Journal of Computational and Applied Mathematics 424 (2023) 115008

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Arc length preserving G^2 Hermite interpolation of circular arcs

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ARTICLE INFO

Article history:
Received 5 July 2022
Received in revised form 15 November 2022

MSC:
65D05
65D07
65D17

Keywords:
Geometric interpolation
Circular arc
Arc length
Pythagorean–hodograph curve
solution selection

ABSTRACT

In this paper, the problem of interpolation of two points, two corresponding tangent directions and curvatures, and the arc length sampled from a circular arc (circular arc data) is considered. Planar Pythagorean–hodograph (PH) curves of degree seven are used since they possess enough free parameters and are capable of interpolating the arc length in an easy way. A general approach using the complex representation of PH curves is presented first and the strong dependence of the solution on the general data is demonstrated. For circular arc data, a complicated system of nonlinear equations is reduced to a numerical solution of only one algebraic equation of degree 6 and a detailed analysis of the existence of admissible solutions is provided. In the case of several solutions, some criteria for selecting the most appropriate one are described and an asymptotic analysis is given. Numerical examples are included which confirm theoretical results.

1. Introduction

Interpolation of local planar geometric data, such as points, tangent directions and curvatures, by parametric polynomial curves is a standard problem in Computer Aided Geometric Design (CAGD) and a common way to construct parametric objects from given discrete data. If such interpolants are joined together, they form geometrically continuous splines of order k (or G^k continuous splines), where k depends on the type of the interpolated data ($k = 0$ if only positions of points are given, $k = 1$ if in addition also tangent directions are provided, etc.). For a detailed survey of geometric interpolation methods the reader is referred to [1] or [2]. However, there are not many results also concerning the interpolation of some global geometric data, such as the arc length. This becomes extremely important when the arc length of the approximant is prescribed in advance (in planar motion design, e.g.), some global shape control is needed or methods relying on optimization of curve energies, such as bending energy, are to be developed. It turned out that there is a specific class of curves which are of great help to solve such kinds of problems, the so-called polynomial Pythagorean–hodograph (PH) curves introduced in [3] and comprehensively described in [4]. They are distinguished by possessing a polynomial arc length function. This implies several nice properties, which will be explained in detail later. Some recent results concerning interpolation of G^1 data by PH quintic curves preserving an arc length are in [5], and confirm the advantage of PH curves if interpolation of global geometric data is considered. The author studied the interpolation of two points together with the corresponding tangent directions and the prescribed arc length. A detailed analysis of the interpolation problem was done, and a simple algorithm relying only on the solution of a certain quadratic equation was described. As already guessed in [5], it is hard to believe that the generalization to the interpolation of G^2 data with the

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https://doi.org/10.1016/j.cam.2022.115008

prescribed arc length would possess such a simple solution as in the quintic case. However, some results considering general G^2 data without an arc length were obtained in [6,7] and recently in [8]. In this paper, we intend to extend these results to the interpolation of G^2 data and the arc length arising from a circular arc (circular arc data) by PH curves of degree seven. It should be noted that even in this case no relevant literature is available. Most of the approximation techniques namely consider the interpolation of local geometric data only and use the remaining parameters to minimize the distance between the interpolant and the circular arc, minimize the deviation of the curvature, etc. The results of this type can be found in [9–24], if we mention just the most important and recent ones. Although the proposed algorithms provide good approximations of circular arcs if the Hausdorff distance is considered as a measure of the error, they do not include an arc length in interpolation data. Our goal is to provide interpolants of G^2 circular arc data which possess the required arc length and retain a small Hausdorff distance. Note that they will be determined numerically since the approach requires solving an algebraic equation of degree 6.

The paper is organized as follows. In Section 2 some basic properties of complex representation of PH curves are given. Special attention is given to PH curves of degree 7 which are presented in detail, and all quantities needed for a solution of the interpolation problem are derived. In the next section, the arc length preserving interpolation of G^2 data is considered and the system of nonlinear equations for the general case is presented. Two examples of such interpolation are given showing that the solution of the problem highly depends on prescribed data. In Section 4 the interpolation of circular arc data in canonical position is considered. The system of nonlinear equations derived for the general case is simplified, and a new, simpler system of two nonlinear equations involving the angle α arising from the circular arc as a parameter is provided. A detailed analysis of the solvability is done. Two cases are considered. For the first one, the absence of real solutions is confirmed, and for the second one, the existence of at least two real admissible solutions is first proved for any $\alpha \in (0, \pi/2]$. Then a simple (numerical) procedure to check the existence of up to four admissible solutions is described for a particular chosen $\alpha \in (0, \pi/2]$. In Section 5 the existence of four solutions is confirmed for any α small enough and asymptotic expansions of solutions are provided. In Section 6 some criteria for selecting the most appropriate solution are described, and in the next section, several numerical examples together with the numerical confirmation of the approximation order are given. The paper is concluded by Section 8.

2. Preliminaries

Planar PH curves are an important subclass of planar parametric polynomial curves. A regular planar parametric polynomial curve $\boldsymbol{p} : [0, 1] \to \mathbb{R}^2$ is a PH curve if $\|\boldsymbol{p}'\|$ is a polynomial, where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^2$. This characterization implies several important geometric properties of PH curves [4]: rational unit tangent, normal, curvature and offset,... Furthermore, the arc length function of a PH curve is the polynomial. All these properties make them useful for the interpolation of local geometric data and for interpolating global geometric quantities, such as an arc length. Let $\boldsymbol{p} = (x, y)^T$ be a PH curve of degree n where x' and y' are relatively prime polynomials. It is known [25] that polynomials x' and y' can be expressed in terms of two polynomials u and v as $x' = u^2 - v^2$ and $y' = 2\,u\,v$ which implies $x'^2 + y'^2 = (u^2 + v^2)^2$ and consequently the arc length function is a polynomial $u^2 + v^2$. It is often better to use a complex representation of PH curves [26]. If $n = 2\,m + 1$ and

$$\boldsymbol{w} = \sum_{k=0}^{m} B_k^m \boldsymbol{w}_k, \quad \boldsymbol{w}_k = u_k + \mathrm{i}\, v_k,\ k = 0, 1, \ldots, m, \tag{1}$$

where $B_k^m(t) = \binom{m}{k} t^k (1-t)^{m-k}$, $k = 0, 1, \ldots, m$, are Bernstein basis polynomials, then the integral of $\boldsymbol{p}' = \boldsymbol{w}^2$ is a PH curve $\boldsymbol{p}$ of degree n. Furthermore, its unit tangent vector $\boldsymbol{g}$, the curvature κ and the arc length s are given by

$$\boldsymbol{g}(t) = \frac{\boldsymbol{w}^2(t)}{\sigma(t)}, \quad \kappa(t) = 2\frac{\mathrm{Im}\left(\overline{\boldsymbol{w}}(t)\boldsymbol{w}'(t)\right)}{\sigma^2(t)}, \quad s(t) = \int_0^t \sigma(\tau)\mathrm{d}\tau, \tag{2}$$

where $\sigma = |\boldsymbol{w}|^2$. Since we will consider PH curves of degree 7, we shall start with a complex cubic polynomial $\boldsymbol{w}$ with complex Bernstein coefficients $\boldsymbol{w}_k = u_k + \mathrm{i}\, v_k$, $k = 0, 1, 2, 3$. Integration of its square results in a PH curve $\boldsymbol{p}$ of degree 7, which can be written in Bernstein-Bézier form as

