affine disc in $\mathbb{B}^n$. In particular, a conformal harmonic disc $f : \mathbb{D} \to \mathbb{B}^n$ with $f(0) = 0$ satisfies $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ (see Corollary 2.7).

In Section 2 we give precise statements of the mentioned results. Theorem 2.1 is proved in Section 3. We introduce a new idea into the subject, connecting it to Lempert's seminal work [1981] on complex geodesics of the Kobayashi metric on bounded convex domains in $\mathbb{C}^n$. Theorem 2.2 is proven in Section 4. In Section 5 we apply Theorem 1.1 to estimate the gradient of a quasiconformal harmonic self-map of the disc in terms of its second Beltrami coefficient at the reference point; see Theorem 5.1.

Motivated by these result, we introduce in Section 6 an intrinsic pseudometric on any domain in $\mathbb{R}^n$, $n \geq 3$ (and more generally on any Riemannian manifold of dimension at least three) in terms of conformal minimal discs, in analogy to Kobayashi's definition of his pseudometric on complex manifolds in terms

of holomorphic discs. This provides the basis for a new hyperbolicity theory of such domains and of Riemannian manifolds.

2. The main results

Given a differentiable map $f:\mathbb{D}\to\mathbb{R}^n$, we denote by f_x and f_y its partial derivatives with respect to x and y, where $z=x+\mathrm{i}y\in\mathbb{D}$. The gradient $\nabla f=(f_x,\,f_y)$ is an $n\times 2$ matrix representing the differential df. The map f is said to be *conformal* at $z\in\mathbb{D}$ if

$$|f_x(z)|=|f_y(z)| \quad\text{and}\quad f_x(z)\cdot f_y(z)=0. \tag{2-1}$$

Here, the dot stands for the Euclidean inner product on $\mathbb{R}^n$, and $|\boldsymbol{x}|$ is the Euclidean norm of $\boldsymbol{x}\in\mathbb{R}^n$. If f is an immersion at z then (2-1) holds if and only if df_z preserves angles. It follows from (2-1) that f has rank zero at any branch point. We denote by $|\nabla f|$ the Euclidean norm of the gradient:

$$|\nabla f(z)|^2=|f_x(z)|^2+|f_y(z)|^2,\quad z\in\mathbb{D}.$$

If f is conformal at z then clearly $\|df_z\|=\sqrt{2}^{-1}|\nabla f(z)|=|f_x(z)|=|f_y(z)|$. The map $f=(f_1,\dots,f_n):\mathbb{D}\to\mathbb{R}^n$ is harmonic if and only if every component f_k is a harmonic function on $\mathbb{D}$, meaning that the Laplacian $\Delta f_k=\partial^2 f_k/\partial x^2+\partial^2 f_k/\partial y^2$ vanishes identically.

We denote by $\mathbb{B}^n$ the unit ball of $\mathbb{R}^n$:

$$\mathbb{B}^n=\left\{\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n:|\boldsymbol{x}|^2=\sum_{k=1}^n x_k^2<1\right\}. \tag{2-2}$$

Our first main result is the following; it is proved in Section 3.

Theorem 2.1. *Let $f:\mathbb{D}\to\mathbb{B}^n$ for $n\ge 2$ be a harmonic map. If f is conformal at a point $z\in\mathbb{D}$ and $\theta\in\left[0,\frac{\pi}{2}\right]$ denotes the angle between the vector $f(z)$ and the plane $\Lambda=df_z(\mathbb{R}^2)\subset\mathbb{R}^n$, then*

$$\|df_z\|=\frac{1}{\sqrt{2}}|\nabla f(z)|\le\frac{1-|f(z)|^2}{1-|z|^2}\frac{1}{\sqrt{1-|f(z)|^2\sin^2\theta}}, \tag{2-3}$$

with equality if and only if f is a conformal diffeomorphism of $\mathbb{D}$ onto the affine disc $\Sigma=(f(z)+\Lambda)\cap\mathbb{B}^n$. (When $f(z)=0$ or $df_z=0$, the angle θ does not matter.)

Note that the number $R=\sqrt{1-|f(z)|^2\sin^2\theta}$ is the radius of the affine disc Σ. In dimension $n=2$ we have $\theta=0$, so Theorem 1.1 is a special case of Theorem 2.1. Without assuming that f is conformal at z or that $f(z)=0$, inequality (2-3) fails for some harmonic diffeomorphisms of the disc as shown by Example 4.1.

For a fixed value of $|f(z)|\in[0,1)$, the maximum of the right-hand side of (2-3) over angles $\theta\in\left[0,\frac{\pi}{2}\right]$ equals $\sqrt{1-|f(z)|^2}/(1-|z|^2)$ and is attained precisely at $\theta=\frac{\pi}{2}$, i.e., when the vector $f(z)$ is orthogonal to $\Lambda=df_z(\mathbb{R}^2)$, unless $f(z)=0$ when it is independent of θ. It turns out that this weaker estimate holds for all harmonic maps $\mathbb{D}\to\mathbb{B}^n$ without any conformality assumption. The following result is proved in Section 4.

Theorem 2.2. *For every harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ we have that*

$$\frac{1}{\sqrt{2}}|\nabla f(z)| \leq \frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{2-4}$$

Equality holds for some $z \in \mathbb{D}$ if $f(z)$ is orthogonal to the two-plane $\Lambda = df_z(\mathbb{R}^2)$ and f is a conformal diffeomorphism onto the affine disc $(f(z)+\Lambda) \cap \mathbb{B}^n$. In particular, if $f(z) = 0$ then

$$|\nabla f(z)| \leq \frac{\sqrt{2}}{1-|z|^2},$$

with equality if and only if f is a conformal diffeomorphism onto the linear disc $\Lambda \cap \mathbb{B}^n$.

The proof of estimate (2-4) relies on Parseval's inequality, using the hypothesis that the L^1-norm of $|f|^2 = \sum_{k=1}^n f_k^2$ on the circles $\{|z| = r\}$ for $0 < r < 1$ is bounded by 1. We find it surprising that this simple approach gives an optimal estimate in certain cases indicated in the theorem. Except in these cases, we do not know whether there exist harmonic maps $\mathbb{D} \to \mathbb{B}^n$ reaching (near) equality in (2-4).

The precise upper bound on the size of the gradient $\|df_0\|$ of a nonconformal harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ with a given center $f(0) = \boldsymbol{x} \in \mathbb{B}^n \setminus \{0\}$ for $n \geq 2$ in terms of the distortion of f at 0 is unknown; see [Brevig et al. 2021; Kovalev and Yang 2020] for $n = 2$. On the other hand, for $n = 1$ the *harmonic Schwarz lemma* (see [Axler et al. 2001, Theorem 6.26]) says that any harmonic function $f : \mathbb{B}^m \to (-1, +1)$ for $m \geq 2$ satisfies the sharp estimate

$$|\nabla f(0)| \leq \frac{2\,\mathrm{Vol}(\mathbb{B}^{m-1})}{\mathrm{Vol}(\mathbb{B}^m)}.$$

For $m = 2$ the inequality reads $|\nabla f(0)| \leq \frac{4}{\pi}$, and a simple proof in this case was given by Kalaj and Vuorinen [2012, Theorem 1.8].

Let us mention a consequence of Theorem 2.1 related to the Schwarz lemma for holomorphic discs in the ball of the complex Euclidean space,

$$\mathbb{B}^n_{\mathbb{C}} = \left\{ \boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |\boldsymbol{z}|^2 = \sum_{k=1}^n |z_k|^2 < 1 \right\} \tag{2-5}$$

(see [Rudin 1980, Section 8.1]). The following corollary to Theorem 2.1 shows that the extremal holomorphic discs in $\mathbb{B}^n_{\mathbb{C}}$ are precisely those extremal orientation-preserving conformal harmonic discs $\mathbb{D} \to \mathbb{B}^n_{\mathbb{C}}$ which parametrize affine complex discs.

Corollary 2.3. *Let $f : \mathbb{D} \to \mathbb{B}^n_{\mathbb{C}}$ be a harmonic map which is conformal at a point $z \in \mathbb{D}$. If $\Lambda = df_z(\mathbb{R}^2)$ is a complex line in $\mathbb{C}^n$, then equality holds in (2-3) for this z if and only if f is a biholomorphic or antibiholomorphic map onto the affine complex disc $(f(z)+\Lambda) \cap \mathbb{B}^n_{\mathbb{C}}$.*

The Cayley–Klein metric. A differential geometric interpretation of the classical Schwarz–Pick lemma is that holomorphic maps $\mathbb{D} \to \mathbb{D}$ are distance decreasing in the Poincaré metric on $\mathbb{D}$, and isometries coincide with holomorphic and antiholomorphic automorphisms of $\mathbb{D}$ (see [Kobayashi 2005]). The analogous conclusion holds for holomorphic maps $\mathbb{D} \to \mathbb{B}^n_{\mathbb{C}}$ with the Kobayashi metric on the complex

ball $\mathbb{B}^n_{\mathbb{C}}$ (2-5), where orientation-preserving isometric embeddings are precisely holomorphic embeddings onto affine complex discs in $\mathbb{B}^n_{\mathbb{C}}$.

