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On the Exact Regions Determined by Kendall's Tau and Other Concordance Measures

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Abstract. We determine the upper and lower bounds for possible values of Kendall's tau of a bivariate copula given that the value of its Spearman's footrule or Gini's gamma is known, and show that these bounds are always attained.

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1. Introduction

One of the most important notions in statistics is the notion of dependence of random variables. When we measure the dependence, we often try to describe it with a single real number. The most commonly used measure is Pearson's correlation coefficient, which measures linear dependence. For the random pair (X, Y) Pearson's correlation coefficient depends not only on the degree of association between X and Y but also on the marginal distributions of the pair (X, Y). If we want to measure only the degree of association, we need measures that do not depend on the marginals of the random vector, but only on the copula connecting its components. This is often done with the help of a concordance measure, or its generalization, weak concordance measure.

Intuitively, two continuous random variables X and Y are in concordance when large values of X occur simultaneously with large values of Y. More precisely, two realizations (x_1, y_1) and (x_2, y_2) of the random vector (X, Y) are concordant when $(x_2 - x_1)(y_2 - y_1) > 0$ and they are discordant when $(x_2 - x_1)(y_2 - y_1) < 0$. We can measure the concordance of a pair of random variables (X, Y) in various ways, see [26]. The most commonly used concordance measures are Spearman's rho, Kendall's tau, Gini's gamma and

Blomqvist's beta (denoted, respectively, by ρ, τ, γ and β), and a weak concordance measure Spearman's footrule (denoted by ϕ). These measures have been studied intensively since their introduction. Recent references for bivariate concordance measures include [8–10,16,20,24,25] and their multivariate generalizations were studied in [1,5,29,30], to name just a few.

Given their widespread use in a variety of practical applications, it is natural to compare different concordance measures in terms of the values that they can attain. In particular, if a value of one measure is known, we may ask what are the possible values of the other measures. In this paper, we study the possible values of Kendall's tau, if the value of some other (weak) concordance measure is given.

The investigation of the above question was started by Daniels [3] and Durbin and Stuart [7], who compared Spearman's rho and Kendall's tau and gave some estimates for the values of the two measures. The exact region of all possible pairs of values $(\tau(C), \rho(C))$, $C \in \mathcal{C}$, was only determined recently in [27]. The regions determined by Blomqvist's beta and the other three concordance measures (Spearman's rho, Kendall's tau, and Gini's gamma) are given in [23] as an exercise for the reader, while the region determined by Blomqvist's beta and Spearman's footrule is given in [16]. The region determined by Spearman's footrule and Gini's gamma is given in [17], and the region determined by Spearman's footrule and Spearman's rho is considered in [18]. Observe that the only remaining regions involving Kendall's tau are the regions for Kendall's tau with respect to Gini's gamma and Spearman's footrule. These two regions are determined in the present paper.

The paper is structured as follows. In Sect. 2, we give some basic definitions that will be used throughout the paper and recall the known exact regions involving Kendall's tau. In Sect. 3 we determine the exact region between Kendall's tau and Spearman's footrule by showing that the two measures satisfy

$$\frac{4}{3}\phi(C) - \frac{1}{3} \leqslant \tau(C) \leqslant \frac{2}{3}\phi(C) + \frac{1}{3}$$

for any copula C, and in Sect. 4, we show that the exact region between Kendall's tau and Gini's gamma is determined by the inequalities

$$\max\left\{\frac{2}{3}\gamma(C) - \frac{1}{3}, 2\gamma(C) - 1\right\} \leqslant \tau(C) \leqslant \min\left\{\frac{2}{3}\gamma(C) + \frac{1}{3}, 2\gamma(C) + 1\right\}.$$

In both cases the bounds are attained. In Sect. 5, we give the similarity measure between Kendall's tau and other (weak) concordance measures.

2. Preliminaries

Let $\mathbb{I} = [0, 1]$ be the unit interval and $R = [u_1, u_2] \times [v_1, v_2]$ a rectangle contained in $\mathbb{I}^2$ with $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. Given a real function $C\colon \mathbb{I}^2 \to \mathbb{R}$, we define the C-volume of rectangle R by $V_C(R) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1)$. A (bivariate) *copula* is a function $C : \mathbb{I}^2 \to \mathbb{I}$ with the following properties:

(i) $C(0,v) = C(u,0) = 0$ for all $u, v \in \mathbb{I}$ (C is *grounded*),
(ii) $C(u,1) = u$ and $C(1,v) = v$ for all $u, v \in \mathbb{I}$ (C has *uniform marginals*), and
(iii) $V_C(R) \geqslant 0$ for every rectangle $R \subseteq \mathbb{I}^2$ (C is $2-$*increasing*).