$$\boldsymbol{p}(t) = \sum_{k=0}^{7} B_k^7(t)\boldsymbol{p}_k,$$

where

$$\begin{aligned}
&\boldsymbol{p}_1 = \boldsymbol{p}_0 + \frac{1}{7}\boldsymbol{w}_0^2, \quad \boldsymbol{p}_2 = \boldsymbol{p}_1 + \frac{1}{7}\boldsymbol{w}_0\boldsymbol{w}_1, \quad \boldsymbol{p}_3 = \boldsymbol{p}_2 + \frac{1}{7}\frac{3\boldsymbol{w}_1^2 + 2\boldsymbol{w}_0\boldsymbol{w}_2}{5},\\
&\boldsymbol{p}_4 = \boldsymbol{p}_3 + \frac{1}{7}\frac{9\boldsymbol{w}_1\boldsymbol{w}_2 + \boldsymbol{w}_0\boldsymbol{w}_3}{10},\\
&\boldsymbol{p}_5 = \boldsymbol{p}_4 + \frac{1}{7}\frac{3\boldsymbol{w}_2^2 + 2\boldsymbol{w}_1\boldsymbol{w}_3}{5}, \quad \boldsymbol{p}_6 = \boldsymbol{p}_5 + \frac{1}{7}\boldsymbol{w}_2\boldsymbol{w}_3, \quad \boldsymbol{p}_7 = \boldsymbol{p}_6 + \frac{1}{7}\boldsymbol{w}_3^2,
\end{aligned} \tag{3}$$

and $\boldsymbol{p}_0$ is a free complex integration constant. By (1) and (2) we obviously have

$$\boldsymbol{g}(0)=\left(\frac{\boldsymbol{w}_0}{|\boldsymbol{w}_0|}\right)^2,\quad \boldsymbol{g}(1)=\left(\frac{\boldsymbol{w}_3}{|\boldsymbol{w}_3|}\right)^2,$$

and by using some basic properties of $\boldsymbol{w}$ we also get

$$\kappa(0)=6\,\mathrm{Im}\left(\frac{\overline{\boldsymbol{w}}_0\boldsymbol{w}_1}{|\boldsymbol{w}_0|^4}\right),\quad \kappa(1)=-6\,\mathrm{Im}\left(\frac{\overline{\boldsymbol{w}}_3\boldsymbol{w}_2}{|\boldsymbol{w}_3|^4}\right).$$

Furthermore, the total arc length L of $\boldsymbol{p}$ is

$$L=\int_0^1|\boldsymbol{w}(\tau)|^2\,\mathrm{d}\tau=\frac{1}{7}\Big(|\boldsymbol{w}_0|^2+\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_1)+\frac{2}{5}\,\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_2)+\frac{1}{10}\,\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_3)+\frac{3}{5}\,|\boldsymbol{w}_1|^2$$
$$+\frac{9}{10}\,\mathrm{Re}\,(\boldsymbol{w}_1\overline{\boldsymbol{w}}_2)+\frac{2}{5}\,\mathrm{Re}\,(\boldsymbol{w}_1\overline{\boldsymbol{w}}_3)+\frac{3}{5}\,|\boldsymbol{w}_2|^2+\mathrm{Re}\,(\boldsymbol{w}_2\overline{\boldsymbol{w}}_3)+|\boldsymbol{w}_3|^2\Big).$$

These results will now be used in the following section where the nonlinear equations for general G^2 data and the arc length will be derived. We will see that the solutions strongly depend on the data and a general analysis might be quite challenging, which justifies the restriction to the data arising from a circular arc.

3. Nonlinear equations

The construction of parametric polynomial curves is usually based on the interpolation of particular geometric data, such as point positions, tangent directions, curvatures, etc. As was already mentioned in the introduction, in some problems also the interpolation of global geometric data, such as an arc length, is required. We shall follow the approach in [5], where the author considered the problem of G^1 data interpolation by PH quintics of prescribed arc length. The problem can be extended to G^2 data interpolation, but the degree of the interpolating PH curve must be elevated to 7 since PH quintic curves do not possess enough free parameters.

Let us assume the complex representation and suppose we want to interpolate two given endpoints $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, their corresponding tangent directions $\boldsymbol{g}_0, \boldsymbol{g}_1$ and curvatures κ_0 and κ_1. Furthermore, we require that the resulting interpolant has a fixed arc length, say $L > \|\boldsymbol{q}_1-\boldsymbol{q}_0\|$. Since we are looking for an interpolant $\boldsymbol{p}$ among PH curves of degree 7, there are 10 free parameters involved (8 parameters arising from the complex Bernstein coefficients (1) and two of them from a complex integration constant $\boldsymbol{p}_0$). The interpolation conditions provide 9 scalar equations. One could use the remaining parameter for shape control or for optimization of some geometric property, but this would lead to a challenging optimization process. To avoid it, we will use the approach from [5], i.e., we shall assume equal lengths of the tangents of $\boldsymbol{p}$ at the boundary points. This reduces the number of involved free parameters by one and gives some hope that the interpolant is fully determined already by given geometric data. The assumption is not too restrictive, and it is quite natural since it ensures symmetric solutions for symmetric data.

In [5], the author considered the reduction of data to the canonical form, which has previously been used also in [27]. The idea is to consider a new coordinate system which should simplify the analysis of the problem as much as possible. Following the above-mentioned references, we can assume that the given data is of the form $\boldsymbol{q}_0=0, \boldsymbol{q}_1=1, \boldsymbol{g}_0=\exp(\mathrm{i}\,\theta_0)$, $\boldsymbol{g}_1=\exp(\mathrm{i}\,\theta_1)$ and $L>1$, where $\theta_0,\theta_1\in(-\pi,\pi]$. The original data are transformed into canonical one by an appropriate translation, rotation and scaling. Finally, the obtained interpolant is pulled back to the original coordinate system by inverse transformations. Note that translation, rotation and scaling preserve the PH property since a translation does not affect the hodograph at all, a rotation preserves the Euclidean norm of the hodograph and a scaling multiplies it by a scaling factor. Thus, if the above canonical data are assumed, the interpolation conditions become

$$\boldsymbol{e}_1=\frac{1}{7}\left(\boldsymbol{w}_0^2+\boldsymbol{w}_0\boldsymbol{w}_1+\frac{3\boldsymbol{w}_1^2+2\boldsymbol{w}_0\boldsymbol{w}_2}{5}+\frac{9\boldsymbol{w}_1\boldsymbol{w}_2+\boldsymbol{w}_0\boldsymbol{w}_3}{10}+\frac{3\boldsymbol{w}_2^2+2\boldsymbol{w}_1\boldsymbol{w}_3}{5}+\boldsymbol{w}_2\boldsymbol{w}_3+\boldsymbol{w}_3^2\right)-1=0,$$

$$\boldsymbol{e}_2=\boldsymbol{w}_0-d\,\exp\left(\mathrm{i}\,\frac{1}{2}\theta_0\right)=0,\quad \boldsymbol{e}_3=\boldsymbol{w}_3-d\,\exp\left(\mathrm{i}\,\frac{1}{2}\theta_1\right)=0,$$

$$e_4=6\,\mathrm{Im}\left(\frac{\overline{\boldsymbol{w}}_0\boldsymbol{w}_1}{|\boldsymbol{w}_0|^4}\right)-\kappa_0=0,\quad e_5=6\,\mathrm{Im}\left(\frac{\overline{\boldsymbol{w}}_3\boldsymbol{w}_2}{|\boldsymbol{w}_3|^4}\right)+\kappa_1=0,$$

$$e_6=\frac{1}{7}\Big(|\boldsymbol{w}_0|^2+\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_1)+\frac{2}{5}\,\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_2)+\frac{1}{10}\,\mathrm{Re}\,(\boldsymbol{w}_0\overline{\boldsymbol{w}}_3)+\frac{3}{5}\,|\boldsymbol{w}_1|^2+\frac{9}{10}\,\mathrm{Re}\,(\boldsymbol{w}_1\overline{\boldsymbol{w}}_2)+\frac{2}{5}\,\mathrm{Re}\,(\boldsymbol{w}_1\overline{\boldsymbol{w}}_3)$$
$$+\frac{3}{5}\,|\boldsymbol{w}_2|^2+\mathrm{Re}\,(\boldsymbol{w}_2\overline{\boldsymbol{w}}_3)+|\boldsymbol{w}_3|^2\Big)-L=0.$$

The first equation arises from the interpolation of two points, the second and the third one from the interpolation of tangent directions, the next two ensure prescribed curvatures, and the last one prescribes the arc length L.

