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Domains without parabolic minimal submanifolds and weakly hyperbolic domains

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Abstract

We show that if Ω is an m-convex domain in $\mathbb{R}^n$ for some $2 \leqslant m < n$ whose boundary $b\Omega$ has a tubular neighbourhood of positive radius and is not m-flat near infinity, then Ω does not contain any immersed parabolic minimal submanifolds of dimension $\geqslant m$. In particular, if M is a properly embedded non-flat minimal hypersurface in $\mathbb{R}^n$ with a tubular neighbourhood of positive radius, then every immersed parabolic hypersurface in $\mathbb{R}^n$ intersects M. In dimension $n = 3$, this holds if M has bounded Gaussian curvature function. We also introduce the class of weakly hyperbolic domains Ω in $\mathbb{R}^n$, characterised by the property that every conformal harmonic map $\mathbb{C} \to \Omega$ is constant, and we elucidate their relationship with hyperbolic domains, and domains without parabolic minimal surfaces.

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1 | INTRODUCTION

This paper is motivated by two closely related lines of developments in the theory of minimal surfaces. The first one is the circle of results known as halfspace theorems. The second one is the recently introduced hyperbolicity theory for minimal surfaces.

Concerning the first topic, Xavier [42] proved in 1984 that the convex hull of a complete non-flat minimal surface in $\mathbb{R}^3$ with bounded Gaussian curvature equals $\mathbb{R}^3$. The *Strong Halfspace Theorem*

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of Hoffman and Meeks [23, Theorem 2] says that any two proper minimal surfaces in $\mathbb{R}^3$ intersect, unless they are parallel planes. By Pacelli Bessa, Jorge and Oliveira-Filho [8] and Rosenberg [40], the same conclusion holds for a pair of complete immersed minimal surfaces in $\mathbb{R}^3$ with bounded curvature; such surfaces need not be proper in $\mathbb{R}^3$ (see Andrade [6]) unless they are embedded (see Meeks and Rosenberg [38, Theorem 2.1]). Closer to the topic of this paper, Pacelli Bessa, Jorge, and Pessoa proved in [9] that an immersed parabolic minimal surface in $\mathbb{R}^3$ intersects every complete immersed non-flat minimal surface with bounded curvature. Related developments concern the maximum principle at infinity; see the surveys by Meeks and Pérez [32, 34] and the recent paper by Gama et al. [17].

In this paper, we find geometric conditions on the boundary of a domain Ω in $\mathbb{R}^n$ for $n \geqslant 3$, implying that Ω does not contain any immersed (not necessarily proper or complete) parabolic minimal submanifolds of a given dimension; see Theorem 2.1. In particular, we obtain a halfspace theorem for a pair of minimal hypersurfaces in $\mathbb{R}^n$, one of which is parabolic and the other one is properly embedded and has a tubular neighbourhood of positive radius; see Corollary 2.2. When $n = 3$, this coincides with a special case of the aforementioned result of Pacelli Bessa et al. [9, Theorem 1.1]; see Corollary 2.3.

Parabolicity of an open Riemannian manifold is important from the point of view of potential theory. Determining whether such a manifold is parabolic or hyperbolic is a classical question, the so-called type problem. The survey by Grigor'yan [19] is a standard reference. The extrinsic case, considering the type problem for a submanifold of a Riemannian manifold, is natural and interesting as well, in particular when the ambient manifold is a Euclidean space. The case of minimal surfaces in $\mathbb{R}^n$ has been widely studied. For minimal submanifolds of dimension higher than 2, there is not much literature, the reason probably being that every complete minimal submanifold of dimension at least 3 in a Euclidean space is hyperbolic; see Markvorsen and Palmer [31, Theorem 2.1].

The absence of parabolic minimal submanifolds in a given domain indicates that the domain is small or tight for minimal submanifolds. Another measure of smallness is hyperbolicity (as measured by minimal surfaces), a notion introduced in the recent paper by Forstnerič and Kalaj [16], and developed further by Drinovec Drnovšek and Forstnerič [12]. A domain $\Omega \subset \mathbb{R}^n$ is hyperbolic if for every point $p \in \Omega$, there are a neighbourhood $U \subset \Omega$ of p and a constant $c_p < \infty$ such that every conformal harmonic disc $f : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \Omega$ with $f(0) \in U$ satisfies $\|df_0\| \leqslant c_p$. This is a close analogue of Kobayashi hyperbolicity in complex analysis; see Kobayashi [27, 28]. In Section 4, we introduce the class of *weakly hyperbolic domains* — domains $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) without non-constant conformal harmonic maps $\mathbb{C} \to \Omega$. Weak hyperbolicity is an analogue of Brody hyperbolicity, which excludes non-constant holomorphic maps from $\mathbb{C}$ to a given complex manifold. A hyperbolic domain is also weakly hyperbolic; the converse holds on convex domains (see Proposition 4.4) but fails in general. In fact, we show that the class of weakly hyperbolic domains properly contains the other two mentioned classes, and there is a non-hyperbolic domain in $\mathbb{R}^3$ without parabolic minimal surfaces (see Proposition 4.2). On the other hand, it remains an open problem whether there exists a hyperbolic domain containing a parabolic minimal surface.

2 | EXCLUDING PARABOLIC MINIMAL SUBMANIFOLDS

Let $m \geqslant 2$ be an integer. A connected m-dimensional Riemannian manifold (R, g) is said to be *parabolic* if every negative subharmonic function on R is constant. A connected manifold

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immersed into a Euclidean space $\mathbb{R}^n$ is said to be parabolic if it is parabolic in the metric induced from the Euclidean metric on $\mathbb{R}^n$ by the immersion. On a surface R, parabolicity only depends on the conformal class of the metric. Every compact conformal surface punctured at finitely many points is parabolic; an example is $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$. See Grigor'yan [19] for more information. The following result is proved in Section 3.