Any copula C induces a probability measure μ_C on the Borel σ-algebra in $\mathbb{I}^2$. This measure is uniquely determined by the property $\mu_C(R) = V_C(R)$ for all rectangles $R \subseteq \mathbb{I}^2$. Furthermore, measure μ_C is bistochastic in the sense that $\mu_C(B \times \mathbb{I}) = \mu_C(\mathbb{I} \times B) = \lambda(B)$ for any Borel set $B \subseteq \mathbb{I}$, where λ denotes the Lebesgue measure on $\mathbb{I}$. The set of all bivariate copulas will be denoted by $\mathcal{C}$. It is well known that this set is compact in the sup norm. For any copula C its diagonal will be denoted by δ_C, i.e., $\delta_C(u) = C(u,u)$ for all $u \in \mathbb{I}$.

Given two copulas C and D, we denote $C \leqslant D$ if $C(u,v) \leqslant D(u,v)$ for all $(u,v) \in \mathbb{I}^2$. This is the so-called *pointwise order* of copulas. For any copula C, we have $W \leqslant C \leqslant M$, where $W(u,v) = \max\{0, u+v-1\}$ and $M(u,v) = \min\{u,v\}$ are the lower and upper *Fréchet-Hoeffding bounds* for the set of all copulas. Furthermore, any copula C induces reflected copulas C^{σ_1} and C^{σ_2} defined by $C^{\sigma_1}(u,v) = v - C(1-u,v)$ and $C^{\sigma_2}(u,v) = u - C(u,1-v)$ for all $(u,v) \in \mathbb{I}^2$.

Let $h\colon \mathbb{I} \to \mathbb{I}$ be a measure preserving bijection, where $\mathbb{I}$ is equipped with the Lebesgue measure λ. Then, the function defined by

$$C(u,v) = \lambda(\{t \in \mathbb{I}; t \leqslant u, h(t) \leqslant v\}) \tag{1}$$

is a copula whose mass in concentrated on the graph of h, i.e., $\mu_C(\{(t,h(t)); t \in \mathbb{I}\}) = 1$. Particular examples of such copulas are the so-called *shuffles of min.* A shuffle of min

$$C = M(n, J, \pi, \omega)$$

is determined by a positive integer n, a partition $J = \{J_1, J_2, \ldots, J_n\}$ of the interval $\mathbb{I}$ into n pieces, where $J_i = [u_{i-1}, u_i]$ and $0 = u_0 \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_{n-1} \leqslant u_n = 1$, shortly written as $(n-1)$-tuple of splitting points $J = (u_1, u_2, \ldots, u_{n-1})$, a permutation $\pi \in S_n$, written as n-tuple of images $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, and a mapping $\omega : \{1, 2, \ldots, n\} \to \{-1, 1\}$, written as n-tuple of images $\omega = (\omega(1), \omega(2), \ldots, \omega(n))$. The mass of C is concentrated on the diagonals of the squares $J_i \times [v_{\pi(i)-1} \times v_{\pi(i)}]$, where $0 = v_0 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant v_{n-1} \leqslant v_n = 1$. Hence, C is defined by the measure preserving bijection $h_C\colon \mathbb{I} \to \mathbb{I}$ given by

$$h_C(u) = \begin{cases} u + v_{\pi(i)-1} - u_{i-1}; & u \in (u_{i-1}, u_i), \omega(i) = 1, \\ v_{\pi(i)} + u_{i-1} - u; & u \in (u_{i-1}, u_i), \omega(i) = -1, \\ v_i; & u = u_i. \end{cases}$$

Furthermore, C is the copula of uniformly distributed random variables U and V on $\mathbb{I}$ with the property $P(V = h_C(U)) = 1$. For more details see [23, Sect. 3.2.3].

In [26], Scarsini introduced formal axioms for *concordance measures.* These are mappings that assign to each copula a real number in $[-1,1]$ and are meant to measure the degree of concordance/discordance between the components of random vectors. Recall that two observations (x,y) and (x',y')

from a random vector (X, Y) are concordant if $(x - x')(y - y') > 0$ and discordant if $(x - x')(y - y') < 0$. For the formal definition of concordance measures and further details, we refer the reader to [6,23]. Here, we give the properties of concordance measures, which we will need in the sequel: if κ is a concordance measure, then $\kappa(M) = 1$, $\kappa(C^{\sigma_1}) = \kappa(C^{\sigma_2}) = -\kappa(C)$ for any copula $C \in \mathcal{C}$, κ is continuous with respect to the pointwise convergence, and κ is monotone increasing with respect to the pointwise order.