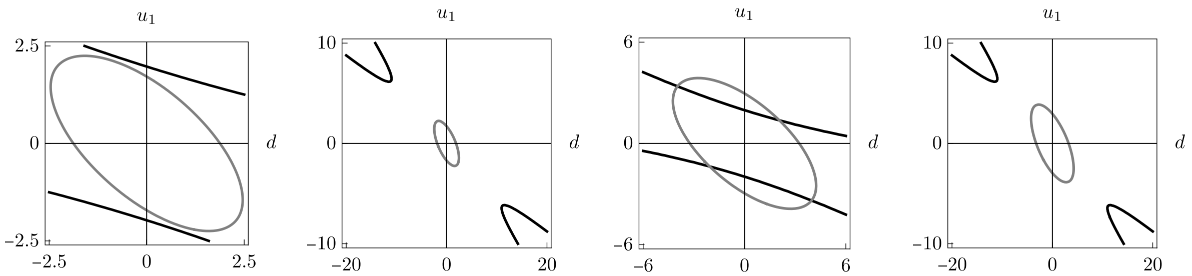

Fig. 1. Ellipses (gray) and hyperbolas (black) arising from the first example with $L = 101/100$ (the first and the second figure from the left) and ellipses and hyperbolas arising from the second example with $L = 3$ (the third and the last figure from the left).

Consider $\boldsymbol{e}_2 = 0$, $\boldsymbol{e}_3 = 0$ and write $\boldsymbol{w}_0 = d(c_0 + \mathrm{i}\, s_0)$, $\boldsymbol{w}_1 = u_1 + \mathrm{i}\, v_1$, $\boldsymbol{w}_2 = u_2 + \mathrm{i}\, v_2$, and $\boldsymbol{w}_3 = d(c_1 + \mathrm{i}\, s_1)$. where $c_i = \cos(\theta_i/2)$, $s_i = \sin(\theta_i/2)$, $i = 0, 1$. Then the above system of complex and real equations can be written as the system of five real equations $35(e_6 + \mathrm{Re}(\boldsymbol{e}_1)) = 0$, $35(e_6 - \mathrm{Re}(\boldsymbol{e}_1)) = 0$, $e_4 = 0$, $e_5 = 0$ and $70\,\mathrm{Im}(\boldsymbol{e}_1) = 0$, or equivalently as

$$\begin{aligned}
&6u_1^2 + 9u_1u_2 + 6u_2^2 + \left(10c_0^2 + 10c_1^2 + c_0c_1\right)d^2 + 10d\,(u_1c_0 + u_2c_1) + 4d\,(u_1c_1 + u_2c_0) - 35(L+1) = 0,\\
&6v_1^2 + 9v_1v_2 + 6v_2^2 + \left(10s_0^2 + 10s_1^2 + s_0s_1\right)d^2 + 10d\,(v_1s_0 + v_2s_1) + 4d\,(v_1s_1 + v_2s_0) - 35(L-1) = 0,\\
&\kappa_0 d^3 + 6s_0u_1 - 6c_0v_1 = 0, \qquad \kappa_1 d^3 - 6s_1u_2 + 6c_1v_2 = 0,\\
&12u_1v_1 + 9u_2v_1 + 9u_1v_2 + 12u_2v_2 + (s_0(20c_0 + c_1) + s_1(c_0 + 20c_1))\,d^2\\
&\;+\; 2\left((5v_1 + 2v_2)c_0 + (2v_1 + 5v_2)c_1 + (5u_1 + 2u_2)s_0 + (2u_1 + 5u_2)s_1\right)d = 0.
\end{aligned} \tag{4}$$

Since the equations arising from G^2 conditions are linear in u_1, u_2, v_1, v_2, some further reduction of the system (4) is definitely possible. However, the analysis of the solvability for general data seems to be extremely complicated, as already guessed in [5]. To justify this, let us consider two particular simple examples showing that the existence of the solution heavily depends on data.

Assume the data $\theta_0 = \pi/2$, $\theta_1 = -\pi/2$, $\kappa_0 = \kappa_1 = 0$ and $L = 101/100$. Plugging corresponding constants in (4) and doing some manipulations with equations reveal that $v_1 = u_1$ and $v_2 = -u_2$. This further implies that either $u_2 = u_1$ or $u_2 = -u_1 - 5\frac{\sqrt{2}}{6}d$. If $u_2 = u_1$, we end up with two biquadratic equations for u_1 and d, namely

$$d^2 + 8\sqrt{2}\,d\,u_1 + 18u_1^2 - 70 = 0, \quad 200d^2 + 200\sqrt{2}\,d\,u_1 + 240u_1^2 - 707 = 0.$$

They represent a hyperbola and an ellipse shown in Fig. 1 (the first figure on the left), and it is easy to show that they do not intersect. Similarly we can check that also for $u_2 = -u_1 - 5\frac{\sqrt{2}}{6}d$ we do not have a solution (a hyperbola and an ellipse in Fig. 1 (the second figure from the left)).

For the second example consider the same data as in the previous one, except that $L = 3$. Similar procedure as before leads to $v_1 = u_1$, $v_2 = -u_2$ and again to $u_2 = u_1$ or $u_2 = -u_1 - 5\frac{\sqrt{2}}{6}d$. If $u_2 = u_1$, we end up with two biquadratic equations for u_1 and d, namely

$$d^2 + 8\sqrt{2}\,d\,u_1 + 18u_1^2 - 70 = 0, \quad 10d^2 + 10\sqrt{2}\,d\,u_1 + 12u_1^2 - 105 = 0,$$

again representing a hyperbola and an ellipse shown in Fig. 1 (the third figure from the left). It is clearly seen that they intersect in four points, thus the system of nonlinear Eqs. (4) has four solutions. The case $u_2 = -u_1 - 5\frac{\sqrt{2}}{6}d$ again implies a hyperbola and an ellipse with no intersections (Fig. 1, the last figure in the row). It is clear that changing also the angles θ_i and curvatures κ_i, $i = 0, 1$, would imply even more complicated examples with solutions relying heavily on the data.

4. Interpolation of circular arc data

The analysis of the system of nonlinear equations determining the PH curve of degree 7, which interpolates given G^2 data with a prescribed arc length is, in general, highly nontrivial task as can be seen from the examples in the previous section. No references are available with at least some progress toward a general solution. Recently, the same interpolation problem was solved in [28] by using PH biarcs of degree 7. However, the problem simplifies due to the fact that biarcs possess more degrees of freedom than a single PH curve. Another special case was considered in [29], where the authors considered arc length preserving G^2 Hermite interpolation of clothoid segments. They have used the fact that the curvature of a clothoid is proportional to the arc length parameter and its arc length has a simple analytic representation. Even with this simplification, they have been able to solve the problem only by applying a numerical method on a reduced system of nonlinear equations (two quadratic and one cubic equation). No detailed analysis of the existence of the solution was provided. In this section, we consider another simplification, the approximation of circular arcs. We consider interpolation of circular arc data, i.e., the data arising from a circular arc. As we will see, we will be able to reduce the problem to one algebraic equation of degree 6 and provide a detailed analysis of the existence of (up to four) admissible solutions. There seem to be no results of this type available in the literature. Thus we may consider our approach as the first approximation scheme of circular arcs involving the interpolation of the arc length.