Theorem 2.1. *Let $2 \leqslant m < n$ be integers. Assume that Ω is a domain in $\mathbb{R}^n$ with $\mathscr{C}^2$ boundary $M = b\Omega$ whose principal curvatures $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{n-1}$ satisfy $\nu_1 + \nu_2 + \cdots + \nu_m \geqslant 0$ at every point, the set of m-flat points $\{p \in M : \nu_j(p) = 0 \text{ for } j = 1, \dots, m\}$ has bounded interior in M and there is an $\epsilon > 0$ such that every point in the ϵ-neighbourhood of M has a unique nearest point in M. Then, Ω does not contain any parabolic immersed minimal submanifolds of dimension $\geqslant m$. In particular, if these conditions hold for $m = 2$, then every conformal harmonic map $R \to \Omega$ from a parabolic conformal surface R is constant.*

The minimal submanifolds in the theorem are not assumed to be complete or proper. The hypothesis on the set of m-flat points of $b\Omega$ cannot be omitted unless $3 \leqslant m = n-1$. A counterexample is a halfspace H of $\mathbb{R}^n$, which contains parabolic minimal submanifolds of dimensions $m \in \{2, \dots, n-2\}$, or $m = 2$ if $n = 3$, contained in hyperplanes parallel to bH. (Note that $\mathbb{R}^m$ with the flat metric is parabolic for $m = 2$, but is hyperbolic if $m \geqslant 3$, carrying the negative subharmonic function $-1/|x|^{n-2}$.) When $m = 2$, the minimal surfaces in the theorem are allowed to have isolated branch points, and in this case, the result is essentially optimal since many open conformal surfaces of hyperbolic type admit non-constant bounded (and even complete) conformal harmonic maps to $\mathbb{R}^3$. This holds in particular for any bordered conformal surface; see [4, Chapter 7] for a survey of this topic.

A domain $\Omega \subset \mathbb{R}^n$ with $\mathscr{C}^2$ boundary whose principal curvatures $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{n-1}$ from the inner side satisfy $\nu_1 + \nu_2 + \cdots + \nu_m \geqslant 0$ at every point of $b\Omega$ is said to be *m-convex*, and if $m = 2$, it is also called *minimally convex*; see [4, Definition 8.1.9] or Definition 3.1. By [4, Theorem 8.1.13], this holds if and only if there exist a neighbourhood $U \subset \mathbb{R}^n$ of $b\Omega$ and a $\mathscr{C}^2$ function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{\rho < 0\}$, $d\rho \neq 0$ on $b\Omega \cap U = \{\rho = 0\}$, and

$$\mathrm{tr}_\Lambda \mathrm{Hess}_\rho(x) \geqslant 0 \quad \text{for every point } x \in b\Omega \text{ and } m\text{-plane } \Lambda \subset T_x b\Omega. \tag{2.1}$$

Here, $\mathrm{tr}_\Lambda \mathrm{Hess}_\rho(x)$ denotes the trace of the restriction to Λ of the Hessian form of ρ at x. It was shown in [15] that a bounded m-convex domain with $\mathscr{C}^2$ boundary in $\mathbb{R}^n$ admits an m-plurisubharmonic defining function, that is, one satisfying condition (2.1) for every point $x \in \overline{\Omega}$ and m-plane $\Lambda \subset \mathbb{R}^n$ (see (3.4)). The main new point shown in this paper is that an unbounded domain Ω as in Theorem 2.1 admits a defining function which is m-plurisubharmonic on $\overline{\Omega}$, obtained by convexifying the signed distance function to $b\Omega$; see Proposition 3.2. This is the key fact of independent interest used in the proof of Theorem 2.1.

Note that m-convex domains play a major role in the theory of minimal submanifolds. In particular, every bounded minimally convex domain contains many properly immersed minimal surfaces parameterised by an arbitrary bordered conformal surface; see [4, Theorems 8.3.1 and 8.3.4] for the orientable case and [3, Theorem 6.9] for the non-orientable one. Such domains also form barriers for minimal submanifolds; see Jorge and Tomi [25] and Gama et al. [17], and the references therein. In [17], the authors proved the maximum principle at infinity in a very general context including parabolic minimal m-varifolds in m-convex domains. Nevertheless, I was unable to deduce Theorem 2.1 from [17, Theorem 1.1], which holds under weaker hypotheses on $b\Omega$ but seemingly stronger hypotheses on the minimal submanifolds.

The case of particular interest is when the boundary $M = b\Omega$ satisfies $\nu_1 + \cdots + \nu_{n-1} = 0$, so it is a minimal hypersurface. If M is not a hyperplane, then the set of its $(n-1)$-flat points has empty interior, and we obtain the following corollary to Theorem 2.1.

Corollary 2.2. *Assume that M is a properly embedded minimal hypersurface in $\mathbb{R}^n$ for $n \geqslant 3$ such that for some $\epsilon > 0$, every point in the ϵ-neighbourhood of M has a unique nearest point in M. Then every immersed parabolic minimal hypersurface $R \to \mathbb{R}^n$ intersects M, unless $n = 3$, M is a plane in $\mathbb{R}^3$, and the image of R is contained in a plane parallel to M.*

The condition in Theorem 2.1 and Corollary 2.1, that the $\mathscr{C}^2$ hypersurface $M = b\Omega$ admits a tubular neighbourhood of positive radius $\epsilon > 0$, is non-trivial when Ω is unbounded, which is the only case of interest. If this holds, one says that M has *positive reach*, a terminology introduced by Federer [14, Sect. 4]. The *reach* of M is the supremum of the numbers $\epsilon > 0$ having this property. A $\mathscr{C}^2$ hypersurface with positive reach has bounded principal curvatures (see the proof of Proposition 3.2). The converse holds in several cases of interest. Federer proved [14, Lemma 4.11] that a graph in $\mathbb{R}^n$ over a domain in $\mathbb{R}^{n-1}$ with sufficiently nice boundary, such that the gradient of the graphing function is Lipschitz, has positive reach. Every compact piece of an embedded $\mathscr{C}^2$ hypersurface has positive reach.

If M is a complete embedded minimal surface in $\mathbb{R}^3$ of finite total Gaussian curvature, then every end of M is a graph over the complement of a disc in $\mathbb{R}^2$ whose graphing function has bounded second-order partial derivatives, and the ends are well separated (see Jorge and Meeks [26]). Hence, every such surface has positive reach. It was shown by Meeks and Rosenberg [38, Theorem 5.3] (see also [34, Corollary 2.6.6]) that a complete embedded minimal surface with bounded curvature in $\mathbb{R}^3$ is proper and has positive reach. This gives the following corollary to Theorem 2.1, originally due to Pacelli Bessa, Jorge, and Pessoa [9, Theorem 1.1].

