Turkish Journal of Mathematics

Volume 47 | Number 7 Article 9

11-9-2023

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Recommended Citation

IRSİC, VESNA; KLAVZAR, SANDI; and TAN, ELİF (2023) "Generalized Pell graphs," *Turkish Journal of Mathematics*: Vol. 47: No. 7, Article 9. https://doi.org/10.55730/1300-0098.3475
Available at: https://journals.tubitak.gov.tr/math/vol47/iss7/9

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1955 – 1973

doi:10.55730/1300-0098.3475

Generalized Pell graphs

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Received: 26.07.2023 • **Accepted/Published Online:** 06.09.2023 • **Final Version:** 09.11.2023

Abstract: In this paper, generalized Pell graphs $\Pi_{n,k}$, $k \geq 2$, are introduced. The special case of $k = 2$ are the Pell graphs Π_n defined earlier by Munarini. Several metric, enumerative, and structural properties of these graphs are established. The generating function of the number of edges of $\Pi_{n,k}$ and the generating function of its cube polynomial are determined. The center of $\Pi_{n,k}$ is explicitly described; if k is even, then it induces the Fibonacci cube Γ_n. It is also shown that $\Pi_{n,k}$ is a median graph, and that $\Pi_{n,k}$ embeds into a Fibonacci cube.

Keywords: Fibonacci cube; Pell graph; generating function; center of graph; median graph; k-Fibonacci sequence

1. Introduction

The Fibonacci sequence is one of the most famous sequences in mathematics. The nth Fibonacci number F_n is defined by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers and their generalizations have many interesting properties and different applications in science and art. There are several generalizations of Fibonacci sequence. One among them is the k-Fibonacci sequence defined by Falcon and Plaza [9] for a positive integer k as

$$F_{n,k} = kF_{n-1,k} + F_{n-2,k},\ n \geq 2, \tag{1.1}$$

with initial values $F_{0,k} = 0$ and $F_{1,k} = 1$. The first few terms of the k-Fibonacci sequence are 0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, k^5+4k^3+3k. If $k=1$, then (1.1) reduces to the Fibonacci sequence $\{F_n\}_{n\geq 0}$, and if $k=2$, then (1.1) reduces to the Pell sequence $\{P_n\}_{n\geq 0}$, where $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. The generating function of the k-Fibonacci sequence is given by

$$F(t) = \sum_{n\geq 0} F_{n,k} t^n = \frac{t}{1-kt-t^2}\,. \tag{1.2}$$

For more on these sequences, we refer the reader to [8]. It should also be noted that the k-Fibonacci numbers defined and used here are not to be confused with the k-generalized Fibonacci numbers (or generalized order-k

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2010 *AMS Mathematics Subject Classification:* 05C75, 05C12, 05C30, 11B39

Fibonacci numbers) that are defined as

$$F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k},\ n \geq k,$$

with the appropriate initial conditions, see [14].

Fibonacci cubes were introduced in 1993 in [10]. They are closely related to the Fibonacci sequence. They have found numerous applications elsewhere and are also extremely interesting in their own right. The state of the art on Fibonacci cubes and related classes of graphs has been collected in the book [7] published in 2023, the research in this direction is still ongoing, see [6, 11, 13]. On the other hand, motivated by the Pell sequence, in 2019 Munarini introduced Pell graphs [16]. Pell graphs have been further investigated in [17]. In this paper, based on the definition (1.1) we introduce generalized Pell graphs such that for each $k \geq 2$ their construction reflects the recursion (1.1).

The rest of the paper is organized as follows. In the next section, we define the concepts discussed in this paper and introduce the required notation. In Section 3 the two-parameter generalized Pell graphs $\Pi_{n,k}$ are formally defined and their fundamental structure is described. Among other results we determine the generating function of the number of edges of $\Pi_{n,k}$ and observe that they are traceable, that is, they contain Hamiltonian paths. In Section 4 we determine the radius and the diameter of $\Pi_{n,k}$. Furthermore, we describe the structure of the center of $\Pi_{n,k}$. Interestingly, if k is even, then the center of $\Pi_{n,k}$ induces the Fibonacci cube Γ_n. In Section 5 additional properties of $\Pi_{n,k}$ are established: the generating function of its cube polynomial, distribution of its degrees, the fact that $\Pi_{n,k}$ is a median graph, and that $\Pi_{n,k}$ embeds into a Fibonacci cube. We conclude the paper with some remarks on a similar project undertaken independently by Došlić and Podrug* and with some open problems.

2. Preliminaries

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is a set of vertices and $E(G)$ is a set of edges consisting of unordered pairs of vertices. The numbers of vertices and edges in G are called the *order* and the *size* of G, respectively. The degree $\deg(u)$ of a vertex $u \in V(G)$ is the number of edges incident with it in G. As usual, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum and the minimum degree of G, respectively. The subgraph induced by $X \subseteq V(G)$ is denoted by $G[X]$.

The *distance* $d(u, v)$ between vertices u and v of a graph G is the number of edges on a shortest u, v-path. The *eccentricity* $\mathrm{ecc}(u)$ of a vertex $u \in V(G)$ is the maximum distance between u and all other vertices of G. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of G are the minimum and the maximum eccentricity of the vertices of G, respectively. The *center* $C(G)$ of G is the set of vertices $u \in V(G)$ with $\mathrm{ecc}(u) = \mathrm{rad}(G)$. The *periphery* of G is defined as the set of vertices $u \in V(G)$ with $\mathrm{ecc}(u) = \mathrm{diam}(G)$.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has vertices $V(G) \times V(H)$ and edges $(g, h)(g', h')$, where either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. The r*-cube* Q_r, $r \geq 1$, is the graph with $V(Q_r) = \{0, 1\}^r$, with an edge between two vertices if and only if they differ in exactly one coordinate. That is, if $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$ are vertices of Q_r, then $xy \in E(Q_r)$ if and only if there exists $j \in [r] = \{1, \ldots, r\}$ such that $x_j \neq y_j$ and $x_i = y_i$ for every $i \neq j$. The r-cube Q_r, $r \geq 2$, can also be described as the Cartesian product $Q_{r-1} \,\square\, K_2$.

*Došlić T, Podrug L. Metallic cubes. Website https://arxiv.org/abs/2307.14054 [accessed 27 July 2023].

Let $\mathcal{F}_n$ denote the set of Fibonacci strings of length n, that is, binary strings of length n that contain no consecutive 1s. Then $\mathcal{F}_0 = \{\varepsilon\}$, $\mathcal{F}_1 = \{0, 1\}$, and if $n \geq 2$, then

$$\mathcal{F}_{n+2} = 0\mathcal{F}_{n+1} + 10\mathcal{F}_n ,$$

where $+$ denotes the disjoint union of sets. Consequently, $|\mathcal{F}_n| = F_{n+2}$. The *Fibonacci cube* Γ_n, $n \geq 1$, is the graph with $V(\Gamma_n) = \mathcal{F}_n$ in which two vertices are adjacent if they differ in a single coordinate. Hence $|V(\Gamma_n)| = F_{n+2}$. Note that the strings $0\mathcal{F}_{n+1}$ in $\mathcal{F}_{n+2}$ induce a subgraph isomorphic to Γ_{n+1} and the strings $10\mathcal{F}_n$ induce a subgragraph isomorphic to Γ_n.