Let the data be sampled from the circular arc with its inner angle equal to 2α. The canonical position and some elementary geometry imply $\alpha = \theta_0 = -\theta_1$, $\kappa_0 = \kappa_1 = -2\sin\alpha$, $L = \alpha\csc\alpha$, the radius of the arc equals $1/(2\sin\alpha)$, and its center is at $(1/2, -1/2\cot\alpha)^T$. Note that in the following we will use two additional standard trigonometric functions $\csc = 1/\sin$ and $\sec = 1/\cos$. We shall further assume that $0 < \alpha \leq \pi/2$, since for practical applications it is enough to construct good approximations of circular arcs up to the semicircle. Similar analysis as in the following could also be done for $\pi/2 < \alpha < \pi$, too.

The circular data are first used to determine constants in the nonlinear system (4). The third and the fourth equation are then solved on v_1 and v_2, i.e.,

$$v_1 = \frac{1}{3}\tan\left(\frac{\alpha}{2}\right)\left(3u_1 - 2d^3\cos\left(\frac{\alpha}{2}\right)\right), \quad v_2 = -\frac{1}{3}\tan\left(\frac{\alpha}{2}\right)\left(3u_2 - 2d^3\cos\left(\frac{\alpha}{2}\right)\right). \tag{5}$$

Combining this with the fifth equation leads to

$$(u_1 - u_2)\left(6(u_1 + u_2) - d(d^2 - 10)\cos\left(\frac{\alpha}{2}\right)\right) = 0.$$

We obviously have two possibilities, $u_2 = -u_1 + \frac{1}{6}d(d^2 - 10)\cos\left(\frac{\alpha}{2}\right)$ or $u_2 = u_1$. Note that (4) implies that if (d, u_1, v_1, u_2, v_2) is a solution, then also $(-d, -u_1, -v_1, -u_2, -v_2)$ is a solution, which, by (3), provides the same interpolant $\boldsymbol{p}$, so it is enough to consider solutions with $d > 0$ only.

4.1. First case

We will prove that there are no real solutions in the case $u_2 = -u_1 + \frac{1}{6}d(d^2 - 10)\cos\left(\frac{\alpha}{2}\right)$. Considering (5) the system of nonlinear Eqs. (4) reduces to

$$\begin{aligned}
&18(4\cos\alpha - 3)u_1^2 + 3d(d^2 - 10)\left(\cos\left(\frac{\alpha}{2}\right) - 2\cos\left(\frac{3\alpha}{2}\right)\right)u_1 \\
&+ \cos^2\left(\frac{\alpha}{2}\right)\left(-3d^6 + 18d^4 - 34d^2 - 420 + 4d^2(d^4 - 6d^2 + 30)\cos\alpha\right) = 0, \\
&36\left(3\sin\left(\frac{3\alpha}{2}\right) - 11\sin\left(\frac{\alpha}{2}\right)\right)u_1^2 + 6\sin\left(\frac{\alpha}{2}\right)\left(5\cos\left(\frac{\alpha}{2}\right) - 3\cos\left(\frac{3\alpha}{2}\right)\right)d(d^2 - 10)u_1 \\
&+ \left(840\alpha - 8d^2(d^4 - 6d^2 + 30)\sin\alpha + d^2(3d^4 - 18d^2 + 34)\sin(2\alpha)\right)\cos\left(\frac{\alpha}{2}\right) = 0.
\end{aligned} \tag{6}$$

Fortunately, the resultant of the polynomials on the left-hand side of (6) with respect to u_1 simplifies to $(504\cos(\alpha/2))^2$ $r(d)$, where

$$r(d) = \left(16\sin^3\alpha\, d^2 + 30((3\cos\alpha - 4)\sin\alpha + \alpha(4\cos\alpha - 3))\right)^2 \tag{7}$$

and the candidates for solutions of (6) with positive d are positive zeros of r.

Lemma 1. *For $\alpha \in (0, \pi/2]$, function r in (7) has precisely one (double) positive zero*

$$d_1 = \frac{\sqrt{30}}{4}\sqrt{\frac{\alpha(3 - 4\cos\alpha) + (4 - 3\cos\alpha)\sin\alpha}{\sin^3\alpha}} > \frac{5}{2}.$$

Proof. By (7) (double) zeros of r are $\pm d_1$. The result of the lemma will follow if we prove that $f(\alpha) > 0$ on $(0, \pi/2]$, where

$$f(\alpha) = \alpha(3 - 4\cos\alpha) + (4 - 3\cos\alpha)\sin\alpha - \frac{10}{3}\sin^3\alpha.$$

Quite clearly $f(0) = 0$ and $f'(\alpha) = \sin\alpha(4\alpha + 6\sin\alpha - 5\sin 2\alpha) \geq 6\sin^2\alpha(1 - \cos\alpha) > 0$, where we have used the known fact that $\alpha > \sin\alpha$ for $\alpha > 0$. Consequently f is positive on $(0, \pi/2]$ and the proof is completed. □

Note that the second equation in (6) is quadratic in u_1 with the discriminant

$$504\left(2\cos\left(\frac{\alpha}{2}\right) + 6\sin\left(\frac{\alpha}{2}\right)\sin\alpha\right)\sin\left(\frac{\alpha}{2}\right)f_\alpha(d), \tag{8}$$

where

$$f_\alpha(d) = 960\alpha + -8d^2(d^4 - 4d^2 + 20)\sin\alpha + d^2(3d^4 - 12d^2 - 4)\sin 2\alpha.$$

Lemma 2. *Function f_α is negative on $[5/2, \infty)$ for all $\alpha \in (0, \pi/2]$.*

Proof. The idea of the proof is similar to the proof of Lemma 1. Let

$$g(\alpha) = f_\alpha(5/2) = 960\alpha - \frac{13625\sin\alpha}{8} + \frac{15275}{64}\sin 2\alpha.$$

Obviously $g(0) = 0$ and $g'(\alpha) = \frac{5}{32}(6110\cos^2\alpha - 10900\cos\alpha + 3089)$. By solving a simple quadratic equation one can conclude that $\alpha_0 \approx 1.2069$ is the unique zero of g' on $(0, \pi/2]$ implying the local minimum $g(\alpha_0) \approx -274.2089$. Since also $g(\pi/2) \approx -195.1605$, function g must be negative on $(0, \pi/2]$. Furthermore, zeros of f'_α are $d_1 = 0$,

$$d_{2,3} = \pm\sqrt{\frac{16 - 12\cos\alpha - h(\alpha)}{12 - 9\cos\alpha}}, \quad d_{4,5} = \pm\sqrt{\frac{16 - 12\cos\alpha + h(\alpha)}{12 - 9\cos\alpha}},$$

where

$$h(\alpha) = \sqrt{-614 + 288\cos\alpha + 90\cos 2\alpha}.$$

Obviously $h^2 < 0$, and $d_{2,3,4,5}$ are complex thus f_α must be monotone on $[5/2, \infty)$. Since $f''_\alpha(0) = -16\sin\alpha(20\sin\alpha + \cos\alpha) < 0$, f_α is decreasing which together with $f_\alpha(5/2) < 0$ implies the result of the lemma. □

From Lemmas 1 and 2 and (8) now follows that the system of nonlinear Eqs. (6) has no real solutions for any $\alpha \in (0, \pi/2]$.