Many of the most important bivariate concordance measures, including Kendall's tau and Gini's gamma, can be expressed with the so-called *concordance function* $\mathcal{Q}$, introduced by Kruskal [19]. If (X_1, Y_1) and (X_2, Y_2) are pairs of continuous random variables, then the concordance function of random vectors (X_1, Y_1) and (X_2, Y_2) depends only on the corresponding copulas C_1 and C_2 and is given by (see [23, Theorem 5.1.1])

$$\mathcal{Q} = \mathcal{Q}(C_1, C_2) = 4 \int_{\mathbb{I}^2} C_2(u, v) dC_1(u, v) - 1. \tag{2}$$

It turns out that the concordance function is symmetric in its arguments, i.e., $\mathcal{Q}(C_1, C_2) = \mathcal{Q}(C_2, C_1)$, and has several other useful properties, see [23, Corollary 5.1.2] and [15, Sect. 3].

The most important concordance measures include Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta. Here, we only define Kendall's tau and Gini's gamma and refer the reader to [23, Sect. 5] for the definition of the other two. With the concordance function at hand, *Kendall's tau* can be defined by

$$\tau(C) = \mathcal{Q}(C, C) = 4 \int_{\mathbb{I}^2} C(u, v) dC(u, v) - 1, \tag{3}$$

and *Gini's gamma* by

$$\gamma(C) = \mathcal{Q}(C, M) + \mathcal{Q}(C, W) = 4 \int_0^1 C(u, u) du + 4 \int_0^1 C(u, 1 - u) du - 2. \tag{4}$$

In [20], Liebscher considered *weak concordance measures*, which are slightly more general mappings than concordance measures (the formal definition can be found in Liebscher's paper). The most important example of a weak concordance measure is *Spearman's footrule* defined by

$$\phi(C) = \frac{1}{2} (3\mathcal{Q}(C, M) - 1) = 6 \int_0^1 C(u, u) du - 2. \tag{5}$$

Spearman's footrule is not a true concordance measure since its range is $[-\frac{1}{2}, 1]$. There is an abundance of information in the literature on all three (weak) concordance measures defined above, including discussions on their statistical meaning. Kendall's tau was investigated in [11,13,14,31], Gini's gamma in [2,12,22], and Spearman's footrule in [4,12,28,30].

Connections between different (weak) concordance measures were investigated in [9,16–18,27]. Here we only give the results which include Kendall's

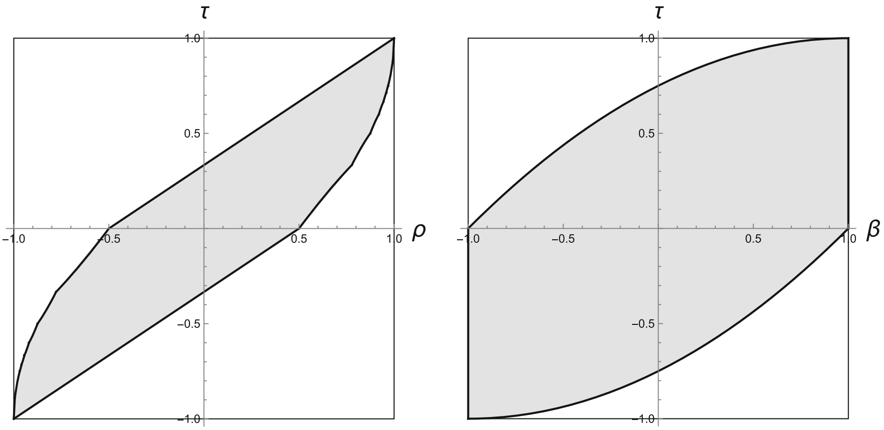

Figure 1. The exact regions determined by Spearman's rho and Kendall's tau (left) and by Blomqvist's beta and Kendall's tau (right)

tau for the sake of completeness. For any copula $C \in \mathcal{C}$ we have

$$\frac{1}{4}(1+\beta(C))^2 - 1 \leq \tau(C) \leq 1 - \frac{1}{4}(1-\beta(C))^2$$

and the bounds are attained (see [23]). The bounds for Kendall's tau with respect to Spearman's rho are given by

$$-\Psi(-\rho(C)) \leq \tau(C) \leq \Psi(\rho(C)),$$

where $\Psi : [-1,1] \to [-1,1]$ is the inverse of $\Phi : [-1,1] \to [-1,1]$,

$$\Phi(x) = \begin{cases} -1; & \text{if } x = -1, \\ \Phi_n(x); & \text{if } x \in [-1+\frac{2}{n}, -1+\frac{2}{n-1}] \text{ for some } n \geq 2, \end{cases}$$

and for every $n \in \mathbb{N}, n \geqslant 2$, $\Phi_n : [-1+\frac{2}{n}, 1] \to [-1,1]$ is a function

$$\Phi_n(x) = -1 - \frac{4}{n^2} + \frac{3}{n} + \frac{3x}{n} - \frac{n-2}{\sqrt{2}n^2\sqrt{n-1}}(n-2+nx)^{3/2}.$$

The bounds are attained, see [27].

