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On metric dimensions of hypercubes

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Abstract

In this note we show two unexpected results concerning the metric, the edge metric and the mixed metric dimensions of hypercube graphs. First, we show that the metric and the edge metric dimensions of Q_d differ by at most one for every integer d. In particular, if d is odd, then the metric and the edge metric dimensions of Q_d are equal. Second, we prove that the metric and the mixed metric dimensions of the hypercube Q_d are equal for every $d \geq 3$. We conclude the paper by conjecturing that all these three types of metric dimensions of Q_d are equal when d is large enough.

Keywords: Edge metric dimension, mixed metric dimension, metric dimension, hypercubes.

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1 Introduction

The metric dimension of connected graphs was introduced about 50 years ago in [6, 22], in connection with modeling navigation systems in networks, although this notion was already known by then for general metric spaces from [1]. Given a connected graph G and two vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between these two vertices is the length of a shortest path connecting v and u. The vertices u, v are *distinguished* or *resolved* by a vertex $x \in V(G)$ if $d_G(u, x) \neq d_G(v, x)$. A given set of vertices S is a *metric generator* for the graph G, if every two vertices of G are distinguished by a vertex of S. The cardinality of the smallest possible metric generator for G is the *metric dimension* of G, which is denoted by $\dim(G)$. The terminology of metric generators was introduced in [11], and the previous two works referred to such sets as *resolving sets* and *locating sets*, respectively. We herewith follow the terminology of [11]. A metric generator for G of cardinality $\dim(G)$ is called a *metric basis*. Although the classical metric dimension is an old topic in graph theory, there are still several open problems that remain unsolved. Recent investigations on this concern are [3, 4, 8, 16]. More results and open questions concerning metric dimension and related variants can be found in the recent surveys [15] and [23].

In order to uniquely identify the edges of a graph, by using vertices, the edge metric dimension of connected graphs was introduced in [10] as follows. Let G be a connected graph and let uv be an edge of G such that $u, v \in V(G)$. The *distance* between a vertex $x \in V(G)$ and the edge uv is defined as, $d_G(uv, x) = \min\{d_G(u, x), d_G(v, x)\}$. It is said that two distinct edges $e_1, e_2 \in E(G)$ are *distinguished* or *resolved* by a vertex $v \in V(G)$ if $d_G(e_1, v) \neq d_G(e_2, v)$. A set $S \subset V(G)$ is called an *edge metric generator* for G if and only if for every pair of edges $e_1, e_2 \in E(G)$, there exists an element of S which distinguishes the edges. The cardinality of a smallest possible edge metric generator of a graph is known as the *edge metric dimension*, and is denoted by $\mathrm{edim}(G)$. After the seminal paper [10], a significant number of researches on such parameter have appeared. Among them, some of the most recent ones are [3, 12, 13, 14, 19]. See also the survey [15] for some other contributions. It is natural to consider comparing the metric and edge metric dimensions of graphs. However, as first proved in [10], and continued in [13, 14], both parameters are not in general comparable since there exist connected graphs G for which $\mathrm{edim}(G) < \dim(G)$, $\mathrm{edim}(G) = \dim(G)$ or $\mathrm{edim}(G) > \dim(G)$.

In order to combine the unique identification of vertices and of edges in only one scheme, the mixed metric dimension of graphs was introduced in [9]. For a connected graph G, a vertex $w \in V(G)$ and an edge $uv \in E(G)$ are *distinguished* or *resolved* by a vertex $x \in V(G)$ if $d_G(w, x) \neq d_G(uv, x)$. A set $S \subset V(G)$ is called a *mixed metric generator* for G if and only if for every pair of elements of the graphs (vertices or edges) $e, f \in E(G) \cup V(G)$, there exists a vertex of S which distinguishes them. The cardinality of a smallest possible mixed metric generator of G is known as the *mixed metric dimension* of G, and is denoted by $\mathrm{mdim}(G)$. Some recent studies on mixed metric dimension of graphs are [20, 21]. Clearly, every mixed metric generator must be a metric generator as well as an edge metric generator, and so, $\mathrm{mdim}(G) \geq \max\{\dim(G), \mathrm{edim}(G)\}$, for any connected graph G. Moreover, since $\dim(G)$ and $\mathrm{edim}(G)$ are in general not comparable (see [13, 14] for more information on this fact), several situations relating these three parameters can be found. That is, there are graphs G with $\mathrm{mdim}(G) \gg \max\{\dim(G), \mathrm{edim}(G)\}$, $\mathrm{mdim}(G) = \dim(G) \gg \mathrm{edim}(G)$, $\mathrm{mdim}(G) = \mathrm{edim}(G) \gg \dim(G)$, or $\mathrm{mdim}(G) = \dim(G) = \mathrm{edim}(G)$.