4.2. Second case

Let us now consider the symmetric case, i.e., $u_1 = u_2$. Eqs. (5) then imply $v_2 = -v_1$ and the first two equations in (4) simplify to

$$3(1+\cos\alpha)d^2 + 8\cos\left(\frac{\alpha}{2}\right)du_1 + 6u_1^2 - 10(1 + \alpha\csc\alpha) = 0, \tag{9}$$

$$4d^6 - 24d^4 + 57d^2 - 12\sec\left(\frac{\alpha}{2}\right)d(d^2-3)u_1 + 9\sec^2\left(\frac{\alpha}{2}\right)u_1^2 + 105\csc^2\left(\frac{\alpha}{2}\right)(1 - \alpha\csc\alpha) = 0, \tag{10}$$

the system of two nonlinear equations for d and u_1. The first equation again represents an ellipse, while the second one is a much more complicated algebraic curve of degree 6 (see Fig. 2). But it is quadratic in u_1 and we can detect its two branches

$$u_1^+(d) = \frac{1}{3}\cos\left(\frac{\alpha}{2}\right)\left(2d^3 - 6d + \sqrt{21}\sqrt{5\csc^2\left(\frac{\alpha}{2}\right)(\alpha\csc\alpha - 1) - d^2}\right), \tag{11}$$

$$u_1^-(d) = \frac{1}{3}\cos\left(\frac{\alpha}{2}\right)\left(2d^3 - 6d - \sqrt{21}\sqrt{5\csc^2\left(\frac{\alpha}{2}\right)(\alpha\csc\alpha - 1) - d^2}\right) \tag{12}$$

(u_1^+ is the black solid part and u_1^- is the black dashed part of the curve in Fig. 2). Let

$$d_{max} = \csc\left(\frac{\alpha}{2}\right)\sqrt{5(\alpha\csc\alpha - 1)}. \tag{13}$$

Then clearly u_1^+ and u_1^- are both defined on $[-d_{max}, d_{max}]$, $u_1^+ \geq u_1^-$ and $u_1^+(\pm d_{max}) = u_1^-(\pm d_{dmax})$. Consequently, (10) is a closed curve on $[-d_{\max}, d_{max}]$. Let us first prove the following observation.

Lemma 3. *For $\alpha \in (0, \pi/2]$ the inequality $d_{max} > \sqrt{10/3}$ holds true.*

Proof. It is easy to see that the inequality from the lemma is equivalent to $f(\alpha) > 0$, where $f(\alpha) = 3(\alpha - \sin\alpha) - \sin\alpha(1 - \cos\alpha)$. Since $f(0) = 0$ and $f'(\alpha) = 8\sin^4\left(\frac{\alpha}{2}\right) > 0$, the proof of the lemma is complete. □

The following lemma guarantees the existence of the solution of the considered interpolation problem.

Lemma 4. *The system of nonlinear Eqs. (9), (10) has at least two real solutions with $d > 0$ for any $\alpha \in (0, \pi/2]$.*

Proof. During the proof, we will refer to Fig. 2. Consider the (d, u_1) plane. Let u_{1e} and u_{1a} be the positive intersections of (9) and (10) with $d = 0$, respectively, and similarly, let d_e and d_a be the positive intersections of the same curves with $u_1 = 0$, respectively. If we show that

$$(u_{1a} - u_{1e})(d_a - d_e) < 0, \tag{14}$$

then curves must intersect in the first quadrant of the chosen coordinate system. But due to the symmetric properties, they must also intersect in the fourth quadrant and the result of the lemma will follow.

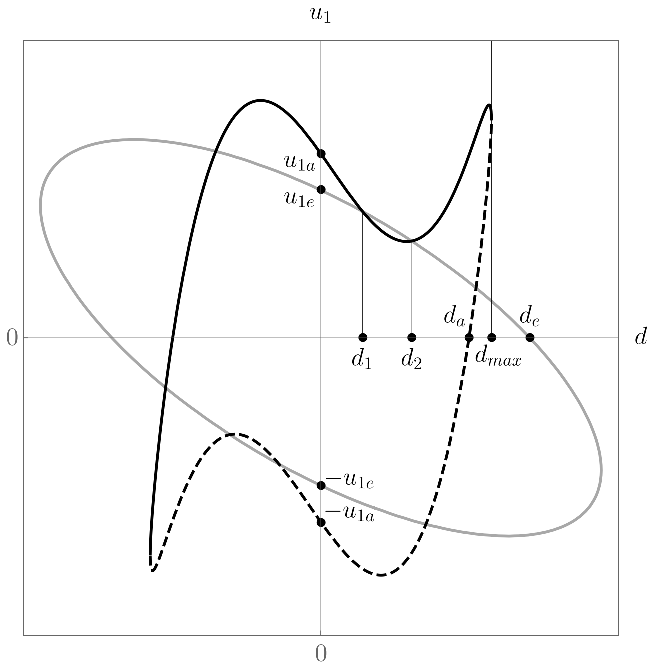

Fig. 2. Curves given by Eqs. (9) (gray, an ellipse) and (10) (black solid and black dotted, an algebraic curve of degree 6) for $\alpha = \pi/2$ with quantities used in the proof of Lemma 4.

Quite clearly, for (14) it is enough to see $u_{1a} > u_{1e}$ and $d_e > d_a$. Let us start by proving the first inequality. If $d = 0$, then (9), (11) and (12) imply

$$u_{1e} = \sqrt{\frac{5}{3}}\sqrt{\alpha\csc\alpha + 1}, \quad u_{1a} = \sqrt{\frac{35}{3}}\cot\left(\frac{\alpha}{2}\right)\sqrt{\alpha\csc\alpha - 1}.$$

Some straightforward calculations reveal that $u_{1a} > u_{1e}$ is equivalent to $f(\alpha) := \cos\alpha\,(4\alpha - 3\sin\alpha) - 4\sin\alpha + 3\alpha > 0$ on $(0, \pi/2]$. Since $f'(\alpha) = 2\sin\alpha(3\sin\alpha - 2\alpha)$, f is strictly increasing on $(0, \alpha^*)$ and strictly decreasing on $(\alpha^*, \pi/2]$, where $\alpha^* \in (\pi/4, \pi/2)$. But $f(0) = 0$, $f(\pi/2) = (3\pi - 8)/2 > 0$, and the conclusion $f > 0$ on $(0, \pi/2]$ follows.

For the second inequality, observe that $d_{max} > d_a$, and it is enough to see that $d_e \geq d_{max}$. Inserting $u_1 = 0$ in (9) and considering (13) lead us to show that $g(\alpha) := \cos\alpha\sin\alpha + 2\sin\alpha - 2\alpha\cos\alpha - \alpha \geq 0$ on $(0, \pi/2]$. But this follows immediately from $g(0) = 0$ and $g'(\alpha) = 2\sin\alpha(\alpha - \sin\alpha) > 0$ on $(0, \pi/2]$. □

Remark 1. Identical proof can be done for the case $\alpha \in (0, \alpha_{max})$, where $\alpha_{max} \approx 2.0682$ is the first positive zero of f defined in the proof of the previous lemma.

From the previous lemma it follows that the considered system of nonlinear equations has an even number of solutions with positive d for any $\alpha \in (0, \pi/2]$. Numerical examples reveal that there might be four of them as indicated in Fig. 2. However, it seems quite difficult to prove the existence of a precise number of solutions in general. Four solutions reduce to three for $\alpha = \alpha_{crit} \approx 2.2337$ and to only two solutions for $\alpha > \alpha_{crit}$ (see Fig. 3). The critical value α_{crit} is a solution of the system (9), (10) and their zero Jacobian. In the following, we will provide an easy (numerical) procedure to check the existence of four solutions with $d > 0$ for a particular $\alpha \in (0, \pi/2]$ and prove their existence for α small enough. The PH interpolant of the degree seven arising from the positive solution d_j, $j = 1, 2, 3, 4$, where $d_1 < d_2 < d_3 < d_4$, will be denoted by $\boldsymbol{p}_j$.

