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On a definition of logarithm of quaternionic functions

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Abstract. For a slice-regular quaternionic function f, the classical exponential function $\exp f$ is not slice-regular in general. An alternative definition of an exponential function, the $*$-exponential $\exp_*$, was given in the work by Altavilla and de Fabritiis (2019): if f is a slice-regular function, then $\exp_* f$ is a slice-regular function as well. The study of a $*$-logarithm $\log_* f$ of a slice-regular function f becomes of great interest for basic reasons, and is performed in this paper. The main result shows that the existence of such a $\log_* f$ depends only on the structure of the zero set of the vectorial part f_v of the slice-regular function $f = f_0 + f_v$, besides the topology of its domain of definition. We also show that, locally, every slice-regular nonvanishing function has a $*$-logarithm and, at the end, we present an example of a nonvanishing slice-regular function on a ball which does not admit a $*$-logarithm on that ball.

1. Introduction

Let $\mathbb{H}$ be the skew field of quaternions and let us denote the 2-sphere of imaginary units of $\mathbb{H}$ by $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$. Consider the natural exponential function $\exp\colon \mathbb{H} \to \mathbb{H} \setminus \{0\}$ defined by the classical power series

$$\exp q = \sum_{n=0}^{+\infty} \frac{q^n}{n!}. \tag{1.1}$$

In the case of quaternions, a satisfactory definition of a (necessarily local) inverse of this exponential function – the logarithm and its different branches – is not a simple task, together with the question of the continuation of the logarithm along curves lying in $\mathbb{H} \setminus \{0\}$ (see [4, 5, 11] and references therein).

Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric domain (see Definition 2.1), and consider the class $\mathcal{SR}(\Omega)$ of all $\mathbb{H}$-valued slice-regular functions defined in Ω (see, e.g., [7]). These functions have proven to be naturally suitable to play the role of holomorphic functions in the quaternionic setting, and have originated a theory that is by now quite rich and well developed (see, e.g., [6] and references therein). Slice-regular functions present several peculiarities, mainly due to the noncommutative setting of quaternions; among these peculiarities, the facts that pointwise product and composition of slice-regular functions

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do not produce slice-regular functions in general. The definition of the $*$-product which is typical for the algebra of polynomials with coefficients in a noncommutative field can be extended to the class of slice-regular functions on an axially symmetric domain $\Omega \subseteq \mathbb{H}$, which naturally becomes an algebra. As for composition, if $f\colon \mathbb{H} \to \mathbb{H}$ is a slice-regular function, even

$$\exp(f(q)) = \sum_{n=0}^{+\infty} \frac{f(q)^n}{n!}$$

turns out not to be slice-regular in general. The $*$-product helps in this situation to find an exponential function which maintains slice-regularity, defined (with obvious notations) as

$$\exp_* f(q) = \sum_{n=0}^{+\infty} \frac{f^{*n}(q)}{n!}. \tag{1.2}$$

This $*$-exponential has many interesting properties typical of an exponential-type function, which can be found, e.g., in [1].

In this paper, we investigate the existence of a slice-regular logarithm $\log_* f$ for a slice-regular function f. This activity finds a deep motivation in the study of quaternionic Cousin problems, that the authors are performing and that will be the object of a forthcoming paper.

We will now briefly outline the path that this paper follows for the tuning of a slice-regular logarithm. Recall that (see [8]) any slice-regular function f defined on an axially symmetric domain Ω can be uniquely written, with respect to the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$, as

$$f = f_0 + f_1 i + f_2 j + f_3 k = f_0 + f_v$$

where f_ℓ ($\ell = 0, 1, 2, 3$) are slice-preserving regular functions, and where

$$f_0(q) = \frac{f(q) + \overline{f(\bar{q})}}{2}$$

denotes the *scalar* part of f and

$$f_v := f - f_0$$

its *vectorial part*. The vectorial part f_v of f plays a fundamental role in the definition of $\log_*$. Indeed, with the adopted notations we have

$$\begin{aligned}
\exp_* f &= \exp_*(f_0 + f_v) = \exp f_0 \exp_* f_v \\
&= \exp f_0 \left(\sum_{m \in \mathbb{N}_0} \frac{(-1)^m (f_v^s)^m}{(2m)!} + \sum_{m \in \mathbb{N}_0} \frac{(-1)^m (f_v^s)^m}{(2m+1)!} f_v \right) \\
&= \exp f_0 \left(\cos(\sqrt{f_v^s}) + \sin(\sqrt{f_v^s}) \frac{f_v}{\sqrt{f_v^s}} \right)
\end{aligned} \tag{1.3}$$

when the symmetrization $f_v^s := f_1^2 + f_2^2 + f_3^2$ of f_v does not vanish, and where the definitions of cos, sin and $\sqrt{f_v^s}$ are the natural ones. A less algebraic, but maybe more enlightening, point of view is the following. To better understand the computation of $\exp_* f_v$ let us notice that, since

$$f_v * f_v = -f_v^s = -f_v * f_v^c$$

holds, outside the zero set of f_v^s, we have

$$\frac{f_v}{\sqrt{f_v^s}} * \frac{f_v}{\sqrt{f_v^s}} = \frac{-f_v^s}{f_v^s} = -1$$

identically. Therefore the vectorial function

$$\frac{f_v}{\sqrt{f_v^s}}$$

can be given the role of an imaginary unit, and therefore

$$\exp_* f_v = \exp_* \left(\frac{f_v}{\sqrt{f_v^s}} \sqrt{f_v^s} \right) = \cos(\sqrt{f_v^s}) + \sin(\sqrt{f_v^s}) \frac{f_v}{\sqrt{f_v^s}}.$$

All this said, we begin by focusing our study of the solutions f of the equation

$$\exp_* f = g$$

to the case of $\exp_* f = 1$ on an axially symmetric domain Ω whose intersection Ω_I with $\mathbb{R} + I\mathbb{R} \cong \mathbb{C}_I$ is "small" for any $I \in \mathbb{S}$. We then proceed to the definition of a local $*$-logarithm for any slice-regular function on such a domain. As one may expect, once the function $\log_* g$ is defined, we can also define the real powers of g, like for example

$$\sqrt[s]{g} := \exp_* \left(\frac{1}{s} \log_* g \right), \tag{1.4}$$

for all $s \in \mathbb{R}$, $s > 0$.

It turns out that the structure of the zeroes of the vectorial part g_v of the slice-regular function $g \colon \Omega \to \mathbb{H}$ in question plays a key role. Roughly speaking, the set $Z(g_v)$ of non-real and nonspherical zeroes of the vectorial part g_v of g (shared with the entire *vectorial (equivalence) class* $[g_v]$ and for this reason denoted $Z([g_v])$, see Definition 5.1) determines the right conditions for the existence of the $*$-logarithm of g in such a domain Ω. In the chosen setting, a slice-regular function $g \colon \Omega \to \mathbb{H}$ belongs to the vectorial class $[0]$ if and only if its vectorial part g_v is equivalent to the null function in Ω, that is, if and only if g belongs to the same vectorial class of its scalar part g_0. This situation is particularly fortunate for our study, as explicitly suggested by formula (1.3).

The set of all slice-regular functions $g \in \mathcal{SR}(\Omega)$ which are in the vectorial class $[0]$ is denoted by $\mathcal{SR}_{[0]}(\Omega) = \mathcal{SR}_{\mathbb{R}}(\Omega)$. In general, $\mathcal{SR}_\omega(\Omega)$ will denote the set of slice-regular functions $g \in \mathcal{SR}(\Omega)$ whose vectorial parts g_v are in the class ω (see Section 5).

For the existence of a $*$-logarithm of a function $g \in \mathcal{SR}(\Omega)$, a sort of slicewise simple-connectedness of the axially symmetric domain Ω is required (but is not in general a sufficient condition): indeed we will require that each of the, at most two, connected components of $\Omega_I = \Omega \cap \mathbb{C}_I$ is simply connected for one (and hence for all) $I \in \mathbb{S}$. Such a domain Ω will be called *a basic domain*. If $W \subseteq \mathbb{H}$ is any subset, then we will set the notation $\mathbb{S}W := \{sw : s \in \mathbb{S},\ w \in W\}$ and use it henceforth.

The main theorem of this paper, stated below and proved in Section 7.1 together with some of its consequences, identifies sufficient conditions for the existence of a $*$-logarithm of a function $g \in \mathcal{SR}(\Omega)$ with respect to the different structures of the vectorial class $[g_v]$ and of its zero set $Z([g_v])$.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}$ be a basic domain and let $g \in \mathcal{SR}_\omega(\Omega)$ be a nonvanishing function. Then the following holds:*

(a) *if $\omega = [0]$, a necessary and sufficient condition for the existence of a $*$-logarithm of g on Ω, $\log_* g \in \mathcal{SR}_{[0]}(\Omega)$, is*

$$g(\Omega \cap \mathbb{R}) \subset (0, +\infty);$$

(b) *if $\omega \neq [0]$, then if $Z(\omega) = \varnothing$ or if $\mathbb{S}Z(\omega) = \Omega$ there are no conditions, and a $*$-logarithm of g on Ω, $\log_* g \in \mathcal{SR}_\omega(\Omega)$, always exists;*

(c) *if $\omega \neq [0]$ and $Z(\omega)$ is discrete, a sufficient condition for the existence of a $*$-logarithm of g on Ω, $\log_* g \in \mathcal{SR}_\omega(\Omega)$, is the validity of both inclusions*

$$\sqrt{g^s}(\Omega \cap \mathbb{R}) \subset (0, +\infty) \tag{1.5}$$

and

$$\frac{g_0}{\sqrt{g^s}}(\Omega) \subset \mathbb{H} \setminus (-\infty, -1], \tag{1.6}$$

where $g^s = g_0^2 + g_v^s$ denotes the symmetrization of g.

Now, if the functions $\mu, \nu \in \mathcal{SR}_{\mathbb{R}}(\mathbb{H})$ are defined by the identities

$$\mu(z^2) = \cos z \qquad \text{and} \qquad \nu(z^2) = \frac{\sin z}{z},$$

then the last formula in (1.3) can be rewritten as

$$\exp_* f = \exp f_0\big(\mu(f_v^s) + \nu(f_v^s) f_v\big).$$

Moreover, for any $I \in \mathbb{S}$ the mapping

$$\mu_I : \mathbb{C}_I \setminus \{k^2\pi^2 : k \in \mathbb{N} \cup \{0\}\} \to \mathbb{C}_I \setminus \{1, -1\}$$

turns out to be a covering map (see Section 4.1). In this setting, we can obtain the second main result of this paper which appears in Section 7.2: Theorem 7.4. It produces a formula for the $*$-logarithms of a nonvanishing slice-regular function g, defined on a basic domain with no real points and whose vectorial part g_v has only one (nonreal) zero.

In the last section, we also show that for the function

$$g(z) = -1 + z^2 i + \sqrt{2} z j + k,$$

which is nonvanishing on the ball $B^4(0, 1.1)$, there is no slice-regular logarithm globally defined in the entire $B^4(0, 1.1)$. Indeed, this function g meets the hypotheses of Theorem 1.1 (c), but does not fulfil the stated sufficient conditions (1.5) and (1.6).

While preparing the final draft of this paper, we became aware that results similar to ours, but suggested by different motivations and involving different techniques, were obtained by Altavilla and de Fabritiis and are now posted on `arXiv` ([2]).