If w is a word over an alphabet Σ and $a \in \Sigma$, then a *run* of as is a maximal subword of w such that all of its letters are a (sometimes called a *block*). (For the research on runs in binary strings and the so-called Fibonacci-run graphs see [1, 4, 5, 19].) A *Pell string* is a word over the alphabet $T = \{0, 1, 2\}$ such that there are no runs of $2s$ of odd length [16]. Equivalently, a Pell string is a word over the alphabet $T' = \{0, 1, 22\}$. Let $\mathcal{P}_n$ denote the set of Pell strings of length n. Then $\mathcal{P}_0 = \{\varepsilon\}$, $\mathcal{P}_1 = \{0, 1\}$ and for $n \geq 0$,

$$\mathcal{P}_{n+2} = 0\mathcal{P}_{n+1} + 1\mathcal{P}_{n+1} + 22\mathcal{P}_n .$$

Thus $|\mathcal{P}_n| = P_{n+1}$. The *Pell graph* Π_n, $n \geq 0$, has $V(\Pi_n) = \mathcal{P}_n$ and two vertices in Π_n are adjacent whenever one of them can be obtained from the other by replacing a 0 with a 1 (or vice versa), or by replacing a factor 11 with 22 (or vice versa). Then $\Pi_0 = K_1$, $\Pi_1 = K_2$, and $|V(\Pi_n)| = P_{n+1}$. Furthermore, the number of edges in Π_n satisfies $|E(\Pi_n)| = |E(\Pi_{n-1})| + |E(\Pi_{n-1})| + |E(\Pi_{n-2})| + P_{n+1} + P_n$. In Figure 1 the first four Pell graphs are drawn. See [16] for more on Pell graphs.

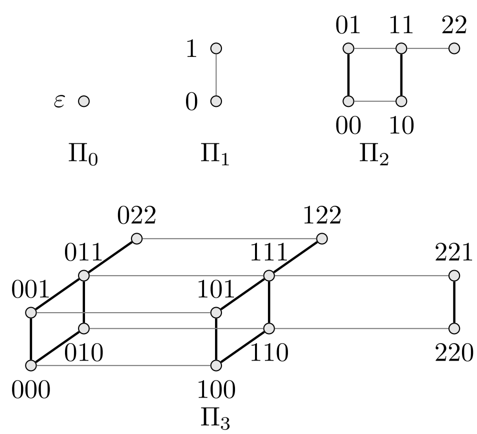

Figure 1. Pell graphs Π_n for $n \in \{0, 1, 2, 3\}$.

3. Generalized Pell graphs and their basic properties

Motivated by the facts from the introduction, we now define generalized Pell graphs as follows.

If $k \geq 2$, then a *generalized Pell string* is a string over the alphabet $\{0, 1, \ldots, k-1, kk\}$. Note that a generalized Pell string with $k = 2$ is a Pell string, and that if α is a generalized Pell string, then each run of ks is of even length. If $n \geq 0$ and $k \geq 2$, then let $\mathcal{F}_{n,k}$ be the set of the generalized Pell strings of length n. Clearly, $\mathcal{F}_{0,k} = \{\varepsilon\}$ and $\mathcal{F}_{1,k} = \{0, 1, \ldots, k-1\}$, while for $n \geq 2$ we have

$$\mathcal{F}_{n,k} = 0\mathcal{F}_{n-1,k} + 1\mathcal{F}_{n-1,k} + \cdots + (k-1)\,\mathcal{F}_{n-1,k} + kk\mathcal{F}_{n-2,k} .$$

Therefore, $|\mathcal{F}_{n,k}| = F_{n+1,k}$, where the values $F_{n,k}$ are defined in (1.1).

Now, if $n \geq 0$ and $k \geq 2$, then the *generalized Pell graph* $\Pi_{n,k}$ has the vertex set $V(\Pi_{n,k}) = \mathcal{F}_{n,k}$ and two vertices being adjacent whenever one of them can be obtained from the other by either replacing an i with an $i+1$ (or vice versa), where $i \in \{0, 1, \ldots, k-2\}$, or by replacing one factor $(k-1)(k-1)$ with kk (or vice versa) in such a way that the new string is again a generalized Pell string. Note that $\Pi_{n,2} = \Pi_n$. In Figure 2 the generalized Pell graphs $\Pi_{n,3}$, $n \in \{0, 1, 2, 3\}$, are shown.

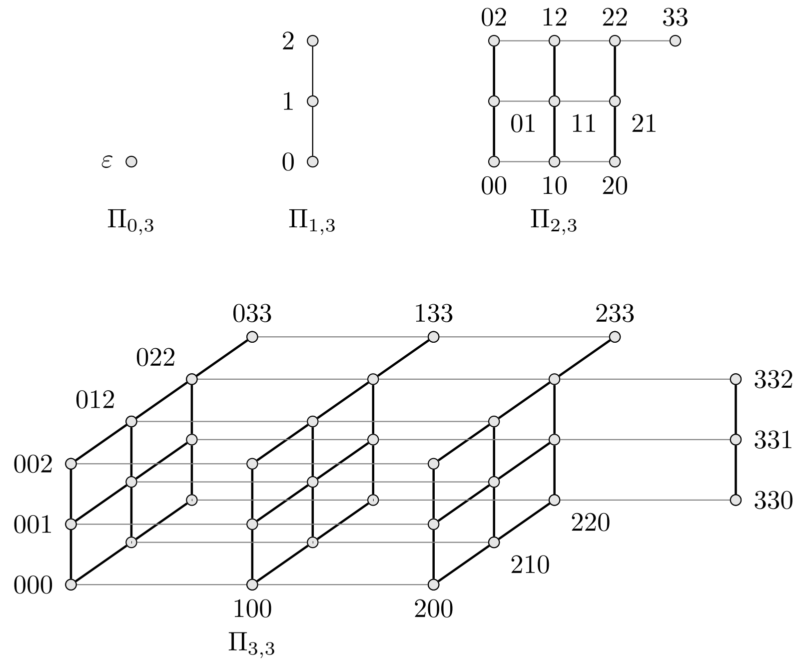

Figure 2. Generalized Pell graphs $\Pi_{n,3}$ for $n \in \{0, 1, 2, 3\}$. To make the figure transparent, not all vertices are labeled.

The way we defined generalized Pell graphs appeared to us as the (most) natural generalization of Pell graphs such that the order of the graph is counted by the k-Fibonacci sequence. But there are other ways to extend Pell graphs, for instance, to use the alphabet $\{0, 1, 22, 33, \ldots, kk\}$. In this case, however, the number of vertices v_n would satisfy $v_n = 2v_{n-1} + (k-1)v_{n-2}$, and not the recursion from (1.1).

The *fundamental decomposition* of the generalized Pell graph $\Pi_{n,k}$ is the following. Note that each of the induced subgraphs $\Pi_{n,k}[j\mathcal{F}_{n-1,k}]$, $j \in \{0, \ldots, k-1\}$, is isomorphic to $\Pi_{n-1,k}$ and it is denoted by $j\Pi_{n-1,k}$. In addition, the induced subgraph $\Pi_{n,k}[kk\mathcal{F}_{n-2,k}]$ is isomorphic to $\Pi_{n-2,k}$ and denoted by $kk\Pi_{n-2,k}$. Then it is straightforward to see that $\Pi_{n,k}[\bigcup_{j=0}^{k-1} j\Pi_{n-1,k}]$ is isomorphic to the Cartesian product of $\Pi_{n-1,k}$ and the path on k vertices. Additionally, each vertex from $kk\Pi_{n-2,k}$ has exactly one neighbor in $(k-1)\Pi_{n-1,k}$. For $n \geq 2$ we formally denote this fundamental decomposition as follows:

$$\Pi_{n,k} = 0\Pi_{n-1,k} \oplus 1\Pi_{n-1,k} \oplus \cdots \oplus (k-1)\,\Pi_{n-1,k} \odot kk\Pi_{n-2,k}, \tag{3.1}$$

with $\Pi_{0,k} = K_1$ and $\Pi_{1,k}$ is the path on k vertices. See Figure 3.

Since the generalized Pell graph $\Pi_{n,k}$ is defined on the vertex set $\mathcal{F}_{n,k}$, the number of vertices of $\Pi_{n,k}$ is $F_{n+1,k}$.