Let us first transform the system of nonlinear Eqs. (9) and (10) to a more appropriate one for the analysis. This can be done by using a Gröebner basis [30] with respect to a particular ordering of the unknowns. The system of nonlinear equations then reads as

$$\begin{aligned} p_1(d) = & -32\sin^6\alpha\, d^{12} + 256\sin^6\alpha\, d^{10} - 1184\sin^6\alpha\, d^8 \\ & - 96\sin^3\alpha(-40\alpha + 9\sin\alpha + 20\sin 2\alpha + 7\sin 3\alpha - 30\alpha\cos\alpha)\, d^6 \\ & + 96\sin^3\alpha(-160\alpha + 99\sin\alpha + 80\sin 2\alpha + 7\sin 3\alpha - 120\alpha\cos\alpha)\, d^4 \\ & + 13440(\alpha - \sin\alpha)\sin^5\alpha\csc^2\left(\frac{\alpha}{2}\right) d^2 \\ & - 1800(6\alpha + 8\alpha\cos\alpha - 2\sin\alpha(3\cos\alpha + 4))^2 = 0, \end{aligned} \tag{15}$$

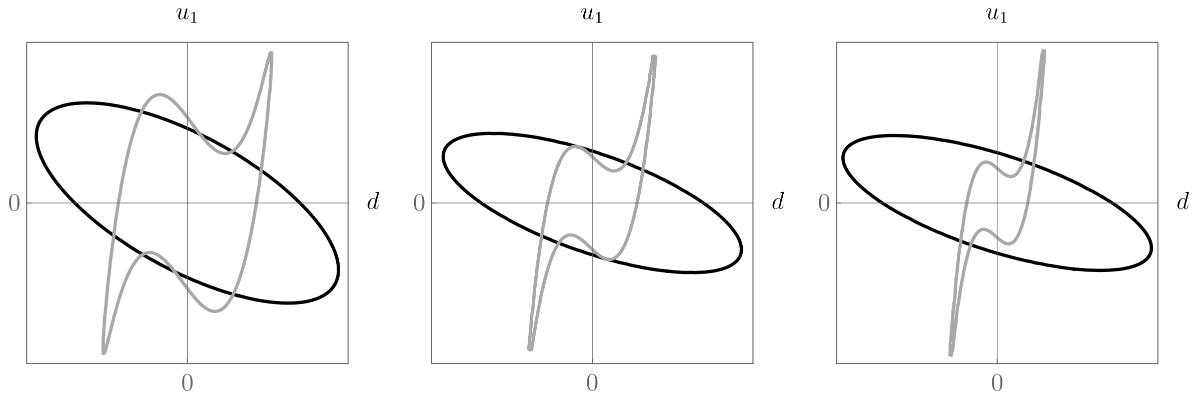

Fig. 3. Four solutions with positive d for $\alpha = 1.8 < \alpha_{crit} \approx 2.2337$ (left) which degenerate into three of them for $\alpha = \alpha_{crit}$ (middle) and transform to two solutions for $\alpha = 2.5 > \alpha_{crit}$ (right).

$$\begin{aligned} p_2(u_1, d) &= 24d\left(d^2-2\right)\sec\left(\frac{\alpha}{2}\right)u_1 + 210\csc^2\left(\frac{\alpha}{2}\right)(\alpha\csc\alpha - 1) \\ &\quad + \sec^2\left(\frac{\alpha}{2}\right)\left(-30\alpha\csc\alpha + 9d^2\cos\alpha + 9d^2 - 30\right) - 8d^6 + 48d^4 - 114d^2 = 0. \end{aligned} \tag{16}$$

Let us analyze the polynomial p_1 first. Since it is even of degree 12, we can reduce its degree to 6 by introducing $p(x) = p_1(\sqrt{x})$. Taylor expansion of p around 0 as a function of α can be obtained with the help of a computer algebra system, namely

$$p(x) = -32q(x)\alpha^6 + \mathcal{O}(\alpha^8), \quad q(x) = x^6 - 8x^5 + 37x^4 - 134x^3 + 284x^2 - 280x + 100. \tag{17}$$

Real zeros of q are

$$x_1 = x_2 = 1 \quad x_3 \approx 2.1842, \quad x_4 \approx 3.2872. \tag{18}$$

Note that x_3 and x_4 can also be written in radicals since they are zeros of some quartic polynomial, but expressions are too complicated to be given here explicitly. We observe that zeros of p must be close to zeros of q at least for α small enough. But it turns out that they play a crucial role in finding the solution of nonlinear system (15) and (16) for general α too.

Lemma 5. *For any $\alpha \in (0, \pi/2]$, the polynomial p has at most four positive zeros on $(0, d_{max}^2)$. If $p(y_i)$, $i = 0, 1, \ldots, 4$, where $y_0 = 0$, $y_1 = 1$, $y_2 = x_3$, $y_3 = x_4$ and $y_4 = d_{max}^2$, are of alternating signs, then p has precisely four positive zeros.*

Proof. First, observe that the fourth derivative of p is of particular simple form, namely $p^{(iv)}(x) = -768\sin^6\alpha\left(15x^2 - 40x + 37\right)$. It is quite clearly negative and consequently, p has at most four real zeros. By Lemma 3 and (18) we have $y_0 < y_1 < y_2 < y_3 < y_4$ and $p(y_i)p(y_{i+1}) < 0$, $i = 0, 1, 2, 3$, implies precisely four zeros due to the continuity of p. □

We are now ready to prove the main theorem of this paper.

Theorem 1. *If $\alpha \in (0, \pi/2]$ then the number of real solutions of the nonlinear system (9), (10) with $d > 0$ is the same as the number of positive zeros of p.*

Proof. Since nonlinear system (9), (10) is equivalent to the system (15), (16), the only candidates for real solutions with $d > 0$ are, by Lemma 5, positive zeros of p. Thus we have to prove that each positive zero of p implies the unique real solution of the system (15), (16). Let z be a positive zero of p. Then $d_z = \sqrt{z}$ is a positive zero of p_1 and the solution of $p_2(u_1, d_z) = 0$ on u_1 provides the desired solution of the system of nonlinear equations. But $p_2(u_1, \cdot)$ is a linear polynomial and it remains to prove that its leading coefficient does not vanish at d_z. It is equivalent to verifying that $z \neq 0, 2$, or equivalently

$$p(0) = -7200f_1(\alpha)^2 < 0, \quad p(2) = -32f_2(\alpha)^2 < 0,$$

where

$$\begin{aligned} f_1(\alpha) &= \cos\alpha\,(4\alpha - 3\sin\alpha) - 4\sin\alpha + 3\alpha, \\ f_2(\alpha) &= 45\alpha - 2\sin^3\alpha - 66\sin\alpha + 60\alpha\cos\alpha + 6\sin\alpha\cos^2\alpha - 45\sin\alpha\cos\alpha. \end{aligned}$$

Thus it is enough to show that $f_1, f_2 > 0$ on $(0, \pi/2]$. The inequality $f_1 > 0$ follows from the fact that $f_1 = f$ from the proof of Lemma 4. To confirm $f_2 > 0$, observe that $f_2(0) = 0$ and $f_2'(\alpha) = -6\sin\alpha f_3(\alpha)$, where $f_3(\alpha) = 10\alpha - 15\sin\alpha + 4\sin\alpha\cos\alpha$. Since $f_3'(\alpha) = 6 - 15\cos\alpha + 8\cos^2\alpha$, f_3' has precisely one zero on $[0, \pi/2]$ and consequently f_3 has at most two zeros there. Since $f_3(0) = 0$, $f_3(\alpha) = -\alpha + \mathcal{O}(\alpha^3)$ and $f_3(\pi/2) = 5(\pi - 3) > 0$, f_3 has the unique zero $\alpha_0 \in (0, \pi/2]$. Thus f_2 is increasing on $(0, \alpha_0)$ and decreasing on $(\alpha_0, \pi/2]$. Since $f_2(0) = 0$ and $f_2(\pi/2) = (45\pi - 136)/2 > 0$, function f_2 must be positive on $(0, \pi/2]$ and the result of the theorem follows. □

The previous theorem provides an efficient and easy way to check the number of polynomial parametric approximants interpolating G^2 data and an arc length arising from a circular arc given by an inner angle 2α. For some practically important angles α, such as $\alpha = \pi/2, \pi/3, \pi/4, \pi/8, \ldots$, the direct application of Lemma 5 confirmed the existence of four zeros of p, except for $\alpha = \pi/2$, where we had to replace $y_2 = x_3$ by $y_2 = 2$. A direct formal proof that precisely four solutions exist for any $\alpha \in (0, \pi/2]$ seems to be quite a difficult task since the analysis of symbolic expressions involving combinations of algebraic and trigonometric terms in Lemma 5 would be needed. However, if α is small enough, the expansion (17) enables us to prove the existence of four solutions in general. This will be done in the following section.