Corollary 2.3. *The image of a non-constant conformal harmonic map $R \to \mathbb{R}^3$ (possibly with branch points) from a parabolic conformal surface intersects every properly embedded non-flat minimal surface M in $\mathbb{R}^3$ with bounded Gaussian curvature.*

Remark 2.4. In [9, Theorem 1.1], Corollary 2.3 is proved under the weaker assumption that M is a complete non-flat immersed minimal surface in $\mathbb{R}^3$ of bounded curvature. (Unlike in the case when M is embedded, such a surface need not be proper in $\mathbb{R}^3$ as shown by Andrade [6].) Their proof for the immersed case is more involved. I thank G. Pacelli Bessa and L. F. Pessoa for having pointed out their work after an earlier version of this preprint was posted.

If one imposes stronger conditions on a minimal surface in a domain $\Omega \subset \mathbb{R}^3$, then the conclusion of Corollary 2.3 holds under weaker conditions on $b\Omega$. For example, if $\Omega \subsetneq \mathbb{R}^3$ is a smoothly bounded minimally convex domain and $R \subset \Omega$ is a complete immersed minimal surface of finite total Gaussian curvature (hence proper in $\mathbb{R}^3$), then R is a plane and Ω is a slab or a halfspace (see [1, Theorem 1.16]). This also follows from the results in [17].

Corollary 2.3 applies to every complete embedded non-flat minimal surface M in $\mathbb{R}^3$ of finite total Gaussian curvature, as well as to many minimal surfaces of infinite total curvature such as the standard helicoid, the helicoids of positive genera constructed by Hoffman, Traizet and White [24], and Riemann's minimal surfaces [35]. It was shown by Meeks, Pérez and Ros [36, Theorem

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1] that every properly embedded minimal surface in $\mathbb{R}^3$ of finite genus has bounded curvature, so Corollary 2.3 holds for such surfaces. Furthermore, every periodic properly embedded minimal surface in $\mathbb{R}^3$ whose fundamental domain is of finite topological type satisfies Corollary 2.3 by results of Meeks and Rosenberg [37]; examples include Scherk's surfaces. We refer to Meeks and Pérez [34] for more information on this topic.

Corollary 2.3 gives a partial negative answer to [13, Problem 1.16], asking whether the complement of a non-flat properly embedded minimal surface in $\mathbb{R}^3$ contains a minimal surface parameterised by $\mathbb{C}$. The case of unbounded curvature (and hence necessarily of infinite genus by Meeks, Pérez and Ros [36, Theorem 1]) remains open.

Problem 2.5. Let M be a properly embedded minimal surface in $\mathbb{R}^3$ with unbounded curvature. Is there a non-constant conformal harmonic map $\mathbb{C} \to \mathbb{R}^3$ whose image avoids M?

3 | PROOF OF THEOREM 2.1

We begin with preliminaries. Recall that a map $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ from an open set $U \subset \mathbb{C}$ is said to be conformal if it preserves angles at every immersion point; equivalently, if $z = x + \imath y$ is a complex coordinate on U, then the partial derivatives of f satisfy $f_x \cdot f_y = 0$ and $|f_x| = |f_y|$. Here, the dot denotes the Euclidean inner product and $|\cdot|$ the Euclidean length. These conditions imply that the rank of f at any point is either two (an immersion point) or zero (a branch point). The analogous definition applies to maps from any conformal surface in local isothermal coordinates. The image of a non-constant conformal harmonic map from an open conformal surface is a minimal surface with isolated branch points; conversely, every minimal surface with isolated branch points is of this form. For background, see, for example, the monographs [4, 11, 39] and the surveys [2, 33, 34].

Given a subset M of $\mathbb{R}^n$, we denote by $\mathrm{dist}(\cdot, M)$ the Euclidean distance function to M:

$$\mathrm{dist}(x, M) = \inf\{|x - p| : p \in M\}, \quad x \in \mathbb{R}^n.$$

For $p \in \mathbb{R}^n$ and $\epsilon > 0$, we let $\mathbb{B}(p, \epsilon) = \{x \in \mathbb{R}^n : |x - p| < \epsilon\}$ and

$$V_\epsilon(M) = \bigcup_{p \in M} \mathbb{B}(p, \epsilon) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, M) < \epsilon\} \tag{3.1}$$

Let M be a properly embedded connected hypersurface in $\mathbb{R}^n$. Such M is orientable and its complement $\mathbb{R}^n \setminus M = M^+ \cup M^-$ consists of a pair of connected domains. The signed distance function to M is defined by

$$\delta_M(x) = \begin{cases} \mathrm{dist}(x, M), & \text{if } x \in M_+ \cup M; \\ -\mathrm{dist}(x, M), & \text{if } x \in M_-. \end{cases} \tag{3.2}$$

If $V \subset \mathbb{R}^n$ is an open neighbourhood of M such that every point $x \in V$ has a unique nearest point $p = \xi(x) \in M$, and if M is of class $\mathscr{C}^r$ for some $r \in \{2, 3, \dots, \infty, \omega\}$, then δ_M is also of class $\mathscr{C}^r$ on V (see Gilbarg and Trudinger [18, Lemma 14.16] or Krantz and Park [29]).

We recall the following notion; see [4, Sections 8.1.4–8.1.5].

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Definition 3.1. An oriented embedded $\mathscr{C}^2$ hypersurface $M \subset \mathbb{R}^n$ is *m-convex* for some $m \in \{1, 2, \dots, n-1\}$ if its principal curvatures $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{n-1}$ satisfy

$$\nu_1(p) + \nu_2(p) + \cdots + \nu_m(p) \geqslant 0 \quad \text{for all } p \in M. \tag{3.3}$$

The hypersurface M is *strongly m-convex* at $p \in M$ if strong inequality holds in (3.3). A point $p \in M$ is *m-flat* if $\nu_j(p) = 0$ for $j = 1, \dots, m$. We say that M *is not m-flat at infinity* if the set $\{p \in M : \nu_j(p) = 0 \text{ for } j = 1, \dots, m\}$ has bounded relative interior in M.