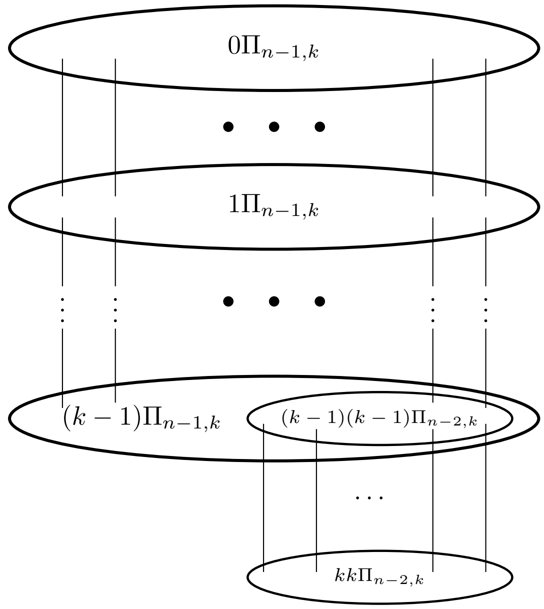

Figure 3. The fundamental decomposition of $\Pi_{n,k}$.

From the fundamental decomposition of $\Pi_{n,k}$ in (3.1), the edges of $\Pi_{n,k}$ are of the following four types:

(i) edges from k copies of $\Pi_{n-1,k}$,

(ii) edges from $\Pi_{n-2,k}$,

(iii) the link edges between the vertices in the k copies of $\Pi_{n-1,k}$, and

(iv) the link edges between the vertices in $kk\Pi_{n-2,k}$ and $(k-1)(k-1)\Pi_{n-2,k}$.

Thus the number of edges can be obtained by the following recurrence relation, for $n \geq 2$

$$|E(\Pi_{n,k})| = k|E(\Pi_{n-1,k})| + |E(\Pi_{n-2,k})| + (k-1)\,F_{n,k} + F_{n-1,k}$$

or (by using the recurrence relation of k-Fibonacci numbers)

$$|E(\Pi_{n,k})| = k|E(\Pi_{n-1,k})| + |E(\Pi_{n-2,k})| + F_{n+1,k} - F_{n,k} \tag{3.2}$$

with $|E(\Pi_{0,k})| = 0$ and $|E(\Pi_{1,k})| = k-1$.

Proposition 3.1 *The generating function of the number of edges in* $\Pi_{n,k}$ *is*

$$\sum_{n\geq 0} |E(\Pi_{n,k})| t^n = \frac{(k-1+t)\,t}{(1-kt-t^2)^2}.$$

Proof Denote the generating function of the sequence of the number of edges in $\Pi_{n,k}$ by $E(t)$. From (3.2), we have

$$\begin{aligned}
E(t) &= \sum_{n\geq 0} |E(\Pi_{n,k})| t^n \\
&= |E(\Pi_{0,k})| + |E(\Pi_{1,k})| t + \sum_{n\geq 2} |E(\Pi_{n,k})| t^n \\
&= (k-1)\,t + \sum_{n\geq 2} \left(k|E(\Pi_{n-1,k})| + |E(\Pi_{n-2,k})| + F_{n+1,k} - F_{n,k}\right) t^n \\
&= (k-1)\,t + k\sum_{n\geq 2} |E(\Pi_{n-1,k})| t^n + \sum_{n\geq 2} |E(\Pi_{n-2,k})| t^n + \\
&\quad \sum_{n\geq 2} F_{n+1,k} t^n - \sum_{n\geq 2} F_{n,k} t^n \\
&= \left(kt + t^2\right) E(t) + (k-1)t + \sum_{n\geq 0} F_{n+1,k} t^n - \sum_{n\geq 0} F_{n,k} t^n \\
&= \left(kt + t^2\right) E(t) + \left(\frac{1}{t} - 1\right) F(t) - 1.
\end{aligned}$$

Thus from this identity and the generating function in (1.2) the proposition follows. □

Proposition 3.2 *The size of* $\Pi_{n,k}$ *is*

$$|E(\Pi_{n,k})| = \sum_{i=0}^{n} F_{i,k}\left(F_{n-i+2,k} - F_{n-i+1,k}\right).$$

Proof From Proposition 3.1, we have

$$E(t) = t^{-1}\left(k - 1 + t\right) F^2(t).$$

From the product of the formal power series, we have

$$F^2(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} F_{i,k} F_{n-i,k} t^n.$$

Hence we can compute as follows:

$$\begin{aligned}
E(t) &= \left((k-1)\,t^{-1} + 1\right)\left(\sum_{n=0}^{\infty}\sum_{i=0}^{n} F_{i,k} F_{n-i,k} t^n\right) \\
&= (k-1)\sum_{n=0}^{\infty}\sum_{i=0}^{n+1} F_{i,k} F_{n-i+1,k} t^n + \sum_{n=0}^{\infty}\sum_{i=0}^{n} F_{i,k} F_{n-i,k} t^n \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n} F_{i,k}\left((k-1)\,F_{n-i+1,k} + F_{n-i,k}\right) t^n \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n} F_{i,k}\left(kF_{n-i+1,k} + F_{n-i,k} - F_{n-i+1,k}\right) t^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} F_{i,k} \left(F_{n-i+2,k} - F_{n-i+1,k} \right) t^n.$$

□

Using the fundamental decomposition (3.1) and the same methods used in the case of Fibonacci cubes (see [7]), the following holds. We omit the proof.

Proposition 3.3 *For every $n \geq 0$ and $k \geq 2$, the graph $\Pi_{n,k}$ has a Hamiltonian path.*

Since $\Pi_{n,k}$ is bipartite and when n is even the partite sets are of different sizes, $\Pi_{n,k}$ has no Hamiltonian cycle if n is even. If n is odd, it is not obvious which graphs $\Pi_{n,k}$ are Hamiltonian and which are not (see Problem 5.9).

4. Radius and diameter

If t is a word over alphabet Σ, then $|t|_i$ denotes the number of occurrences of the letter $i \in \Sigma$ in the word t. A substring consisting of m consecutive letters $i \in \Sigma$ is denoted by i^m.

Proposition 4.1 *If $n \geq 1$ and $k \geq 2$, then*

$$\mathrm{rad}(\Pi_{n,k}) = \left\lfloor \frac{kn}{2} \right\rfloor.$$

Proof Let $t = t_1 \dots t_n \in V(\Pi_{n,k})$. We distinguish two cases.

Case 1 k is even.

Let $|t|_k = 2\ell$. Then $|t|_0 + \dots + |t|_{k-1} = n - 2\ell$. Consider the generalized Pell string t' obtained from t by first exchanging the role of i and $i + \frac{k}{2}$ for every $i \in \{0, \dots, \frac{k}{2} - 1\}$, and then replacing each substring kk with 00. Exchanging the role of i and $i+\frac{k}{2}$ requires $\frac{k}{2}$ consecutive changes in the string, while replacing kk with 00 requires $2k-1$ consecutive changes (for example, $kk \to (k-1)(k-1) \to (k-2)(k-1) \to \dots \to 00$). Since these changes are disjoint, we obtain

$$d(t, t') = (n - 2\ell)\frac{k}{2} + \ell(2k - 1) = \frac{kn}{2} + \ell(k - 1) \geq \frac{kn}{2},$$

since $\ell \geq 0$ and $k \geq 2$.

Case 2 k is odd.

Let $|t|_k = 2\ell$, $|t|_{\frac{k-1}{2}} = m$, and let $p \geq 0$ denote the maximum number of disjoint appearances of the substring $(\frac{k-1}{2})(\frac{k-1}{2})$ in t. Then $|t|_0 + \dots + |t|_{k-1} - |t|_{\frac{k-1}{2}} = n - 2\ell - m$ and since p is the largest possible, $m \leq 2p + \lceil \frac{n-2p}{2} \rceil$.

Consider the generalized Pell string t' obtained from t by consecutively applying the following changes to the string t:

(i) exchange the role of i and $i + \frac{k+1}{2}$ for every $i \in \{0, \dots, \frac{k-3}{2}\}$;

(ii) replace each substring kk with 00;

(iii) replace each of the p disjoint pairs of $(\frac{k-1}{2})(\frac{k-1}{2})$ with kk; and

(iv) replace each remaining $\frac{k-1}{2}$ with 0.