Figure 1 depicts the exact regions determined by Spearman's rho and Kendall's tau and by Blomqvist's beta and Kendall's tau.

3. The Exact Region Determined by τ and ϕ

In this section we will describe the exact region determined by Kendall's tau and Spearman's footrule.

Proposition 1. *Let $h : \mathbb{I} \to \mathbb{I}$ be a bijective measure preserving function. Let $C \in \mathcal{C}$ be a copula defined by (1) with the mass concentrated on the graph of h, i.e. $\mu_C(\{(u,h(u)), u \in \mathbb{I}\}) = 1$. Then*

$$\tau(C) \geq \frac{4}{3}\phi(C) - \frac{1}{3}.$$

Proof. We estimate the integral

$$\begin{aligned} I &= \iint_{\mathbb{I}^2} C(u,v)dC(u,v) = \int_0^1 C(u,h(u))du \\ &\geq \int_0^1 C(\min\{u,h(u)\},\min\{u,h(u)\})du \\ &= \int_{\{u\in\mathbb{I};u<h(u)\}} C(u,u)du + \int_{\{u\in\mathbb{I};u\geq h(u)\}} C(h(u),h(u))du \\ &= \int_0^1 C(u,u)du + \int_0^1 C(h(u),h(u))du \\ &\quad - \left(\int_{\{u\in\mathbb{I};u\geq h(u)\}} C(u,u)du + \int_{\{u\in\mathbb{I};u<h(u)\}} C(h(u),h(u))du\right) \\ &\quad = \int_0^1 C(u,u)du + \int_0^1 C(h(u),h(u))du \\ &\quad\quad - \int_0^1 C(\max\{u,h(u)\},\max\{u,h(u)\})du. \end{aligned} \tag{6}$$

We introduce a new variable $t = h(u)$ into the second to last integral in (6) to get

$$\int_0^1 C(h(u),h(u))du = \int_0^1 C(t,t)dh^{-1}(t) = \int_0^1 C(t,t)dt,$$

since h is bijective and measure preserving. By looking at the mass of the copula C inside the rectangle from the origin to the point $(\max\{u,h(u)\}, \max\{u,h(u)\})$, we obtain

$$\begin{aligned} &C(\max\{u,h(u)\},\max\{u,h(u)\}) \\ &\quad = \lambda(\{t\in\mathbb{I}; t\leqslant \max\{u,h(u)\}, h(t)\leqslant \max\{u,h(u)\}\}) \\ &\quad = \int_0^1 f(t,u)dt, \end{aligned}$$

where

$$f(t,u) = \begin{cases} 1; & \max\{t,h(t)\} \leqslant \max\{u,h(u)\}, \\ 0; & \text{otherwise.} \end{cases}$$

Notice that $f(t,u) + f(u,t) = 1$ almost everywhere on $\mathbb{I}^2$, since

$$\lambda^2(\{(t,u)\in\mathbb{I}^2; \max\{t,h(t)\} = \max\{u,h(u)\}\}) = 0,$$

where λ^2 is the Lebesgue measure on $\mathbb{I}^2$. It follows that the last integral in (6) equals

$$\int_0^1 C(\max\{u,h(u)\},\max\{u,h(u)\})du = \iint_{\mathbb{I}^2} f(t,u)dtdu = \frac{1}{2}.$$

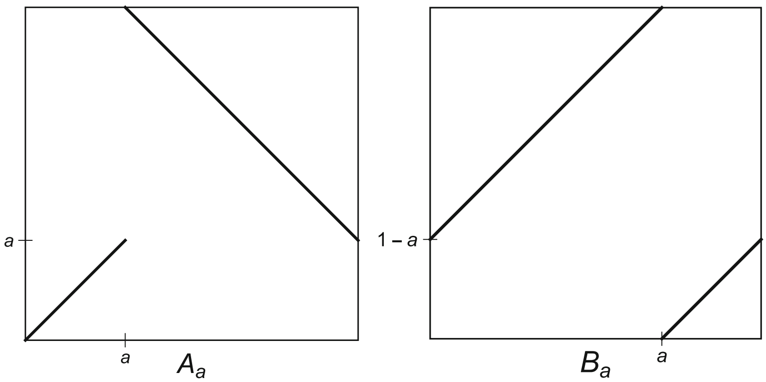

Figure 2. The scatterplots of copulas A_a and B_a from Examples 2 and 3

We finally obtain that

$$I = \iint_{\mathbb{I}^2} C(u,v)dC(u,v) \geq 2\int_0^1 C(u,u)du - \frac{1}{2}$$

and

$$\tau(C) = 4I - 1 \geq 8\int_0^1 C(u,u)du - 3 = 8 \cdot \frac{\phi(C)+2}{6} - 3 = \frac{4}{3}\phi(C) - \frac{1}{3}.$$

□

The following example gives copulas for which the bound of Proposition 1 is attained.