Given a $\mathscr{C}^2$ function $\rho : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{R}^n$, we denote by $\mathrm{Hess}_\rho(x)$ the Hessian of ρ at the point $x \in \Omega$, that is, the quadratic form on $T_x\mathbb{R}^n = \mathbb{R}^n$ represented by the matrix $(\frac{\partial^2 \rho}{\partial x_i \partial x_j}(x))$ of second-order partial derivatives of ρ at x. For $1 \leqslant m < n$, let $G_m(\mathbb{R}^n)$ denote the Grassman manifold of m-planes in $\mathbb{R}^n$. Given $\Lambda \in G_m(\mathbb{R}^n)$, we denote by $\mathrm{tr}_\Lambda \mathrm{Hess}_\rho(x) \in \mathbb{R}$ the trace of the restriction of the Hessian $\mathrm{Hess}_\rho(x)$ to Λ. A $\mathscr{C}^2$ function $\rho : \Omega \to \mathbb{R}$ is said to be *m-plurisubharmonic* if

$$\mathrm{tr}_\Lambda \mathrm{Hess}_\rho(x) \geqslant 0 \quad \text{for all } x \in \Omega \text{ and } \Lambda \in G_m(\mathbb{R}^n). \tag{3.4}$$

If $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_n(x)$ are the eigenvalues of $\mathrm{Hess}_\rho(x)$, then (3.4) is equivalent to

$$\lambda_1(x) + \cdots + \lambda_m(x) \geqslant 0 \quad \text{for every } x \in \Omega.$$

In fact, $\lambda_1(x) + \cdots + \lambda_m(x) = \inf_\Lambda \mathrm{tr}_\Lambda \mathrm{Hess}_\rho(x)$. (See [4, Sect. 8.1] or [22].) The main point is that a function $\rho : \Omega \to \mathbb{R}$ is m-plurisubharmonic if and only if its restriction to every m-dimensional minimal submanifold in Ω is a subharmonic function on the submanifold (see [4, Proposition 8.1.2]).

The notion of an m-plurisubarmonic function extends to upper semicontinuous functions by asking that for any m-dimensional affine subspace L of $\mathbb{R}^n$, the restriction of the function to $L \cap \Omega$ is a subharmonic function (see [4, Definition 8.1.1]). Such a function can be approximated from above on any relatively compact subdomain of Ω by smooth m-plurisubarmonic functions (see [4, Proposition 8.1.6]). In this paper, we shall use continuous m-plurisubharmonic functions obtained by taking the maximum of a smooth m-plurisubharmonic function and a constant, and in this case, smoothing can be obtained globally in a simple way; see the last part of the proof of Proposition 3.2.

Theorem 2.1 is a consequence of the following proposition of independent interest.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\mathscr{C}^r$ boundary for some $r \geqslant 2$ satisfying the assumptions in Theorem 2.1. There is a function $\rho : \overline{\Omega} \to (-\infty, 0]$ of class $\mathscr{C}^r(\overline{\Omega})$ which is m-plurisubharmonic on Ω, it vanishes on $M = b\Omega$ and it satisfies $d\rho \neq 0$ on $\rho^{-1}((-1, 0])$ and for every $t \in (-1, 0)$, the level set $M_t = \{\rho = t\}$ coincides with a level set of the signed distance function δ_M (3.2), M_t is m-convex, it is strongly m-convex at every m-non-flat point and M_t is not m-flat near infinity.*

We point out that, for the proof of Theorem 2.1, it suffices to find a continuous m-plurisubharmonic function $\rho : \overline{\Omega} \to (-\infty, 0]$, which is of class $\mathscr{C}^2$ on an interior collar around $M = b\Omega$ and has the other stated properties. Such a function is given by (3.18).

Proof. By Gilbarg and Trudinger [18, Lemma 14.16] or Krantz and Parks [29], the conditions on $M = b\Omega$ imply that the signed distance function $\delta = \delta_M$ to M (3.2) is of class $\mathscr{C}^r$ on the ϵ-neighbourhood $V_\epsilon(M)$ of M (see (3.1)). We choose the sign so that $\delta < 0$ on

$$C_\epsilon := \Omega \cap V_\epsilon(M) = \{x \in \Omega : \mathrm{dist}(x, M) < \epsilon\}. \tag{3.5}$$

We recall some further properties of δ, referring to Bellettini [7, Theorem 1.18, p. 14] and Gilbarg and Trudinger [18, Section 14.6]. There is a projection $\xi : V_\epsilon(M) \to M$ of class $\mathscr{C}^{r-1}$ such that for every $x \in V_\epsilon(M)$, the point $p = \xi(x) \in M$ is the unique nearest point to x in M. The gradient $\nabla\delta$ has constant norm $|\nabla\delta| = 1$ on $V_\epsilon(M)$, and it has constant value on the intersection of $V_\epsilon(M)$ with the normal line $N_p = p + \mathbb{R} \cdot \nabla\delta(p)$ at $p \in b\Omega = M$. There an orthonormal basis $(v_1, \dots, v_{n-1}, v_n = \nabla\delta(p))$ of $\mathbb{R}^n$ such that the vectors $v_1, \dots, v_{n-1}$ span T_pM, and in this basis, the (symmetric) matrix $A(p)$ of $\mathrm{Hess}_\delta(p)$ is diagonal:

$$A(p)v_j = \nu_j v_j \quad \text{for } j = 1, \dots, n-1; \quad A(p)v_n = 0, \tag{3.6}$$

where the numbers $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{n-1}$ are the principal normal curvatures of M at p from the interior side. For any point $x \in N_p \cap V_\epsilon(M)$, the same basis $(v_1, \dots, v_n)$ diagonalises $\mathrm{Hess}_\delta(x)$, with the corresponding eigenvalues

$$\nu_j(x) = \frac{\nu_j}{1 + \delta(x)\nu_j} \quad \text{for } j = 1, \dots, n-1; \quad A(p)v_n = 0. \tag{3.7}$$

If $\nu_j > 0$ for some j, then the above shows that $1 + \delta(x)\nu_j > 0$ for all $x \in C_\epsilon \cap N_p$. Since $\delta(x)$ assumes all values in $(-\epsilon, 0)$ on the set C_ϵ (3.5), this implies $1 - \epsilon\nu_j \geqslant 0$ and hence $\nu_j \leqslant 1/\epsilon$ for all such j. From this and the hypothesis $\nu_1 + \nu_2 + \cdots + \nu_m \geqslant 0$, it also follows that $\nu_j \geqslant -(m-1)/\epsilon$ for every $j = 1, \dots, n-1$. Summarising, we have that

$$-(m-1)/\epsilon \leqslant \nu_j(p) \leqslant 1/\epsilon \quad \text{for all } j = 1, \dots, n-1 \text{ and } p \in M. \tag{3.8}$$