Each exchange from (i) requires $\frac{k+1}{2}$ consecutive changes in the string, each replacement from (ii) requires $2k-1$ consecutive changes, each replacement from (iii) needs k changes, and each replacement from (iv) needs $\frac{k-1}{2}$ changes. Since these changes are disjoint, we obtain

$$\begin{aligned} d(t,t') &= (n-2\ell-m)\frac{k+1}{2} + \ell(2k-1) + pk + (m-2p)\frac{k-1}{2} \\ &= \frac{kn+n}{2} + \ell(k-2) - m + p. \end{aligned}$$

Using the fact that $\ell \geq 0$, $k \geq 3$, and $m \leq 2p + \lceil \frac{n-2p}{2} \rceil$, we get

$$d(t,t') \geq \frac{kn+n}{2} - \left\lceil \frac{n-2p}{2} \right\rceil - p.$$

If n is even, then this yields $d(t,t') \geq \frac{kn}{2} = \lfloor \frac{kn}{2} \rfloor$, while if n is odd, we get $d(t,t') \geq \frac{kn-1}{2} = \lfloor \frac{kn}{2} \rfloor$.

Thus $\mathrm{rad}(\Pi_{n,k}) \geq \lfloor \frac{kn}{2} \rfloor$. To prove the equality it suffices to find a vertex with eccentricity $\lfloor \frac{kn}{2} \rfloor$. We claim that if k is even, then $t = \left(\frac{k}{2}\right)^n$ is such a vertex, and if k is odd, then $t' = \left(\frac{k-1}{2}\right)^n$ is a required vertex. Indeed, if k is even, then $d(t, 0^n) = \lfloor \frac{kn}{2} \rfloor$ and by the above $d(t,x) \leq \lfloor \frac{kn}{2} \rfloor$ for any other vertex $x \in V(\Pi_{n,k})$, hence $\mathrm{ecc}(t) = \lfloor \frac{kn}{2} \rfloor$. Similarly, if k is odd and n is even, then $d(t', k^n) = \lfloor \frac{kn}{2} \rfloor$, and if k is odd and n is odd, then $d(t', k^{n-1}0) = \lfloor \frac{kn}{2} \rfloor$. Hence, $\mathrm{ecc}(t') = \lfloor \frac{kn}{2} \rfloor$ as claimed. □

The center of the Pell graph Π_n is isomorphic to the Fibonacci cube Γ_n [16, Proposition 5]. It turns out that the same happens for certain generalized Pell graphs (see Theorem 4.2), but not for every $k \geq 2$. Using a computer, we have computed the cardinalities of the center of some small generalized Pell graphs. These are given in Table 1.

Table 1. The cardinality of the center of some of the graphs $\Pi_{n,k}$.

k \ n	1	2	3	4	5	6	7	8	9	10
2	2	3	5	8	13	21	34	55	89	144
3	1	3	2	8	4	20	8	48	16	112
4	2	3	5	8	13					
5	1	3	2	8	4					
6	2	3	5	8	13					
7	1	3	2	8	4					
8	2	3	5	8	13					
9	1	3	2	8	4					

The values in Table 1 indicate that $|C(\Pi_{n,k})|$ depends only on the parity of k, and not its exact value. In the following, we prove that this is indeed the case, and explicitly describe the center of generalized Pell graphs. For this, we introduce the following families of words.

Let $k \geq 2$ be even and set

$$\Theta_n(k) = \left\{ t = t_1 \dots t_n;\ t_i \in \left\{ \frac{k}{2} - 1, \frac{k}{2} \right\} \text{ and } t \text{ contains no two consecutive } \frac{k}{2} - 1 \right\}.$$

It is easy to see that $\Pi_{n,k}[\Theta_n(k)] \cong \Gamma_n$.

For n even define $\Phi_n(a,b)$ to be the set of words of length n over the alphabet $\{aa, ab, ba\}$, where the letter ba never appears before the letter ab. For example, $\underline{ba}aaaa\underline{ab}aa \notin \Phi_{10}(a,b)$, but $ababbaaaba \in \Phi_{10}(a,b)$. Note that a word of length n consists of $n/2$ letters.

For n odd define $\Psi_n(a,b)$ to be the set of words of length n over the alphabet $\{a,b\}$ that start and end with a, contain no substring bb, and have all runs of as of odd length. For example, $abb \notin \Psi_3(a,b)$, $baaba \notin \Psi_5(a,b)$ and $abaaa \in \Psi_5(a,b)$.

Theorem 4.2 *If $k \geq 2$ and $n \geq 2$, then*

$$C(\Pi_{n,k}) = \begin{cases} \Theta_n(k); & k \text{ even}, \\ \Phi_n(\frac{k-1}{2}, \frac{k+1}{2}); & k \text{ odd and } n \text{ even}, \\ \Psi_n(\frac{k-1}{2}, \frac{k+1}{2}); & k \text{ odd and } n \text{ odd}. \end{cases}$$

Consequently,

$$|C(\Pi_{n,k})| = \begin{cases} F_{n+2}; & k \text{ even}, \\ (n+4)2^{\frac{n}{2}-2}; & k \text{ odd and } n \text{ even}, \\ 2^{\frac{n-1}{2}}; & k \text{ odd and } n \text{ odd}. \end{cases}$$

In addition, if k is even, then $\Pi_{n,k}[C(\Pi_{n,k})] \cong \Gamma_n$.

Proof We distinguish between three main cases.

$\underline{k \geq 2 \text{ even}}$:

We are going to prove that $C(\Pi_{n,k}) = \Theta_n(k)$. From this it immediately follows that $\Pi_{n,k}[C(\Pi_{n,k})] \cong \Gamma_n$ and that $|C(\Pi_{n,k})| = F_{n+2}$.

Let $t = t_1 \dots t_n \in V(\Pi_{n,k})$. If t contains the substring kk, then reevaluating the calculation in Case 1 of the proof of Proposition 4.1 for $\ell \geq 1$ yields $d(t,t') \geq \frac{kn}{2} + 1 > \text{rad}(\Pi_{n,k})$, thus such t is not in the center of $\Pi_{n,k}$. From now on we may thus assume that t contains no kk.

If t contains $x \in \{0, \dots, k-1\} \setminus \{\frac{k}{2} - 1, \frac{k}{2}\}$, then consider $t'' \in V(\Pi_{n,k})$ obtained in the following way. First replace x with $k-1$ if $x \leq \frac{k}{2} - 2$, or with 0 if $x \geq \frac{k}{2} + 1$. Next, for each other letter in t, exchange i and $i + \frac{k}{2}$, $i \in \{0, \dots, \frac{k}{2} - 1\}$. Then $d(t,t'') = \left(\frac{k}{2} + 1\right) + (n-1)\frac{k}{2} = \frac{nk}{2} + 1 > \text{rad}(\Pi_{n,k})$.

If t contains only letters $\frac{k}{2} - 1$ and $\frac{k}{2}$, but it also contains at least one substring $(\frac{k}{2} - 1)(\frac{k}{2} - 1)$, then consider $t''' \in V(\Pi_{n,k})$ obtained in the following way. Replace this substring $(\frac{k}{2} - 1)(\frac{k}{2} - 1)$ with kk, and for the other letters in t, exchange $\frac{k}{2} - 1$ with $k-1$ and $\frac{k}{2}$ with 0. Clearly, $d(t,t''') = (1+k) + (n-2)\frac{k}{2} = \frac{nk}{2} + 1 > \text{rad}(\Pi_{n,k})$.

These arguments show that $C(\Pi_{n,k}) \subseteq \Theta_n(k)$. For $t \in \Theta_n(k)$, we prove that $\text{ecc}(t) \leq \frac{nk}{2}$. Let $u = u_1 \dots u_n \in V(\Pi_{n,k})$. If $u_i \in \{0, 1, \dots, k-1\}$, then changing t_i to u_i requires at most $\frac{k}{2}$ steps. If

$u_iu_{i+1} = kk$, then since t contains no two consecutive $\left(\frac{k}{2}-1\right)$s, the change from t_it_{i+1} to kk requires at most $k = 2\cdot\frac{k}{2}$ steps. Thus $d(t,u) \le n\cdot\frac{k}{2} = \mathrm{rad}(\Pi_{n,k})$. Thus $C(\Pi_{n,k}) = \Theta_n(k)$.