2. Preliminary results

Given any quaternion $z \notin \mathbb{R}$, there exist (and are uniquely determined) an imaginary unit $I \in \mathbb{S}$, and two real numbers, $x, y, y > 0$, such that $z = x + Iy$. With this notation, the conjugate of z will be $\bar{z} := x - Iy$ and $|z|^2 = z\bar{z} = \bar{z}z = x^2 + y^2$. Each $I \in \mathbb{S}$ generates (as a real algebra) a copy of the complex plane denoted by $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$. We call such a complex plane a *slice*. The upper half-plane in $\mathbb{C}_I$, namely the set $\mathbb{C}_I^+ := \{x + yI \in \mathbb{C}_I : y > 0\}$ will be called a *leaf*.

Definition 2.1. A domain Ω of $\mathbb{H}$ will be called *axially symmetric*[1] if

$$\Omega = \bigcup_{x+Iy\in\Omega} x + \mathbb{S}y,$$

i.e., if for all $x, y \in \mathbb{R}$ and all $I \in \mathbb{S}$, we have that $x + Iy \in \Omega$ implies that the entire 2-sphere $x + \mathbb{S}y$ is contained in Ω.

The proof of the following facts is straightforward.

Proposition 2.2. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric domain. For all $I \in \mathbb{S}$, we have that*

$$\Omega = \bigcup_{x+Iy\in\Omega_I} x + \mathbb{S}y.$$

Moreover, for all $I \in \mathbb{S}$, the set $\Omega_I \subseteq R + I\mathbb{R}$ is invariant under conjugation, i.e., $\Omega_I = \overline{\Omega_I}$.

A class of natural domains of definition for slice-regular functions is the following one.

Definition 2.3. A domain Ω of $\mathbb{H}$ is called a *slice domain* if, for all $I \in \mathbb{S}$, the subset Ω_I is a domain in $\mathbb{R} + I\mathbb{R}$ and if $\Omega \cap \mathbb{R} \neq \varnothing$. If, moreover, Ω is axially symmetric, then it is called a *symmetric* slice domain.

[1] Some authors use the term "circular".

On the other hand, slice functions (see [10]) are naturally defined on axially symmetric domains which are not necessarily slice domains.

Definition 2.4. An axially symmetric domain Ω of $\mathbb{H} \setminus \mathbb{R}$ is called a *product domain*.

If $\Omega \subseteq \mathbb{H}$ is an axially symmetric domain, then for (one and hence for) all $I \in \mathbb{S}$, the set Ω_I is an open subset of $\mathbb{C}_I$ such that: either it is a connected set that intersects $\mathbb{R}$, or it has two symmetric connected components separated by the real axis, swapped by the conjugation. In the former case, Ω is an axially symmetric slice domain; in the latter case, Ω is a product domain.

Proposition 2.5. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric domain. Then Ω is either a symmetric slice domain or it is a product domain.*

The following class of domains will play a key role in this paper.

Definition 2.6. A domain Ω of $\mathbb{H}$ is called a *basic domain* if it is axially symmetric and if, for (one and hence for) all $I \in \mathbb{S}$, the single connected component or both the connected components of Ω_I are simply connected. A basic domain is also a *basic neighbourhood* of any of its points.

The following examples show that being a simply connected domain and being a basic domain are distinct notions in general.

Example 2.7. For any given pair of positive real numbers $0 < r < R$, the axially symmetric domain $A_{r,R} = \{q \in \mathbb{H} : r < |q| < R\}$ is simply connected but the domain of the slice $\mathbb{C}_I$ obtained as $A_{r,R} \cap \mathbb{C}_I$ is not simply connected for any $I \in \mathbb{S}$. Hence $A_{r,R}$ is not a basic domain.

Example 2.8. The axially symmetric domain $\mathbb{H} \setminus \mathbb{R}$ is not simply connected, but the intersection of $\mathbb{H} \setminus \mathbb{R}$ with any slice $\mathbb{C}_I$ has two connected components, and each one is simply connected. Hence $\mathbb{H} \setminus \mathbb{R}$ is a basic domain.

We will now recall a unified definition of the class of slice-regular functions on axially symmetric domains, valid both for slice domains and for product domains (see, e.g., [9]). If $\iota^2 = -1$, consider the complexification $\mathbb{H}_\mathbb{C} = \mathbb{H} + \iota\mathbb{H}$ of the skew field $\mathbb{H}$ and set $z = x + \iota y \mapsto \bar{z} = x - \iota y$ to be the *natural involution* of $\mathbb{H}_\mathbb{C}$. For any $J \in \mathbb{S}$, let the map

$$\phi_J : \mathbb{H}_\mathbb{C} \to \mathbb{H}$$

be defined by

$$\phi_J(x + \iota y) = x + Jy.$$

Notice that the map ϕ_J, when restricted to $\mathbb{R}_\mathbb{C} = \mathbb{R} + \iota\mathbb{R} \cong \mathbb{C}$, is an isomorphism between $\mathbb{R} + \iota\mathbb{R}$ and $\mathbb{R} + J\mathbb{R} = \mathbb{C}_J$.

If $\Omega \subseteq \mathbb{H}$ is an axially symmetric domain, then the intersection $\Omega_i = \Omega \cap (\mathbb{R} + i\mathbb{R}) = \Omega \cap \mathbb{C}_i$ defines a domain of the complex plane that is invariant under standard complex conjugation. With respect to the established notations, the subset $\Omega_\iota = \{x + \iota y \in \mathbb{H} + \iota\mathbb{H} : x + iy \in \Omega_i\}$ is called the *image of* Ω_i *in* $\mathbb{H}_\mathbb{C}$, and is invariant under the natural involution, i.e., $\Omega_\iota = \overline{\Omega_\iota}$. We are now in a position to recall the following definitions.

Definition 2.9. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric open set, let $\Omega_i = \Omega \cap (\mathbb{R} + i\mathbb{R})$ and let Ω_ι be the image of Ω_i in $\mathbb{H}_\mathbb{C}$.

A function $F : \Omega_\iota \to \mathbb{H}_\mathbb{C}$ is called a *stem function* if $F(\bar{z}) = \overline{F(z)}$ for all $z \in \Omega_\iota$. For each stem function $F : \Omega_\iota \to \mathbb{H}_\mathbb{C}$, there exists a unique $f : \Omega \to \mathbb{H}$ such that the diagram

$$\begin{array}{ccc} \Omega_\iota & \xrightarrow{F} & \mathbb{H}_\mathbb{C} \\ \downarrow{\scriptstyle \phi_J} & & \downarrow{\scriptstyle \phi_J} \\ \Omega & \xrightarrow{f} & \mathbb{H} \end{array}$$

commutes for all $J \in \mathbb{S}$. The function f is called the *slice function* induced by F and denoted by $\mathcal{I}(F)$.

Let $f = \mathcal{I}(F)$, $g = \mathcal{I}(G)$ be the slice functions induced by the stem functions F, G respectively. The $*$-*product* of f and g is defined as the slice function $f * g := \mathcal{I}(FG)$.

We will use a definition of slice-regularity (and $*$-product) that involve stem functions, and that is valid for any axially symmetric domain of $\mathbb{H}$. When restricted to symmetric slice domains, it coincides with the definition of slice-regularity initially presented in [7, Definition 1.2].

Definition 2.10. Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric open set. A slice function $f : \Omega \to \mathbb{H}$, induced by a stem function $F : \Omega_\iota \to \mathbb{H}_\mathbb{C}$, is called *slice-regular* if F is holomorphic. The set of all slice-regular functions on Ω is denoted by $\mathcal{SR}(\Omega)$.

A slice function $f : \Omega \to \mathbb{H}$ is said to be *slice-preserving* if and only if $\forall I \in \mathbb{S}$, $\forall z \in \Omega_I := \Omega \cap \mathbb{C}_I$ we have that $f(z) \in \mathbb{C}_I$. The set of all slice-regular functions, which are slice-preserving in Ω, will be denoted as $\mathcal{SR}_\mathbb{R}(\Omega)$.

The next proposition recalls two well-known technical results that will be extensively used in the sequel (see, e.g., [9]).

Proposition 2.11. *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric open set, and let $f, g \in \mathcal{SR}(\Omega)$ be two slice-regular functions. Then*

(a) *the $*$-product $f * g$ is a slice-regular function on Ω;*

(b) *if f is slice-preserving, then $f * g = fg = g * f$, i.e., the $*$-product coincides with the pointwise product.*

Let us now define the *imaginary unit function*

$$\mathcal{I} : \mathbb{H} \setminus \mathbb{R} \to \mathbb{S}$$

by setting $\mathcal{I}(q) = I$ if $q \in \mathbb{C}_I$. The function $\mathcal{I}$ is slice-regular and slice-preserving, but it is not an open mapping and it is not defined on any slice domain.

Consider now an axially symmetric open set Ω and $f \in \mathcal{SR}(\Omega)$. We have already defined the splitting $f = f_0 + f_v$, where the scalar part f_0 of f is a slice-preserving function.

Definition 2.12. The function $f \in \mathcal{SR}(\Omega)$ is a *vectorial function* if $f = f_v$. The set of vectorial functions on Ω will be denoted by $\mathcal{SR}^v(\Omega)$. We have

$$\mathcal{SR}(\Omega) = \mathcal{SR}_{\mathbb{R}}(\Omega) \oplus \mathcal{SR}^v(\Omega).$$

Given a standard basis of $\mathbb{H}$, the vectorial part can be decomposed further ([8], [3, Proposition 3.12], compare [1, Proposition 2.1]), as we see in the following proposition.

Proposition 2.13. *Let $\{1, i, j, k\}$ be the standard basis of $\mathbb{H}$ and assume Ω is an axially symmetric domain of $\mathbb{H}$. Then the map*

$$(\mathcal{SR}_{\mathbb{R}}(\Omega))^4 \ni (f_0,\ f_1,\ f_2,\ f_3) \mapsto f_0 + f_1 i + f_2 j + f_3 k \in \mathcal{SR}(\Omega)$$

is bijective.

In the sequel, all bases of $\mathbb{H} \cong \mathbb{R}^4$ will be orthonormal (and positively oriented) with respect to the standard scalar product of $\mathbb{R}^4$. Proposition 2.13 implies that, given any $f, g \in \mathcal{SR}(\Omega)$, there exist and are unique $f_0, f_1, f_2, f_3, g_0, g_1, g_2, g_3 \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that

$$f = f_0 + f_1 i + f_2 j + f_3 k = f_0 + f_v,$$
$$g = g_0 + g_1 i + g_2 j + g_3 k = g_0 + g_v.$$

With the above given notation, if we call *regular conjugate* of f the function

$$f^c = f_0 - f_v,$$

then we have

$$f_0 = \frac{f + f^c}{2}.$$

Furthermore, using Definition 2.9 and Proposition 2.11, we obtain the following expression for the $*$-product of f and g:

$$f * g := f_0 g_0 - f_1 g_1 - f_2 g_2 - f_3 g_3 + f_0 g_v + g_0 f_v + \frac{f_v * g_v - g_v * f_v}{2}. \tag{2.7}$$

We now set

$$f^s := f_0^2 + f_1^2 + f_2^2 + f_3^2 = f * f^c = f^c * f$$

and call f^s the *symmetrization* of f.