The metric dimension of hypercube graphs has attracted the attention of several researchers from long ago. For instance, the work of Lindström [17] is probably one of the oldest ones, and for some recent ones we suggest the works [7, 18, 24]. Surprisingly, for other related invariants there has been comparatively little research on hypercube graphs, although one can find some interesting recent results on this topic such as those that appeared in [5, 7]. It is our goal to present some results on the close connections that exist among the metric, the edge metric and the mixed metric dimensions of hypercube graphs.

The d-dimensional hypercube, denoted by Q_d, with $d \in \mathbb{N}$, is a graph whose vertices are represented by d-dimensional binary vectors, *i.e.*, $u = (u_1, \ldots, u_2) \in V(Q_d)$ where $u_i \in \{0, 1\}$ for every $i \in \{1, \ldots, d\}$. Two vertices are adjacent in Q_d if their vectors differ in exactly one coordinate. Hypercubes can be also seen as the d times Cartesian product of the graph P_2, that is, $Q_d \cong P_2 \Box P_2 \Box \cdots \Box P_2$, or recursively, $Q_d \cong Q_{d-1} \Box P_2$. The distance between two vertices in Q_d represents the total number of coordinates in which their vectors differ. The hypercube Q_d is bipartite, and has 2^d vertices and $d \cdot 2^{d-1}$ edges. We remark that Q_2 is the cycle C_4 and that Q_4 can be also seen as the torus graphs $C_4 \Box C_4$.

2 Results

Our first contribution is to relate the metric generators with the edge metric generators of bipartite graphs.

Lemma 2.1. *Let G be a connected bipartite graph. Then, every metric generator for G is also an edge metric generator.*

Proof. Let S be an arbitrary metric generator for G. We will show that S is an edge metric generator as well.

Let $e_1 = x_1 y_1$ and $e_2 = x_2 y_2$ be two arbitrary distinct edges of G. Since G is bipartite and e_1, e_2 are distinct, one can w.l.o.g. assume that x_1, x_2 (with $x_1 \neq x_2$) belong to one of the bipartition sets and y_1, y_2 to the other one. Hence the distance between $u = x_1$ and $v = x_2$ is even.

Now, as u and v are distinct, there must be a vertex $s \in S$ that distinguishes them, *i.e.* $d(s, u) \neq d(s, v)$. We may assume that $d(s, u) + 1 \leq d(s, v)$. Since u and v are on even distance, it follows that distances $d(s, u)$ and $d(s, v)$ are of same parity, otherwise we encounter a closed walk of odd length in G, which is not possible in a bipartite graph. This implies $d(s, u) + 2 \leq d(s, v)$, and now we easily derive

$$d(e_1, s) \leq d(u, s) < d(v, s) - 1 \leq d(e_2, s).$$

In particular, $d(e_1, s) < d(e_2, s)$ implies that e_1, e_2 are distinguished by $s \in S$. Since the choice of these two edges was arbitrary, we conclude that S is also an edge metric generator. $\Box$

It is then natural to think in the opposite direction with regard to the result above. In particular, we ask if an edge metric generator for a bipartite graph is also a metric generator. In contrast with the result above, achieving this seems to be a challenging task. However, we have at least managed to show a weaker result for an infinite family of bipartite graphs, namely the hypercubes Q_d. That is, when d is odd, every edge metric generator for Q_d is indeed a metric generator, and when d is even, every edge metric generator is “almost” a metric generator.

From now on we denote by α_i the vector of dimension d whose i^{th}-coordinate is 1, and the remaining coordinates are 0. Also, by "$\oplus$" we represent the standard (binary) XOR operation. Notice that, for any vertex $u \in V(Q_d)$, $u \oplus \alpha_i$ means switching the i^{th}-coordinate of u from 0 to 1, or vice versa.

Lemma 2.2. *Let S be an edge metric generator of Q_d. If there exist two distinct vertices u and v not distinguished by S, then they must be antipodal in Q_d and d is even. If d is odd, then S is also a metric generator of Q_d.*

Proof. Suppose that $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$ are not antipodal. Then, $u_i = v_i$ for some i. Let Q^0_{d-1} and Q^1_{d-1} be the half-cubes regarding the dimension i. Notice that u and v belongs to a same half-cube, say Q^0_{d-1}. Let e_u and e_v be the edges corresponding to the component i (in Q_d) incident with u and v, respectively. In other words, as $u\oplus\alpha_i$ and $v\oplus\alpha_i$ are the neighbours of u and v in Q^1_{d-1}, we have $e_u = (u, u\oplus\alpha_i)$ and $e_v = (v, v \oplus \alpha_i)$. We claim that the edges e_u and e_v are not distinguished by S. To see this, observe that if $s \in S$ belongs to Q^0_{d-1}, then

$$d(s, e_u) = d(s, u) = d(s, v) = d(s, e_v).$$

Also, if $s \in S$ belongs to Q^1_{d-1}, then

$$d(s, e_u) = d(s, u \oplus \alpha_i) = d(s, v \oplus \alpha_i) = d(s, e_v).$$

We hence derive that the edges e_u and e_v are not distinguished by S, which is a contradiction.