That is, the hypersurface $M = b\Omega$ has bounded principal curvatures. The worst case estimate in the first inequality in (3.8) occurs only when $\nu_1 = -(m-1)/\epsilon$ and $\nu_j = 1/\epsilon$ for $j = 2, \dots, m$, so we also conclude that

$$\sum_{\nu_j(p) \leqslant 0} \nu_j(p) \geqslant -\frac{m-1}{\epsilon} \quad \text{for all } p \in M. \tag{3.9}$$

Consider the family of domains

$$\Omega_t = \{x \in C_\epsilon : \delta(x) < t\} \cup (\Omega \setminus C_\epsilon) \quad \text{for } t \in (-\epsilon, 0].$$

As t increases towards 0, the domains Ω_t increase to $\Omega_0 = \Omega$. The tangent space to the hypersurface $b\Omega_t = \{\delta = t\}$ at the point $x = p + t\nabla\delta(p)$ is spanned by the same vectors $v_1, \dots, v_{n-1}$ as T_pM (see (3.6)), and the numbers $\nu_j(x)$ in (3.7) for $j = 1, \dots, n-1$ are the principal curvatures of $b\Omega_t$ at x. As $\delta(x)$ decreases (we move away from M), each of the curvatures $\nu_j(x)$ strictly increases unless $\nu_j(p) = 0$ (in whihc case $\nu_j(x) = 0$), and it does not change the sign. This implies that for every $x = p + t\nabla\delta(p) \in b\Omega_t$ with $t \in (-\epsilon, 0)$, we have

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$$\nu_1(x) \leqslant \nu_2(x) \leqslant \cdots \leqslant \nu_{n-1}(x), \quad \nu_j(x) \geqslant \nu_j(p) \ \text{ for } j = 1, \dots, n-1, \ \text{ and} \tag{3.10}$$

$$H(x) := \nu_1(x) + \nu_2(x) + \cdots + \nu_m(x) \geqslant \nu_1(p) + \nu_2(p) + \cdots + \nu_m(p) \geqslant 0, \tag{3.11}$$

where the first inequality in (3.11) is equality if and only if the point $p \in M$ is m-flat (see Definition 3.1). Indeed, the function H increases as we move away from $M = b\Omega$ into Ω, and it vanishes on $N_p \cap C_\epsilon$ if and only if the point $p \in M$ is m-flat. We conclude that for $t \in (-\epsilon, 0)$, the hypersurface $M_t = b\Omega_t$ is m-convex, and it is strongly m-convex at $x = p + t\nabla\delta(p) \in C_\epsilon$ if and only if the point $p = \xi(x) \in M$ is not m-flat. In particular, since M is not m-flat near infinity by the assumption, M_t is not m-flat near infinity for any $t \in (-\epsilon, 0)$.

From (3.7)–(3.10), it follows that for every $x \in C_{\epsilon/2}$ (see (3.5)), we have that

$$-\frac{m-1}{\epsilon} \leqslant \nu_j(x) \leqslant \frac{2}{\epsilon} \ \text{ for } j = 1, \dots, n-1, \text{ and} \tag{3.12}$$

$$\sum_{\nu_j(x) \leqslant 0} \nu_j(x) \geqslant -\frac{m-1}{\epsilon}. \tag{3.13}$$

Despite the estimate (3.11), the function δ need not be m-plurisubharmonic on C_ϵ due to the zero eigenvalue of Hess δ in the normal direction determined by $\nabla\delta$. However, we now show that a suitable convexification of δ on a collar $C_{\epsilon_0} \subset \Omega$ in (3.5) for some $0 < \epsilon_0 < \epsilon/2$ extends to an m-plurisubharmonic function $\rho : \Omega \to (-\infty, 0)$ such that every level set

$$M_t = \{\rho = t\} \ \text{ for } t \in (-\epsilon_0, 0)$$

coincides with a level set $\{\delta = \tau\}$ for some $\tau = \tau(t) \in (-\epsilon, 0)$. The argument is similar to [15, proof of Theorem 1.1], but we are now dealing with an unbounded domain Ω, and we obtain uniform estimates of the quantities involved.

Choose a number $\alpha > (m-1)/\epsilon$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth, convex, increasing function with $h(0) = 0$, $\dot{h}(0) = 1$, and $\ddot{h}(0) = \alpha$. Then,

$$h(t) < 0 \ \text{ and } \ 0 \leqslant \dot{h}(t) < 1 \ \text{ for } \ -\infty < t < 0. \tag{3.14}$$

Choose numbers ϵ_0, ϵ_0' with $0 < \epsilon_0 < \epsilon_0' < \epsilon/2$ such that

$$\ddot{h}(t) > \frac{m-1}{\epsilon} \ \text{ for } -\epsilon_0' \leqslant t \leqslant 0. \tag{3.15}$$

Consider the function $h \circ \delta : C_\epsilon \to (-\infty, 0)$. We have that

$$\mathrm{Hess}_{h\circ\delta} = (\dot{h}\circ\delta)\mathrm{Hess}_\delta + (\ddot{h}\circ\delta)\nabla\delta \cdot (\nabla\delta)^T. \tag{3.16}$$

(Here, $(\nabla\delta)^T$ is the transpose of $\nabla\delta$, and $\nabla\delta \cdot (\nabla\delta)^T$ is an $n \times n$ matrix.) Recall that for $p \in M$, the orthonormal vectors $v_1, \dots v_{n-1}, v_n = \nabla\delta(p)$ diagonalise $\mathrm{Hess}_\delta(p)$, where $T_pM = \mathrm{span}\{v_1, v_2, \dots, v_{n-1}\}$. Note that T_pM lies in the kernel of the matrix $\nabla\delta \cdot (\nabla\delta)^T$ while v_n is an eigenvector with eigenvalue 1. The same basis then diagonalises $\mathrm{Hess}_{h\circ\delta}(x)$ at every point $x = p + \delta(x)v_n \in N_p \cap C_\epsilon$, the eigenvalues corresponding to $v_1, \dots, v_{n-1}$ get multiplied by $\dot{h}(\delta(x)) \geqslant 0$ and the eigenvalue in the normal direction v_n equals $\ddot{h}(\delta(x))$. Summarising, the eigenvalues of