3. Basic properties of the exponential

If $\exp q$ is the (quaternionic) exponential mapping defined in (1.1), then for every $k \in \mathbb{Z}$, we define its restriction to the cylinder $\{q : \mathrm{Im}(q) \in \mathbb{S}(k\pi, (k+1)\pi)\}$ to be

$$\exp_k \colon \{q : \mathrm{Im}(q) \in \mathbb{S}(k\pi, (k+1)\pi)\} \to \mathbb{H} \setminus \mathbb{R}.$$

For any $k \in \mathbb{Z}$ the function $\exp_k$ is a bijective slice-regular slice-preserving function with a slice-regular and slice-preserving inverse, namely

$$\log_k q = \log |q| + \mathcal{I}(q) \arg_{\mathcal{I}(q),k}(q),$$

where $\arg_{\mathcal{I}(q),k} \in (k\pi, (k+1)\pi)$ denotes the argument of q in the complex plane $\mathbb{C}_{\mathcal{I}(q)}$. The mapping $\log_0$ is called the *principal branch* of the logarithm and can be extended to

$$\log_0 \colon \mathbb{H} \setminus (-\infty, 0] \to \{q : \mathrm{Im}(q) \in \mathbb{S}[0, \pi)\}$$

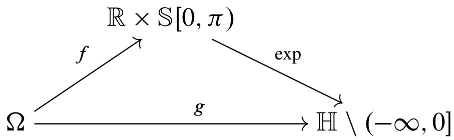
as the inverse of the extension of

$$\exp_0 \colon \{q : \mathrm{Im}(q) \in \mathbb{S}[0, \pi)\} \to \mathbb{H} \setminus (-\infty, 0].$$

Let us turn our attention to the problem of computing the logarithm of a function g, defined on a domain Ω of $\mathbb{H}$. For any continuous function $g \colon \Omega \to \mathbb{H} \setminus (-\infty, 0]$, one can define

$$f := \log_0 \circ g,$$

so that the diagram



commutes. In these hypotheses, for any $z \in \Omega$, we have the equality $\exp(f(z)) = g(z)$ by definition, but even if g is slice-regular, no regularity on the function f can be argued. If, in addition Ω, is axially symmetric and $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ is a slice-regular and slice-preserving function, then f is a well-defined slice-regular and slice-preserving function too. Indeed, (see Proposition 2.11) the equality $\exp_* f = \exp f = g$ holds on Ω for $f = \log_0 \circ g$ and we say that the function f is a logarithmic function of g (in Ω).

We have thus shown that the following proposition holds.

Proposition 3.1. *Let $\Omega \subseteq \mathbb{H}$ be a symmetric slice domain. If $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ is such that*

$$g(\Omega) \subset \mathbb{H} \setminus (-\infty, 0],$$

then the function

$$f = \log_0 \circ g$$

is the (slice-regular and slice-preserving) principal logarithm of g.

Let us point out that if $f \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, with $\Omega \subseteq \mathbb{H}$ any symmetric slice domain, then $(\exp_* f)(x_0) = (\exp f)(x_0) > 0$ for any $x_0 \in \Omega \cap \mathbb{R}$. Hence the condition

$$g(\Omega \cap \mathbb{R}) \subset (0, +\infty) \tag{3.8}$$

is a necessary condition for a slice-preserving function $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ to have a slice-preserving logarithm (see also [1]).

4. The $*$-exponential of a quaternionic function

In this section, we shortly recall some results from [1], which are necessary to explain our definition of $*$-logarithm.

The $*$-exponential map of a slice-regular function $f \in \mathcal{SR}(\Omega)$, with Ω an axially symmetric domain, is defined for any $z \in \Omega$ as in (1.2) by

$$\exp_* f(z) = \sum_{k \geq 0} \frac{f^{*k}(z)}{k!}$$

in such a way that $\exp_* f \in \mathcal{SR}(\Omega)$. The equality $\exp_*(f + g) = \exp_* f * \exp_* g$ does not hold in general as stated in Theorem 4.3 (see also [1, Theorem 4.14]), which we premise a crucial definition to.

Definition 4.1. Let $f_v \in \mathcal{SR}^v(V)$ and $g_v \in \mathcal{SR}^v(V)$, where $V \subset \mathbb{H}$ is an axially symmetric domain in $\mathbb{H}$. We say that f_v and g_v are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(V)$ if and only if there exist $a, b \in \mathcal{SR}_{\mathbb{R}}(V)$, with a or b not identically zero in V, such that $af_v + bg_v = 0$ in V. If $V \subset \mathbb{H}$ is an axially symmetric open set in $\mathbb{H}$, then f_v and g_v are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(V)$ if and only if they are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(V_\lambda)$ for each connected component V_λ of V.

Remark 4.2. Real isolated zeroes and isolated spherical zeroes can be factored out of a slice-regular function (see, e.g., [6, 9]). As a consequence for any vectorial function $f_v \in \mathcal{SR}^v(\Omega)$ on an axially symmetric open set Ω and for every axially symmetric open set $V \Subset \Omega$, there exists a nonidentically zero, slice-regular and slice-preserving function $a \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that

$$f_v = a\widetilde{f_v}$$

with $\widetilde{f_v} \in \mathcal{SR}^v(\Omega)$ having neither real nor spherical zeroes on V. Of course f_v and $\widetilde{f_v}$ are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(\Omega)$.

Theorem 4.3 ([1]). *Assume that the axially symmetric domain Ω intersects the real axis (i.e., it is a symmetric slice domain). Take $f, g \in \mathcal{SR}(\Omega)$. If*

$$\exp_*(f + g) = \exp_* f * \exp_* g \tag{4.9}$$

then either

(i) *f_v and g_v are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(\Omega)$, or*

(ii) *there exist $n, m, p \in \mathbb{Z} \setminus \{0\}$ such that $f^s = n^2\pi^2$, $g^s = m^2\pi^2$, $2(f_1 g_1 + f_2 g_2 + f_3 g_3) = (p^2 - n^2 - m^2)\pi$ and $n + m \cong p \mod 2$.*

Vice versa, if either (i) *or* (ii) *are satisfied, then* (4.9) *holds.*

Hence equality (4.9) holds if there exist $a, b \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ such that $a f_v + b g_v = 0$ with $a \not\equiv 0$ or $b \not\equiv 0$. In particular, this implies

$$\exp_* 0 = \exp_* f * \exp_* -f \equiv 1$$

so that, for every $f \in \mathcal{SR}(\Omega)$, the slice-regular function $\exp_* f$ is a nonvanishing function in Ω. If $f = f_0 + f_v$, then

$$\exp_* f = \exp f_0 \exp_* f_v. \tag{4.10}$$

Moreover,

$$\exp_* f^c = (\exp_* f)^c$$

whence

$$\begin{aligned}(\exp_* f)^s &= \exp_* f * (\exp_* f)^c \\ &= \exp_* f * \exp_* f^c = \exp_*(f + f^c) \\ &= \exp(2 f_0)\end{aligned}$$

and, from (4.10),

$$\exp_* f = \exp f_0 \left(\sum_{m \in \mathbb{N}_0} \frac{(-1)^m (f_v^s)^m}{(2m)!} + \sum_{m \in \mathbb{N}_0} \frac{(-1)^m (f_v^s)^m}{(2m+1)!} f_v \right).$$

Following [1, Remark 4.8] we will use the notations

$$\mu(z) := \sum_{m \in \mathbb{N}_0} \frac{(-1)^m z^m}{(2m)!}, \qquad \nu(z) := \sum_{m \in \mathbb{N}_0} \frac{(-1)^m z^m}{(2m+1)!}. \tag{4.11}$$

Both functions μ and ν are entire slice-regular and slice-preserving functions in $\mathbb{H}$, in symbols $\mu, \nu \in \mathcal{SR}_{\mathbb{R}}(\mathbb{H})$. Furthermore,

$$\mu(z^2) = \cos z \qquad \text{and} \qquad \nu(z^2) = \frac{\sin z}{z}, \tag{4.12}$$

where, in general,

$$\cos_* f = \sum_{m \in \mathbb{N}_0} \frac{(-1)^m f^{*(2m)}}{(2m)!} \qquad \text{and} \qquad \sin_* f = \sum_{m \in \mathbb{N}_0} \frac{(-1)^m f^{*(2m+1)}}{(2m+1)!}$$

for $f \in \mathcal{SR}(\Omega)$. Notice that also $\cos_*$ and $\sin_*$ are entire slice-regular and slice-preserving functions in $\mathbb{H}$. More in detail (see again [1, Corollary 4.7]), given a basic domain Ω and a slice-regular function $f:\Omega \to \mathbb{H}$, such that f_v^s is not identically zero and f_v has only real or spherical zeroes, then, in Ω,

$$\exp_* f = \exp f_0 \left(\cos(\sqrt{f_v^s}) + \sin(\sqrt{f_v^s}) \frac{f_v}{\sqrt{f_v^s}} \right), \tag{4.13}$$

where $\sqrt{f_v^s}$ is defined in the obvious way, being f_v^s a slice-preserving function. Indeed, we will refer to (4.13) as the *polar representation* for $\exp_* f$. The reader can find more details about the definition of square roots in [1, Proposition 3.1 and Corollary 3.2] (see also [5]).

4.1. Properties of the function μ

Let us first list some properties of the function μ, defined by (4.11), which are essential to define the logarithm of a slice-regular function. Since we have the identity $\mu(q^2) = \cos(q)$, for any $q \in \mathbb{H}$, we first define the branches μ_k of μ using the branches of the inverse of the function cos, i.e., the inverses of

$$\cos_k : \{q : \mathrm{Re}(q) \in (k\pi, (k+1)\pi)\} \to \mathbb{H}((-\infty, -1] \cup [1, +\infty)),$$

denoted by $\arccos_k$. To this end consider first the domains

- $D_0 := \{q : \mathrm{Re}(q) \in [0, \pi)\}$, $D_{-1} := \{q : \mathrm{Re}(q) \in (-\pi, 0]\}$,
- $D_k := \{q : \mathrm{Re}(q) \in (k\pi, (k+1)\pi)\}$ for $k \in \mathbb{Z} \setminus \{0, -1\}$.

Notice that the domains D_k, $k \neq 0, -1$, lie entirely either in the right half-space $\{q : \mathrm{Re}(q) > 0\}$ or in the left half-space $\{q : \mathrm{Re}(q) < 0\}$, so the squaring map p_2, $p_2(q) = q^2$, is injective on each D_k and hence bijective onto $p_2(D_k)$ with an inverse $\sqrt{\ }$.

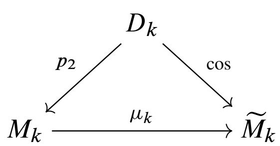
For all $k \in \mathbb{Z}$ define the domains M_k, $\widetilde{M}_k$ to be

$$M_k := p_2(D_k), \qquad \widetilde{M}_k := \mu(M_k) = \cos(D_k), \qquad \mu_k := \mu|_{M_k}$$

and observe that

- $0 \in M_0 = M_{-1}$,
- $\widetilde{M}_0 = \widetilde{M}_{-1} = \mathbb{H} \setminus (-\infty, -1]$,
- $\widetilde{M}_k = \mathbb{H} \setminus ((-\infty, -1] \cup [1, +\infty))$, $k \neq 0, -1$.