Based on the above arguments we conclude that u and v are antipodal, *i.e.* $d(u, v) = d$. Hence, every vertex x of S satisfies $d(u, x) + d(x, v) = d$. As every vertex $s \in S$ must be equally distanced from u and v, we conclude that $d(u, s) = d(s, v) = d/2$, and consequently, d must be even. This establishes the main claim.

Finally, observe that if d is odd, then no vertex is equally distanced from two antipodal vertices of Q_d, and therefore, S is a metric generator of Q_d. □

Next lemma will ensure that enlarging an edge metric generator of Q_d with one chosen element, we get a metric generator of Q_d.

Lemma 2.3. *Let S be an edge metric generator of Q_d and let s be an arbitrary element of S. Then, $S \cup \{s \oplus \alpha_1\}$ is a metric generator of Q_d.*

Proof. If S is a metric generator of Q_d, then $S\cup\{s\oplus\alpha_1\}$ is so too, and we are done. Thus, we assume that S is not a metric generator of Q_d. Then, by Lemma 2.2, d is even and there must exist antipodal vertices u and v such that $d(u, x) = d(v, x) = d/2$ for every $x \in S$. This will not be the case for $s \oplus \alpha_1$, as $|d(u, s \oplus \alpha_1) - d(v, s \oplus \alpha_1)| = 2$. Therefore, we conclude that $S \cup \{s \oplus \alpha_1\}$ is a metric generator of Q_d. □

Since Q_d is a bipartite graph, the two previous lemmas give us the following consequence.

Theorem 2.4. *Let $d \geq 1$. Then*

$$\operatorname{edim}(Q_d) \leq \dim(Q_d) \leq \operatorname{edim}(Q_d) + 1,$$

with the second inequality being tight only if d is even.

Proof. The lower bound holds by Lemma 2.1. The upper bound and its possible tightness (for more than one case) follows by Lemmas 2.2 and 2.3. □

Notice that the upper bound $\dim(Q_d) \leq \operatorname{edim}(Q_d) + 1$ is indeed tight for the case Q_4, since $4 = \dim(Q_4) = \operatorname{edim}(Q_4) + 1$, as proved in [10].

We now turn our attention to relating the metric dimension with the mixed metric dimension of hypercubes. To this end, we will need the following two results. We must remark that the first of next two lemmas already appeared in [18]. We include its proof for completeness.

Lemma 2.5. *If S is a metric generator (in particular, a metric basis) of Q_d and $s \in S$, then $(S \setminus \{s\}) \cup \{s'\}$ is also a metric generator (in particular, a metric basis) of Q_d, where $s' \in V(Q_d)$ is the antipodal vertex of s.*

Proof. If $s \in S$ distinguishes some pair of vertices x and y of Q_d, then s' distinguishes such pair as well, since $d(x, s') = d - d(x, s)$ and $d(y, s') = d - d(y, s)$. This also means that no metric basis of Q_d contains two antipodal vertices. Thus, if S is a metric generator (or a metric basis) of Q_d, then $S \setminus \{s\} \cup \{s'\}$ is a metric generator (or a metric basis) as well. □

Lemma 2.6. *If S is a metric generator of Q_d, then there is at most one index $i \in \{1, \ldots, d\}$ such that all the vertices from S have the same value at the i^{th} coordinate.*

Proof. Suppose that there exist two different indices i and j such that all vertices from S have the same value at the i^{th} and j^{th} coordinates. First, let us consider the case when there are zeroes at such coordinates. Other cases can be shown by using similar arguments. Now, let $x \in V(Q_d)$ be a vertex having zeroes at all coordinates, except at the i^{th}, and let y be a vertex having zeroes at all positions except at the j^{th}. Then, $d(x, s) = d(y, s)$ for any vertex $s \in S$, a contradiction. □

The mixed metric dimension of hypercubes Q_1 and Q_2 are 2 and 3, respectively. This can be derived from results for paths and cycles from [9]. This gives us that $\dim(Q_d) < \operatorname{mdim}(Q_d)$, for $d \in \{1, 2\}$. For all higher dimensions the mixed metric dimension is equal to the metric dimension as we next show.