In the same spirit, we shall now interpret Theorem 2.1 as the distance-decreasing property of conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ with respect to the *Cayley–Klein metric*[1] on $\mathbb{B}^n$:

$$\mathcal{CK}(\boldsymbol{x}, \boldsymbol{v}) = \frac{\sqrt{1-|\boldsymbol{x}|^2 \sin^2 \phi}}{1-|\boldsymbol{x}|^2}|\boldsymbol{v}|, \quad \boldsymbol{x} \in \mathbb{B}^n,\ \boldsymbol{v} \in \mathbb{R}^n, \tag{2-6}$$

where $\phi \in \left[0, \frac{\pi}{2}\right]$ is the angle between the vector $\boldsymbol{x}$ and the line $\mathbb{R}\boldsymbol{v}$. Equivalently,

$$\mathcal{CK}(\boldsymbol{x}, \boldsymbol{v})^2 = \frac{(1-|\boldsymbol{x}|^2)|\boldsymbol{v}|^2 + |\boldsymbol{x}\cdot\boldsymbol{v}|^2}{(1-|\boldsymbol{x}|^2)^2} = \frac{|\boldsymbol{v}|^2}{1-|\boldsymbol{x}|^2} + \frac{|\boldsymbol{x}\cdot\boldsymbol{v}|^2}{(1-|\boldsymbol{x}|^2)^2}. \tag{2-7}$$

Let $G_2(\mathbb{R}^n)$ denote the Grassmann manifold of two-planes in $\mathbb{R}^n$. We define a Finsler pseudometric $\mathcal{M} : \mathbb{B}^n \times G_2(\mathbb{R}^n) \to \mathbb{R}_+$ by

$$\mathcal{M}(\boldsymbol{x}, \Lambda) = \frac{\sqrt{1-|\boldsymbol{x}|^2 \sin^2 \theta}}{1-|\boldsymbol{x}|^2}, \quad \boldsymbol{x} \in \mathbb{B}^n,\ \Lambda \in G_2(\mathbb{R}^n), \tag{2-8}$$

where $\theta \in \left[0, \frac{\pi}{2}\right]$ is the angle between $\boldsymbol{x}$ and Λ. At $\boldsymbol{x} = \boldsymbol{0}$ we have $\mathcal{M}(\boldsymbol{0}, \Lambda) = 1$ for all $\Lambda \in G_2(\mathbb{R}^n)$. Assume now that $\boldsymbol{x} \neq \boldsymbol{0}$. Let $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ be a vector having angle $\phi \in \left[0, \frac{\pi}{2}\right]$ with the line $\mathbb{R}\boldsymbol{x}$. The angle θ between $\boldsymbol{x}$ and any 2-plane Λ containing $\boldsymbol{v}$ satisfies $0 \le \theta \le \phi$, and the maximum of θ over all such Λ equals ϕ. Hence, (2-6) gives

$$\frac{\mathcal{CK}(\boldsymbol{x}, \boldsymbol{v})}{|\boldsymbol{v}|} = \min\{\mathcal{M}(\boldsymbol{x}, \Lambda) : \Lambda \in G_2(\mathbb{R}^n),\ \boldsymbol{v} \in \Lambda\}, \tag{2-9}$$

$$\mathcal{M}(\boldsymbol{x}, \Lambda) = \max\left\{\frac{\mathcal{CK}(\boldsymbol{x}, \boldsymbol{v})}{|\boldsymbol{v}|} : \boldsymbol{v} \in \Lambda\right\}. \tag{2-10}$$

Inequality (2-3) in Theorem 2.1 is obviously equivalent to

$$\mathcal{M}(f(z), df_z(\mathbb{R}^2))|df_z(\xi)| = \frac{\sqrt{1-|f(z)|^2 \sin^2 \theta}}{1-|f(z)|^2}|df_z(\xi)| \le \frac{|\xi|}{1-|z|^2}, \tag{2-11}$$

where $\theta \in \left[0, \frac{\pi}{2}\right]$ is the angle between $f(z)$ and the 2-plane $\Lambda = df_z(\mathbb{R}^2)$. By (2-9) the left-hand side of (2-11) is bigger than or equal to $\mathcal{CK}(f(z), df_z(\xi))$. Equality holds if and only if the angle ϕ between the line $f(z)\mathbb{R}$ and the vector $df_z(\xi) \in \Lambda$ equals θ; clearly this holds if and only if $df_z(\xi)$ is tangent to the diameter of the affine disc $\Sigma = (f(z) + \Lambda) \cap \mathbb{B}^n$ through the point $f(z)$. This and the addition concerning equality in (2-3) give the following corollary to Theorem 2.1. Note that $\mathcal{P}_{\mathbb{D}}(z, \xi) := |\xi|/(1-|z|^2)$ is the Poincaré metric on the disc.

[1]The Beltrami–Calvin–Klein model of hyperbolic geometry was introduced by Arthur Cayley [1859] and Eugenio Beltrami [1868], and it was developed by Felix Klein [1871; 1873]. The underlying space is the n-dimensional unit ball, geodesics are straight line segments with ideal endpoints on the boundary sphere, and the distance between points on a geodesic is given by a cross ratio. This is a special case of the Hilbert metric on convex domains in $\mathbb{R}^n$ and $\mathbb{RP}^n$, introduced by David Hilbert [1895]. These are examples of projectively invariant metrics discussed by many authors; see the surveys by S. Kobayashi [1977; 1984], W. M. Goldman [2019], and J. G. Ratcliffe [1994].

Corollary 2.4. *If $f : \mathbb{D} \to \mathbb{B}^n$ is a conformal harmonic map then for every point $z \in \mathbb{D}$ and tangent vector $\xi \in \mathbb{R}^2$ we have*

$$\mathcal{CK}(f(z), df_z(\xi)) \le \frac{|\xi|}{1-|z|^2} = \mathcal{P}_{\mathbb{D}}(z, \xi). \tag{2-12}$$

Equality holds for some $z \in \mathbb{D}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$ if and only if f is a conformal diffeomorphism onto the affine disc

$$\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$$

and the vector $df_z(\xi)$ is tangent to the diameter of Σ through the point $f(z)$.

This shows in particular that every linear conformal embedding $f : \mathbb{D} \to \Sigma$ onto a proper affine disc in $\mathbb{B}^n$ is geodesic on each diameter $(-1, +1) \ni r \mapsto f(re^{it}) \in \Sigma$ for every fixed $t \in \mathbb{R}$. However, distances between points of different rays strictly decrease from the Poincaré metric on $\mathbb{D}$ to the Cayley–Klein metric on the disc $\Sigma \subset \mathbb{B}^n$.

Remark 2.5. The Cayley–Klein metric (2-7) is the restriction of the Kobayashi metric on the unit ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points $\boldsymbol{x} \in \mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ of the real ball and tangent vectors in $T_{\boldsymbol{x}}\mathbb{R}^n \cong \mathbb{R}^n$. A direct geometric argument was given by Lempert [1993, proof of Theorem 3.1]. The Cayley–Klein metric also equals $1/\sqrt{n+1}$ times the Bergman metric on $\mathbb{B}^n_{\mathbb{C}}$ restricted to $\mathbb{B}^n$ and real tangent vectors; see [Krantz 1992, Proposition 1.4.22]. (On the ball of $\mathbb{C}^n$, most holomorphically invariant metrics coincide up to scalar factors.) The Cayley–Klein metric equals the Poincaré metric $|\boldsymbol{v}|/(1-|\boldsymbol{x}|^2)$ on $\mathbb{B}^n$ on vectors $\boldsymbol{v}$ parallel to the base point $\boldsymbol{x} \in \mathbb{B}^n$, but is strictly smaller on other vectors. While the Poincaré metric on $\mathbb{B}^n$ is conformally equivalent to the Euclidean metric, the Cayley–Klein metric is not.

We now extend Corollary 2.4 to more general minimal surfaces. A *conformal surface* is a topological surface M together with a *conformal atlas*, i.e., an atlas whose transition maps between charts are conformal diffeomorphisms between plane domains. Every surface admits a conformal structure. Indeed, every topological surface admits a smoothing, and a conformal structure on a smooth surface is determined by the choice of a Riemannian metric in view of the existence of local isothermal coordinates (see [Osserman 1969] or [Alarcón et al. 2021, Theorem 1.8.6]). Oriented conformal surfaces are Riemann surfaces. There is a well-defined notion of a harmonic function on a conformal surface. Indeed, a Riemannian metric g defines the metric Laplacian Δ_g and hence g-harmonic functions satisfying $\Delta_g h = 0$. The Laplacians associated to any two Riemannian metrics in the same conformal class on a surface differ by a positive multiplicative function (see [Alarcón et al. 2021, Corollary 1.8.2]), and hence the notion of a harmonic function is independent of the choice of metric in a given conformal class.

A conformal surface M is said to be *hyperbolic* if its universal conformal covering space is the disc $\mathbb{D}$. Let $h : \mathbb{D} \to M$ be a universal conformal covering map. Since conformal automorphisms of $\mathbb{D}$ are isometries of the Poincaré metric $\mathcal{P}_{\mathbb{D}} = |dz|/(1-|z|^2)$, there is a unique Riemannian metric $\mathcal{P}_M$ on M (a Kähler metric if M is a Riemann surface) such that h is a local isometry. This Poincaré metric $\mathcal{P}_M$ is a complete metric of constant Gaussian curvature -4 (see [Kobayashi 2005, p. 48, Example 2]), which agrees with the Kobayashi metric if M is a Riemann surface. This leads to the following generalization of Corollary 2.4.

Theorem 2.6 (metric and distance decreasing property of conformal harmonic maps). *Let M be a connected hyperbolic conformal surface endowed with the Poincaré metric $\mathcal{P}_M$. Every conformal harmonic map $f : M \to \mathbb{B}^n$ $(n \geq 3)$ satisfies the estimate*

$$\mathcal{CK}(f(p), df_p(\xi)) \leq \mathcal{P}_M(p, \xi), \quad p \in M,\ \xi \in T_pM. \tag{2-13}$$

If equality holds in (2-13) for some point $p \in M$ and vector $0 \neq \xi \in T_pM$, or if f preserves the distance on a pair of distinct points in M, then $M = \mathbb{D}$ and f is a conformal diffeomorphism onto an affine disc in $\mathbb{B}^n$.

Proof. Assume first that M is orientable and hence a Riemann surface. Choose a holomorphic covering map $h : \mathbb{D} \to M$ and a point $z \in \mathbb{D}$ with $h(z) = p$. The conformal harmonic map $\tilde{f} = f \circ h : \mathbb{D} \to \mathbb{B}^n$ then satisfies $\tilde{f}(z) = f(p)$ and $d\tilde{f}_z = df_p \circ dh_z$. Let $\eta \in \mathbb{R}^2$ be such that $dh_z(\eta) = \xi$. Then $\mathcal{P}_M(p, \xi) = \mathcal{P}_{\mathbb{D}}(z, \eta)$ by the definition of the metric $\mathcal{P}_M$, and $d\tilde{f}_z(\eta) = df_p(\xi)$. From (2-12) it follows that

$$\mathcal{CK}(f(p), df_p(\xi)) = \mathcal{CK}(\tilde{f}(z), d\tilde{f}_z(\eta)) \leq \frac{|d\tilde{f}_z(\eta)|}{1 - |\tilde{f}(z)|^2} = \frac{|df_p(\xi)|}{1 - |f(p)|^2},$$

which gives (2-13). If $\xi \neq 0$ and equality holds, then by Corollary 2.4 the map $\tilde{f} = f \circ h : \mathbb{D} \to \mathbb{B}^n$ is a conformal diffeomorphism onto an affine disc in $\mathbb{B}^n$, and hence $h : \mathbb{D} \to M$ is a biholomorphism.