By definition, for each $k \in \mathbb{Z}$, the diagram



commutes. The choice of domains D_k, $k \neq 0, -1$, is such that both cos and p_2 are bijective, hence so is μ_k. To see that also μ_0 and μ_{-1} are bijective it remains to show that they are bijective when restricted to the imaginary axis. In this case, since both cos and p_2 are even, we have for $q \in \mathrm{Im}(\mathbb{H})$

$$\cos(q) = \cos(-q) \qquad \text{and} \qquad p_2(q) = p_2(-q).$$

Moreover, for each $I \in \mathbb{S}$, $k = 0, -1$, the restrictions

$$\cos\colon I[0, +\infty) \to [1, +\infty)$$

and

$$p_2\colon I[0, +\infty) \to (-\infty, 0]$$

are injective, which implies that the induced maps μ_0, μ_{-1} are bijective.

The points $k\pi$, $k \in \mathbb{Z}$, are branching points for the complex cosine, which implies that the points $k^2\pi^2$ are branching points for μ, except the point 0, which is contained in M_0 and where $\mu_0'(0) = -1/2 \neq 0$.

We can summarize these considerations in the following proposition.

Proposition 4.4. *For each k the function $\mu_k\colon M_k \to \widetilde{M}_k$ is bijective with the inverse μ_k^{-1}. In particular, $\mu_0(0) = 1$ and the function μ_0 maps a neighbourhood of* 0 *bijectively to a neighbourhood of* 1*. The mapping*

$$\mu\colon \mathbb{H} \setminus \{k^2\pi^2 : k \in \mathbb{N} \cup \{0\}\} \to \mathbb{H} \setminus \{\pm 1\}$$

is a slice-covering map*, i.e.,*

$$\mu_I\colon \mathbb{C}_I \setminus \{k^2\pi^2 : k \in \mathbb{N} \cup \{0\}\} \to \mathbb{C}_I \setminus \{\pm 1\}$$

is a covering map for every $I \in \mathbb{S}$. Furthermore, any map μ_I extends to a local diffeomorphism across the point 0*.*

It turns out that for $k \neq 0, -1$ we have

$$\arccos_k := \sqrt{} \circ \mu_k^{-1}\colon \widetilde{M}_k \to D_k,$$

and, for $k = 0, -1$, we have

$$\arccos_k := \sqrt{} \circ \mu_k^{-1}\colon \widetilde{M}_k \setminus [1, +\infty) \to D_k \setminus \{q : \mathrm{Re}(q) = 0\}.$$

5. Globally defined vectorial class

Formula (4.9) shows how crucial it is for two slice-regular functions to have linearly dependent vectorial parts. This motivates the following.

Definition 5.1. Let $f_v \in \mathcal{SR}^v(U)$ and $g_v \in \mathcal{SR}^v(U')$, where $U, U' \subset \mathbb{H}$ are axially symmetric domains in $\mathbb{H}$ such that $U \cap U' \neq \varnothing$. Take $p \in U \cap U'$; we say that f_v and g_v are *equivalent at* p, in symbols $f_v \sim_p g_v$, if there exists an axially symmetric neighbourhood of p, $V_p \subset U \cap U'$, such that f_v and g_v are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(V_p)$ in V_p. We will denote by $[f_v]_p$ the $\sim_p$ equivalence class whose representative is f_v.

It is easy to verify that the relation $\sim_p$ is an equivalence relation at each point p; The definition above immediately implies that if $f_v \sim_p g_v$ then $f_v \sim_q g_v$ for every $q \in \mathbb{S}p =: \mathbb{S}_p$. Moreover, we get the following remark.

Remark 5.2. For each equivalence class $[f_v]_p$ we can choose a local representative $\widetilde{f_v}$ having neither real nor spherical zeroes (see Remark 4.2).

Definition 5.3. By $\mathcal{V}_p$ we denote the set of all $\sim_p$ equivalence classes of vectorial functions at p, namely

$$\mathcal{V}_p := \big\{[f_v]_p : f_v \in \mathcal{SR}^v(U),\ U \text{ axially symmetric neighbourhood of } p\big\}.$$

Definition 5.4. Let U be an axially symmetric open set and

$$\mathcal{V}_U := \{\mathcal{V}_p : p \in U\}$$

be the set of all equivalence classes of vectorial functions with respect to equivalence relations $\sim_p$, with $p \in U$. A *vectorial class* ω_U on U is defined to be any function

$$\omega_U : U \to \mathcal{V}_U$$

such that

- for all $p \in U$, it holds $\omega_U(p) \in \mathcal{V}_p$;
- if $p, q \in U$, if $\omega_U(p) = [f_v]_p$ with $f_v \in \mathcal{SR}^v(V_p)$ for an axially symmetric domain $V_p \subset U$ containing p, if $\omega_U(q) = [g_v]_q$ with $g_v \in \mathcal{SR}^v(V_q)$ for an axially symmetric domain $V_q \subset U$ containing q, then $[f_v]_{\widetilde{p}} = [g_v]_{\widetilde{p}}$ for all $\widetilde{p} \in V_p \cap V_q$.

We denote by $\mathcal{V}(U)$ the set of all vectorial classes over U.

If $f_v \in \mathcal{SR}^v(U)$ then it obviously defines the vectorial class on U

$$p \mapsto [f_v]_p, \quad p \in U$$

which we denote by $[f_v]_U$ and call *principal vectorial class (associated to f_v)* on U.

Notice that $\mathcal{V}(U)$ is not a ring over $\mathcal{SR}_{\mathbb{R}}(U)$ (it is not possible to define the sum of two classes); furthermore, if $[f_v]_U = [g_v]_U$ and $\widetilde{f_v}$ and $\widetilde{g}_v$ are representatives on U without real or spherical zeroes, then $\widetilde{f_v}^s$ identically zero in U implies $\widetilde{g}_v^s$ identically zero in U.

Definition 5.5. Let $U, U' \subset \mathbb{H}$ be two axially symmetric open sets such that $U \cap U' \neq \varnothing$. If $f_v \in \mathcal{SR}^v(U)$ and $g_v \in \mathcal{SR}^v(U')$ are linearly dependent over $\mathcal{SR}_{\mathbb{R}}(U \cap U')$ in $U \cap U'$, then they define a vectorial class $[f_v \vee g_v]_{U \cup U'} \in \mathcal{V}(U \cup U')$ by

$$\begin{cases} [f_v] & \text{in } U \setminus U', \\ [f_v] = [g_v] & \text{in } U \cap U', \\ [g_v] & \text{in } U' \setminus U. \end{cases}$$

Definition 5.6. Let $V \subset U \subset \mathbb{H}$ be axially symmetric open sets and let $\omega_U \in \mathcal{V}(U)$ be a vectorial class on U. The restriction morphism

$$\mathrm{res}_{V,U} \colon \mathcal{V}(U) \to \mathcal{V}(V)$$

is defined by

$$\mathrm{res}_{V,U}(\omega_U) := \omega_U|_{U \cap V} =: \omega_V.$$

Proposition 5.7. *The collection $\{U, \mathcal{V}(U)\}$ of vectorial classes over all axially symmetric domains $U \subset \mathbb{H}$ together with the families of restriction morphisms $\mathrm{res}_{V,U} \colon \mathcal{V}(U) \to \mathcal{V}(V)$, $V \subset U$, is a presheaf.*

Proof. It is immediate that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{V}(U)}$. It is also immediate that $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ holds for axially symmetric domains $W \subset V \subset U$, since vectorial classes are functions. ■

Proposition 5.8. *The presheaf from Proposition 5.7 is a sheaf and will be denoted by $\mathcal{V}$.*

Proof. Let U be an axially symmetric domain and $\varpi_U, \omega_U \in \mathcal{V}(U)$. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of U with axially symmetric open sets.

(i) Locality. If we have $\varpi_{U_\alpha} = \omega_{U_\alpha}$ for all $\alpha \in \Lambda$, then by definition $\varpi_U = \omega_U$.

(ii) Gluing. Let the vectorial classes ω_{α,U_α}, $\alpha \in \Lambda$ be such that

$$\omega_{\alpha,U_\alpha}|_{U_\alpha \cap U_\beta} = \omega_{\beta,U_\beta}|_{U_\beta \cap U_\alpha}, \quad \alpha, \beta \in \Lambda.$$

The function defined by

$$(\omega_U)|_{U_\alpha} := \omega_{\alpha,U_\alpha}, \quad \alpha \in \Lambda,$$

is a vectorial class on U. ■

Remark 5.9. Vectorial classes $\mathcal{V}(U)$ are sections of the sheaf $\mathcal{V}_U$.

Let Ω be an axially symmetric domain and f_v a vectorial function on Ω. Then, being slice-regular, its symmetrization f_v^s is either identically[2] 0 or has isolated real or spherical zeroes.

Proposition 5.10. *Let Ω be an axially symmetric domain, $f_v = f_1 i + f_2 j + f_3 k$ a vectorial function on Ω and assume that f_v^s is not identically zero on Ω. Let z_0 be a real zero of f_v^s. Then it is a real zero of f_v and there exists $k > 0$ such that*

$$(q - z_0)^{-k} f_v =: g_v,$$

[2] This does not imply that f_v is identically zero. Consider for example $f_v \in \mathcal{SR}(\mathbb{H} \setminus \mathbb{R})$ defined as $f_v(x + Iy) = Ii + j$; then $f_v^s = I^2 + 1 \equiv 0$, $f_v \not\equiv 0$ (and f_v has a zero on every sphere).

$g_v \in \mathcal{SR}_{[f_v]}(\Omega)$ and $g_v(z_0) \neq 0$. Similarly, if f_v has a spherical zero $\mathbb{S}_{z_0} = \{a + Ib : I \in \mathbb{S}\}$ of multiplicity $k > 0$, then

$$(q^2 - 2q\,\mathrm{Re}(z_0) + |z_0|^2)^{-k} f_v =: g_v,$$

$g_v \in \mathcal{SR}_{[f_v]}(\Omega)$ and $g_v(q) \neq 0$ for all $q \in \mathbb{S}_{z_0}$, except maybe at one point.

Proof. First notice that z_0 is a real zero of $f_v \not\equiv 0$ if and only if it is a common zero of $f_l, l = 1, 2, 3$. If z_0 is a real zero of $f_v^s \not\equiv 0$ then $f_1^2(z_0) + f_2^2(z_0) + f_3^2(z_0) = 0$ which implies that z_0 is a common zero of all the components of f_v of multiplicity k for some $k \in \mathbb{N}$, since $f_l(z_0) \in \mathbb{R}, l = 1, 2, 3$. Therefore we may factor out a slice-preserving factor $(q - z_0)^k$ from f_1, f_2, f_3 and hence the function $(q - z_0)^{-k} f_v$ is nonvanishing on a neighbourhood of z_0. In other words, one can locally write $f_v = \lambda w$, where w does not have real zeroes and $\lambda \not\equiv 0$ is a slice-preserving function. If f_v has a spherical zero $\mathbb{S}_{z_0} = \{a + Ib : I \in \mathbb{S}\}$, then $f_l(a + Ib) = a_l + Ib_l, l = 1, 2, 3$ for any $I \in \mathbb{S}$. For $i = I$ we have that $f_2(z_0)j + f_3(z_0)k = x_1 j + x_2 k$ and $f_1(z_0)i = a_1 i - b_1$, hence the condition $f_1(z_0)i + f_2(z_0)j + f_3(z_0)k = 0$ implies $a_1 = b_1 = 0$ and, analogously, $a_l = b_l = 0$ for $l = 2, 3$, hence f_1, f_2, f_3 all have $\mathbb{S}_{z_0}$ as a spherical zero. If the spherical zero is of multiplicity k, then we can factor out a term $(q^2 - 2q\,\mathrm{Re}(z_0) + |z_0|^2)^k$ from f_l, with $l = 1, 2, 3$. ∎

Definition 5.11. Let ω be a vectorial class on an axially symmetric domain Ω. Define

$$\mathcal{SR}_\omega(\Omega) = \big\{g \in \mathcal{SR}(\Omega) : [g_v]_p \in \omega(p), \forall\, p \in \Omega\big\}.$$

For the case $\omega = [0]$, notice that by definition $\mathcal{SR}_{[0]}(\Omega) = \mathcal{SR}_{\mathbb{R}}(\Omega)$.