$k \ge 3$ odd and $n \ge 2$ even:

We first prove that $C(\Pi_{n,k}) = \Phi_n(\frac{k-1}{2}, \frac{k+1}{2})$. Let $a = \frac{k-1}{2}$ and $b = \frac{k+1}{2}$.

Let $t = t_1 \dots t_n \in V(\Pi_{n,k})$, $|t|_k = 2\ell$, $|t|_a = m$, and let $p \ge 0$ denote the number of appearances of the substring aa in t which originate from the letter aa.

First we prove that if $t \in \Phi_n(a,b)$, then $t \in C(\Pi_{n,k})$. Since $t \in \Phi_n(a,b)$, p equals the number of times the letter aa is used in t. Since in the words ab and ba both a and b appear an equal number of times, we know that each of a and b appears in pairs ab and ba exactly $\frac{1}{2}(n-2p)$ times, and thus $m = 2p + \frac{1}{2}(n-2p)$.

If $u \in V(\Pi_{n,k})$, then

$$d(t,u) \le pk + \frac{1}{2}(n-2p)\frac{k-1}{2} + \frac{1}{2}(n-2p)\frac{k+1}{2} = \frac{nk}{2} = \mathrm{rad}(\Pi_{n,k}),$$

since replacing aa with kk requires k steps, replacing a with i, $i \ne k$, requires at most $\frac{k-1}{2}$ steps, replacing bb with kk requires $k-2 \le 2\frac{k+1}{2}$ steps, replacing b with i, $i \ne k$, requires at most $\frac{k+1}{2}$ steps, and replacing ab or ba with kk requires at most $k-1 < \frac{k-1}{2} + \frac{k+1}{2}$ steps. This shows that $t \in C(\Pi_{n,k})$.

Next we prove that if $t \notin \Phi_n(a,b)$, then $t \notin C(\Pi_{n,k})$. We consider the following cases.

Case 1. t contains kk.

Let t' be as in the proof of Proposition 4.1. Then since $\ell \ge 1$ and $k \ge 3$, $d(t,t') \ge \frac{nk}{2} + 1 > \mathrm{rad}(\Pi_{n,k})$.

Case 2. t does not contain kk.

Since n is even, letters in t can be paired as $t_{2i-1}t_{2i}$ for $i \in [\frac{n}{2}]$. We will call this partition the *pair-partition of* t.

Case 2.1. t contains x, $x \in \{0,\dots,\frac{k-5}{2},\frac{k+3}{2},\dots,k-1\}$.

Let t' be as in the proof of Proposition 4.1, except that x is replaced by $k-1$ if $x \le \frac{k-5}{2}$ and by 0 if $x \ge \frac{k+3}{2}$. Then since $\ell = 0$, and replacing x with $k-1$ or 0 requires at least $\frac{k+3}{2}$ steps, we obtain

$$\begin{aligned} d(t,t') &\ge (n-m-1)\frac{k+1}{2} + \frac{k+3}{2} + pk + (m-2p)\frac{k-1}{2} \\ &= \frac{nk+n}{2} + 1 - m + p. \end{aligned}$$

Recall from the proof of Proposition 4.1 that $m \le 2p + \frac{n-2p}{2}$, thus

$$d(t,t') \ge \frac{nk}{2} + 1 > \mathrm{rad}(\Pi_{n,k}).$$

Case 2.2. t contains only $\frac{k-3}{2}, a, b$, and $\frac{k-3}{2}$ appears at least once or bb appears in the pair-partition of t.

Let t'' be constricted from the pair-partition of t by the following:

1. replace each pair aa with kk (requires k steps),
2. replace each pair consisting of a and $\frac{k-3}{2}$ with kk (requires $k+1>k$ steps),
3. replace each pair bb with 00 (requires $k+1>k$ steps),
4. replace each pair consisting of a and b with $k-1$ and 0 (requires at least k steps),
5. replace each pair consisting of b and $\frac{k-3}{2}$ with 0 and $k-1$ (requires at least $k+1>k$ steps),
6. replace each pair $(\frac{k-3}{2})(\frac{k-3}{2})$ with kk (requires $k+2>k$ steps).

Since $\frac{k-3}{2}$ appears at least once or bb appears as some $t_{2i-1}t_{2i}$, $i \in [\frac{n}{2}]$, in t, we get

$$d(t,t'') \geq (k+1) + \left(\frac{n}{2}-1\right)k = \frac{nk}{2} + 1 > \mathrm{rad}(\Pi_{n,k}).$$

Case 2.3. t contains only a and b, bb does not appear in the pair-partition of t, and ba appears at least once before ab in the pair-partition of t.

Let q be the number of times aa appears in the pair-partition of t. Then since ba appears before ab at least once, $p \geq q+1$. The number of bs in t is $\frac{1}{2}(n-2q)$ and $m = 2q + \frac{1}{2}(n-2q) = \frac{n}{2}+q$. Let t' be obtained from t as in the proof of Proposition 4.1. Then we get

$$d(t,t') = \frac{1}{2}(n-2q)\frac{k+1}{2} + pk + (m-2p)\frac{k-1}{2} \geq \frac{nk}{2} + 1 > \mathrm{rad}(\Pi_{n,k}).$$

Since we found a vertex of $\Pi_{n,k}$ that is at a distance strictly more than $\mathrm{rad}(\Pi_{n,k})$ from t in each case, it follows that $t \notin C(\Pi_{n,k})$. Thus $C(\Pi_{n,k}) = \Phi_n(a,b)$.

It remains to prove that $|\Phi_n(a,b)| = (n+4)2^{\frac{n}{2}-2}$. For $n \geq 4$ even, we have that $|\Phi_n(a,b)| = 2|\Phi_{n-2}(a,b)| + 2^{\frac{n-2}{2}}$ since a word from $\Phi_n(a,b)$ either starts with aa and is followed by a word from $\Phi_{n-2}(a,b)$, or starts with ab and is followed by a word from $\Phi_{n-2}(a,b)$, or starts with ba and is followed by a word over the alphabet $\{aa, ba\}$. Then the claim follows by induction.

<u>$k \geq 3$ odd and $n \geq 3$ odd</u>:

We first prove that $C(\Pi_n, k) = \Psi_n(\frac{k-1}{2}, \frac{k+1}{2})$. Again, let $a = \frac{k-1}{2}$ and $b = \frac{k+1}{2}$.

Let $t = t_1 \ldots t_n \in V(\Pi_{n,k})$, $|t|_k = 2\ell$, $|t|_a = m$, and let $p \geq 0$ denote the maximum number of disjoint appearances of the substring aa in t.

Suppose that $t \in \Psi_n(a,b)$. Let r denote the number of runs of as in t. Then since each run of as is of odd length we get $p = \frac{m-r}{2}$, and since there is no bb, $t_1 \neq b$, and $t_n \neq b$, there is exactly $r-1$ bs in t, so we have $m = n-r+1$.

If $u \in V(\Pi_{n,k})$, then

$$d(t,u) \leq pk + (m-2p)\frac{k-1}{2} + (r-1)\frac{k+1}{2} = \frac{nk-1}{2} = \mathrm{rad}(\Pi_{n,k}),$$

since replacing aa with kk requires k steps, replacing a with i, $i \neq k$, requires at most $\frac{k-1}{2}$ steps, replacing b with i, $i \neq k$, requires at most $\frac{k+1}{2}$ steps, and replacing ab or ba with kk requires at most $k-1 < \frac{k-1}{2} + \frac{k+1}{2}$ steps. This shows that $t \in C(\Pi_{n,k})$.

Now suppose that $t \notin \Phi_n(a,b)$. To see that $t \notin C(\Pi_{n,k})$, we consider the following cases.

Case 1. t contains kk.