5. Asymptotic analysis

Let us now consider α small enough. Using (17) and considering some additional terms in the expansion, we get

$$p(y_0) = -3200\alpha^6 + \mathcal{O}(\alpha^8), \quad p(y_1) = 64\alpha^{10} + \mathcal{O}(\alpha^{12}), \quad p(y_2) \approx -52.3867\alpha^8 + \mathcal{O}(\alpha^{10}),$$
$$p(y_3) \approx 2292.89\alpha^8 + \mathcal{O}(\alpha^{10}), \quad p(y_4) = -\frac{156800}{729}\alpha^6 + \mathcal{O}(\alpha^8).$$

Consequently, p has four positive zeros by Lemma 5. The leading terms constants can also be written in a closed form, thus their numerical values can be computed with arbitrary precision.

In the following, we will find asymptotic expansions of positive zeros of p, which provide asymptotic expansions of real solutions of the system of nonlinear Eqs. (9), (10). Let z_i, $i = 1, 2, 3, 4$, be a positive zero of p. Then (17) suggests the expansion of z_i as

$$z_i = x_i + \sum_{j=1}^{\infty} c_{i,j}\alpha^j, \quad i = 1, 2, 3, 4.$$

Constants $c_{i,j}$ can now be found as a solution of the system of equations for $c_{i,j}$ arising from the condition that terms in the expansion of $p(z_i)$ vanish for all α. Let us demonstrate the procedure for $i = 2$, since the solution z_2 will later turn out as the most appropriate one. The expansion of $p(z_2)$ reads as

$$p(z_2) = -1248c_{2,1}^2\alpha^8 + 64c_{2,1}\left(23c_{2,1}^2 - 39c_{2,2} - 2\right)\alpha^9$$
$$- 32\left(12c_{2,1}^4 - 3\left(46c_{2,2} + 9\right)c_{2,1}^2 + 78c_{2,3}c_{2,1} + 39c_{2,2}^2 + 4c_{2,2} - 2\right)\alpha^{10} + \ldots$$

The requirement that the coefficients at α^j, $j = 8, 9, 10$, vanish, leads to the triangular system of nonlinear equations with solutions $c_{2,1} = 0$, $\quad c_{2,2} = (-2 \pm \sqrt{82})/39$. Since $z_2 > 1$, we must take $c_2 = (-2 + \sqrt{82})/39$. Considering more terms in the expansion, we can similarly compute additional constants $c_{2,j}$ but we will skip the details. Recall that $d_2 = \sqrt{z_2}$, so the asymptotic expansion of d_2 is

$$d_2 = 1 - \frac{1}{78}\left(2 - \sqrt{82}\right)\alpha^2 + \frac{\left(37966 + 10579\sqrt{82}\right)}{19456632}\alpha^4 + \cdots \tag{19}$$

Together with (16) we get the asymptotic expansions

$$u_{1,2} = u_{2,2} = 1 + \frac{1}{312}\left(73 - 4\sqrt{82}\right)\alpha^2 - \frac{\left(3071515 - 636632\sqrt{82}\right)}{311306112}\alpha^4 + \cdots \tag{20}$$

and finally from (5) also

$$v_{1,2} = -v_{2,2} = \frac{\alpha}{6} + \frac{\left(371 - 36\sqrt{82}\right)}{1872}\alpha^3 - \frac{\left(8660963 - 1179080\sqrt{82}\right)}{1037687040}\alpha^5 + \cdots \tag{21}$$

Similarly we compute asymptotic expansions for the other three solutions d_1, d_3, d_4 and consequently also expansions for $u_{1,j}$, $u_{2,j}$, $v_{1,j}$ and $v_{2,j}$, $j = 1, 3, 4$. Either in the non-asymptotic or in the asymptotic approach we obtain several solutions. In the next section, we will provide suggestions on choosing the most appropriate one.

6. Solution selection

Multiple solutions are regularly observed fact when one is dealing with interpolation by PH curves. Usually, some of them are more appropriate for applications (without undesirable loops, e.g.) than others. This was observed already in the early papers dealing with interpolation by PH curves [27,31]. There are several suggestions on how to choose the most appropriate solution, but none of them can be considered a universal one. Quite standard measure of fairness is the absolute rotation index [4, p. 532], which is defined as

$$R_{abs} = \int_0^1 |\kappa(t)| \, \|\boldsymbol{p}'(t)\| \mathrm{d}t. \tag{22}$$

It was successfully used in [5] to identify more appropriate solutions. We can use the same criterion here for the general interpolation of G^2 data. However, for the circular arc data, it seems reasonable to observe the deviation of the curvature of the interpolant from the (constant) curvature of the corresponding circular arc in the L^2 norm. Since the curvature of the circular arc in the chosen canonical position is $-2\sin\alpha$, the error becomes

$$E_\kappa = \int_0^1 (\kappa(t) + 2\sin\alpha)^2 \, \mathrm{d}t. \tag{23}$$

Note that the (numerical) evaluation of E_κ for a PH curve is quite simple since its curvature κ is a rational function.

For the asymptotic case explained in the previous section is promising to choose the solution which provides a curve with the best approximation properties, such as the minimal Hausdorff distance. Since the approximation of a circular arc is considered, one can use the radial distance d_{rad} as the error measure [32]. It is a special type of parametric distance considered in [11] and later in [19] where the authors have proved that it coincides with the Hausdorff distance in the case of circular arc approximation. Let $\boldsymbol{p} = (\boldsymbol{p}_x, \boldsymbol{p}_y)^T$ be a PH curve of degree seven approximating the circular arc $\boldsymbol{c}$ given by some small inner angle 2α in the canonical position. Then the radial distance is defined as

$$d_{rad}(\boldsymbol{p}; \alpha) = \max_{t\in[0,1]} \left| \sqrt{\left(\boldsymbol{p}_x(t) - \frac{1}{2}\right)^2 + \left(\boldsymbol{p}_y(t) + \frac{1}{2}\cot\alpha\right)^2} - \frac{1}{2\sin\alpha} \right|.$$

i.e., the distance between the point $\boldsymbol{p}(t)$ and the intersection of the line passing through the center of the circular arc $(1/2, -1/2\cot\alpha)^T$ and $\boldsymbol{p}(t)$ with the circular arc. Using the asymptotic expansions (19)–(21) we can derive

$$d_{rad}(\boldsymbol{p}; \alpha) = c\,\alpha^r + \mathcal{O}\left(\alpha^{r+1}\right), \tag{24}$$

where c is some positive constant and $r \in \mathbb{N}$ is the asymptotic approximation order. Since $\boldsymbol{p}$ interpolates two points, two tangent directions, two curvatures and an arc length, the expected approximation order is 7. Indeed, for the solution $d = d_2$ which implies the PH interpolant $\boldsymbol{p}_2$ we have

$$d_{rad}(\boldsymbol{p}_2; \alpha) = \frac{47773 - 5264\sqrt{82}}{318898944}\alpha^7 + \mathcal{O}\left(\alpha^8\right) \approx 3.3068 \times 10^{-7}\alpha^7 + \mathcal{O}\left(\alpha^9\right), \tag{25}$$

while d_1, d_3 and d_4 imply interpolants $\boldsymbol{p}_1$, $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$, respectively, with inferior leading term constant or much lower approximation order. More precisely,

$$d_{rad}(\boldsymbol{p}_1; \alpha) = \frac{47773 + 5264\sqrt{82}}{318898944}\alpha^7 + \mathcal{O}\left(\alpha^8\right) \approx 2.9928 \times 10^{-4}\alpha^7 + \mathcal{O}\left(\alpha^9\right), \tag{26}$$

$$d_{rad}(\boldsymbol{p}_3; \alpha) = 0.0173\alpha + \mathcal{O}\left(\alpha^3\right), \quad d_{rad}(\boldsymbol{p}_4; \alpha) = 0.1246\alpha + \mathcal{O}\left(\alpha^3\right). \tag{27}$$

7. Numerical examples

In all numerical examples, canonical data will be considered. Since we know that there are always several solutions of the problem, we can find them numerically either by applying the continuation method [33] or by using Lemma 5 which provides excellent starting values for an iterative algorithm (such as the Newton–Raphson method) to find positive real zeros of p. The selection criterion (23) (for general circular arc data) or (24) (for circular data with α small enough) is then used to identify the most pleasant interpolant.