If $f, g \in \mathcal{SR}_\omega(\Omega)$ then also $f * g = g * f \in \mathcal{SR}_\omega(\Omega)$, because the last term in formula (2.7) vanishes. In particular, since $\mathcal{SR}_{\mathbb{R}}(\Omega) \subseteq \mathcal{SR}_\omega(\Omega)$ for any ω, if $f \in \mathcal{SR}_\omega(\Omega)$ and $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$, then $f * g \in \mathcal{SR}_\omega(\Omega)$.

Remark 5.2 suggests now the following.

Definition 5.12. Let Ω be an axially symmetric domain and let $\omega \in \mathcal{V}(\Omega)$. Let $U \subset \Omega$ be an axially symmetric open set and let $w \in \mathcal{SR}_\omega(\Omega)$ be the vectorial part of a slice-regular function. Then w is called *minimal on U* if it has neither real nor spherical zeroes on U.

We have shown that in the case $f_v^s \not\equiv 0$, spherical and real zeroes of the vectorial part are precisely the common zeroes of the components of f_v. The vectorial function $w(z) = z^2 i + \sqrt{2}zj + k$ is an example of a minimal representative; it has an isolated zero on the unitary sphere $\mathbb{S}$, namely $z_0 = \frac{k-i}{\sqrt{2}} j$, and its symmetrization $w^s(z) = (z^2 + 1)^2$ vanishes on $\mathbb{S}$. Notice furthermore, that $z^2 + 1$ is not a common factor of the components of w.

For all $f_v \in \mathcal{SR}^v(\Omega)$, the factorization $f_v = \lambda w$ with $w \in \mathcal{SR}_{[f_v]}(\Omega)$ minimal and $\lambda \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ is unique up to a multiplication by a slice-preserving nonvanishing function. If w_α, w_β are two minimal representatives of the same vectorial class on an axially symmetric subset $U \subset \Omega$, then $aw_\alpha = bw_\beta$ for some $a, b \in \mathcal{SR}_{\mathbb{R}}(U)$ and by minimality both

a and b are nonvanishing on U; moreover the zero sets of w_α and w_β coincide. Therefore, given a vectorial class ω on an axially symmetric domain Ω, we can define the zero set $Z(\omega)$ of ω.

Definition 5.13. Let Ω be an axially symmetric domain and let $\omega \in \mathcal{V}(\Omega)$ be a vectorial class. If $\omega \not\equiv 0$, let w be a minimal representative of ω on an axially symmetric open set $U \subset \Omega$. Define $Z(\omega) \cap \mathrm{U} = w^{-1}(0)$. Then the zero set $Z(\omega)$ of ω is defined to be the union of all zeroes $w^{-1}(0)$ where w runs over minimal representatives of ω on open axially symmetric subsets U of Ω.

If $\omega = [0]$, then we define $Z([0]) = \varnothing$.

Proposition 5.14. *Let Ω be an axially symmetric domain and let $\omega \in \mathcal{V}(\Omega)$ be a vectorial class. If w is a local minimal representative of ω on an axially symmetric domain $U \subset \Omega$, then*

(i) *if $w^s \not\equiv 0$, then $Z(\omega) \subset \Omega$ is a discrete set of nonreal quaternions;*

(ii) *if $w^s \equiv 0$ but $w \not\equiv 0$, we have $\mathbb{S}Z(\omega) = \Omega$, there is precisely one zero of $Z(\omega)$ on each sphere and, moreover, $\Omega \subset \mathbb{H} \setminus \mathbb{R}$.*

Proof. Let w be a local minimal representative of $\omega \not\equiv [0]$ on a basic domain $U \subset \Omega$. Then w^s is slice-preserving and hence it is either identically equal to 0 or has isolated real or spherical zeroes (or no zeroes). If w^s is not identically equal to 0, the same holds for any other minimal representative by the identity principle and then obviously the set $Z(\omega)$ is either discrete or empty.

Assume that $w^s \equiv 0$ but $w \not\equiv 0$. Recall that, for any other representative $\widetilde{w}$ we have $\widetilde{w}^s \equiv 0$, by the identity principle. The identity principle implies that $\Omega \subset \mathbb{H} \setminus \mathbb{R}$ is a product domain. Indeed, if Ω is a slice domain, then on the real axis the symmetrization w^s is a sum of squares of real numbers and hence, if it is identically 0, then by the identity principle also $w \equiv 0$ in an axially symmetric domain containing $\Omega \cap \mathbb{R}$, and hence in the entire slice domain Ω; contradiction.

Now, $w^s \equiv 0$ on the product domain Ω implies that w has a zero on each sphere, and can have neither a sphere of zeroes nor a real zero, since it is a minimal representative of ω.

Since $w^s = w * w^c$ and $w^c = -w$, the equation $w^s(z_0) = 0$ implies that either $w(z_0) = 0$ or if $w(z_0) \neq 0$, $w^c(z) = -w(z) = 0$ for $z = w(z_0)^{-1} z_0 w(z_0) \in \mathbb{S}_{z_0}$. If there were two distinct zeroes on $\mathbb{S}_{z_0}$ then extension formula would imply that $w(\mathbb{S}_{z_0}) = 0$, which contradicts the assumption that w is minimal. ■

If $f_v = \lambda w$ is the (local) decomposition of f_v with w minimal and λ a slice-preserving function, then $f_v^s = \lambda^2 w^s$. If w^s is nonvanishing on a basic domain $U \subset \Omega$, then one can define square roots of f_v^s and w^s (denoted as $\sqrt{f_v^s}$ and $\sqrt{w^s}$) (see [1, Proposition 3.1], and next sections) and find that $\sqrt{f_v^s} = \pm\lambda\sqrt{w^s}$. Therefore we can state the following proposition.

Proposition 5.15. *Let Ω be a basic domain, let $\omega \neq [0]$ be a vectorial class on Ω with $Z(\omega) = \varnothing$ and let $f_v \in \mathcal{SR}_\omega(\Omega)$. If w is a minimal representative of ω in Ω, then the* normalized *vectorial function*

$$f_v/\sqrt{f_v^s} \in \{\pm w/\sqrt{w^s}\}$$

is minimal and such that

$$(f_v/\sqrt{f_v^s})^s = 1$$

in Ω.

Proof. After the premises to this statement, the proof is straightforward. ∎

6. Local definition of $\log_*$

We now reach the heart of the problem: if Ω is an axially symmetric domain of $\mathbb{H}$, given $g \in \mathcal{SR}(\Omega)$ not vanishing in Ω and $z \in \Omega$ an arbitrary point, find an open axially symmetric neighbourhood U of z and a function $f \in \mathcal{SR}(U)$ such that

$$\exp_* f = g \quad \text{on } U.$$

The assumption that $g \in \mathcal{SR}(\Omega)$ is a nonvanishing function in Ω is intrinsic with the problem, since, where defined, the function $\exp_* f$ is nonvanishing. We will find necessary and sufficient conditions on g to define a local logarithmic function of g.

Let us assume henceforth that Ω is a basic domain in $\mathbb{H}$. After writing $g = g_0 + g_v$ and $f = f_0 + f_v$ as in the previous section, we will proceed by steps.

6.1. Case 0: $g \in \mathcal{SR}(\Omega)$ is a constant function

To avoid confusion, the constant function q_0 will be denoted by C_{q_0}.

Consider first the case $q_0 = 1$. Then the principal branch of the logarithm can be defined, because the function $\exp_0$ is a bijection between $\{q : \mathrm{Im}(q) \in \mathbb{S}[0, \pi)\}$ and $\mathbb{H} \setminus (-\infty, 0]$ and so we can define

$$\log_{*,0,0} := \log_0 C_1 = 0,$$

in the whole $\mathbb{H}$. Choose a point $z_0 \in \mathbb{H}$ and let ω be any vectorial class with $z_0 \notin \mathbb{S}Z(\omega)$. Let w be one of the two normalized minimal nonzero representatives of ω (see Proposition 5.15) defined on a basic neighbourhood U_{z_0} of z_0. Then, for all $n \in \mathbb{Z}$, the function

$$\log_{*,0,2nw} C_1 := 2\pi n w \tag{6.14}$$

also satisfies $\exp_*(\log_{*,0,2nw} C_1) = 1$ (see formula (4.13)). If, moreover, $U_{z_0} \subset \mathbb{H} \setminus \mathbb{R}$ is a product domain then the imaginary unit function $\mathcal{I}$ is a well-defined slice-preserving function and hence we have the possibilities

$$\log_{*,m,nw} C_1 := m\pi \mathcal{I} + n\pi w,$$

on U_{z_0}, where $m, n \in \mathbb{Z}$ are such that $m + n \equiv 0 \bmod 2$. Notice that if U_{z_0} is a basic slice domain, then the only possibilities are those appearing in formula (6.14).

For any constant function C_{q_0}, $q_0 \in \mathbb{H} \setminus (-\infty, 0]$, the situation is completely analogous, and we have

$$\log_{*,0,2nw} C_{q_0} := \log_0 q_0 + 2n\pi w.$$

(for any $n \in \mathbb{Z}$) on a basic slice neighbourhood U_{z_0} of z_0, and on a product neighbourhood U_{z_0} we have

$$\log_{*,m,nw} C_{q_0} := \log_0 q_0 + m\pi \mathcal{I} + n\pi w,$$

where $m, n \in \mathbb{Z}$ are such that $m + n \equiv 0 \bmod 2$.

Consider now the constant function C_{-1}. Define, for $n \in \mathbb{Z}$,

$$\log_{*,0,2nw} C_{-1} := (2n + 1)\pi w.$$

This function satisfies $\exp_*(\log_{*,0,2nw} C_{-1}) = -1$ and on a basic product neighbourhood U_{z_0} of a point $z_0 \in \mathbb{H} \setminus \mathbb{R}$ we also have

$$\log_{*,m,nw} C_{-1} := \pi w + (m\pi \mathcal{I} + n\pi w) = \pi w + \log_{*,m,nw} C_1,$$

for $n + m \equiv 0 \bmod 2$. With the notation of the previous section, for any constant function C_{q_0}, $q_0 \in \mathbb{H} \setminus \{0\}$, we have

$$\log_{*,m,nw} C_{q_0} \in \mathcal{SR}_\omega(U).$$

Remark 6.1. Once a slice-regular logarithm of two slice-regular functions $g, h \in \mathcal{SR}(U)$ is defined in a basic domain U, one can always add to each logarithm a vectorial function $2n\pi w$ (with w any normalized minimal representative of a vectorial class ω in the basic domain U with $Z(\omega) \cap U = \varnothing$), but for the price of losing the property $\exp_*(\log_* g + \log_* h) = g * h$. Indeed, notice that, for example, the equality

$$\exp_*(\log_{*,2m_1,2n_1 w_1} C_1 + \log_{*,2m_2,2n_2 w_2} C_1) = 1$$

is not necessarily valid, if $w_2 \notin [w_1]$ (compare [1, Theorem 4.14 (ii)]). The property $\exp_*(\log_* g + \log_* h) = g * h$ is still valid for functions $g, h \in \exp_*^{-1}(\mathcal{SR}_\mathbb{R}(U)) \cap (\mathcal{SR}_\omega(U))$.