$\mathrm{Hess}_{h\circ\delta}(x)$ at any point $x \in C_\epsilon$ equal

$$\dot h(\delta(x))\nu_1(x), \dots, \dot h(\delta(x))\nu_{n-1}(x), \quad \ddot h(\delta(x)). \tag{3.17}$$

If $x \in C_{\epsilon_0'}$ then $-\epsilon_0' < \delta(x) < 0$, and hence by (3.13), (3.14) and (3.15), we have that

$$\sum_{\nu_j(x)\leqslant 0} \dot h(\delta(x))\nu_j(x) + \ddot h(\delta(x)) > 0.$$

Together with (3.10) and (3.11), it follows that the sum of any m numbers in the list (3.17) is non-negative at every point $x \in C_{\epsilon_0'}$, that is, the function $h\circ\delta$ is m-plurisubharmonic on $C_{\epsilon_0'}$. Hence, the continuous function $\rho_0 : \Omega \to [h(-\epsilon_0), 0)$ given by

$$\rho_0(x) = \begin{cases} h(\delta(x)), & \text{if } x \in C_{\epsilon_0}; \\ h(-\epsilon_0), & \text{if } x \in \Omega \setminus C_{\epsilon_0} \end{cases} \tag{3.18}$$

is m-plurisubharmonic. Indeed, near bC_{ϵ_0}, we have that $\rho_0 = \max\{h\circ\delta, h(-\epsilon_0)\}$, and the maximum of two m-plurisubharmonic functions is m-plurisubharmonic (see [21, Sect. 6]).

To get a smooth m-plurisubharmonic function $\rho \in \mathscr{C}^r(\Omega)$ which agrees with ρ_0 on a smaller collar $C_{\epsilon_1} \subset C_{\epsilon_0}$, we choose numbers $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$, pick a smooth increasing convex function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(t)$ is a negative constant for $t \leqslant h(-\epsilon_2)$ and $\chi(t) = t$ for $t \geqslant h(-\epsilon_1)$ and set $\rho = \chi\circ\rho_0$. Clearly, ρ is well defined on $\overline{\Omega}$ and of class $\mathscr{C}^r$, and it is constant on $\Omega \setminus C_{\epsilon_2}$. Using the formula (3.16) for points in C_{ϵ_0} gives

$$\mathrm{Hess}_{\chi\circ\rho_0} = (\dot\chi\circ\rho_0)\mathrm{Hess}_{\rho_0} + (\ddot\chi\circ\rho)\nabla\rho_0 \cdot (\nabla\rho_0)^T.$$

Since $\dot\chi\circ\rho_0 \geqslant 0$ and $\ddot\chi\circ\rho \geqslant 0$, ρ_0 is m-plurisubharmonic on C_{ϵ_0}, and the matrix $\nabla\rho_0 \cdot (\nabla\rho_0)^T$ is non-negative definite, we infer that $\rho = \chi\circ\rho_0$ is of class $\mathscr{C}^r$ and m-plurisubharmonic on $\overline{\Omega}$. Clearly, ρ satisfies the conclusion of the proposition for all $t \in (h(-\epsilon_1), 0)$. For such t, we have $\chi(t) = t$ and hence the hypersurface $M_t = \{\rho = t\} = \{h\circ\delta = t\}$ equals

$$M_t = \{\delta = \tau\} \text{ for } \tau = h^{-1}(t) \in (-\epsilon_1, 0). \tag{3.19}$$

Replacing ρ by $c\rho$ with $c = -1/h(-\epsilon_1) > 0$, this holds for $t \in (-1, 0)$. □

Proof of Theorem 2.1. Since an m-convex domain is also k-convex for any $m < k \leqslant n-1$, and every k-flat point is also m-flat, it suffices to show that the domain Ω in the theorem does not contain any parabolic minimal surfaces of dimension m.

Let $\rho : \Omega \to (-\infty, 0)$ be an m-plurisubharmonic function given by Proposition 3.2. If $f : R \to \Omega$ is an immersed minimal submanifold of dimension m, then $\rho\circ f$ is a negative function on R which is subharmonic in the Riemannian metric induced by f from the Euclidean metric on $\mathbb{R}^n$. If in addition, this minimal surface is connected and parabolic, then this function is constant. Thus, either $f(R) \subset M_t = \{\rho = t\}$ for some $t \in (-1, 0)$, or $f(R) \subset \Omega \setminus C_{\epsilon_1}$ where $\epsilon_1 > 0$ is as in (3.19) and C_{ϵ_1} is given by (3.5).

Assume that the first possibility holds. The minimal m-dimensional submanifold $f(R) \subset M_t$ is clearly contained in the set of points where the hypersurface M_t is not strongly m-convex. By Proposition 3.2, M_t is m-flat at every point where it fails to be strongly m-convex, and the set of its

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m-flat points has bounded interior. Since $f(R)$ is parabolic, it cannot be contained in a bounded set, so this case is impossible.

Therefore, $f(R) \subset \Omega \setminus C_{\epsilon_1}$, so the distance between $f(R)$ and $b\Omega$ is at least $\epsilon_1 > 0$. If we translate Ω for the distance $\epsilon_1/2$ in any direction, the same argument (with the same constants) applies to the translated domain Ω' and its boundary $b\Omega'$. Since $f(R) \subset \Omega'$, we infer that $\mathrm{dist}(f(R), b\Omega') \geqslant \epsilon_1$. Since this holds for every translate Ω' of Ω for $\epsilon_1/2$, we conclude that $\mathrm{dist}(f(R), b\Omega) \geqslant 3\epsilon_1/2$. Repeating this argument shows that the distance from $f(R)$ to $b\Omega$ must be infinite, a contradiction.