For a nonorientable hyperbolic conformal surface M we obtain the same conclusion by passing to its orientable two-sheeted conformal cover. The statement concerning distances is an immediate consequence. Note that if the distances agree for a pair of distinct points in M and their images in $\mathbb{B}^n$, then the differential df_p has operator norm 1 at some point $p \in M$ in the given pair of metrics. □

On the disc with the Poincaré metric $\mathcal{P}_{\mathbb{D}} = |dz|/(1 - |z|^2)$, the Poincaré distance equals

$$\operatorname{dist}_{\mathcal{P}}(z, w) = \frac{1}{2} \log\left(\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|}\right), \quad z, w \in \mathbb{D}. \tag{2-14}$$

The Cayley–Klein distance function on the ball $\mathbb{B}^n$ coincides up to a scalar factor $\sqrt{n+1}$ with the restriction to $\mathbb{B}^n$ of the Bergman distance function on the complex ball $\mathbb{B}^n_{\mathbb{C}}$ or, equivalently, with the restriction to $\mathbb{B}^n$ of the Kobayashi distance function on $\mathbb{B}^n_{\mathbb{C}}$. The following explicit formula for the Kobayashi distance between a pair of points $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n_{\mathbb{C}}$ can be found in [Krantz 1992, p. 437]; here, $\boldsymbol{z} \cdot \overline{\boldsymbol{w}} = \sum_{k=1}^n z_k \overline{w}_k$:

$$\operatorname{dist}(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{2} \log\left(\frac{|1 - \boldsymbol{z} \cdot \overline{\boldsymbol{w}}| + \sqrt{|\boldsymbol{z} - \boldsymbol{w}|^2 + |\boldsymbol{z} \cdot \overline{\boldsymbol{w}}|^2 - |\boldsymbol{z}|^2|\boldsymbol{w}|^2}}{|1 - \boldsymbol{z} \cdot \overline{\boldsymbol{w}}| - \sqrt{|\boldsymbol{z} - \boldsymbol{w}|^2 + |\boldsymbol{z} \cdot \overline{\boldsymbol{w}}|^2 - |\boldsymbol{z}|^2|\boldsymbol{w}|^2}}\right). \tag{2-15}$$

As said before, the same formula applied to points in $\mathbb{B}^n$ gives the Cayley–Klein distance. Taking $w = 0$ and $\boldsymbol{w} = 0$ in the above formulas, we obtain

$$\operatorname{dist}_{\mathcal{P}}(z, 0) = \frac{1}{2} \log\left(\frac{1 + |z|}{1 - |z|}\right) \ (z \in \mathbb{D}), \quad \operatorname{dist}(\boldsymbol{z}, \boldsymbol{0}) = \frac{1}{2} \log\left(\frac{1 + |\boldsymbol{z}|}{1 - |\boldsymbol{z}|}\right) \ (\boldsymbol{z} \in \mathbb{B}^n).$$

Together with Theorem 2.6 this implies the following corollary.

Corollary 2.7. *If $f : \mathbb{D} \to \mathbb{B}^n$, $n \geq 3$, is a conformal harmonic map with $f(0) = \mathbf{0}$, then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Equality at one point $z \in \mathbb{D} \setminus \{0\}$ implies that f is a conformal parametrization of a linear disc obtained by intersecting $\mathbb{B}^n$ with a plane through the origin, and hence equality holds at all points.*

3. Proof of Theorem 2.1

It suffices to prove Theorem 2.1 for $z = 0$. Indeed, with f and z as in the theorem, let $\phi_z \in \mathrm{Aut}(\mathbb{D})$ be such that $\phi_z(0) = z$. The harmonic map $g = f \circ \phi_z : \mathbb{D} \to \mathbb{B}^n$ is then conformal at the origin. Since $|\phi_z'(0)| = 1 - |z|^2$, inequality (2-3) follows from the same estimate for g at $z = 0$. On the image side, the hypotheses and the statement of the theorem are invariant under postcomposition of maps $\mathbb{D} \to \mathbb{B}^n$ by elements of the orthogonal group O_n.

We begin with an explicit description of conformal parametrizations of proper affine discs in $\mathbb{B}^n$. Fix a point $\boldsymbol{q} \in \mathbb{B}^n$ and a linear two-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\boldsymbol{q} + \Lambda) \cap \mathbb{B}^n$. Let us identify conformal parametrizations $\mathbb{D} \to \Sigma$ sending 0 to $\boldsymbol{q}$. Let $\boldsymbol{p} \in \Sigma$ be the closest point to the origin. If $n = 2$ then $\boldsymbol{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n \geq 3$. Up to an orthogonal rotation, we may assume

$$\boldsymbol{p} = (0, 0, p, 0, \ldots, 0) \quad \text{and} \quad \Sigma = \{(x, y, p, 0, \ldots, 0) : x^2 + y^2 < 1 - p^2\}. \tag{3-1}$$

Let $\boldsymbol{q} = (b_1, b_2, p, 0, \ldots, 0) \in \Sigma$, and let θ denote the angle between $\boldsymbol{q}$ and Σ. Set

$$c = \sqrt{1 - p^2} = \sqrt{1 - |\boldsymbol{q}|^2 \sin^2\theta}, \quad a = \frac{b_1 + \mathrm{i}b_2}{c} \in \mathbb{D}, \quad |a| = \frac{|\boldsymbol{q}| \cos\theta}{c}. \tag{3-2}$$

We orient Σ by the tangent vectors ∂_x, ∂_y in the parametrization (3-1). Every orientation-preserving conformal parametrization $f : \mathbb{D} \to \Sigma$ with $f(0) = \boldsymbol{q}$ is then of the form

$$f(z) = \left(c\Re \frac{\mathrm{e}^{\mathrm{i}t} z + a}{1 + \bar{a}\mathrm{e}^{\mathrm{i}t} z}, c\Im \frac{\mathrm{e}^{\mathrm{i}t} z + a}{1 + \bar{a}\mathrm{e}^{\mathrm{i}t} z}, p, 0, \ldots, 0 \right), \quad z \in \mathbb{D}, \tag{3-3}$$

for some $t \in \mathbb{R}$. (Here, $\Re$ and $\Im$ stand for the real and imaginary parts of a complex number. If $n = 2$ then $p = 0$ and $c = 1$, and the same holds if we drop all coordinates except the first two. Orientation-reversing conformal parametrizations are obtained by replacing $z = x + \mathrm{i}y$ with $\bar{z} = x - \mathrm{i}y$. By a rotation in the (x, y)-plane, we may further assume that $b_2 = 0$ and $f(0) = (b_1, 0, p, 0, \ldots, 0)$; in this case $a \in [0, 1)$. By also allowing rotations on the disc $\mathbb{D}$, we can take $t = 0$ in (3-3).) Using the complex coordinate $x + \mathrm{i}y$ in the plane $df_0(\mathbb{R}^2) = \mathbb{R}^2 \times \{0\}^{n-2}$, the map (3-3) can be written in the form

$$f(z) = \left(c \frac{\mathrm{e}^{\mathrm{i}t} z + a}{1 + \bar{a}\mathrm{e}^{\mathrm{i}t} z}, p, 0, \ldots, 0 \right) = (h(z), p, 0, \ldots, 0).$$

From (3-2) it follows that

$$\begin{aligned} |h'(0)| = c(1 - |a|^2) = \frac{c^2 - c^2|a|^2}{c} &= \frac{1 - |\boldsymbol{q}|^2 \sin^2\theta - |\boldsymbol{q}|^2 \cos^2\theta}{c} \\ &= \frac{1 - |\boldsymbol{q}|^2}{\sqrt{1 - |\boldsymbol{q}|^2 \sin^2\theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2\theta}}. \end{aligned}$$

Since $\|df_0\| = |h'(0)|$, this gives equality in (2-3) at $z = 0$.

Theorem 2.1 now follows immediately from the following lemma.

Lemma 3.1. *Let $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ be the disc (3-3). If $g : \mathbb{D} \to \mathbb{B}^n$ is a harmonic disc such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$, then $\|dg_0\| \leq \|df_0\|$, with equality if and only if $g(z) = f(e^{is}z)$ or $g(z) = f(e^{is}\bar{z})$ for some $s \in \mathbb{R}$ and all $z \in \mathbb{D}$.*

The proof of Lemma 3.1 uses ideas from Lempert's seminal paper [1981] concerning complex geodesics of the Kobayashi metric in convex domains in $\mathbb{C}^n$; see Remark 3.2.

Proof. Let p, c and a be as in (3-2) related to the map f in (3-3), where $\boldsymbol{q} = f(0)$. Precomposing f by a rotation in $\mathbb{C}$, we may assume that $t = 0$ in (3-3). For simplicity of notation we assume that $n = 3$; the proof for $n \neq 3$ is exactly the same. If $n = 2$, we delete the remaining components and take $c = 1$.

Consider the holomorphic disc $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times i\mathbb{R}^3$ given by

$$F(z) = \left(c\frac{z+a}{1+\bar{a}z}, -ci\frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}. \tag{3-4}$$

Then, $f = \Re F$. Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is as in the lemma. Up to replacing $g(z)$ by $g(e^{is}z)$ or $g(e^{is}\bar{z})$ for a suitable $s \in \mathbb{R}$, we may assume that

$$dg_0 = r\,df_0 \quad \text{for some } r > 0. \tag{3-5}$$

We must prove that $r \leq 1$, and that $r = 1$ if and only if $g = f$.

Let $G : \mathbb{D} \to \Omega$ be the unique holomorphic map with $\Re G = g$ and $G(0) = F(0)$. In view of the Cauchy–Riemann equations, condition (3-5) implies

$$G'(0) = r\,F'(0), \tag{3-6}$$

where the prime denotes the complex derivative. It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$, and its value at $z = 0$ equals

$$\lim_{z\to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0). \tag{3-7}$$

The bounded harmonic map $g : \mathbb{D} \to \mathbb{B}^3$ has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for its holomorphic extension G; see [Garnett 1981].