Remark 6.1 suggests restricting our considerations to the sets $\exp_*^{-1}(\mathcal{SR}_\omega(U)) \cap (\mathcal{SR}_\omega(U))$. According to Proposition 5.14 and Definition 5.13 we have the following four different possibilities with respect to the vectorial classes and the structure of their zero sets.

6.2. Case 1: $g \in \mathcal{SR}_\mathbb{R}(\Omega)$ is slice-preserving, i.e., $g_v \equiv 0$

Let us now consider the general case of a nonvanishing slice-regular and slice-preserving function $g = g_0$. In this case, the involved regular functions behave like holomorphic functions on each slice, but at the same time topological obstructions near the real axis complicate the problem of finding a logarithmic function.

We assume that the necessary condition expressed by formula (3.8), i.e., $g_0(\Omega \cap \mathbb{R}) \subset (0, +\infty)$, holds. Then, since $g \neq 0$, one can locally define a logarithmic function of g in the following way. Consider a point $z_0 \in \Omega$.

If $z_0 \notin \mathbb{R}$, then we have the following possibilities:

- $g(z_0) \in \mathbb{H} \setminus (-\infty, 0]$, then a logarithmic function of g can be defined in a neighbourhood of z_0 since the function $\exp_0$ is a bijection between $\mathbb{R} \times \mathbb{S}[0, \pi)$ and $\mathbb{H} \setminus (-\infty, 0]$; indeed, locally, for all $m \in \mathbb{Z}$, we can define

$$\log_{*,2m,0} g := \exp_0^{-1} \circ g + 2m\pi \mathcal{I}.$$

- $g(z_0) \in (-\infty, 0)$. In this case, a logarithmic function can be locally defined for $-g$ as in the previous point. And then we can exploit the equality

$$\log_{*,2m,0} g = \log_{*,2m,0}(-g) + \pi \mathcal{I} = \exp_0^{-1} \circ (-g) + (2m+1)\pi \mathcal{I}.$$

If $z_0 \in \mathbb{R}$, then by hypothesis $g(z_0) > 0$ and we have the only possibility

$$\log_{*,0,0} g := \exp_0^{-1} \circ g$$

since the function $\mathcal{I}$ cannot be defined on the real axis.

Remark 6.2. Condition (3.8) is necessary if we want the logarithm of a slice-preserving function to be slice-preserving. If not, then this condition is no longer needed. Indeed, consider any normalized minimal representative w of any vectorial class defined on an axially symmetric neighbourhood U of z_0 which is nonvanishing on U and assume that $g(z_0) < 0$ for some $z_0 \in \mathbb{R}$. Then $\log_{*,0,0}(-g)$ is defined and

$$f = \log_{*,0,(2n+1)w}(-g) = \log_{*,0,0}(-g) + (2n+1)\pi w$$

satisfies

$$\begin{aligned} \exp_* f &= -g(-1)^0(\mu((2n+1)^2\pi^2) + \nu((2n+1)^2\pi^2)(2n+1)\pi w) \\ &= -g(-1)^{2n+1} = g. \end{aligned}$$

Remark 6.3. The above considerations imply that given a nonvanishing slice-regular and slice-preserving function $g \in \mathcal{SR}_{\mathbb{R}}(\Omega)$ (not necessarily satisfying condition (3.8), that $g(\Omega \cap \mathbb{R}) \subset (0, +\infty)$), one can always locally define a slice-preserving logarithmic function of at least one of the two functions $g, -g$ or both, depending on the domain of definition.

6.3. Case 2: $g \in \mathcal{SR}(\Omega)$ with $g_v \not\equiv 0$, $g_v^s \equiv 0$

Consider now $g = g_0 + g_v$ such that g is nonvanishing and g_v is not identically 0 but g_v^s is (which implies g_0 is nonvanishing). Then Ω is a product domain since otherwise

g_v would be identically 0 because of the identity principle (Proposition 5.14). Therefore $\log_{*,2m,0} g_0$ can be locally defined on a basic neighbourhood U of any point of Ω, for all $m \in \mathbb{Z}$. The class $[g_v] = \omega$ does not have a normalized minimal representative, therefore in this case we use the notation $\log_{*,m,0\cdot[g_v]}$ to indicate that the resulting function is in the class $\mathcal{SR}_\omega(\Omega)$ but there are no periods in any minimal representative of $[g_v]$.

In general, whenever $g = g_v$ with $g_v^s \equiv 0$, the equality

$$\exp_* g = \exp_* g_v = 1 + g_v$$

holds, since $g_v^{*2} = -(g_v)^s = 0$ and then g_v^{*k} vanishes for all $k \geq 2$. In these cases we put

$$\log_*(1+g) = \log_{*,0,0\cdot[g_v]}(1+g_v) := g_v = g.$$

Assume now $g = g_0 + g_v$ with g_0 nonvanishing and $g_v^s \equiv 0$ in Ω. Then one can write $g = g_0(1 + \frac{g_v}{g_0})$; hence, from $\exp_*(\log_{*,0,\cdot[g_v]} g) = g$, one concludes that

$$\log_{*,0,\cdot[g_v]} g = \log_{*,0,0} g_0 + \log_{*,0,\cdot[g_v]}(1 + g_v/g_0) = \log_{*,0,\cdot[g_v]} g_0 + \frac{g_v}{g_0};$$

more generally, on a product domain $U \subseteq \Omega$,

$$\log_{*,m,0} g := \log_{*,m,0} g_0 + \frac{g_v}{g_0}, \quad m \in \mathbb{Z},$$

which completely describes all possible solutions for $\exp_* f = g$ with the given assumptions for g, namely g not vanishing, $g_v \not\equiv 0$ and $g_v^s \equiv 0$ in Ω.

Example 6.4. Consider the function

$$z = x + Iy \mapsto \Psi(x + Iy) := Ii + j;$$

clearly $\Psi = \Psi_0 + \Psi_1 i + \Psi_2 j + \Psi_3 k$ is well defined, slice-regular in $\Omega = \mathbb{H} \setminus \mathbb{R}$ and constant on any slice and $\Psi|_{\Omega_{-k}} \equiv 0$. Moreover, since $\Psi_0 = 0$, $\Psi_1 = \mathcal{I}$, $\Psi_2 = 1$ and $\Psi_3 = 0$, then $\Psi_v^s = 0$. Hence

$$\exp_* \Psi = 1 + \Psi.$$

Notice that $\Psi_1 = \mathcal{I} \in \mathcal{SR}_\mathbb{R}(\mathbb{H} \setminus \mathbb{R})$ and cannot be extended continuously to $\mathbb{H}$. Consider now the function $g(z) = z + \Psi(z)$; clearly g is a nonvanishing slice-regular and slice-preserving function in $\Omega = \mathbb{H} \setminus \mathbb{R}$. Furthermore, $g_0 = \mathrm{Id}$ and $g_v = \Psi$ and so, for any $z \in \mathbb{H} \setminus \mathbb{R}$, we have

$$\begin{aligned}(\log_{*,k} g)(z) &= [\log_{*,k}(\mathrm{Id} + \Psi)](z) = [\log_{*,k}(g_0 + g_v)](z)\\ &= \log_{*,k} z + \frac{\Psi(z)}{z}\\ &= \log(|z|) + [\arg_{\mathcal{I}(z)}(z) + 2k\pi]\mathcal{I}(z) + \frac{\Psi(z)}{z},\end{aligned}$$

where log represents the usual real natural logarithm.

6.4. Case 3: $g \in \mathcal{SR}(\Omega)$ and $z_0 \in \Omega$ such that $Z([g_v]) \cap \{\mathbb{S}_{z_0}\} = \varnothing$

The condition $Z([g_v]) \cap \{\mathbb{S}_{z_0}\} = \varnothing$ implies the following: either $g_v \neq 0$ on $\mathbb{S}_{z_0}$ or there is a factorization $g_v = \lambda \widetilde{w}$ with $\widetilde{w} \neq 0$ on $\mathbb{S}_{z_0}$. Hence the function $h := \sqrt{\widetilde{w}^s}$ is locally well defined on a basic open neighbourhood U of z_0 and satisfies $h^2 = \widetilde{w}^s$. Put $\sqrt{g_v^s} := \lambda\sqrt{\widetilde{w}^s}$. The normalized vectorial function

$$\frac{g_v}{\sqrt{g_v^s}} = \frac{\widetilde{w}}{\sqrt{\widetilde{w}^s}} =: w$$

is thus well defined in U. Similarly, the function $\pm\sqrt{g^s}$ is well defined in U. If U intersects the real axis, we choose the sign so that $\sqrt{g^s}(U \cap \mathbb{R}) \subset (0, +\infty)$. Then $f_0 := \log_{*,0,0}(\sqrt{g^s})$ is well defined. If U does not intersect the real axis then we define f_0 in accordance to the next formula,

$$\begin{aligned} f_0 &:= \log_{*,2m,0}(\sqrt{g^s}), && \text{if } \sqrt{g^s}(\mathbb{S}_{z_0}) \subset \mathbb{H} \setminus (-\infty, 0), \\ f_0 &:= \log_{*,2m+1,0}(-\sqrt{g^s}), && \text{if } \sqrt{g^s}(\mathbb{S}_{z_0}) \subset (-\infty, 0), \end{aligned} \tag{6.15}$$

with $m \in \mathbb{Z}$. Notice also that the image of a sphere $\mathbb{S}_z$ by a slice-preserving function is always a sphere centred on the real axis.

For $f = f_0 + f_v = \log_* g$ following formula (4.13), we want the following listed equalities to hold:

$$\begin{aligned} \frac{f_v}{\sqrt{f_v^s}} &= \frac{g_v}{\sqrt{g_v^s}}, \\ \cos_* \sqrt{f_v^s} &= \frac{g_0}{\sqrt{g^s}}, \\ \sin_* \sqrt{f_v^s} &= \frac{\sqrt{g_v^s}}{\sqrt{g^s}}. \end{aligned}$$

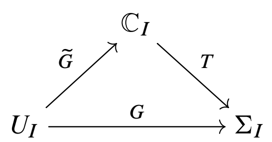
For each $I \in \mathbb{S}$ define the complex manifold Σ_I to be the regular set $s^{-1}(1)$ for $s\colon \mathbb{C}_I^2 \to \mathbb{C}_I$, $s(u, v) = u^2 + v^2$. It is not difficult to show that the mapping

$$T\colon \mathbb{C}_I \to \Sigma_I, \qquad T(q) = (\cos q, \sin q)$$

is a covering map and by construction we have

$$G := \left(\frac{g_0}{\sqrt{g^s}}, \frac{\sqrt{g_v^s}}{\sqrt{g^s}}\right)\colon U_I \to \Sigma_I.$$

There exists a lift $\widetilde{G}$ such that the diagram



commutes, i.e., $T \circ \widetilde{G} = G$.