If $m = 2$, the same argument shows that there are no non-constant conformal harmonic maps $f : R \to \Omega$ (possibly with branch points) from any parabolic conformal surface R. (In this case, parabolicity does not depend on f but only on the conformal structure of R.) □

Remark 3.3 (Concerning the paper [15]). By [15, Theorem 1.1], every bounded domain Ω in $\mathbb{R}^n (n \geqslant 3)$, whose boundary is of class $\mathscr{C}^{r,\alpha}$ for some $r \geqslant 2$ and $0 < \alpha \leqslant 1$ and is m-convex for some $m \in \{1, \dots, n-1\}$, admits an m-plurisubharmonic defining function of class $\mathscr{C}^{r,\alpha}$. The proof in the cited paper refers to the result of Li and Nirenberg [30], saying that the signed distance function to a hypersurface of class $\mathscr{C}^{r,\alpha}$ for such (r, α) is of the same class $\mathscr{C}^{r,\alpha}$ near the hypersurface. By using the result of Gilbarg and Trudinger [18, Lemma 14.16] and Krantz and Parks [29], we see that [15, Theorem 1.1] also holds in smoothness classes $\mathscr{C}^r$ for $r = 2, 3, \dots, \infty, \omega$.

The second remark is that the last statement in [15, Theorem 1.1], concerning strongly m-convex domains, is one of several special cases of a result by Harvey and Lawson [20, Theorem 5.12]. I wish to thank the authors for the relevant communication.

4 | WEAKLY HYPERBOLIC DOMAINS

As mentioned in the introduction, the second motivation for this paper comes from the recently introduced hyperbolicity theory for domains in $\mathbb{R}^n$ ($n \geqslant 3$) in terms of minimal surfaces. We recall the background.

In 2021, Forstnerič and Kalaj [16] defined on any domain Ω in $\mathbb{R}^n$ for $n \geqslant 3$ a Finsler pseudometric $g_\Omega : T\Omega = \Omega \times \mathbb{R}^n \to \mathbb{R}_+ = [0, +\infty)$, called the *minimal pseudometric*, and the associated pseudodistance $\mathrm{dist}_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ obtained by integrating g_Ω, by using conformal harmonic discs $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \Omega$, in the same way as the Kobayashi–Royden pseudometric and pseudodistance are defined on any complex manifold by using holomorphic discs; see [27, 28, 41]. We recall the definition. Let $\mathrm{CH}(\mathbb{D}, \Omega)$ denote the space of conformal harmonic discs $f : \mathbb{D} \to \Omega$, possibly with branch points. Denote by $z = x + \imath y$ the complex coordinate on $\mathbb{D}$ and by f_x the partial derivative of f on x. The minimal pseudometric at $p \in \Omega$ on a vector $v \in \mathbb{R}^n$ is defined by

$$g_\Omega(p, v) = \inf\left\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = p,\ f_x(0) = rv\right\} \geqslant 0.$$

It follows from definitions that dist_Ω is the biggest pseudometric on Ω such that every conformal harmonic map $R \to \Omega$ from a conformal surface R is distance decreasing with respect to the Poincaré pseudometric on R. Thus, dist_Ω describes the fastest possible rate of growth of hyperbolic minimal surfaces in Ω. A domain Ω is said to be *hyperbolic* if dist_Ω is a distance function, and *complete hyperbolic* if $(\Omega, \mathrm{dist}_\Omega)$ is a complete metric space. Every bounded domain is hyperbolic but not necessarily complete hyperbolic. On the unit ball of $\mathbb{R}^n$, the minimal metric coincides with the Beltrami–Cayley–Klein metric, one of the classical models of hyperbolic geometry; see [16]. We refer to [12] for the basic hyperbolicity theory for domains in Euclidean spaces.

We now introduce the following weaker notion of hyperbolicity, which is of interest on unbounded domains where hyperbolicity remains poorly understood.

Definition 4.1. A domain Ω in $\mathbb{R}^n$, $n \geqslant 3$, is *weakly hyperbolic* (for minimal surfaces) if every conformal harmonic map $\mathbb{C} \to \Omega$ is constant.

Weak hyperbolicity is an analogue of Brody hyperbolicity in complex analysis: a complex manifold X is said to be *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant. Every Kobayashi hyperbolic manifold is clearly also Brody hyperbolic; the converse holds on compact complex manifolds (see Brody [10]) but it fails in general (see [10, p. 219]).

Likewise, every hyperbolic domain in $\mathbb{R}^n$ is weakly hyperbolic but the converse fails in general. For the first claim, assume that $\Omega \subset \mathbb{R}^n$ is not weakly hyperbolic. Pick a non-constant conformal harmonic map $f : \mathbb{C} \to \Omega$ and a point $a \in \mathbb{C}$ where $df_a \neq 0$. The conformal harmonic discs $f_r : \mathbb{D} \to \Omega$ given by $f_r(z) = f(a + rz)$ for $r > 0$ satisfy $f_r(0) = f(a)$ and $d(f_r)_0 = r df_a$. Letting $r \to +\infty$, we get $g_\Omega(f(a), v) = 0$ for all $v \in df_a(\mathbb{C})$, so Ω is not hyperbolic.

The failure of the converse implication is given by part (b) of the following proposition. Part (c) shows that a weakly hyperbolic domain may contain parabolic minimal surfaces whose universal conformal covering surface is the disc.

Proposition 4.2.

(a) *There is a non-hyperbolic domain in $\mathbb{R}^3$ which does not contain any parabolic minimal surfaces.*
(b) *In particular, there is a weakly hyperbolic domain in $\mathbb{R}^3$ which is not hyperbolic.*
(c) *For every parabolic domain $D \subsetneq \mathbb{R}^2 = \mathbb{C}$ which omits at least two points of $\mathbb{C}$, there is a weakly hyperbolic but non-hyperbolic domain $\Omega_D \subset \mathbb{R}^3$ containing $D \times \{0\}$ as a proper minimal surface.*

Proof. Given a domain $D \subset \mathbb{R}^2$, we define the domain $\Omega_D \subset \mathbb{R}^3$ by

$$\Omega_D = \left\{(x, y, z) \in \mathbb{R}^3 : |z| < 1,\ z^2(x^2 + y^2) < 1,\ (x, y) \in D \text{ if } z = 0\right\}.$$

We claim that the following assertions hold.

(i) The minimal distance in Ω_D between any pair of points $p, q \in D \times \{0\}$ vanishes.
(ii) The image of every non-constant conformal harmonic map $R \to \Omega_D$ from a parabolic conformal surface R is contained in $D \times \{0\}$.