Let us first consider the following circular arc data:

$$\alpha := \theta_0 = -\theta_1 = \pi/2, \quad \kappa_0 = \kappa_1 = -2, \quad L = \pi/2. \tag{28}$$

We know from Lemma 5 that four admissible solutions exist. The error (23) is taken as the selection criterion. Its corresponding values for the approximants $\boldsymbol{p}_i$ arising from the positive solutions d_i, $i = 1, 2, 3, 4$, are 4.2527×10^{-2}, 8.6586×10^{-8}, 2.4235×10^6 and 34.0648, respectively. The approximant $\boldsymbol{p}_2$ corresponding to $d_2 \approx 1.2756$ is clearly the most appropriate, which is also confirmed in Fig. 4. The Hausdorff distance between the chosen interpolating curve and the circular arc is approximately 1.2850×10^{-5} and it is attained at the middle of the arc. Thus the constructed PH curve

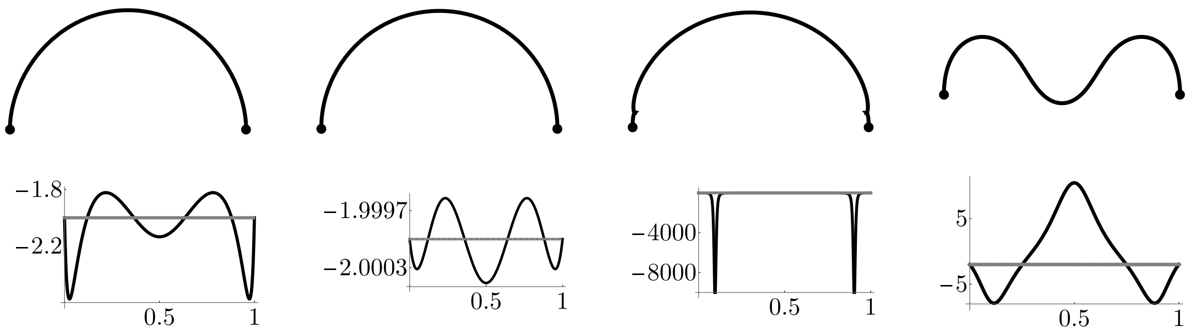

Fig. 4. Plots of semicircle approximants $\boldsymbol{p}_i$, $i = 1, 2, 3, 4$, interpolating data (28) (top) together with corresponding curvature profiles (bottom). The gray horizontal line is the curvature of the approximated semicircle. Note that $\boldsymbol{p}_3$ possesses two almost invisible tiny loops.

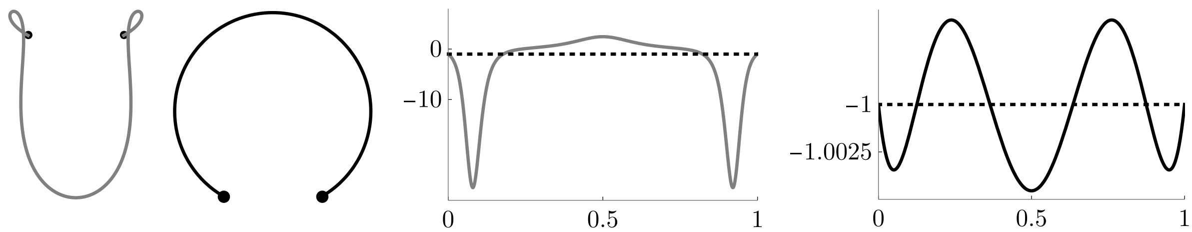

Fig. 5. Plots of interpolants of data (29) (the first and the second on the left) and corresponding curvature profiles. Dotted lines are curvatures of the circular arc from which the data were taken.

Table 1
Radial distances and estimated approximation orders for G^2 interpolants $\boldsymbol{p}_2$ (the second and the third column) and $\boldsymbol{p}_3$ (the fourth and the fifth column).

α	$d_{rad}(\boldsymbol{p}_2; \alpha)$	r	$d_{rad}(\boldsymbol{p}_3; \alpha)$	r
$\pi/2$	1.2850×10^{-5}	–	1.3865×10^{-2}	–
$\pi/4$	6.8517×10^{-8}	7.55	1.3143×10^{-2}	0.08
$\pi/8$	4.9016×10^{-10}	7.13	6.7687×10^{-3}	0.96
$\pi/16$	3.7474×10^{-12}	7.03	3.3944×10^{-3}	1.00
$\pi/32$	2.9119×10^{-14}	7.01	1.6980×10^{-3}	1.00

of the degree 7 can be considered a remarkably accurate approximation of the semicircle preserving the arc length. For the G^2 approximation of the whole circle, just consider the spline approximant build by the constructed interpolant and its rotation.

As the second example, let us consider the data from the circular arc with $\alpha > \pi/2$. Let

$$\alpha := \theta_0 = -\theta_1 = 5\pi/6, \quad \kappa_0 = \kappa_1 = -1, \quad L = 5\pi/3. \tag{29}$$

The system of nonlinear Eqs. (9) and (10) has only two admissible solutions. According to (23), the first one is clearly rejected since the error is $E_\kappa \approx 61.3568$, much higher than the error of the second one $E_\kappa \approx 9.0995 \times 10^{-6}$. This is evidently confirmed also in Fig. 5 where approximants together with their curvature profiles are shown. The Hausdorff distance of the second interpolant and the circular arcs is 1.6607×10^{-3}, which is less than 0.2% relatively to the radius.

Note that we could take also a greater value of α. Numerical examples confirm admissible solutions of the system of nonlinear equations for any $\alpha < \pi$, i.e., for the data arising from circular arcs up to almost the whole circle.

Let us conclude this section with a numerical evaluation of the approximation orders (25)–(27). For the sake of simplicity we will consider just interpolants $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$. They are computed for $\alpha_n = \pi/2^n$, $n = 1, 2, \ldots, 5$, and the corresponding Hausdorff errors e_n are determined. From $e_n \approx c\alpha_n^r$ one easily concludes that $r \approx \log(e_n/e_{n+1})/\log 2$. Numerical results are collected in Table 1 and they confirm theoretical values established in the previous section.

8. Closure

PH curves of degree seven are promising objects for interpolating G^2 local data and preserving an arc length. Since the problem turns out to be quite complicated for general data, a relaxation to the circular arc data was done and a detailed analysis was provided. It turned out that the above-mentioned curves provide excellent approximants of circular arcs, and they preserve a prescribed arc length. An algorithm for the construction of such curves was provided. It basically requires just solving an algebraic equation of degree six. An asymptotic analysis reveals that the approximation order is seven.

For future work, it would be nice to make some progress in studying the interpolation of general data. This requires some deeper analysis of a general system of nonlinear Eqs. (4). Another approach to solve the same problem would be using the PH quintic biarcs, a generalization of cubic biarcs studied already in an early paper by [34] and recently in [35].

Data availability

No data was used for the research described in the article.

Acknowledgments

The author is grateful to anonymous reviewers for their valuable suggestions and observations, which resulted in an improved revision of the original manuscript.

The author was supported in part by the program P1-0288 and the grants J1-9104, N1-0137 and J1-3005 by Slovenian Research Agency.

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