If U_I is simply connected, $U \cap \mathbb{R}$ is an open interval and $G(U \cap \mathbb{R}) \subset S^1 \subset \mathbb{R}^2 \subset (\mathbb{C}_I)^2$, since all the functions are slice-preserving. The only possibility that both $\sin(z)$ and $\cos(z)$ are real, is, that z is real. Therefore, for any lift $\widetilde{G}$, the restriction $\widetilde{G}|_{U\cap\mathbb{R}}$ is real-valued and hence satisfies the reflection property $\widetilde{G}(z) = \widetilde{G}(\bar{z})$. If U_I has connected components $U_{I,n}$, $n = 1, 2$, then first define the function $\widetilde{G}$ on $U_{I,1}$ to be an arbitrary lift of $G|_{U_{I,1}}$ and extend the definition to $U_{I,2}$ by reflection property. Define

$$\sqrt{f_v^s} := \widetilde{G}$$

on U_I. The reflection property guarantees that $\sqrt{f_v^s}$ has a slice-preserving extension to U.

In the case $U \cap \mathbb{R} \neq \varnothing$, the final formula is

$$\log_{*,0,2nw} g = \log_{*,0,0}(\sqrt{g^s}) + (\sqrt{f_v^s} + 2n\pi)\frac{g_v}{\sqrt{g_v^s}}. \tag{6.16}$$

If $U \cap \mathbb{R} = \varnothing$ we also have periodicity in the scalar part and the formula is

$$\log_{*,m,nw} g = \log_{*,0,0}(\sqrt{g^s}) + m\pi\mathcal{I} + (\sqrt{f_v^s} + n\pi)\frac{g_v}{\sqrt{g_v^s}}, \tag{6.17}$$

where $m, n \in \mathbb{Z}$ are such that $m + n \equiv 0 \bmod 2$ and the logarithm $f_0 := \log_{*,0,0}(\sqrt{g^s})$ is chosen in accordance with (6.15).

Notice that, contrary to the previous cases 1 and 2 (and the next case, case 4), in case 3 one cannot specify the "principal branch", unless one chooses a specific point in the domain and specific normalized minimal representative.

6.5. Case 4: $g \in \mathcal{SR}(\Omega)$ and $z_0 \in \Omega$ are such that $z_0 \in \mathbb{S}Z([g_v])$

Without loss of generality we assume that $z_0 \in Z([g_v])$, since the logarithmic function is to be defined on a basic neighbourhood of z_0. We have the following two possibilities:

(i) z_0 is a nonreal isolated zero of g_v,

(ii) z_0 is a nonreal isolated zero and $\mathbb{S}_{z_0}$ is a spherical zero of g_v.

Let us first consider case (i). Since $g^s(z_0) = g_0^2(z_0) \neq 0$, we define

$$\sqrt{g^s} := g_0\sqrt{1 + \frac{g_v^s}{g_0^2}}$$

with $\sqrt{}$ defined using the principal branch of the logarithm (see formula (1.4)). The function $\sqrt{g^s}$ is a slice-preserving and slice-regular function with $g_0(z_0) = \sqrt{g^s}(z_0)$. This function is well defined in a neighbourhood of $\mathbb{S}_{z_0}$. Define

$$f_v^s := \mu^{-1}\left(\left(\sqrt{1 + \frac{g_v^s}{g_0^2}}\right)^{-1}\right)$$

where $\mu^{-1} = \mu_0^{-1}$ is the inverse function of μ from a neighbourhood of 1 to a neighbourhood of 0, so that $f_v^s(z_0) = 0$ (see Section 4.1). This is equivalent to the choice of the principal branch of arccos denoted by $\arccos_0$ indeed

$$f_v^s = \left(\arccos_0 \frac{g_0}{\sqrt{g^s}}\right)^2.$$

Recall that the function μ is locally invertible near 0 because $\mu'(0) = (-1/2)\nu(0) = -1/2$. If the function $-\sqrt{g^s}$ is chosen instead, f_v^s cannot be defined since the function μ has branching points at

$$\mu^{-1}(-1) = \{(2k+1)^2\pi^2 : k \in \mathbb{N}\}.$$

Remark 6.5. The isolated nonreal zeroes of the vectorial part force the choice of the function $\sqrt{g^s}$ to be such that $g_0(z)/\sqrt{g^s}(z) = 1$ for every zero z of g_v^s.

For the definition of f_0 we have to calculate a logarithm of $\sqrt{g^s}$ depending on the two cases as in (6.15),

$$\begin{aligned} f_0 &:= \log_{*,2m,0}(\sqrt{g^s}), && \text{if } \sqrt{g^s}(\mathbb{S}_{z_0}) \subset \mathbb{H} \setminus (-\infty, 0), \\ f_0 &:= \log_{*,2m+1,0}(-\sqrt{g^s}), && \text{if } \sqrt{g^s}(\mathbb{S}_{z_0}) \subset (-\infty, 0), \end{aligned}$$

with $m \in \mathbb{Z}$. Since $f_v^s(z_0) = 0$ in both cases, we have $\mu(f_v^s)(z_0) = \nu(f_v^s)(z_0) = 1$.

Define the vectorial function f_v to be

$$f_v = \exp_*(-f_0)\frac{g_v}{\nu(f_v^s)}.$$

Then $f = f_0 + f_v$ solves $\exp_* f = g$. The complete formula is

$$\log_{*,m,0\cdot[g_v]} g = \log_{*,m,0}((-1)^m\sqrt{g^s}) + \frac{g_v}{\nu\left(\mu^{-1}\left(1/\sqrt{1+\frac{g_v^s}{g_0^2}}\right)\right)\sqrt{g^s}}, \tag{6.18}$$

where m depends on the values of $\sqrt{g^s}$ as in (6.15) and ν is defined in formulae (4.11) and (4.12). Notice that the period in the imaginary directions appears from the definition of the branches of logarithm for the slice-preserving part.

If, in addition, $\mathbb{S}_{z_0}$ is also a spherical zero of g_v, the necessary condition, namely that $g_0(z) = \sqrt{g^s}(z)$ for every zero of g_v^s, is fulfilled on the whole sphere $\mathbb{S}_{z_0}$, hence the same formula applies to case (ii).

Remark 6.6. In the case where the zero z_0 has even multiplicity, the square root $\sqrt{g_v^s}$ is well defined and we could follow the construction for case 3 and get formulae (6.16) or (6.17); instead the vectorial part

$$(\sqrt{f_v^s} + 2k\pi)\frac{g_v}{\sqrt{g_v^s}}$$

has a pole unless we choose $k = 0$. In addition, we must also have $\sqrt{f_v^s}(\mathbb{S}_{z_0}) = 0$ and this implies that

$$\arccos \frac{g_0}{\sqrt{g^s}}(z_0) = 0,$$

which at the end gives formula (6.18).

Remark 6.7. Let $f, g, w \in \mathcal{SR}_\omega(U)$ for U a basic domain in $\mathbb{H}$ and let w be a normalized representative of ω on U. Assume that $\forall\, m, n \in \mathbb{Z}$, $\log_{*,m,nw} fg$, $\log_{*,m,nw} f$ and $\log_{*,m,nw} g$, all exist. Since there is no "principal branch" in w, there is no reason that the equality

$$\log_{*,m,nw} fg = \log_{*,m_0,n_0w} f + \log_{*,m-m_0,(n-n_0)w} g$$

should hold; in general we have

$$\log_{*,m,nw} fg = \log_{*,m_0,n_0w} f + \log_{*,m-m_0,(n-n_0)w} g + 2k\pi w.$$

7. Global definition of $\log_*$ and proof of Theorem 1.1

In this section, we prove Theorem 1.1, namely we consider the global problem of determining the logarithmic function of a given slice-regular function, with the requirement that the logarithmic function defines the same vectorial class as the original function: if Ω is a basic domain of $\mathbb{H}$, given $g \in \mathcal{SR}(\Omega)$ not vanishing in Ω, find $f \in \mathcal{SR}_{[g_v]}(\Omega)$ such that

$$\exp_* f = g \quad \text{on } \Omega.$$

A classical result in complex analysis states that it is not possible to define $\log z^2$ on $\mathbb{C} \setminus \{0\}$ and hence it is also not possible to define a logarithmic function of $p_2(q) = q^2$ on $\mathbb{H} \setminus \{0\}$, although the function p_2 satisfies the necessary condition (3.8).

7.1. Proof of Theorem 1.1

The proof of Theorem 1.1 is presented according to the four cases as in Section 6. Here we recall the statement, before proving it.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{H}$ be a basic domain and let $g \in \mathcal{SR}_\omega(\Omega)$ be a nonvanishing function. Then the following holds:*

(a) *if $\omega = [0]$, a necessary and sufficient condition for the existence of a $*$-logarithm of g on Ω, $\log_* g \in \mathcal{SR}_{[0]}(\Omega) = \mathcal{SR}_{\mathbb{R}}(\Omega)$, is*

$$g(\Omega \cap \mathbb{R}) \subset (0, +\infty);$$

(b) *if $\omega \neq [0]$, then if $Z(\omega) = \varnothing$ or if $\mathbb{S}Z(\omega) = \Omega$ there are no conditions, and a $*$-logarithm of g on Ω, $\log_* g \in \mathcal{SR}_\omega(\Omega)$, always exists;*

(c) *if* $\omega \neq [0]$ *and* $Z(\omega)$ *is discrete, a sufficient condition for the existence of a* $*$-*logarithm of* g *on* Ω, $\log_* g \in \mathcal{SR}_\omega(\Omega)$, *is the validity of both inclusions*

$$\sqrt{g^s}(\Omega \cap \mathbb{R}) \subset (0, +\infty) \tag{1.5}$$

and

$$\frac{g_0}{\sqrt{g^s}}(\Omega) \subset \mathbb{H} \setminus (-\infty, -1], \tag{1.6}$$

where $g^s = g_0^2 + g_v^s$ *denotes the symmetrization of* g.

7.1.1. Proof of Theorem 1.1 (a). The conditions in (a) correspond to case 1 presented in Section 6.2. Assume that Ω is a basic product domain. This implies that the imaginary unit function $\mathcal{I}$ is well defined. In each leaf $\mathbb{C}_I^+$ the set $\Omega_I^+ := \mathbb{C}_I^+ \cap \Omega$ is simply connected. Assume that $g \in \mathcal{SR}_\mathbb{R}(\Omega)$ is a nonzero function. Then $g_I^+ := g|_{\Omega_I^+} \to \mathbb{C}_I$ is holomorphic and therefore it has a holomorphic logarithm $f_I^+ := \log g_I^+$. Because g is also slice-preserving, we can define $\log_{-I} g_{-I}^+(z) = \overline{\log_I(g_I^+(\overline{z}))}$ and extend the logarithm to Ω. Denote this extension by $f = \log_* g$. Similarly, the whole family of logarithmic functions $f_k = \log_*((-1)^k g) + k\pi\mathcal{I}$ is also well defined. Notice that it is essential for this construction that the imaginary unit function $\mathcal{I}$ exists.