To prove the first claim (i), assume that $p = (a, b, 0)$ and $q = (c, d, 0)$. Set $p_k = (a, b, 1/k)$ and $q_k = (c, d, 1/k)$ for $k = 2, 3, \ldots$. It is obvious that $\lim_{k\to\infty} \mathrm{dist}_{\Omega_D}(p, p_k) = 0$ and $\lim_{k\to\infty} \mathrm{dist}_{\Omega_D}(q, q_k) = 0$. The sequence of linear discs $\{(x, y, 1/k) : x^2 + y^2 < k^2\} \subset \Omega_D$ shows that $\lim_{k\to\infty} \mathrm{dist}_{\Omega_D}(p_k, q_k) = 0$. Hence, $\mathrm{dist}_{\Omega_D}(p, q) = 0$.

To prove the second claim (ii), assume that R is a parabolic conformal surface and $f = (f_1, f_2, f_3) : R \to \Omega_D$ is a non-constant conformal harmonic map. We have that $|f_3| < 1$, so $f_3 = c$ is constant. If $c \neq 0$, then $f_1^2 + f_2^2 < 1/c^2 < +\infty$, so f_1 and f_2 are constant as well, a contradiction. Thus, $c = 0$ and hence $f(R) \subset D \times \{0\}$.

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Taking D to be a bounded domain in $\mathbb{R}^2$, property (ii) implies that Ω_D does not contain any parabolic minimal surfaces. Since Ω_D is non-hyperbolic by property (i), such a domain satisfies condition (a) in the proposition. Part (b) is an immediate consequence.

Let $D \subset \mathbb{C}$ be a parabolic domain which omits at least two points of $\mathbb{C}$. By (ii), the image of every non-constant conformal harmonic map $\mathbb{C} \to \Omega_D$ is contained in $D \times \{0\}$, which contradicts Picard's theorem. Thus, Ω_D is weakly hyperbolic but it contains the parabolic minimal surface $D \times \{0\}$. This proves part (c) of the proposition. □

Problem 4.3. Is there a hyperbolic domain $\Omega \subset \mathbb{R}^3$ containing a parabolic minimal surface? In particular, if $D \subset \mathbb{R}^2 = \mathbb{C}$ is a parabolic domain which omits at least two points of $\mathbb{C}$, is the domain $D \times (-\epsilon, \epsilon) \subset \mathbb{R}^3$ hyperbolic for some (or all) $\epsilon > 0$?

Note that the domain $(\mathbb{C} \setminus \{0, 1\}) \times \mathbb{D}$ in $\mathbb{C}^2$ is Kobayashi hyperbolic. The difference between the two cases is that the projection of a complex curve to any factor in a product of complex manifolds is holomorphic, while the projection of a conformal harmonic map is harmonic but not conformal in general. It is easily seen that $\mathbb{C} \setminus \{0, 1\}$ contains many harmonic images of $\mathbb{C}$ which are conformal at some point, but not everywhere.

The situation is much simpler for convex domains as shown by the following proposition.

Proposition 4.4. *For a convex domain Ω in $\mathbb{R}^n$, $n \geqslant 3$, the following are equivalent.*

(a) *Ω is complete hyperbolic.*
(b) *Ω is hyperbolic.*
(c) *Ω is weakly hyperbolic.*
(d) *Ω does not contain any affine 2-plane.*
(e) *Ω does not contain any parabolic minimal surface.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ and $(e) \Rightarrow (d)$ are trivial. The implication $(d) \Rightarrow (a)$ was proved in [12, Theorem 5.1], which establishes the equivalence of properties (a)–(d). It remains to prove $(d) \Rightarrow (e)$. It was shown in [12, proof of Theorem 5.1] that a convex domain Ω in $\mathbb{R}^n$ satisfying condition (d) is contained in the intersection of $n-1$ halfspaces $H_j = \{\ell_j < c_j\}$ $(j = 1, \dots, n-1)$, where $\ell_1, \dots, \ell_{n-1} : \mathbb{R}^n \to \mathbb{R}$ are linearly independent linear functionals and c_j are constants. If $f : R \to \Omega$ is a conformal harmonic map from a connected parabolic conformal surface R, then $\ell_j \circ f : R \to (-\infty, c_j)$ is a bounded from above harmonic function for every $j = 1, \dots, n-1$, hence constant. Thus, $f(R)$ lies in a real line and hence f is constant. □

A property of a domain directly opposite to weak or strong hyperbolicity is *flexibility* for minimal surfaces, introduced in [13, Definition 1.1]. A domain $\Omega \subset \mathbb{R}^n$ for $n \geqslant 3$ is said to be flexible if, given an open conformal surface M, a compact set $K \subset M$ whose complement has no relatively compact connected components, and a conformal harmonic immersion $f : U \to \Omega$ from an open neighbourhood of K, we can approximate f uniformly on K by conformal harmonic immersions $M \to \Omega$. A flexible domain admits many conformal harmonic images of $\mathbb{C}$, so it is not weakly hyperbolic. In particular, the domains in Theorem 2.1 for $m = 2$ are not flexible. A halfspace is neither (weakly) hyperbolic nor flexible.

It is not surprising that the situation regarding Corollary 2.3 is quite different in $\mathbb{R}^4$. As shown in [13, Example 1.9], a domain in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4), given by

$$\Omega = \{x_4 > -a|x_2| + b|x_3| \quad \text{for some } a > 0 \text{ and } b \in \mathbb{R}\},$$

is flexible. Taking $b > 0$, the complementary domain $\Omega' = \mathbb{R}^4 \setminus \overline{\Omega}$ is of the same type with the reversed roles of x_2 and x_3, and with x_4 replaced by $-x_4$, so it is also flexible. By Alarcón and López [5], each of these domains contains a properly immersed conformal minimal surface parameterised by an arbitrary open Riemann surface. (In fact, their result holds for any concave wedge in $\mathbb{R}^3$ obtained by intersecting Ω with the hyperplane $x_3 = 0$.) This gives many pairs of disjoint properly immersed minimal surfaces in $\mathbb{R}^4$ of any given conformal type. There also exist pairs of disjoint catenoids in $\mathbb{R}^4$ whose ends are asymptotic to a pair of orthogonal 2-planes in $\mathbb{R}^4$, so their closures in $\mathbb{RP}^4$ are disjoint as well. On the other hand, a pair of complex algebraic curves in $\mathbb{C}^2$ intersect, or their closures in $\mathbb{CP}^2$ intersect at infinity. (Note that every complex curve is also a minimal surface.)

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