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by $\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \sum_{i=1}^n z_i w_i$ for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n$. Note that on vectors in $\mathbb{R}^n$ this is the Euclidean inner product. For each $z = e^{it} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex and f is real-valued, we have

$$\Re\langle F(z) - G(z), f(z)\rangle = \langle f(z) - g(z), f(z)\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \tag{3-8}$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$. It is at this point that strong convexity of the ball $\mathbb{B}^3$ is used in an essential way.

We now consider the map $\tilde{f}$ on the circle $b\mathbb{D}$ given by

$$\tilde{f}(z) = z|1+\bar{a}z|^2 f(z), \quad |z| = 1. \tag{3-9}$$

An explicit calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{1}{2}c(1+a^2+4(\Re a)z+(1+\bar{a}^2)z^2) \\ \frac{1}{2}c(\mathrm{i}(1-a^2)+4(\Im a)z+\mathrm{i}(\bar{a}^2-1)z^2) \\ p(z+a)(1+\bar{a}z) \end{pmatrix}. \tag{3-10}$$

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the quadratic holomorphic polynomial map on the right-hand side above. Since $|1+\bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, (3-8) implies

$$h(z) := \Re\langle F(z) - G(z), |1+\bar{a}z|^2 f(z)\rangle = \langle f(z) - g(z), |1+\bar{a}z|^2 f(z)\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$. From (3-9) we see that

$$h(z) = \Re\left\langle \frac{F(z)-G(z)}{z}, \tilde{f}(z)\right\rangle \quad a.e.\ z \in b\mathbb{D}. \tag{3-11}$$

Since the maps $(F(z)-G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, formula (3-11) provides an extension of h from $b\mathbb{D}$ to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$. Inserting the value (3-7) into (3-11) gives

$$h(0) = \Re\langle F'(0) - G'(0), \tilde{f}(0)\rangle = (1-r)\Re\langle F'(0), \tilde{f}(0)\rangle \geq 0,$$

with equality if and only if $f = g$. Applying this argument to the linear map $g(z) = f(0) + rdf_0(z)$ $(z \in \mathbb{D})$ for a small $r > 0$ we get $\Re\langle F'(0), \tilde{f}(0)\rangle > 0$. It follows that $r \leq 1$, with equality if and only if $g = f$. □

Remark 3.2. The main point in the above proof is that the complexification of a conformal proper affine disc in $\mathbb{B}^n$ is a *stationary disc* in the tube $\mathcal{T}_{\mathbb{B}^n} = \mathbb{B}^n \times \mathrm{i}\mathbb{R}^n$. In Lempert's terminology [1981], a proper holomorphic disc $F : \mathbb{D} \to \Omega$ in a smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$, extending continuously to $\overline{\mathbb{D}}$, is a stationary disc if, denoting by $\nu : b\mathbb{D} \to \mathbb{C}^n$ the unit normal vector field to $b\Omega$ along the boundary circle $F(b\mathbb{D}) \subset b\Omega$, there is a positive continuous function $q > 0$ on $b\mathbb{D}$ such that the function $zq(z)\overline{\nu(z)}$ extends from the circle $|z| = 1$ to a holomorphic function $\tilde{f}(z)$ on $\mathbb{D}$. Lempert [1981] showed that every stationary disc F in a bounded strongly convex domain is the unique Kobayashi extremal disc through the point $F(a)$ in the tangent direction $F'(a)$ for every $a \in \mathbb{D}$. In our case, a suitable holomorphic function $\tilde{f}$ is given by (3-9) and (3-10). Lempert's theory also works on tubes over bounded strongly convex domains (see [Jarnicki and Pflug 2013, Section 11.1]); however, our proof of Theorem 2.1 does not depend on this information.

4. Proof of Theorem 2.2

Precomposing the given harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ in Theorem 2.2 by a holomorphic automorphism of the disc $\mathbb{D}$, we see that it suffices to prove estimate (2-4) for $z = 0$.

Assume first that $f : \mathbb{D} \to \mathbb{R}$ is a harmonic function on $\mathbb{D}$. Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be the holomorphic function on $\mathbb{D}$ with $\Re F = f$ and $F(0) = f(0) \in \mathbb{R}$. Writing $z = re^{it}$ with $0 \le r < 1$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} f(re^{it})^2 &= \tfrac{1}{4}(a_0 + a_1 re^{it} + r^2 e^{2it} + \cdots + a_0 + \bar{a}_1 re^{-it} + \bar{a}_2 r^2 e^{-2it} + \cdots)^2 \\ &= a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} r^{2k}|a_k|^2 + \cdots, \end{aligned}$$

where each of the remaining terms in the series contains a power e^{mit} for some $m \in \mathbb{Z} \setminus \{0\}$. Integrating around the circle $|z| = r$ for $0 < r < 1$ annihilates all such terms and yields

$$\int_0^{2\pi} f(re^{it})^2 \frac{dt}{2\pi} = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} r^{2k}|a_k|^2.$$

Clearly, $a_0 = f(0)$. Writing $z = x + iy$, we have that $a_1 = F'(0) = F_x(0) = f_x(0) - i f_y(0)$ by the Cauchy–Riemann equations. Therefore,

$$a_0^2 = f(0)^2 \quad \text{and} \quad |a_1|^2 = f_x(0)^2 + f_y(0)^2 = |\nabla f(0)|^2,$$

and hence

$$\int_0^{2\pi} f(re^{it})^2 \frac{dt}{2\pi} = |f(0)|^2 + \frac{1}{2}|\nabla f(0)|^2 r^2 + \frac{1}{2}\sum_{k=2}^{\infty} r^{2k}|a_k|^2. \tag{4-1}$$

Suppose now that $f = (f_1, \ldots, f_n) : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map. Then, $\sum_{j=1}^n f_j(re^{it})^2 < 1$ for all $0 \le r < 1$ and $t \in \mathbb{R}$. Integrating this inequality and taking into account the identity (4-1) for each component f_j of f gives

$$\int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = |f(0)|^2 + \frac{1}{2}|\nabla f(0)|^2 r^2 + \frac{1}{2}\sum_{k=2}^{\infty} r^{2k}|a_k|^2 < 1.$$

Letting r increase to 1 gives $|f(0)|^2 + \frac{1}{2}|\nabla f(0)|^2 \le 1$, with equality if and only if all higher-order coefficients in the Fourier expansion of f vanish. The latter holds if and only if f is a linear disc. This gives the estimate (2-4).

Note that (2-4) holds if the L^2-Hardy norm of f is at most 1. This does not necessarily imply that there is a harmonic disc in $\mathbb{B}^n$ reaching equality in (2-4). However, equality is attained if $f(0)$ is orthogonal to the two-plane $df_0(\mathbb{R}^2)$. In this case we may assume that $f(0) = (0, 0, p, 0, \ldots, 0)$ for some $0 \le p < 1$ and $df_0(\mathbb{R}^2) = \mathbb{R}^2 \times \{0\}^{n-2}$. The affine disc

$$\Sigma = \{(x, y, p, 0, \ldots, 0) : x^2 + y^2 < 1 - p^2\}$$

of radius $c = \sqrt{1 - p^2}$ is then orthogonal to $f(0)$, proper in $\mathbb{B}^n$, and its conformal linear parametrization f has gradient of size $c\sqrt{2}$ at the origin, so $|f(0)|^2 + \frac{1}{2}|\nabla f(0)|^2 = p^2 + c^2 = 1$. (Compare with (3-1) and (3-3).) This completes the proof of Theorem 2.2. □

We now show by examples that the inequality (2-3) fails in general for some nonconformal harmonic maps, and even for harmonic diffeomorphisms of the disc.

Example 4.1. Let U be the harmonic function on the disc $\mathbb{D}$ given by

$$U(z) = \Im \frac{2}{\pi} \log \frac{1+z}{1-z} = \frac{2}{\pi} \arctan \frac{2y}{1-x^2-y^2}. \tag{4-2}$$

This is the extremal harmonic function whose boundary value equals $+1$ on the upper unit semicircle and -1 on the lower semicircle, and we have that $\nabla U(0) = \frac{4}{\pi}(0, 1)$ and $|\nabla U(0)| = \frac{4}{\pi}$. For every $c \in \mathbb{R}$ the harmonic map

$$f(z) = \frac{1}{\sqrt{1+|c|^2}}(c + \mathrm{i}U(z)), \qquad z \in \mathbb{D},$$

clearly takes the unit disc into itself. For $c = 1$ we have

$$f(0) = \frac{1}{\sqrt{2}}, \quad \nabla f(0) = \frac{2\sqrt{2}}{\pi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad |\nabla f(0)| = \frac{2\sqrt{2}}{\pi} \approx 0.9, \quad \sqrt{2}(1-|f(0)|^2) = \frac{\sqrt{2}}{2} \approx 0.7.$$

Hence, inequality (2-3) fails in this example. On the other hand, $\sqrt{2}\sqrt{1-|f(0)|^2} = 1$, so inequality (2-4) holds, as it should by Theorem 2.2.