Next, assume that Ω is a basic slice domain. Then in each leaf $\mathbb{C}_I^+$ the set $\Omega_I^+ := \mathbb{C}_I^+ \cap \Omega$ is simply connected and the intersection $\Omega_\mathbb{R} := \Omega \cap \mathbb{R}$ is connected. Assume that $g \in \mathcal{SR}_\mathbb{R}(\Omega)$ is a nonzero function satisfying $g(\Omega \cap \mathbb{R}) \subset (0, \infty)$. Let Ω_0 be a connected component of $g^{-1}(g(\Omega) \cap (\mathbb{H} \setminus (-\infty, 0]))$ which contains the set $\Omega_\mathbb{R}$. Since the image $g(\Omega_0)$ does not intersect the negative real axis, the function $f_0 = \log_{*,0,0} g$ is well defined on Ω_0 and it is the unique logarithm as explained in Section 6.

If $\Omega = \Omega_0$ the problem is solved so assume that $\Omega \neq \Omega_0$. Then Ω_0 is an open neighbourhood of an interval $\Omega_\mathbb{R}$. The set $\Omega_1 := \Omega \setminus \Omega_\mathbb{R}$ is also connected and basic, but $\Omega_{1,I} := \Omega_1 \cap \mathbb{C}_I$ has two connected components, $\Omega_{1,I\pm}$. Choose the component $\Omega_{1,I+}$. Since it is simply connected, the function g has a complex logarithm f_+ on $\Omega_{1,I+}$. On the intersection of their domains of definition (which is an open connected set), the functions f_0 and f_+ differ by $2\pi k I$, $f_0 = f_+ + 2k\pi I$. Redefine f_+ to be $f_+ + 2k\pi I$ and define f_- to be the Schwarz reflection of the function of f_+. Since f_0 is slice-preserving, $f_0(\overline{z}) = \overline{f_0(z)}$, the reflected function coincides with f_0 on the intersection of domains of definition and hence defines a function f on Ω_I, which satisfies $f(\overline{z}) = \overline{f(z)}$. By the extension formula, the function f can be extended to a slice-preserving function on Ω. ∎

7.1.2. Proof of Theorem 1.1 (b). The first condition in (b), $\omega \neq 0$, $Z(\omega) = \varnothing$, corresponds to case 3 presented in Section 6.4. The function g^s is nonvanishing, the function g_v^s has isolated real or spherical zeroes with even multiplicities and Ω is a basic domain, which are precisely the conditions of [1, Proposition 1.6], which states, that under these conditions, the square roots $\sqrt{g^s}$ and $\sqrt{g_v^s}$ can be globally defined on Ω. Moreover, the normalized vectorial class

$$\frac{g_v}{\sqrt{g_v^s}} =: w$$

is globally well defined and nonzero on Ω. Therefore, formulae (6.16) and (6.17) are globally valid and the logarithm exists.

The second condition in (b), $\omega \neq 0$, $\mathbb{S}Z(\omega) = \Omega$ and hence $\Omega \subset \mathbb{H} \setminus \mathbb{R}$, corresponds to case 2 presented in subsection 6.3. As already mentioned, when in case 2, the basic domain Ω does not intersect the real axis and g_0 is not vanishing in Ω. Then, for $m \in \mathbb{Z}$, one can define

$$\log_{*,2m,0\cdot[g_v]} g := \log_{*,2m,0} g_0 + \frac{g_v}{g_0}$$

since from the previous considerations $\log_{*,2m,0} g_0$ is well defined on Ω. ∎

7.1.3. Proof of Theorem 1.1 (c). The condition in (c), $\omega \not\equiv 0$, $Z(\omega)$ is discrete, corresponds to case 4 presented in Section 6.5. The logarithm $\log_* \sqrt{g^s}$ exists by case 1. The assumptions imply that $\mu_0^{-1}(g_0/\sqrt{g^s})$ is well defined on Ω and that $g_0(z) = \sqrt{g^s}(z)$ for every zero z of g_v^s, because -1 is not in the image of $g_0/\sqrt{g^s}$. Hence the logarithm is given by formula (6.18). ∎

Remark 7.2. Notice that in the hypotheses of case (c) of Theorem 1.1, the stated sufficient conditions are always fulfilled on "small" basic product domains that are neighbourhoods of a (nonreal) $z_0 \in Z(\omega)$ (for instance on any set $\mathbb{S}B^4(z_0, r)$ with small enough $r > 0$).

As, by definition, every set $\mathcal{SR}_\omega(\Omega)$ contains also the set $\mathcal{SR}_\mathbb{R}(\Omega)$, Theorem 1.1 (b) yields the following.

Corollary 7.3. *Let Ω be a basic domain, $g \in \mathcal{SR}_\mathbb{R}(\Omega)$ and let ω be a vectorial class in Ω with $Z(\omega) \cap \Omega = \emptyset$. Then there exists a logarithmic function of g in the class $\mathcal{SR}_\omega(\Omega)$, denoted by* $\log_* g$.

7.2. The case of one isolated nonreal zero

For the case of a slice-regular function defined on a basic product domain, and whose vectorial part has only an isolated zero, we can – as announced – produce a formula for the $*$-logarithms.

Theorem 7.4. *Let $g \in \mathcal{SR}_\omega(\Omega)$ be a nonvanishing function and Ω be a basic product domain. Let $Z(g_v) \cap \Omega = \{z_0\}$ and let $\sqrt{g^s}$ be such that $\sqrt{g^s}(z_0) = g_0(z_0)$. Then there exists a logarithmic function f of g, $f \in \mathcal{SR}_\omega(\Omega)$, given by the formula*

$$f = \log_{*,m,0\cdot[g_v]} g = \log_{*,m,0}((-1)^m \sqrt{g^s}) + \frac{g_v}{\nu(\mu^{-1}(g_0/\sqrt{g^s}))\sqrt{g^s}},$$

where $\log_{*,m,0} = \log_{*,0,0} + m\pi\mathcal{I}$ *for $m \in \mathbb{Z}$ and*

(a) *m is even if $\sqrt{g^s}(\mathbb{S}_{z_0}) \subset \mathbb{H} \setminus (-\infty, 0)$ or odd if $\sqrt{g^s}(\mathbb{S}_{z_0}) \subset (-\infty, 0)$,*

(b) *the function*

$$\mu^{-1}(g_0/\sqrt{g^s})$$

is the lift of the function $g_0/\sqrt{g^s}$ with respect to the mapping μ such that

$$\mu^{-1}(g_0/\sqrt{g^s})(z_0) = 0.$$

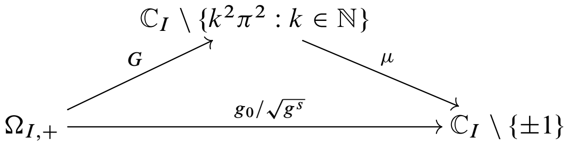
Proof. The logarithm of $\sqrt{g^s}$ exists by Theorem 1.1 (a), because $\Omega \subset \mathbb{H} \setminus \mathbb{R}$. The function $\sqrt{g^s}$ is such that not only $\sqrt{g^s}(z_0) = g_0(z_0)$ but also $\sqrt{g^s}(z) = g_0(z)$ for every $z \in \mathbb{S}_{z_0}$. Indeed, on $\mathbb{S}_{z_0}$ we have

$$\sqrt{g^s}(z) = \sqrt{g_0^2(z)}$$

and since $\sqrt{g_0^2(z_0)} = g_0(z_0)$, the same holds on the whole sphere $\mathbb{S}_{z_0}$. We have to show that the lift of the function $g_0/\sqrt{g^s}$ via μ can be defined on Ω. First observe that $(g_0/\sqrt{g^s})^{-1}(1) \cap \Omega = \mathbb{S}_{z_0}$. Let I be such that $z_0 \in \mathbb{C}_{I,+}$ and choose an arc $l_{I,+}$ connecting z_0 to the boundary of $\Omega_{I,+} := \Omega \cap \mathbb{C}_{I,+}$. Let $l_{I,-}$ be the reflected arc. The domain $\Omega_{I,+} \setminus l_{I,+}$ is simply connected and

$$(g_0/\sqrt{g^s})(\Omega_{I,+} \setminus l_{I,+}) \subset \mathbb{C}_I \setminus \{\pm 1\}.$$

Since the map μ is a slice-covering map from $\mathbb{H} \setminus \{k^2\pi^2 : k \in \mathbb{N}\}$ to $\mathbb{H} \setminus \{\pm 1\}$, the lift G of $g_0/\sqrt{g^s}$ exists,



and can be chosen in such a way that $\lim_{z \to z_0} G(z) = 0$. Cover the arc $l_{I,+}$ with a (possibly infinite) chain of discs D_i such that $z_0 \in D_0$ and each D_i intersects only D_{i-1} and D_{i+1} and intersections are connected. Let $G_0 := \mu_0^{-1}(g_0/\sqrt{g^s})$ near z_0. The lifts G and G_0 coincide on $D_0 \setminus l_{I,+}$ and hence define a lift on the union $D_0 \cup (\Omega_{I,+} \setminus l_{I,+})$ which we also denote by G. Since z_0 is the only point with value $g_0/\sqrt{g^s}(z) = 1$ in $\Omega_{I,+}$, we can choose a lift G_1 on D_1 so that it matches G_0 on $D_0 \cap D_1$. Since $D_1 \cap (D_0 \cup \Omega_{I,+})$ is connected, by the lifting property, the lift G_1 also matches the lift G on $D_1 \setminus l_{I,+}$. Repeating this procedure extends the lift G to $\Omega_{I,+}$. Notice that $G(z) \neq k^2\pi^2$, $k \in \mathbb{N}$ and so $\nu(G) \neq 0$. Extend the definition of G to $\Omega_{I,-}$ by Schwarz reflection and then use the extension formula to get a slice-preserving function, which we denote – with a slight abuse of notation – by $\mu^{-1}(g_0/\sqrt{g^s})$. The logarithm is now given by formula (6.18). ■

Example 7.5. Consider the function

$$g(z) = -1 + z^2 i + \sqrt{2} z j + k$$

defined on $\mathbb{H}$. Because $g^s = 1 + (z^2 + 1)^2$, the zeroes of g^s on $\mathbb{C}_I$ are $z_{1,2} = \sqrt{\pm I - 1}$ and $z_{3,4} = -\sqrt{\pm I - 1}$. Hence these zeroes lie on the sphere with radius $r = 2^{1/4} > 1.1$, so g is nonvanishing in the ball $\Omega = B^4(0, 1.1)$. A simple calculation shows that g^s maps Ω_I to a cardioid-shaped domain in the right half-plane of $\mathbb{H}$, so the image misses the negative real axis, hence there exists a unique logarithmic function of g^s, namely $\log_0 g^s$. Since the domain Ω intersects the real axis, the necessary condition for logarithm of $\sqrt{g^s}$

to exist is (3.8), $\sqrt{g^s}(\Omega \cap \mathbb{R}) \subset (0, +\infty)$, therefore the only possibility for the definition of f_0 is to take the principal branch of the square root and set

$$f_0 := \log_{*,0,0} \sqrt{g^s} = \frac{1}{2} \log_{*,0,0} g^s.$$

The vectorial function g_v has only $z_0 = \frac{k-i}{\sqrt{2}} j$ as the unique (double) zero and the symmetrization of g_v is $g_v^s = (z^2 + 1)^2$ on Ω. Unfortunately, $g(z_0) = g_0(z_0) = -1$ and $\sqrt{g^s}(z_0) = 1$ and hence condition (1.6) no longer holds, which makes it impossible to define the functions f_v and f_v^s near the point z_0, because -1 is a branching point for μ^{-1}. Notice that the function g meets the hypotheses of Theorem 1.1 (c), but does not fulfil one of the stated sufficient conditions, namely condition (1.6).

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