Example 2. Let $a \in [0,1]$ and let A_a be a shuffle of min

$$A_a = M(2,(a),(1,2),(1,-1)).$$

The scatterplot of copula A_a is shown in Fig. 2 (left). Notice that $A_0 = W$ and $A_1 = M$. We have

$$\delta_{A_a}(u) = \begin{cases} u; & 0 \leqslant u \leqslant a, \\ a; & a \leqslant u \leqslant \frac{a+1}{2}, \\ 2u-1; & \frac{a+1}{2} \leqslant u \leqslant 1, \end{cases}$$

and

$$h_{A_a}(u) = \begin{cases} u; & 0 \leqslant u < a, \\ a+1-u; & a \leqslant u \leqslant 1. \end{cases}$$

It follows that

$$\phi(A_a) = -\tfrac{3}{2}a^2 + 3a - \tfrac{1}{2} \quad \text{and} \quad \tau(A_a) = -2a^2 + 4a - 1,$$

so that $\tau(A_a) = \frac{4}{3}\phi(A_a) - \frac{1}{3}$ and the point $(\phi(A_a), \tau(A_a))$ lies on the line segment AB, where $A(-\frac{1}{2}, -1)$ and $B(1,1)$. Every point on this line segment is attained by some A_a with $a \in [0,1]$.

In next example, we give copulas that will correspond to the points on the lower bound of the region determined by Kendall's tau and Spearman's footrule.

Example 3. Let $a \in [\frac{1}{2}, 1]$ and let B_a be a shuffle of min

$$B_a = M(2, (a), (2, 1), (1, 1)).$$

The scatterplot of copula B_a is shown in Fig. 2 (right). Notice that $B_1 = M$. We have

$$\delta_{B_a}(u) = \begin{cases} 0; & 0 \leqslant u \leqslant 1 - a, \\ u + a - 1; & 1 - a \leqslant u \leqslant a, \\ 2u - 1; & a \leqslant u \leqslant 1, \end{cases}$$

and

$$h_{B_a}(u) = \begin{cases} u + 1 - a; & 0 \leqslant u < a, \\ u - a; & a \leqslant u < 1, \\ 1; & u = 1. \end{cases}$$

It follows that

$$\phi(B_a) = 6a^2 - 6a + 1 \quad \text{and} \quad \tau(B_a) = 4a^2 - 4a + 1,$$

so that $\tau(B_a) = \frac{2}{3}\phi(B_a) + \frac{1}{3}$ and the point $(\phi(B_a), \tau(B_a))$ lies on the line segment CB, where $C(-\frac{1}{2}, 0)$ and $B(1, 1)$. Every point on this line segment is attained by some B_a with $a \in [\frac{1}{2}, 1]$.

We can now describe the exact region determined by τ and ϕ.

Theorem 4. *The exact region determined by Kendall's tau and Spearman's footrule of all points* $\{(\phi(C), \tau(C)) \in [-\frac{1}{2}, 1] \times [-1, 1];\ C \in \mathcal{C}\}$ *is a triangular region given by*

$$\frac{4}{3}\phi(C) - \frac{1}{3} \leqslant \tau(C) \leqslant \frac{2}{3}\phi(C) + \frac{1}{3}.$$

Proof. We will first prove that the lower bound from Proposition 1 holds for any copula C.

Let $\varepsilon > 0$. Since τ is a concordance measure, there exists $\delta > 0$ such that for every copula C' with $\sup_{(u,v)\in\mathbb{I}^2} |C(u, v) - C'(u, v)| < \delta$ we have $|\tau(C) - \tau(C')| < \varepsilon$. By [21] there exists a shuffle of min C' with $\sup_{(u,v)\in\mathbb{I}^2} |C(u, v) - C'(u, v)| < \min\{\varepsilon, \delta\}$. Hence $|\phi(C) - \phi(C')| < 6\varepsilon$ by the definition of ϕ and

$$\tau(C) > \tau(C') - \varepsilon \geq \frac{4}{3}\phi(C') + \frac{1}{3} - \varepsilon > \frac{4}{3}\phi(C) + \frac{1}{3} - 9\varepsilon$$

by Proposition 1. By sending ε to 0, we obtain the desired lower bound.