With some more effort we can show that inequality (2-3) fails for harmonic diffeomorphisms of the unit disc onto itself. Consider the sequence φ_n, $n \in \mathbb{N}$, of orientation-preserving homeomorphisms of the interval $[0, 2\pi]$ onto itself, defined by

$$\varphi_n(t) = \begin{cases} \dfrac{\pi}{2\pi - 1/n} t & \text{if } t \in \left[0, 2\pi - \frac{1}{n}\right], \\ 2(\pi - n\pi^2) + n\pi t & \text{if } t \in \left[2\pi - \frac{1}{n}, 2\pi\right]. \end{cases}$$

Let $\phi_n : \mathbb{T} \to \mathbb{T}$ be the associated sequence of homeomorphisms of the circle $\mathbb{T} = b\mathbb{D}$ given by $\phi_n(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\varphi_n(t)}$ for $t \in [0, 2\pi]$. Denote by

$$f_n(z) = P[\phi_n](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}t} - z|^2} \phi_n(\mathrm{e}^{\mathrm{i}t})\, dt, \qquad z \in \mathbb{D},$$

the Poisson extension of ϕ_n. By the Radó–Kneser–Choquet theorem (see [Duren 2004, Section 3.1]), f_n is a harmonic diffeomorphism of $\mathbb{D}$ for every $n \in \mathbb{N}$. As $n \to \infty$, the sequence f_n converges uniformly on compacts in $\mathbb{D}$ to the harmonic map $f = P[\phi_0]$, where $\phi_0(\mathrm{e}^{\mathrm{i}t}) = \lim_{n\to\infty} \phi_n(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t/2}$ for $t \in [0, 2\pi)$. Further, we have

$$\lim_{n\to\infty} \frac{|\nabla f_n(0)|}{1-|f_n(0)|^2} = \frac{|\nabla f(0)|}{1-|f(0)|^2}.$$

A calculation shows that

$$\frac{1}{\sqrt{2}} \frac{|\nabla f(0)|}{1-|f(0)|^2} = \frac{\sqrt{|A|^2 + |B^2|}}{1-|C|^2},$$

where

$$A = \frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}t/2} \cos t\, dt = -\frac{4\mathrm{i}}{3\pi}, \quad B = \frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}t/2} \sin t\, dt = \frac{8}{3\pi}, \quad C = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}t/2}\, dt = \frac{2\mathrm{i}}{\pi}.$$

Hence,

$$\frac{1}{\sqrt{2}} \frac{|\nabla f(0)|}{1-|f(0)|^2} = \frac{2\sqrt{10}}{3\pi(1-4/\pi^2)} \approx 1.1.$$

This shows that (2-3) fails for harmonic diffeomorphisms of the unit disc onto itself.

Example 4.2. Let $U(x, y)$ be the function (4-2). The harmonic map $f(x, y) = (U(y, x), U(x, y))$ takes the disc $\mathbb{D}$ onto the square $P = \{(x, y) \in \mathbb{R}^2 : |x| < 1,\ |y| < 1\}$ and $df_0(0, 0) = \frac{4}{\pi}\mathrm{Id}$. In particular, f is conformal at $(0, 0)$ and $\|df_0\| = \frac{4}{\pi} \approx 1.27$. On the other hand, a conformal diffeomorphism of $\mathbb{D}$ onto P mapping the origin to itself has the derivative at the origin of absolute value ≈ 1.08. Hence, the Schwarz–Pick lemma in Theorem 1.1 fails for maps from the disc to more general domains in $\mathbb{C}$. In particular, while the domain $\mathbb{C} \setminus \{0, 1\}$ is Kobayashi hyperbolic, one can find nonconstant harmonic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ whose differential is nonvanishing and conformal at the origin.

Problem 4.3. Assume that $D \subsetneq \mathbb{R}^2$ is a simply connected domain such that, for some point $p \in D$, the supremum of the norm $\|df_0\|$ of the differential of f at $0 \in \mathbb{D}$ over all harmonic maps $f : \mathbb{D} \to D$ with $f(0) = p$ which are conformal at 0 is attained by a conformal diffeomorphism of $\mathbb{D}$ onto D. Does it follow that D is a disc?

5. A Schwarz–Pick lemma for quasiconformal harmonic maps

In this section we apply the Schwarz–Pick lemma for harmonic self-maps of the disc, given by Theorem 1.1, to provide an estimate of the gradient of an orientation-preserving harmonic map $f : \mathbb{D} \to \mathbb{D}$ in terms of its *second Beltrami coefficient*

$$\omega(z) = \frac{\overline{(f_{\bar z})}}{f_z}, \quad z \in \mathbb{D}. \tag{5-1}$$

Here, $f_z = \frac{1}{2}(f_x - \mathrm{i} f_y)$ and $f_{\bar z} = \frac{1}{2}(f_x + \mathrm{i} f_y)$. If the map f is harmonic then ω is a holomorphic function; see (5-2). This is not the case for the Beltrami coefficient μ from the Beltrami equation $f_{\bar z} = \mu(z) f_z$. The number $|\mu(z)| = |\omega(z)|$ measures the dilatation of df_z; in particular, $\mu(z) = \omega(z) = 0$ if and only if f is conformal at z. We refer to [Ahlfors 1966; Duren 2004; Lehto and Virtanen 1973; Hengartner and Schober 1986] for background on the theory of quasiconformal maps.

The main question is to find the optimal estimate on $\|df_0\|$ for a harmonic map $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ and with a given value of $|\omega(0)| = |\mu(0)|$. A related problem was studied by Kovalev and Yang [2020] and Brevig et al. [2021], where the reader can find references to earlier works. Here we prove the following result.

Theorem 5.1. *Assume that f is an orientation-preserving harmonic map of the unit disc into itself, and let $\omega(z)$ denote its second Beltrami coefficient (5-1). Then we have the inequality*

$$\|df_z\| \le \frac{2(|\omega(z) f(z)^2| + \Re(\omega(z) f(z)^2))}{(1 - |\omega(z)|^2)(1 - |z|^2)} + \frac{1 + |\omega(z)|}{1 - |\omega(z)|} \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

If f is conformal at a point z, i.e., $\omega(z) = 0$, this estimate coincides with the Schwarz–Pick inequality (1-1) in Theorem 1.1.

Proof. It suffices to prove the inequality in the theorem for $z = 0$. For other points, we obtain it replacing f by $f \circ \phi_z$ for $\phi_z \in \mathrm{Aut}(\mathbb{D})$. However, we cannot reduce to the case $f(0) = 0$ since postcompositions by automorphisms of $\mathbb{D}$ are not allowed. The main idea is to construct from f a new harmonic map $\tilde f : \mathbb{D} \to \mathbb{D}$ which is conformal at 0, to which we then apply the Schwarz–Pick lemma given by Theorem 1.1.

Let us write $f = g + \bar{h}$, where g and h are holomorphic functions on $\mathbb{D}$. Then,

$$f_z(z) = g'(z), \quad f_{\bar{z}}(z) = \overline{h'(z)}, \quad \omega(z) = \frac{h'(z)}{g'(z)}. \tag{5-2}$$

Recall that the second Beltrami coefficient ω (5-1) is holomorphic. It follows that

$$\|df\| = |g'|(1 + |\omega|). \tag{5-3}$$

Since f is orientation preserving, we have that $|g'(z)| \geq |h'(z)|$ for all $z \in \mathbb{D}$. Let

$$a = g'(0) \quad \text{and} \quad b = h'(0). \tag{5-4}$$

We may assume that $|a| + |b| > 0$, for otherwise the estimate is trivial. Since $|f(z)| < 1$ for all $z \in \mathbb{D}$, the complex harmonic function

$$\tilde{f}(z) := \frac{\bar{a} f - \bar{b} \bar{f}}{|a| + |b|}, \quad z \in \mathbb{D}, \tag{5-5}$$

clearly maps the unit disc into itself. We have $\tilde{f} = \tilde{g} + \overline{\tilde{h}}$, where

$$\tilde{g} = \frac{\bar{a} g - \bar{b} h}{|a| + |b|} \quad \text{and} \quad \tilde{h} = \frac{a h - b g}{|a| + |b|} \tag{5-6}$$

are holomorphic functions on $\mathbb{D}$. Since

$$\tilde{h}'(0) = \frac{a h'(0) - b g'(0)}{|a| + |b|} = 0, \tag{5-7}$$

the second Beltrami coefficient $\tilde{\omega}$ of $\tilde{f}$ vanishes at $z = 0$, and hence $\tilde{f}$ is conformal at 0. Our Schwarz–Pick lemma (see Theorem 1.1) gives

$$\|d\tilde{f}_0\| \leq 1 - |\tilde{f}(0)|^2. \tag{5-8}$$

Taking into account (5-3), (5-4), (5-6), and (5-7), we have

$$\|d\tilde{f}_0\| = |\tilde{g}'(0)| = |g'(0)| - |h'(0)|.$$

Together with (5-5), (5-4), and (5-8) this gives the estimate

$$\begin{aligned} |g'(0)| - |h'(0)| &\leq 1 - \frac{|g'(0)\overline{f(0)} - h'(0) f(0)|^2}{(|g'(0)| + |h'(0)|)^2} \\ &= \frac{2|g'(0)| \cdot |h'(0)| \cdot |f(0)|^2 + 2\Re(\overline{g'(0)} h'(0) f(0)^2)}{(|g'(0)| + |h'(0)|)^2} + 1 - |f(0)|^2. \end{aligned}$$

In view of (5-2), this inequality can be written in the form

$$(1 - |\omega(0)|)|g'(0)| \leq \frac{2(|\omega(0)||f(0)|^2 + \Re(\omega(0) f(0)^2))}{(1 + |\omega(0)|)^2} + 1 - |f(0)|^2.$$

From (5-3) we see that

$$|g'(0)| = \frac{\|df_0\|}{1 + |\omega(0)|}.$$

Inserting this into the expression on the left-hand side of the previous inequality gives

$$\|df_0\|\frac{1-|\omega(0)|}{1+|\omega(0)|} \le \frac{2|\omega(0)f(0)^2|+2\Re(\omega(0)f(0)^2)}{(1+|\omega(0)|)^2}+1-|f(0)|^2,$$

which is clearly equivalent to

$$\|df_0\| \le \frac{2|\omega(0)f(0)^2|+2\Re(\omega(0)f(0)^2)}{1-|\omega(0)|^2}+\frac{1+|\omega(0)|}{1-|\omega(0)|}(1-|f(0)|^2). \qquad \square$$

6. An intrinsic pseudometric defined by conformal harmonic discs

In this section we introduce an intrinsic Finsler pseudometric g_D on any domain D in $\mathbb{R}^n$, $n \ge 3$, and more generally on any Riemannian manifold of dimension at least three, in terms of conformal minimal discs $\mathbb{D} \to D$. The definition is modeled on Kobayashi's definition of his pseudometric on complex manifolds, which uses holomorphic discs. The pseudometric g_D and the associated pseudodistance $\rho_D : D \times D \to \mathbb{R}_+$ are the largest ones having the property that any conformal harmonic map $M \to D$ from a hyperbolic conformal surface with the Poincaré metric is metric and distance decreasing. On the ball $\mathbb{B}^n$, we have that $g_{\mathbb{B}^n}$ coincides with the Cayley–Klein metric; see Theorem 6.2. The same definition of g_D applies in any Riemannian manifold of dimension at least three; see Remark 6.6. This provides the basis for hyperbolicity theory of domains in Euclidean spaces and, more generally, of Riemannian manifolds, in terms of minimal surfaces.