Let t' be as in the proof of Proposition 4.1. Then since $\ell \geq 1$ and $k \geq 3$, $d(t,t') \geq \frac{nk-1}{2} + 1 > \mathrm{rad}(\Pi_{n,k})$.

Case 2. t does not contain kk.

Letters in t can be partitioned into pairs $t_{2i-1}t_{2i}$ for $i \in [\frac{n-1}{2}]$, and a singleton t_n. We will call this partition the *pair-partition of* t.

Case 2.1. t contains x, $x \in \{0, \ldots, \frac{k-5}{2}, \frac{k+3}{2}, \ldots, k-1\}$.

Let t' be as in the proof of Proposition 4.1, except that x is replaced by $k-1$ if $x \leq \frac{k-5}{2}$ and by 0 if $x \geq \frac{k+3}{2}$. Then since $\ell = 0$, and replacing x with $k-1$ or 0 requires at least $\frac{k+3}{2}$ steps, we obtain

$$\begin{aligned} d(t,t') &\geq (n-m-1)\frac{k+1}{2} + \frac{k+3}{2} + pk + (m-2p)\frac{k-1}{2} \\ &= \frac{nk+n}{2} + 1 - m + p. \end{aligned}$$

Recall from the proof of Proposition 4.1 that $m \leq 2p + \frac{n-2p+1}{2}$, thus

$$d(t,t') \geq \frac{nk-1}{2} + 1 > \mathrm{rad}(\Pi_{n,k}).$$

Case 2.2. t contains only $\frac{k-3}{2}, a, b$, but $\frac{k-3}{2}$ appears at least once or bb appears at least once or $t_1 = b$ or $t_n = b$.

If $t_1 = b$ and $t_n \neq b$, then without loss of generality exchange the role of t_i and t_{n-i} for all $i \in [\frac{n-1}{2}]$. Let t'' be constricted from the pair-partition of t by the following:

1. replace each pair aa with kk (requires k steps),
2. replace each pair consisting of a and $\frac{k-3}{2}$ with kk (requires $k+1 > k$ steps),
3. replace each pair bb with 00 (requires $k+1 > k$ steps),
4. replace each pair consisting of a and b with $k-1$ and 0 (requires at least k steps),
5. replace each pair consisting of b and $\frac{k-3}{2}$ with 0 and $k-1$ (requires at least $k+1 > k$ steps),
6. replace each pair $(\frac{k-3}{2})(\frac{k-3}{2})$ with kk (requires $k+2 > k$ steps),
7. replace t_n with 0 if $t_n = b$ and with $k-1$ otherwise (requires at least $\frac{k-1}{2}$ steps, but $\frac{k+1}{2}$ if $t_n \in \{\frac{k-3}{2}, b\}$).

Since $\frac{k-3}{2}$ appears at least once or bb appears as some pair or $t_n = b$, we get

$$d(t,t'') \geq \frac{n-1}{2}k + \frac{k-1}{2} + 1 = \frac{nk-1}{2} + 1 > \mathrm{rad}(\Pi_{n,k}).$$

Case 2.3. t contains only a and b, bb does not appear in t, $t_1 \neq b$, $t_n \neq b$, but not all runs of as are of odd length.

Let r be the number of runs of as. Then since not all runs of as are of odd length, $p \geq \frac{m-r+1}{2}$.

The number of bs in t is $r-1$ and $m = n-r+1$. Let t' be obtained from t as in the proof of Proposition 4.1. Then we get

$$d(t,t') = (r-1)\frac{k+1}{2} + pk + (m-2p)\frac{k-1}{2} \geq \frac{nk-1}{2} + \frac{1}{2} > \text{rad}(\Pi_{n,k}).$$

Since we found a vertex of $\Pi_{n,k}$ that is at distance strictly more than $\text{rad}(\Pi_{n,k}) = \frac{nk-1}{2}$ from t in each case, it follows that $t \notin C(\Pi_{n,k})$. Thus $C(\Pi_{n,k}) = \Psi_n(a,b)$.

It remains to show that $|\Psi_n(a,b)| = 2^{\frac{n-1}{2}}$. For $n \geq 3$ odd, we have that $|\Psi_n(a,b)| = 2|\Psi_{n-2}(a,b)|$ since the word can either start by aa or ab, and in both cases, it needs to be followed by a word from $\Psi_{n-2}(a,b)$ (in particular, it needs to start with a). Thus the claim can be proved using induction on n. □

We have excluded the case $n = 1$ from Theorem 4.2. But since $\Pi_{1,k}$ is isomorphic to the path on k vertices, its center is isomorphic to either K_1 or K_2. An example of a generalized Pell graph with its center is presented in Figure 4.

Proposition 4.3 *If $n \geq 1$ and $k \geq 2$, then*

$$\text{diam}(\Pi_{n,k}) = nk - \left\lceil \frac{n}{2} \right\rceil.$$

Proof If n is even, then $d(0^n, k^n) = n(k-1) + \frac{n}{2} = nk - \frac{n}{2}$. If n is odd, then $d(0^n, k^{n-1}(k-1)) = n(k-1) + \frac{n-1}{2} = nk - \frac{n+1}{2}$. Thus $\text{diam}(\Pi_{n,k}) \geq nk - \lceil \frac{n}{2} \rceil$. We need to prove that this is also the upper bound.

For $t = t_1 \dots t_n \in V(\Pi_{n,k})$, $\text{ecc}(t) \leq n(k-1) + \lfloor \frac{n}{2} \rfloor$, since each t_i can contribute at most $k-1$ to the distance by itself, and at most one more in a pair with t_{i-1} or t_{i+1} (but these pairs need to be disjoint). □

Additionally, it is not difficult to see that if n is even, the periphery of $\Pi_{n,k}$ consists only of vertices obtained by using strings 00 and kk. If n is odd, the periphery is formed by vertices consisting of strings 00, kk, and exactly one additional occurrence of either 0 or $k-1$.

5. Additional properties

5.1. The cube polynomial

The cube polynomial of a graph G is denoted by $C_G(x)$, and is the generating function $C_G(x) = \sum_{i \geq 0} c_i(G) x^i$, where $c_i(G)$ counts the number of induced i-cubes in G. This polynomial was introduced in [3], see also [2, 18]. Clearly, $c_0(G) = |V(G)|$ and $c_1(G) = |E(G)|$.

The first few cube polynomials of $\Pi_{n,k}$ are listed below:

$$C_{\Pi_{0,k}}(x) = 1$$

$$C_{\Pi_{1,k}}(x) = k + (k-1)\,x$$

$$C_{\Pi_{2,k}}(x) = \left(k^2+1\right) + \left(2k^2-2k+1\right) x + \left(k^2-2k+1\right) x^2$$

$$C_{\Pi_{3,k}}(x) = k^3 + 2k + \left(3k^3 - 3k^2 + 4k - 2\right) x + \left(3k^3 - 6k^2 + 5k - 2\right) x^2 + \left(k^3 - 3k^2 + 3k - 1\right) x^3$$