Next we prove the upper bound. For any copula C we can estimate

$$\tau(C) = 4\iint_{\mathbb{I}^2} C(u, v)dC(u, v) - 1 \leqslant 4\iint_{\mathbb{I}^2} M(u, v)dC(u, v) - 1.$$

Since the concordance function is symmetric we obtain

$$\tau(C) \leqslant 4\iint_{\mathbb{I}^2} C(u, v)dM(u, v) - 1 = 4\int_0^1 C(u, u)du - 1$$

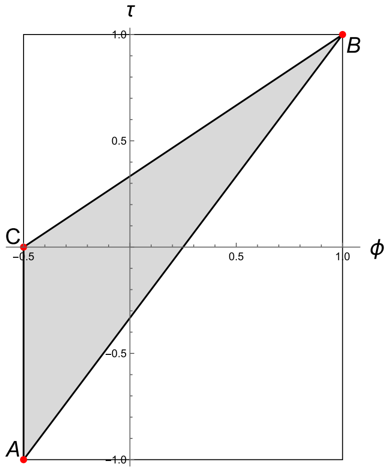

Figure 3. The exact region determined by Kendall's tau and Spearman's footrule

$$= 4 \cdot \frac{\phi(C) + 2}{6} - 1 = \frac{2}{3}\phi(C) + \frac{1}{3}$$

as claimed.

So, any copula C satisfies the given inequalities and all the points on the upper and lower boundary of the region are attained by copulas from Examples 2 and 3. It remains to be shown that all the points in between are attained as well.

Fix $p \in [-\frac{1}{2}, 1]$. By the above there exist copulas C_ℓ and C_u such that $\phi(C_\ell) = \phi(C_u) = p$, $\tau(C_\ell) = \frac{4}{3}p - \frac{1}{3}$, and $\tau(C_u) = \frac{2}{3}p + \frac{1}{3}$. Define a map $v\colon [0,1] \to [-\frac{1}{2}, 1] \times [-1, 1]$ by $v(t) = (\phi(C_t), \tau(C_t))$, where $C_t = tC_u + (1-t)C_\ell$. Then $v(0) = (p, \frac{4}{3}p - \frac{1}{3})$ and $v(1) = (p, \frac{2}{3}p + \frac{1}{3})$. Furthermore, $\phi(C_t) = t\phi(C_u) + (1-t)\phi(C_\ell) = p$ for all $t \in [0,1]$, so the image of v is contained in the line segment connecting the points $v(0)$ and $v(1)$. Since ϕ and τ are weak concordance measures, the map v is also continuous, hence, its image is connected. This implies that the image of v is precisely the line segment connecting the points $v(0)$ and $v(1)$. Since p was arbitrary, this concludes the proof. □

The exact region determined by Kendall's tau and Spearman's footrule is a triangle with vertices $A(-\frac{1}{2}, -1)$, $B(1,1)$, and $C(-\frac{1}{2}, 0)$, shown in Fig. 3.

4. The Exact Region Determined by τ and γ

In this section, we will describe the exact region determined by Kendall's tau and Gini's gamma. We first provide examples of copulas that will correspond to the points on the boundary of the region.

Example 5. Let $a \in [0, \frac{1}{2}]$ and define shuffles of min

$$C_a = M(2, (a), (1,2), (-1,-1)),$$

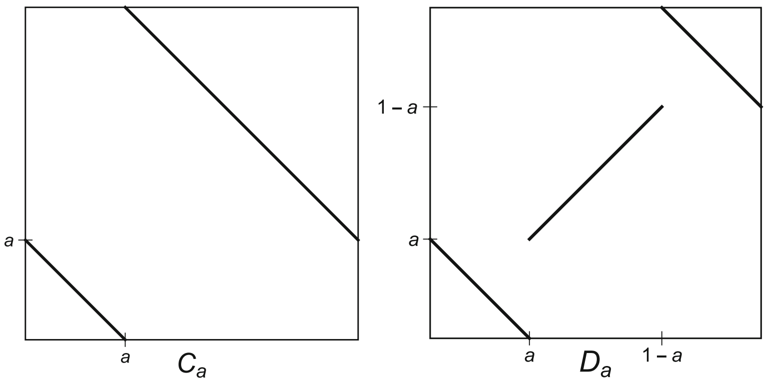

Figure 4. The scatterplots of copulas C_a and D_a from Example 5

$$D_a = M(3, (a, 1-a), (1, 2, 3), (-1, 1, -1)).$$

The scatterplot of copulas C_a and D_a are shown in Fig. 4. We have

$$C_a(u,u) = \begin{cases} 0; & 0 \leqslant u \leqslant \frac{a}{2}, \\ 2u-a; & \frac{a}{2} \leqslant u \leqslant a, \\ a; & a \leqslant u \leqslant \frac{a+1}{2}, \\ 2u-1; & \frac{a+1}{2} \leqslant u \leqslant 1, \end{cases}$$