We begin by introducing a Finsler pseudometric on the bundle of two-planes over a domain $D \subset \mathbb{R}^n$, analogous to the metric $\mathcal{M}$ on the ball; see (2-8). A *conformal frame* in $\mathbb{R}^n$ is a pair $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $|\boldsymbol{u}| = |\boldsymbol{v}|$ and $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. We denote by CF_n the space of all conformal frames on $\mathbb{R}^n$, including $(\boldsymbol{0}, \boldsymbol{0})$. Given a domain $D \subset \mathbb{R}^n$, let $\mathrm{CH}(\mathbb{D}, D)$ denote the space of conformal harmonic maps $\mathbb{D} \to D$ (i.e., such that (2-1) holds at every point of $\mathbb{D}$). Define the function $\mathcal{M}_D : D \times \mathrm{CF}_n \to \mathbb{R}_+$ by

$$\mathcal{M}_D(\boldsymbol{x}, (\boldsymbol{u}, \boldsymbol{v})) = \inf\Big\{\frac{1}{r} : \exists f \in \mathrm{CH}(\mathbb{D}, D),\ f(0) = \boldsymbol{x},\ f_x(0) = r\boldsymbol{u},\ f_y(0) = r\boldsymbol{v}\Big\}. \tag{6-1}$$

Clearly, $\mathcal{M}_D$ is homogeneous and rotation-invariant, in the sense that for any $c \in \mathbb{R}$ and orthogonal rotation R in the two-plane $\Lambda = \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ we have for every $\boldsymbol{x} \in D$ that

$$\mathcal{M}_D(\boldsymbol{x}, (c\boldsymbol{u}, c\boldsymbol{v})) = |c|\mathcal{M}_D(\boldsymbol{x}, (\boldsymbol{u}, \boldsymbol{v})), \quad \mathcal{M}_D(\boldsymbol{x}, (R\boldsymbol{u}, R\boldsymbol{v})) = \mathcal{M}_D(\boldsymbol{x}, (\boldsymbol{u}, \boldsymbol{v})). \tag{6-2}$$

Thus, $\mathcal{M}_D$ is determined by its values on unitary conformal frames $(\boldsymbol{u}, \boldsymbol{v})$ with $|\boldsymbol{u}| = |\boldsymbol{v}| = 1$ and hence on $D \times G_2(\mathbb{R}^n)$, where $G_2(\mathbb{R}^n)$ is the Grassmann manifold of two-planes in $\mathbb{R}^n$. Precisely, for a two-plane $\Lambda \in G_2(\mathbb{R}^n)$ we set $\mathcal{M}_D(\boldsymbol{x}, \Lambda) = \mathcal{M}_D(\boldsymbol{x}, (\boldsymbol{u}, \boldsymbol{v}))$, where $(\boldsymbol{u}, \boldsymbol{v})$ is any unitary conformal frame spanning Λ. Note that

$$\mathcal{M}_D(\boldsymbol{x}, \Lambda) = \inf\left\{\frac{1}{\|df_0\|} : f \in \mathrm{CH}(\mathbb{D}, D),\ f(0) = \boldsymbol{x},\ df_0(\mathbb{R}^2) = \Lambda\right\}. \tag{6-3}$$

By shrinking the disc $\mathbb{D}$ and using rotations and translations on $\mathbb{R}^n$, we see that the function $\mathcal{M}_D$ is upper semicontinuous on $D \times \mathrm{CF}_n$. Obviously, $\mathcal{M}_{\mathbb{R}^n} \equiv 0$. On the ball $\mathbb{B}^n$, we have that $\mathcal{M}_{\mathbb{B}^n}(\boldsymbol{x}, \Lambda)$ is given by (2-8) according to Theorem 2.1.

We also introduce a Finsler pseudometric $g_D : D \times \mathbb{R}^n \to \mathbb{R}_+$, called the *minimal metric* on D, whose value at a point $\boldsymbol{x} \in D$ on a tangent vector $\boldsymbol{u} \in T_{\boldsymbol{x}} D = \mathbb{R}^n$ is given by

$$\begin{aligned} g_D(\boldsymbol{x}, \boldsymbol{u}) &= \inf\left\{ \frac{1}{r} > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, D),\ f(0) = \boldsymbol{x},\ f_x(0) = r\boldsymbol{u} \right\} \\ &= |\boldsymbol{u}| \cdot \inf\{\mathcal{M}_D(\boldsymbol{x}, \Lambda) : \Lambda \in G_2(\mathbb{R}^n),\ \boldsymbol{u} \in \Lambda\}. \end{aligned} \tag{6-4}$$

It follows that every conformal harmonic map $f : \mathbb{D} \to D$ satisfies

$$g_D(f(z), df_z(\xi)) \le \mathcal{P}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \quad z \in \mathbb{D},\ \xi \in \mathbb{R}^2, \tag{6-5}$$

and g_D is the biggest pseudometric on D with this property. For $z = 0$ this follows directly from the definition, and for any other point $z \in \mathbb{D}$ we precompose f by a conformal automorphism of $\mathbb{D}$ mapping 0 to z. The same holds if $\mathbb{D}$ is replaced by any hyperbolic conformal surface; see the proof of Theorem 2.6.

By integrating g_D we get the *minimal pseudodistance* $\rho_D : \Omega \times \Omega \to \mathbb{R}_+$:

$$\rho_D(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\gamma} \int_0^1 g_D(\gamma(t), \dot\gamma(t))\, dt, \quad \boldsymbol{x}, \boldsymbol{y} \in \Omega. \tag{6-6}$$

The infimum is over all piecewise smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \boldsymbol{x}$ and $\gamma(1) = \boldsymbol{y}$. Obviously, ρ_Ω satisfies the triangle inequality, but it need not be a distance function. In particular, $\rho_{\mathbb{R}^n}$ vanishes identically.

There is another natural procedure to obtain the pseudodistance ρ_D in (6-6), which is motivated by Kobayashi's definition of his pseudodistance on complex manifolds [1967]. Fix a pair of points $\boldsymbol{x}, \boldsymbol{y} \in D$. To any finite chain of conformal harmonic discs $f_i : \mathbb{D} \to D$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \boldsymbol{x}, \quad f_{i+1}(0) = f_i(a_i) \ \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \boldsymbol{y}, \tag{6-7}$$

we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \ge 0.$$

The i-th term in the sum is the Poincaré distance from 0 to a_i in $\mathbb{D}$. The pseudodistance $\rho_D(\boldsymbol{x}, \boldsymbol{y})$ is defined to be the infimum of the numbers obtained in this way. The proof that the two definitions yield the same result is similar to the one given for the Kobayashi pseudodistance by Royden [1971, Theorem 1]; see [Drinovec Drnovšek and Forstnerič 2023, Theorem 3.1] for the details.

The following proposition says that the minimal pseudodistance ρ_D gives an upper bound for growth of conformal minimal surfaces in the domain D.

Proposition 6.1. *Every conformal harmonic map $M \to D$ from a hyperbolic conformal surface is distance decreasing in the Poincaré distance on M and the pseudodistance ρ_D, and ρ_D is the biggest pseudodistance on D for which this holds.*

Proof. Let M be a hyperbolic conformal surface and $h : \mathbb{D} \to M$ be a conformal universal covering. Choose a conformal harmonic map $f : M \to D$ and a pair of points $p, q \in M$. Let $a, b \in \mathbb{D}$ be such that $h(a) = p$ and $h(b) = q$. Precomposing h by an automorphism of the disc, we may assume that $a = 0$.

Then, $g := f \circ h : \mathbb{D} \to D$ is a conformal harmonic disc with $g(0) = f(p)$ and $g(b) = f(q)$, and it follows from the definition of ρ_D that

$$\rho_D(f(p), f(q)) = \rho_D(g(0), g(b)) \le \frac{1}{2} \log \frac{1+|b|}{1-|b|}.$$

The infimum of the right-hand side over all points $b \in \mathbb{D}$ with $h(b) = q$ equals the Poincaré distance between p and q in M, so we see that f is distance decreasing.

Suppose now that τ is a pseudodistance on D such that every conformal harmonic map $\mathbb{D} \to D$ is distance decreasing with the Poincaré metric on $\mathbb{D}$. Let $f_i : \mathbb{D} \to D$ and $a_i \in \mathbb{D}$ for $i = 1, \dots, k$ be a chain as in (6-7) connecting the points $\boldsymbol{x}, \boldsymbol{y} \in D$. Then,

$$\tau(\boldsymbol{x}, \boldsymbol{y}) \le \sum_{i=1}^{k} \tau(f_i(0), f_i(a_i)) \le \sum_{i=1}^{k} \frac{1}{2} \log \frac{1+|a_i|}{1-|a_i|}.$$

Taking the infimum over all such chains gives $\tau(\boldsymbol{x}, \boldsymbol{y}) \le \rho_D(\boldsymbol{x}, \boldsymbol{y})$. □

We have already observed that, on the ball $\mathbb{B}^n$ $(n \ge 3)$, the Finsler metric $\mathcal{M}_{\mathbb{B}^n}$ is given by (2-8). From (2-9) and (6-4) it follows that $g_{\mathbb{B}^n}$ equals the Cayley–Klein metric $\mathcal{CK}$ (2-6):

Theorem 6.2. *On the ball $\mathbb{B}^n$, $n \ge 3$, we have*

$$g_{\mathbb{B}^n} = \mathcal{CK}, \quad \rho_{\mathbb{B}^n} = \operatorname{dist}_{\mathcal{CK}}.$$

Hilbert [1895] defined a metric on any convex domain in $\mathbb{RP}^n$ that generalizes the Cayley–Klein metric on the ball. Hilbert metrics are examples of projectively invariant metrics which have been studied by many authors; see the surveys by Kobayashi [1977; 1984] and Goldman [2019]. Kobayashi [1977] discussed the analogy between his metric and Hilbert's metric. Lempert [1987] established an explicit connection, and then in [Lempert 1993, Theorem 3.1] proved that the Hilbert metric $\mathcal{H}_D$ on any bounded convex domain $D \subset \mathbb{R}^n$ is the restriction to D of the Kobayashi metric on the elliptic tube $D^* \subset D \times \mathrm{i}\mathbb{R}^n \subset \mathbb{C}^n$ obtained as follows; see [Lempert 1993, p. 441]. Every affine line segment $L \subset D$ with endpoints on bD is the diameter of a unique complex disc in $D \times \mathrm{i}\mathbb{R}^n$, and D^* is the union of all such discs. The elliptic tube over the ball $\mathbb{B}^n$ is the complex ball $\mathbb{B}^n_{\mathbb{C}}$, and the metric $g_{\mathbb{B}^n}$ agrees with the Hilbert metric $\mathcal{H}_{\mathbb{B}^n} = \mathcal{CK}$ according to Theorem 6.2.