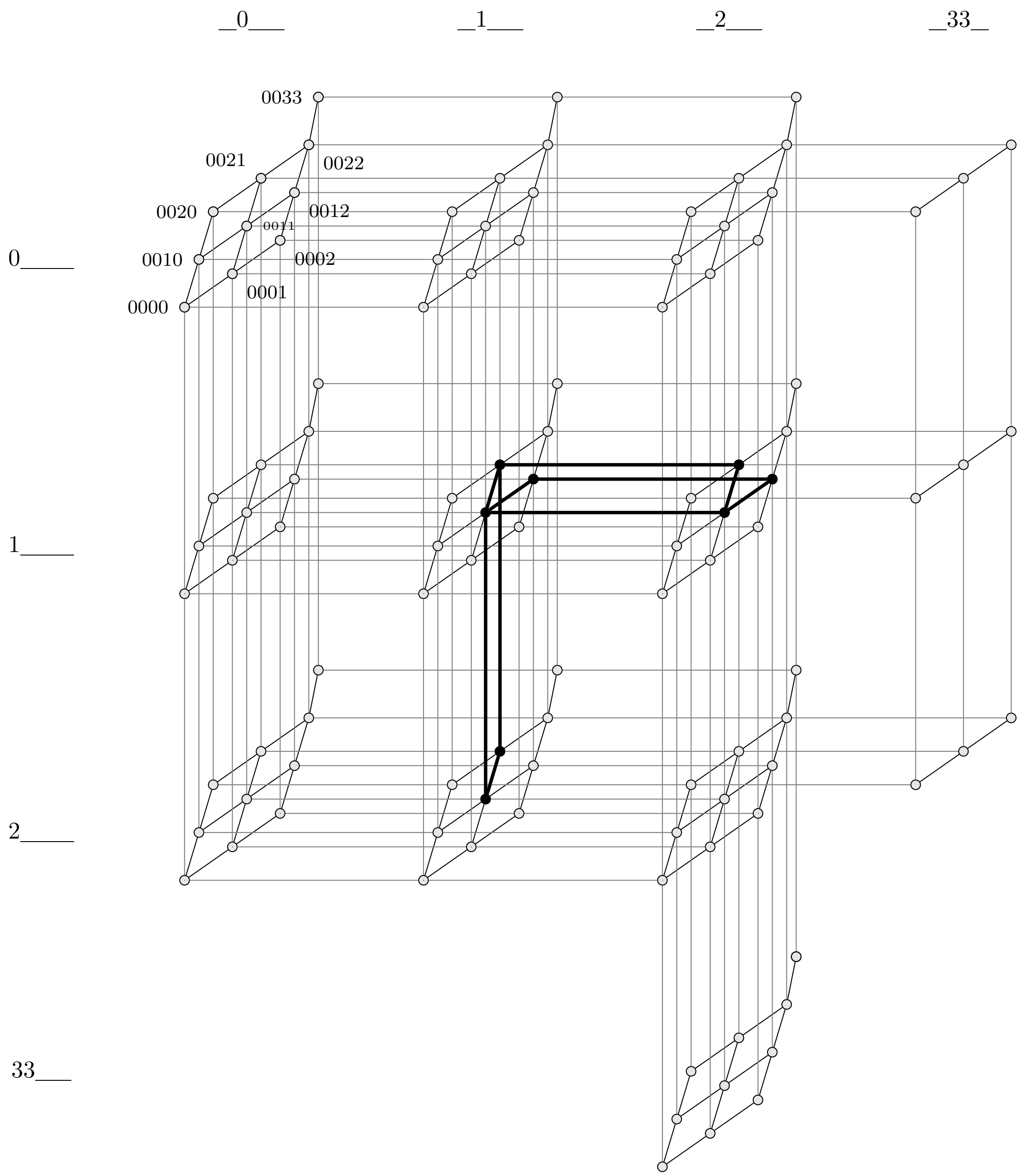

Figure 4. The graph $\Pi_{4,3}$ with its center marked in black. Notice that in this case, the center is isomorphic to a Fibonacci cube (however, this is not the case in general for k odd).

$$C_{\Pi_{4,k}}(x) = \left(k^4 + 3k^2 + 1\right) + \left(4k^4 - 4k^3 + 9k^2 - 6k + 2\right) x + \left(6k^4 + 15k^2 - 12k^3 - 12k + 4\right) x^2 + \left(4k^4 - 12k^3 + 15k^2 - 10k + 3\right) x^3 + \left(k^4 - 4k^3 + 6k^2 - 4k + 1\right) x^4$$

The next result follows from the recursive structure of $\Pi_{n,k}$.

Proposition 5.1 *The cube polynomials $C_{\Pi_{n,k}}(x)$ satisfy the recurrence relation*

$$C_{\Pi_{n,k}}(x) = (k + (k-1)\,x)\, C_{\Pi_{n-1,k}}(x) + (1+x)\, C_{\Pi_{n-2,k}}(x),\ n \geq 2,$$

with the initial values $C_{\Pi_{0,k}}(x) = 1$ and $C_{\Pi_{1,k}}(x) = k + (k-1)\,x$.

From the recurrence relation of the cube polynomials, we can derive the following result using standard methods.

Proposition 5.2 *The generating function of the sequence $\left\{C_{\Pi_{n,k}}(x)\right\}_{n\geq 0}$ is*

$$\sum_{n\geq 0} C_{\Pi_{n,k}}(x) t^n = \frac{1}{1 - (k + (k-1)\,x)\,t - (1+x)\,t^2}.$$

From the generating function of the cube polynomials, we get the following result.

Proposition 5.3 *For $n \geq 0$, the cube polynomial $C_{\Pi_{n,k}}(x)$ is of degree n and*

$$C_{\Pi_{n,k}}(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} (k + (k-1)\,x)^{n-2i} (1+x)^i.$$

5.2. Distribution of the degrees

The distribution of degrees of Pell graphs has been studied in [16]. If $t \in V(\Pi_n)$, then

$$\deg(t) = |t|_0 + |t|_1 + \frac{1}{2}|t|_2 + e,$$

where e is the number of pairwise disjoint occurences of 11 in t. In particular, for $n \geq 1$, $\Delta(\Pi_n) = 2n-1$ and $\delta(\Pi_n) = \left\lceil \frac{n}{2} \right\rceil$, see [16, Proposition 27]. Since generalized Pell graphs with $k=2$ are isomorphic to Pell graphs, we will only consider graphs $\Pi_{n,k}$ for $k \geq 3$.

Proposition 5.4 *If $n \geq 1$, $k \geq 3$ and $t \in V(\Pi_{n,k})$, then*

$$\deg(t) = |t|_0 + 2 \cdot \sum_{i=1}^{k-1} |t|_i + \frac{1}{2}|t|_k - r,$$

where r is the number of runs of $(k-1)$s in t.

Proof Let $t = t_1 \dots t_n \in V(\Pi_{n,k})$. If $t_i = 0$, its contribution to the degree of t is 1, while if $t_i \in [k-2]$, it contributes 2 (since $k \geq 3$). It is also easy to see that each occurrence of kk contributes 1. Note that if $t_i = t_{i+1} = t_{i+2} = t_{i+3} = k$ is the start of a run of ks in t, then only the pairs $t_i t_{i+1}$ and $t_{i+2} t_{i+3}$ will result in a valid generalized Pell string after kk is exchanged with $(k-1)(k-1)$.

Lastly, let r be the number of runs of $(k-1)$s in t and let $q_1, \dots, q_r$ be the lengths of these runs. Observe that if $t_i = k-1$, it contributes 1 to the degree of t (by exchanging $k-1$ with $k-2$). However, each occurrence of $(k-1)(k-1)$ contributes an additional 1 to the degree of t (by exchanging it with kk). Note that any pair of consecutive $(k-1)$s yields a generalized Pell string, thus a run of $(k-1)$s of length q_j contributes $q_j - 1$. Then occurrences of $k-1$ in t altogether contribute

$$\sum_{i=1}^{r} q_i + \sum_{i=1}^{r} (q_i - 1) = 2\sum_{i=1}^{r} q_i - r = 2|t|_{k-1} - r.$$

Combining the observed contributions yields the desired formula. □

Corollary 5.5 *If $n \geq 1$ and $k \geq 3$, then $\Delta(\Pi_{n,k}) = 2n$ and $\delta(\Pi_{n,k}) = \lceil \frac{n}{2} \rceil$.*

Proof Let $t \in V(\Pi_{n,k})$. Since $\sum_{i=0}^{k} |t|_i = n$, it clearly holds

$$\left\lceil \frac{n}{2} \right\rceil \leq \deg(t) \leq 2n.$$

The left bound is attained for example by the vertex k^n if n is even, and by the vertex $k^{n-1}(k-1)$ if n is odd. The right bound is attained for example by the vertex 1^n (since $k \geq 3$). □

Corollary 5.6 *The number of vertices of $\Pi_{n,k}$ of degree $\Delta(\Pi_{n,k}) - 1$ is*

$$n(n-1)^{k-2} + \sum_{\ell=1}^{n} (n - \ell + 1)(n - \ell)^{k-2}.$$

Proof If $t \in V(\Pi_{n,k})$, then $\deg(t) = 2n - 1$ if either $|t|_0 = 1$ and $|t|_{k-1} = |t|_k = 0$, or $|t|_0 = |t|_k = 0$, $|t|_{k-1} \geq 1$ and t contains only one run of $(k-1)$s. A simple counting argument shows the formula. □

5.3. Median graphs

A *median* of a triple x, y, z of vertices of a graph G is a vertex u that simultaneously lies on a shortest x,y-path, a shortest x,z-path, and a shortest y,z-path. G is a *median graph* if each triple of vertices has a unique median [15].