$$C_a(u,1-u) = \begin{cases} u; & 0 \leqslant u \leqslant a, \\ a; & a \leqslant u \leqslant 1-a, \\ 1-u; & 1-a \leqslant u \leqslant 1, \end{cases}$$

$$D_a(u,u) = \begin{cases} 0; & 0 \leqslant u \leqslant \frac{a}{2}, \\ 2u-a; & \frac{a}{2} \leqslant u \leqslant a, \\ u; & a \leqslant u \leqslant 1-a, \\ 1-a; & 1-a \leqslant u \leqslant 1-\frac{a}{2}, \\ 2u-1; & 1-\frac{a}{2} \leqslant u \leqslant 1, \end{cases}$$

$$D_a(u,1-u) = \begin{cases} u; & 0 \leqslant u \leqslant \frac{1}{2}, \\ 1-u; & \frac{1}{2} \leqslant u \leqslant 1, \end{cases}$$

$$h_{C_a}(u) = \begin{cases} a-u; & 0 \leqslant u \leqslant a, \\ a+1-u; & a < u < 1, \\ 1; & u = 1, \end{cases}$$

and

$$h_{D_a}(u) = \begin{cases} a-u; & 0 \leqslant u \leqslant a, \\ u; & a < u < 1-a, \\ 2-a-u; & 1-a \leqslant u \leqslant 1. \end{cases}$$

It follows that

$$\gamma(C_a) = -6a^2 + 6a - 1 \quad \text{and} \quad \tau(C_a) = -4a^2 + 4a - 1,$$
$$\gamma(D_a) = -2a^2 + 1 \quad \text{and} \quad \tau(D_a) = -4a^2 + 1,$$

so that $\tau(C_a) = \frac{2}{3}\gamma(C_a) - \frac{1}{3}$, $\gamma(C_a) \in [-1, \frac{1}{2}]$, and $\tau(D_a) = 2\gamma(D_a) - 1$, $\gamma(D_a) \in [\frac{1}{2}, 1]$. The point $(\gamma(C_a), \tau(C_a))$ lies on the line segment AB, and the point $(\gamma(D_a), \tau(D_a))$ lies on the line segment BC, where $A(-1, -1)$, $B(\frac{1}{2}, 0)$ and $C(1, 1)$. Every point on these line segments is attained by some C_a or D_a with $a \in [0, \frac{1}{2}]$.

We can now describe the exact region determined by τ and γ.

Theorem 6. *The exact region determined by Kendall's tau and Gini's gamma of all points* $\{(\gamma(C), \tau(C)) \in [-1, 1] \times [-1, 1];\ C \in \mathcal{C}\}$ *is given by*

$$\max\left\{\frac{2}{3}\gamma(C) - \frac{1}{3}, 2\gamma(C) - 1\right\} \leqslant \tau(C) \leqslant \min\left\{\frac{2}{3}\gamma(C) + \frac{1}{3}, 2\gamma(C) + 1\right\}.$$

Proof. For any copula C, we have

$$\int_0^1 C^{\sigma_2}(u, u)du = \frac{1}{2} - \int_0^1 C(u, 1-u)du,$$

hence

$$\begin{aligned}\gamma(C) &= 4\int_0^1 C(u,u)du + 4\left(\frac{1}{2} - \int_0^1 C^{\sigma_2}(u,u)du\right) - 2 \\ &= 4 \cdot \frac{\phi(C) + 2}{6} - 4 \cdot \frac{\phi(C^{\sigma_2}) + 2}{6} = \\ &= \frac{2}{3}(\phi(C) - \phi(C^{\sigma_2})).\end{aligned}$$

Since τ is a concordance measure, we infer

$$\tau(C) = \frac{2}{3}\tau(C) + \frac{1}{3}\tau(C) = \frac{2}{3}\tau(C) - \frac{1}{3}\tau(C^{\sigma_2}).$$

Using Theorem 4 to estimate both terms on the right-hand side, we obtain

$$\begin{aligned}\tau(C) &\leqslant \frac{2}{3}\left(\frac{2}{3}\phi(C) + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{4}{3}\phi(C^{\sigma_2}) - \frac{1}{3}\right) \\ &= \frac{4}{9}(\phi(C) - \phi(C^{\sigma_2})) + \frac{1}{3} = \frac{2}{3}\gamma(C) + \frac{1}{3}.\end{aligned} \tag{7}$$

On the other hand, we may use Theorem 4 and the inequality $\phi(C) \geq -\frac{1}{2}$ to estimate

$$\begin{aligned}\tau(C) = -\tau(C^{\sigma_2}) &\leqslant -\frac{4}{3}\phi(C^{\sigma_2}) + \frac{1}{3} = \frac{4}{3} \cdot \left(-\frac{1}{2}\right) - \frac{4}{3}\phi(C^{\sigma_2}) + 1 \\ &\leqslant \frac{4}{3}(\phi(C) - \phi(C^{\sigma_2})) + 1 = 2\gamma(C) + 1.\end{aligned} \tag{8}$$