While Hilbert's metric is invariant under projective linear transformations, the minimal metric is invariant (at least in an obvious way) only under the conformal group (see Proposition 6.5); hence it is expected that the two metrics differ on most convex domains. We give an explicit example on ellipsoids.

Example 6.3. Let (x, y, z) be coordinates on $\mathbb{R}^3$. Consider the ellipsoid

$$D_a = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + \frac{1}{a^2}(y^2 + z^2) < 1 \right\}, \quad a > 0.$$

Note that $D_a \subset \mathbb{B}^3$ if and only if $0 < a \le 1$, and $D_1 = \mathbb{B}^3$. We will show that for $0 < a < 1$ the Hilbert metric on D_a does not agree with the minimal metric at the origin $\mathbf{0} \in \mathbb{R}^3$. Since the x-axis intersects D_a in the interval $(-1, 1)$, the Hilbert length of the vector $\boldsymbol{e}_1 = (1, 0, 0)$ equals 1. Pick a

two-plane $\Lambda \subset \mathbb{R}^3$ containing the vector $\boldsymbol{e}_1$. Due to rotational symmetry of D_a in the (y, z)-coordinates the value of $\mathcal{M}_{D_a}(\mathbf{0}, \Lambda)$ (6-3) does not depend on the choice of Λ, so we may take $\Lambda = \{z = 0\}$. Let $f = (f_1, f_2, f_3) : \mathbb{D} \to D_a$ be a conformal harmonic disc with $f(0) = \mathbf{0}$ and $df_0(\mathbb{R}^2) = \{z = 0\}$. Replacing f by $f(e^{it}z)$ for a suitable $t \in \mathbb{R}$ gives $f_x(0) = r\boldsymbol{e}_1$ and $f_y(0) = \pm r\boldsymbol{e}_2$ with $r = \|df_0\| > 0$. The projection $h = (f_1, f_2) : \mathbb{D} \to \mathbb{R}^2$ maps $\mathbb{D}$ into the ellipse $E_a = \{x^2 + y^2/a^2 < 1\}$, $h(0) = \mathbf{0}$, and h is conformal at 0. For $0 < a < 1$ we have $E_a \subsetneq \mathbb{D}$. Theorem 1.1 implies that $r = \|dh_0\| < 1$; equality is excluded since in that case we would have $h(\mathbb{D}) = \mathbb{D}$. By a normal families argument we also have that $\sup_f \|df_0\| < 1$. It follows that $\mathcal{M}_{D_a}(\mathbf{0}, \Lambda) > 1$ for every such Λ, and hence

$$g_{D_a}(\mathbf{0}, \boldsymbol{e}_1) = \mathcal{M}_{D_a}(\mathbf{0}, \Lambda) > 1 = \mathcal{H}_{D_a}(\mathbf{0}, \boldsymbol{e}_1) \quad \text{if } 0 < a < 1.$$

Problem 6.4. On which bounded convex domains $D \subset \mathbb{R}^n$, $n \geq 3$ (besides the ball) does the Hilbert metric coincide with the minimal metric g_D? Is the ball the only such domain?

Denote by $\mathcal{R}_n$ the Lie group of transformations $\mathbb{R}^n \to \mathbb{R}^n$ generated by the orthogonal group O_n, translations, and dilations by positive numbers. Elements of $\mathcal{R}_n$ are called *rigid transformations* of $\mathbb{R}^n$. Postcomposition of any conformal harmonic map $f : M \to \mathbb{R}^n$ by a rigid transformation of $\mathbb{R}^n$ is again a conformal harmonic map, and it is well known that $\mathcal{R}_n$ is the largest group of diffeomorphisms of $\mathbb{R}^n$ having this property. This gives the following.

Proposition 6.5. *Given a domain $D \subset \mathbb{R}^n$, $n \geq 3$, and a map $R \in \mathcal{R}_n$, the restriction $R|_D : D \to D' = R(D)$ is an isometry of pseudometric spaces $(D, \rho_D) \to (D', \rho_{D'})$.*

Remark 6.6. The intrinsic pseudometric g_D and the associated pseudodistance ρ_D can be defined in the very same way on an arbitrary Riemannian manifold $(D, \tilde{g})$ of dimension at least three. The Riemannian metric $\tilde{g}$ determines the class of conformal harmonic maps $\mathbb{D} \to D$, which coincide with conformal minimal discs in D.

Hyperbolic domains in $\mathbb{R}^n$. We now introduce the notion of (complete) hyperbolic domains in $\mathbb{R}^n$, in analogy with Kobayashi hyperbolic complex manifolds.

Definition 6.7. A domain $D \subset \mathbb{R}^n$ $(n \geq 3)$ is *hyperbolic* if the pseudodistance ρ_D is a distance function on D, and is *complete hyperbolic* if (D, ρ_D) is a complete metric space.

Example 6.8. (a) The ball $\mathbb{B}^n \subset \mathbb{R}^n$ $(n \geq 3)$ is complete hyperbolic. Indeed, the Cayley–Klein metric (2-6) is complete, so the conclusion follows from Theorem 6.2.

(b) Every bounded domain $D \subset \mathbb{R}^n$ is hyperbolic. Indeed, if B is a ball containing D then $\rho_D(\boldsymbol{x}, \boldsymbol{y}) \geq \rho_B(\boldsymbol{x}, \boldsymbol{y})$ for any pair $\boldsymbol{x}, \boldsymbol{y} \in D$, and B is complete hyperbolic by (a). However, a bounded domain need not be complete hyperbolic. For example, if bD is strongly concave at $\boldsymbol{p} \in bD$, there is a conformal linear disc $\Sigma \subset D \cup \{\boldsymbol{p}\}$ containing $\boldsymbol{p}$, and it is easily seen that $\boldsymbol{p}$ is at finite ρ-distance from D.

(c) The half-space $\mathbb{H}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is not hyperbolic, and the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on all planes $x_n = \text{const}$. However, every point on $b\mathbb{H}^n = \{x_n = 0\}$ is at infinite minimal distance from points in $\mathbb{H}^n$ [Drinovec Drnovšek and Forstnerič 2023, Lemma 5.2].

By using the expression for the metric (2-8) on the ball we can determine the asymptotic rate of growth of the Finsler metric $\mathcal{M}_D$, and hence of the distance function ρ_D, on any bounded strongly convex domain $D \subset \mathbb{R}^n$ with $\mathscr{C}^2$ boundary. Let $\delta = \delta(\boldsymbol{x}) = 1 - |\boldsymbol{x}|$ denote the distance from a point $\boldsymbol{x} \in \mathbb{B}^n \setminus \{0\}$ to the sphere $b\mathbb{B}^n$, and let $\Lambda \subset \mathbb{R}^n$ be a 2-plane forming an angle θ with $\boldsymbol{x}$. As $\boldsymbol{x}$ approaches the sphere radially, we have

$$\mathcal{M}_{\mathbb{B}^n}(\boldsymbol{x}, \theta) := \mathcal{M}_{\mathbb{B}^n}(\boldsymbol{x}, \Lambda) \approx \frac{\sqrt{\cos^2\theta + 2\delta \sin^2\theta}}{2\delta},$$

in the sense that the quotient of the two sides converges to 1 as $\delta \to 0$. In particular,

$$\mathcal{M}_{\mathbb{B}^n}\Big(\boldsymbol{x}, \frac{\pi}{2}\Big) \approx \frac{1}{\sqrt{2\delta}} \qquad \text{and} \qquad \mathcal{M}_{\mathbb{B}^n}(\boldsymbol{x}, \theta) \approx \frac{\cos\theta}{2\delta} \quad \text{for } \theta \in \Big[0, \frac{\pi}{2}\Big).$$

Assume now that $D \subset \mathbb{R}^n$ is a bounded strongly convex domain with $\mathscr{C}^2$ boundary. There is a collar $U \subset \mathbb{R}^n$ around bD such that every point $\boldsymbol{x} \in U \cap D$ has a unique closest point $\pi(\boldsymbol{x}) \in bD$. Comparison with inscribed and circumscribed balls to D passing through the point $\pi(\boldsymbol{x})$ shows that there are constants $0 < c < C$ such that

$$c\frac{\sqrt{\cos^2\theta + 2\delta \sin^2\theta}}{2\delta} \le \mathcal{M}_D(\boldsymbol{x}, \Lambda) \le C\frac{\sqrt{\cos^2\theta + 2\delta \sin^2\theta}}{2\delta} \tag{6-8}$$

for $\boldsymbol{x} \in U \cap D$, where $\delta = |\boldsymbol{x} - \pi(\boldsymbol{x})| = \operatorname{dist}(\boldsymbol{x}, bD)$ and θ is the angle between the 2-plane Λ and the normal vector $N_{\boldsymbol{x}} = \delta^{-1}(\pi(\boldsymbol{x}) - \boldsymbol{x})$ to bD at $\pi(\boldsymbol{x}) \in bD$. The upper bound uses comparison with inscribed balls, so it holds on any domain with $\mathscr{C}^2$ boundary, while the lower bound uses comparison with circumscribed ball, and hence it depends on strong convexity of D. These estimates are analogous to the asymptotic boundary estimates of the Kobayashi metric in bounded strongly pseudoconvex domains in $\mathbb{C}^n$ and are due to Graham [1975]. (There is a large subsequent literature on this subject.) These estimates show in particular that the distance function ρ_D induced by $\mathcal{M}_D$ is complete, thereby giving the following.