From [12] we know that Fibonacci cubes are median graphs and from [16] Pell graphs also belong to this family of graphs. This property extends to all generalized Pell graphs as the next result asserts.

Proposition 5.7 *If $n \geq 1$ and $k \geq 2$, then $\Pi_{n,k}$ is a median graph.*

Proof (Sketch) We proceed by induction, the result being clear for $n = 1$ and all k since $\Pi_{1,k}$ is isomorphic to the path with k vertices. Suppose $n \geq 2$ and consider $\Pi_{n,k}$. The part of its fundamental decomposition (3.1)

$$X = 0\Pi_{n-1,k} \oplus 1\Pi_{n-1,k} \oplus \cdots \oplus (k-1)\,\Pi_{n-1,k}$$

is isomorphic to the Cartesian product of $\Pi_{n-1,k}$ by the path on k vertices. As the factors are median graphs by the induction hypothesis, and the Cartesian product operation preserves the property of being median; X is also median. Finally, we can observe that $\Pi_{n,k}$ is obtained from X by the so-called convex peripheral expansion (cf. [15]), hence $\Pi_{n,k}$ is median as well. □

5.4. Subgraph of a Fibonacci cube

It is known [16, Theorem 7] that the Pell graph Π_n is a subgraph of the Fibonacci cube Γ_{2n-1}, written as $\Pi_n \subseteq \Gamma_{2n-1}$. We prove an analogous result for generalized Pell graphs. Notice that for $k = 2$, Proposition 5.8 states the same as the existing result for Pell graphs.

Proposition 5.8 *If $n \geq 1$ and $k \geq 2$, then*

$$\Pi_{n,k} \subseteq \Gamma_{(2k-2)n-1}.$$

Proof We define a mapping $\varphi\colon \Pi_{n,k} \to \Gamma_{(2k-2)n-1}$ in the following way. First, consider a mapping $\varphi'\colon \Pi_{n,k} \to \Gamma_{(2k-2)n}$ that maps a vertex $t = t_1 \dots t_n \in V(\Pi_{n,k})$ in the following way:

$$i \mapsto (10)^{k-1-i}0^{2i}$$

$$kk \mapsto 010^{2k-4}$$

Clearly $\varphi'(t)$ is of length $(2k-2)n$, and contains no 11, thus $\varphi'(t) \in V(\Gamma_{(2k-2)n})$. Observe also that $\varphi'(t)$ always ends with 0. Deleting the ending 0 gives $\varphi(t) \in V(\Gamma_{(2k-2)n-1})$. By definition, φ is injective. Thus it remains to show that it maps edges to edges.

Let $pq \in E(\Pi_{n,k})$, where $p = p_1 \dots p_n$ and $q = q_1 \dots q_n$. If p and q differ in only one letter, then without loss of generality, $p_i = \ell$, $q_i = \ell + 1$, and $p_j = q_j$ for all $j \in [n] \setminus \{j\}$, for some $1 \leq \ell \leq k-2$ and $1 \leq i \leq n$. Thus $\varphi(p)_{(2k-2)(i-1)+2(k-\ell-2)+1} = 1$, $\varphi(q)_{(2k-2)(i-1)+2(k-\ell-2)+1} = 0$ and $\varphi(p)_j = \varphi(q)_j$ for all $j \in [(2k-2)n-1] \setminus \{(2k-2)(i-1) + 2(k-\ell-2) + 1\}$. So $\varphi(p)\varphi(q) \in E(\Gamma_{(2k-2)n-1})$.

Otherwise, it must hold for some i, $1 \leq i \leq n-1$, that $p_i = p_{i+1} = k-1$, $q_i = q_{i+1} = k$, and $p_j = q_j$ for all $j \in [n] \setminus \{i, i+1\}$. Thus $\varphi(q)_{(2k-2)(i-1)+2} = 1$, $\varphi(p)_{(2k-2)(i-1)+2} = 0$, and $\varphi(p)_j = \varphi(q)_j$ for all $j \in [(2k-2)n-1] \setminus \{(2k-2)(i-1)+2\}$. Hence again $\varphi(p)\varphi(q) \in E(\Gamma_{(2k-2)n-1})$. □

Concluding remarks

In the final stages of preparing our paper, we learned that Došlić and Podrug † had independently proposed a second generalization of Pell graphs. Their motivation was much as ours, that is, to construct graphs that reflect (1.1), and doing so, they defined graphs denoted by Π_n^k. While these graph have the same order and

†Došlić T, Podrug L. Metallic cubes. Website https://arxiv.org/abs/2307.14054 [accessed 27 July 2023].

size as the generalized Pell graphs $\Pi_{n,k}$ from this paper, their structure is significantly different. For instance, if n and k are both odd, then the center of Π_n^k consists of a single vertex, while we have seen in Theorem 4.2 that the center of $\Pi_{n,k}$ contains $2^{\frac{n-1}{2}}$ vertices. Additional structural differences include:

- For $k = 2$, $\Pi_{n,2} \cong \Pi_n$, but Π_n^2 is not isomorphic to the Pell graph.
- For $k \geq 2$ and $n \geq 3$, $\mathrm{diam}(\Pi_{n,k}) = nk - \lceil \frac{n}{2} \rceil < nk - 1 = \mathrm{diam}(\Pi_n^k)$.
- For $n \geq 4$, graphs $\Pi_{n,k}$ contain strictly more vertices of degree $2n-1$ than graphs Π_n^k (combining Corollary 5.6 and an observation that Π_n^k contains $2n(n-1)^{k-2} + (n-1)(n-2)^{k-2}$ vertices of degree $2n-1$, which are vertices t with $|t|_0 = 1$, $|t|_{a-1} = |t|_a = 0$, or $|t|_{a-1} = 1$, $|t|_0 = |t|_a = 0$, or exactly one occurrence of $0(a-1)$ and otherwise containing only letters from $[k-2]$).

We conclude the paper with some problems that appear interesting for further investigation.

As mentioned in Proposition 3.3, it is easy to see that graphs $\Pi_{n,k}$ are traceable. The existence of a Hamiltonian cycle seems more complicated.

Problem 5.9 *Characterize graphs $\Pi_{n,k}$ that are Hamiltonian.*

In Proposition 5.8 we prove that $\Pi_{n,k}$ is a subgraph of a sufficiently large Fibonacci cube. However, it is not clear if the result is optimal.

Problem 5.10 *Does there exist a function $f(n,k) < (2k-2)n - 1$ such that for every $n \geq 1$ and $k \geq 2$, $\Pi_{n,k} \subseteq \Gamma_{f(n,k)}$?*

It would be of interest to know the (edge) connectivity of generalized Pell graphs. For both of them we suspect that they are equal to the minimum degree (for every $n \geq 1$ and $k \geq 2$).

Problem 5.11 *Determine $\kappa(\Pi_{n,k})$ and $\kappa'(\Pi_{n,k})$.*

Acknowledgements

This work has been supported by TÜBİTAK and the Slovenian Research Agency under grant numbers 122N184 and BI-TR/22-24-20, respectively. Vesna Iršič and Sandi Klavžar also acknowledge the financial support from the Slovenian Research Agency (research core funding P1-0297 and projects J1-2452, N1-0285, Z1-50003). Vesna Iršič also acknowledges the financial support from the ERC KARST project.

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