Inequalities (7) and (8) imply

$$\tau(C) \leqslant \min\left\{\frac{2}{3}\gamma(C) + \frac{1}{3}, 2\gamma(C) + 1\right\}, \tag{9}$$

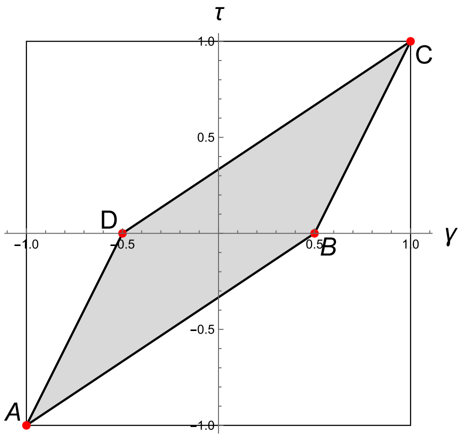

Figure 5. The exact region determined by Kendall's tau and Gini's gamma

which proves the upper bound for $\tau(C)$. We obtain the lower bound if we apply inequality (9) to C^{σ_2}

$$\begin{aligned}\tau(C) = -\tau(C^{\sigma_2}) &\geq -\min\left\{\frac{2}{3}\gamma(C^{\sigma_2}) + \frac{1}{3}, 2\gamma(C^{\sigma_2}) + 1\right\} \\ &= -\min\left\{-\frac{2}{3}\gamma(C) + \frac{1}{3}, -2\gamma(C) + 1\right\} \\ &= \max\left\{\frac{2}{3}\gamma(C) - \frac{1}{3}, 2\gamma(C) - 1\right\}.\end{aligned}$$

Example 5 implies that any point on the lower boundary is attained by either A_a or B_a with $a \in [0, \frac{1}{2}]$. Furthermore, by an analogous argument as above, any point on the upper boundary is attained by either $A_a^{\sigma_2}$ or $B_a^{\sigma_2}$ with $a \in [0, \frac{1}{2}]$. The proof that any point in between is also attained is similar as the corresponding part of the proof of Theorem 4. □

The exact region determined by Kendall's tau and Gini's gamma is a parallelogram with vertices $A(-1, -1)$, $B(\frac{1}{2}, 0)$, $C(1, 1)$, and $D(-\frac{1}{2}, 0)$, shown in Fig. 5.

5. Concordance Similarity Measure Between Kendall's Tau and Other Concordance Measures

In paper [17] the authors introduce the (κ_1, κ_2)-*similarity measure* between (weak) concordance measures κ_1 and κ_2 as

$$\kappa sm(\kappa_1, \kappa_2) = 1 - \frac{A(\kappa_1, \kappa_2)}{(1 - \kappa_1(W))(1 - \kappa_2(W))},$$

where $A(\kappa_1, \kappa_2)$ is the area of the exact region determined by κ_1 and κ_2. They also compute concordance similarity measure between the pairs Kendall's

Table 1. Concordance similarity measure between Kendall's tau and other (weak) concordance measures

κ	ρ	γ	ϕ	β
$\kappa sm(\tau,\kappa)$	0.7114	0.7500	0.7500	0.3333

tau and Spearman's rho $\kappa sm(\tau,\rho) = 0.7114$ and between Kendall's tau and Blomqvist's beta $\kappa sm(\tau,\beta) = \frac{1}{3} = 0.3333$.

We now compute concordance similarity measure between the pairs Kendall's tau and Spearman's footrule and between Kendall's tau and Gini's gamma. We have

$$A(\tau,\phi) = \text{area } \Delta A(-\tfrac{1}{2},-1)B(1,1)C(-\tfrac{1}{2},0) = \tfrac{3}{4},$$

so $\kappa sm(\tau,\phi) = 1 - \frac{1}{3}A(\tau,\phi) = \frac{3}{4} = 0.75$, and

$$A(\tau,\gamma) = \text{area } A(-1,-1)B(\tfrac{1}{2},0)C(1,1)D(-\tfrac{1}{2},0) = 1,$$

so $\kappa sm(\tau,\gamma) = 1 - \frac{1}{4}A(\tau,\gamma) = \frac{3}{4} = 0.75$. Table 1 gives all the values

Thus, knowing the value of Blomqvist's beta gives us in average very little information about possible values of Kendall's tau. On the other hand, knowing the value of Spearman's rho, Spearman's footrule, or Gini's gamma gives us more information about possible values of Kendall's tau. The amount of information is in average almost the same for all three measures.

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