Theorem 6.9. *Every bounded strongly convex domain in $\mathbb{R}^n$, $n \ge 3$, with $\mathscr{C}^2$ boundary is complete hyperbolic in the minimal metric.*

Remark 6.10. Since the first version of this paper was posted on arXiv in February 2021, progress on the subject of minimal hyperbolicity was made by Drinovec Drnovšek and Forstnerič [2023], whose paper we will henceforth abbreviate as [DDF 2023]. Besides establishing basic characterizations of (complete) hyperbolicity, they proved that a convex domain in $\mathbb{R}^n$ is hyperbolic if and only if it is complete hyperbolic if and only if it does not contain any affine 2-plane [DDF 2023, Theorem 5.1]. They also showed that every bounded strongly minimally convex domain in $\mathbb{R}^n$, $n \ge 3$, is complete hyperbolic [DDF 2023, Theorem 9.2]. This is a considerable generalization of Theorem 6.9, whose proof relies on the lower bound for $\mathcal{M}_\Omega$ (and hence g_Ω) given by another Finsler pseudometric $\mathcal{F}_\Omega : \Omega \times G_2(\mathbb{R}^n) \to \mathbb{R}_+$ defined in terms of minimal plurisubharmonic functions; see [DDF 2023, Section 7]. A discussion of this class of domains and functions can be found in [Alarcón et al. 2019; 2021, Chapter 8]. Finally, they established a localization theorem for the minimal pseudometric analogous to the results for the Kobayashi pseudometric; see [DDF 2023, Section 8].

The following problem remains open; an affirmative answer is known for the case when M is a plane; see [DDF 2023, Lemma 5.2].

Problem 6.11. Let M be an embedded minimal surface in $\mathbb{R}^3$. Is the minimal distance from $\mathbb{R}^3 \setminus M$ to M infinite? Is the complement of a catenoid in $\mathbb{R}^3$ complete hyperbolic?

Extremal minimal discs. Another important and natural question is the following.

Problem 6.12. Let $D \subset \mathbb{R}^n$ be a bounded strongly convex domain with smooth boundary. Is there a unique (up to a conformal reparametrization) extremal conformal harmonic disc through any given point $\boldsymbol{x} \in D$ tangent to a given two-plane $\Lambda \in G_2(\mathbb{R}^n)$ at $\boldsymbol{x}$?

Theorem 2.1 gives an affirmative answer on the ball, and this is the only domain for which the answer seems to be known. By the seminal result of Lempert [1981; 1987], the analogous result holds for the extremal holomorphic discs for the Kobayashi metric in any smoothly bounded strongly convex domain $D \subset \mathbb{C}^n$.

We now describe a condition which implies an affirmative answer to this problem. It explores a comparison between the Finsler pseudometric $\mathcal{M}_D$ (6-1) on a domain $D \subset \mathbb{R}^n$ and a Kobayashi-type pseudometric on the tube $\mathcal{T}_D = \mathbb{D} \times \mathrm{i}\mathbb{R}^n \subset \mathbb{C}^n$. To this end, we recall a few basic facts from the theory of minimal surfaces; see [Alarcón et al. 2021, Chapter 2] or [Osserman 1969].

A holomorphic map $F = (F_1, \ldots, F_n) : \mathbb{D} \to \mathbb{C}^n$ satisfying

$$\sum_{i=1} F_i'(z)^2 = 0 \quad \text{for all } z \in \mathbb{D}$$

is called a *holomorphic null map*. The analogous definition applies with the disc replaced by any open Riemann surface, considering the above equation in local holomorphic coordinates. (The map F need not be an immersion.) The complex cone

$$\boldsymbol{A}^{n-1} = \left\{ \boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^n z_i^2 = 0 \right\} \tag{6-9}$$

is called the *null cone*, and its elements are *null vectors*. Hence, a holomorphic map F is null if and only if the complex derivative $F'(z)$ lies in $\boldsymbol{A}^{n-1}$ for every z. It is a basic fact that the real and imaginary parts of a holomorphic null map $M \to \mathbb{C}^n$ are conformal harmonic maps $M \to \mathbb{R}^n$; conversely, every conformal harmonic map $\mathbb{D} \to \mathbb{R}^n$ from the disc is the real part of a holomorphic null map $\mathbb{D} \to \mathbb{C}^n$; see [Alarcón et al. 2021, Theorem 2.3.4]. Given a domain $D \subset \mathbb{R}^n$, we denote by $\mathrm{HN}(\mathbb{D}, \mathcal{T}_D)$ the space of all holomorphic null maps $F = (F_1, \ldots, F_n) : \mathbb{D} \to \mathcal{T}_D$. Define a pseudometric on $(\boldsymbol{z}, \boldsymbol{w}) \in \mathcal{T}_D \times \boldsymbol{A}^{n-1}$ by

$$\mathcal{N}_D(\boldsymbol{z}, \boldsymbol{w}) = \inf\left\{ \frac{1}{|a|} : \exists F \in \mathrm{HN}(\mathbb{D}, \mathcal{T}_D),\ F(0) = \boldsymbol{z},\ F'(0) = a\boldsymbol{w} \right\}. \tag{6-10}$$

Here, a may be a complex number. Clearly, $\mathcal{N}_D(\boldsymbol{z}, \boldsymbol{w})$ is bigger than or equal to the Kobayashi pseudonorm of the vector $\boldsymbol{w} \in T_{\boldsymbol{z}}(\mathcal{T}_D)$, since in the definition of the latter one uses all holomorphic discs as opposed to just null discs. Note that for each conformal frame $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{CF}_n$ the vectors $\boldsymbol{u} \pm \mathrm{i}\boldsymbol{v} \in \mathbb{C}^n$ are null vectors; conversely, the real and imaginary component of a null vector $\boldsymbol{w} \in \boldsymbol{A}^{n-1}$ form a conformal frame. The

aforementioned correspondence between conformal harmonic discs in D and holomorphic null discs in $\mathcal{T}_D$ shows that for all $\boldsymbol{x} \in D$, $\boldsymbol{y} \in \mathbb{R}^n$, and $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{CF}_n$ we have

$$\mathcal{N}_D(\boldsymbol{x} + \mathrm{i}\boldsymbol{y}, \boldsymbol{u} \pm \mathrm{i}\boldsymbol{v}) = \mathcal{M}_D(\boldsymbol{x}, (\boldsymbol{u}, \boldsymbol{v})). \tag{6-11}$$

This shows in particular that every extremal conformal harmonic disc in D is the real part of an extremal holomorphic null disc in the tube $\mathcal{T}_D$. Therefore, the correspondence between the extremal conformal minimal discs in the ball $\mathbb{B}^n \subset \mathbb{R}^n$ and the Kobayashi geodesics in the tube $\mathcal{T}_{\mathbb{B}^n}$, used in the proof of Lemma 3.1, extends to any bounded strongly convex domain $D \subset \mathbb{R}^n$ with $\mathscr{C}^2$ boundary satisfying the following condition. The notion of a stationary holomorphic disc was explained in Remark 3.2.

Definition 6.13. A domain $D \subset \mathbb{R}^n$ satisfies *Condition N* if for every point $\boldsymbol{x} \in D$ and null vector $0 \neq \boldsymbol{w} \in \boldsymbol{A}^{n-1}$ there is a stationary holomorphic null disc in the tube $\mathcal{T}_D$ through the point $\boldsymbol{x} + \mathrm{i}0$ in the direction $\boldsymbol{w}$.

Our proof of Theorem 2.1 implies the following.

Theorem 6.14. *If D is a bounded strongly convex domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary and satisfying Condition N, then for every point $\boldsymbol{x} \in D$ and two-plane $\Lambda \in G_2(\mathbb{R}^n)$ there exists an extremal conformal harmonic disc $f : \mathbb{D} \to D$ with $f(0) = \boldsymbol{x}$ and $df_0(\mathbb{R}^2) = \Lambda$. Such an f is unique up to a rotation of $\mathbb{D}$.*

Proof. Let $0 \neq \boldsymbol{w} = \boldsymbol{u} - \mathrm{i}\boldsymbol{v} \in \boldsymbol{A}^{n-1}$ be such that $\Lambda = \mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$. By Condition N there is a stationary holomorphic null disc $F : \mathbb{D} \to \mathcal{T}_D$ with $F(0) = \boldsymbol{x} + \mathrm{i}0$ and $F'(0) = \alpha \boldsymbol{w}$ for $\alpha \in \mathbb{C}$, and F is unique up to rotations of $\mathbb{D}$ by Lempert's theorem [1981, Theorem 2]. The real part $f = \Re F : \mathbb{D} \to D$ is then a conformal harmonic disc as in the theorem. □

Problem 6.15. Which bounded strongly convex domains in $\mathbb{R}^n$, besides the ball, satisfy Condition N?

Complex geodesics of the Kobayashi metric in tubes over convex domains $D \subset \mathbb{R}^n$ were studied by Zając [2015; 2016], Pflug and Zwonek [2018], and Zwonek [2022]. It would be of interest to see whether these works can be used to give information on the validity of Condition N. The fact that Condition N holds on the ball $\mathbb{B}^n$ may simply be a lucky coincidence which makes our analysis work on this most symmetric domain.

Acknowledgements

Forstnerič was partially supported by the research program P1-0291 and grants J1-9104, J1-3005, and N1-0237 from ARRS, Republic of Slovenia. Kalaj was partially supported by a research fund of the University of Montenegro. We wish to thank Antonio Alarcón for remarks which improved the presentation, Dmitry Khavinson for information regarding harmonic self-maps of the disc, László Lempert and Stefan Nemirovski for information regarding projectively invariant metrics, Joaquín Pérez for consultation on the state of the art concerning growth of conformal minimal surfaces, and Sylwester Zając and Włodzimierz Zwonek for communications concerning geodesics of the Kobayashi metric in tube domains. Forstnerič also thanks Miodrag Mateljević with whom he discussed this problem in 2016. Finally, we thank the referee for a careful reading of the paper and suggestions which improved the presentation.

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Received 1 Dec 2021. Revised 5 Jul 2022. Accepted 11 Aug 2022.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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Analysis & PDE

Volume 17 No. 3 2024