Theorem 2.7. *Let $d \geq 3$. Then*

$$\dim(Q_d) = \operatorname{mdim}(Q_d).$$

Proof. First, $\{(1, 1, 1), (0, 1, 0), (0, 0, 1)\}$ and $\{(1, 1, 1, 1), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ are mixed metric bases for Q_3 and Q_4, respectively. Thus, the equality follows for these cases since $\dim(Q_3) = 3$ and $\dim(Q_4) = 4$. It remains to check the equality for $d \geq 5$.

Let S be a metric basis for Q_d with $d \geq 5$. By Lemma 2.1, S is an edge metric generator of Q_d. In this sense, in Q_d we only need to distinguish those pairs of elements, one of them being a vertex and the other one, an edge. For this, let u be an arbitrary vertex and let $e = xy$ be an arbitrary edge of Q_d.

Suppose first that u is not a vertex of e. As $d(u, x)$ and $d(u, y)$ are of different parity, we may assume that u and x are on even distance. Now, let s_i be a vertex from S that distinguishes u and x. Similarly, as in Lemma 2.1, notice that $d(s_i, u)$ and $d(s_i, x)$ are of the same parity, and as they are different, we have that $|d(s_i, u) - d(s_i, x)| \geq 2$. So, if $d(s_i, u) < d(s_i, x)$, then we derive

$$d(s_i, u) < d(s_i, u) + 1 \leq d(s_i, x) - 1 \leq d(s_i, e),$$

and if $d(s_i, x) < d(s_i, u)$, then we have

$$d(s_i, e) \leq d(s_i, x) < d(s_i, u).$$

Thus, in both cases e and u are distinguished by a vertex from S.

So all the pairs of elements (vertices and edges) considered in the upper part are distinguished by an arbitrary metric basis. To conclude the proof, we need to construct a metric basis of cardinality $|S|$ that will also distinguish incident vertices and edges.

Suppose now that u is an endpoint of e, say $u = x$. To distinguish u and e there needs to be a vertex $s \in S$ which is from the half-cube Q_{d-1} that contains vertex y and does not contain vertex x. To distinguish all such pairs there must be at least one vertex from the mixed metric generator in every half-cube Q_{d-1}. For any index $i \in \{1, \ldots, d\}$, there exists a vertex from a mixed metric generator having 0 on the i^{th} coordinate, and a vertex from a mixed metric generator having 1 on the i^{th} coordinate. In other words, a mixed metric basis does not have a column of zeroes or a column of ones at an arbitrary index i (if we arrange all vectors of the mixed metric basis as a matrix with such vectors as the rows of such matrix).

We have started from an arbitrary metric basis S. Since Q_d is a vertex transitive graph, we may assume that the vertex $s_1 = (0, 0, \ldots, 0)$ (all coordinates equal to 0) is in S. If S does not contain a column of zeroes, then S is also a mixed metric basis. Otherwise, by Lemma 2.6, there exists only one such column, say at index i_0. By Lemma 2.5, we know that we can replace any of the vertices from the set S with its antipodal vertex and the incurred set $S' = S \setminus \{s\} \cup \{s'\}$ is a metric basis too, since the column at index i_0 (all zeroes) ensures that no two vertices in S are antipodal to each other. Moreover, in view of Lemma 2.1, S is an edge metric generator as well.

There exist at least four different vertices $s_1 = (0, 0, \ldots, 0)$, s_2, s_3 and s_4 in the set S, since $\dim(Q_d) \geq 4$, for $d \geq 5$. We construct four sets S'_i in the following way:

$$\begin{aligned} S'_1 &= (S \setminus \{s_1\}) \cup \{s'_1\}, \quad & S'_2 &= (S \setminus \{s_2, s_3\}) \cup \{s'_2, s'_3\}, \\ S'_3 &= (S \setminus \{s_2\}) \cup \{s'_2\}, \quad & S'_4 &= (S \setminus \{s_1, s_3\}) \cup \{s'_1, s'_3\}, \end{aligned}$$

and consider the next situations:

(I): If S'_1 is not a mixed metric generator, then there is a column of ones in S'_1 at some index i_1.

(II): If S'_2 is not a mixed metric generator, then there is a column of zeroes in S'_2 at some index i_2.

(III): If S'_3 is not a mixed metric generator, then there is a column of zeroes in S'_3 at some index i_3.

(IV): If S'_4 is not a mixed metric generator, then there is a column of ones in S'_4 at some index i_4.

Observe that all these indices i_0, i_1, i_2, i_3, and i_4 are different. If none of the four sets S'_i defined above is a mixed metric generator, then the initial set S looks as follows.

	i_0	i_1	i_2	i_3	i_4	$\dots$
s_1 :	0	0	0	0	0	$\dots$
s_2 :	0	1	1	1	1	$\dots$
s_3 :	0	1	1	0	0	$\dots$
s_4 :	0	1	0	0	1	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$s_{\lvert S \rvert}$:	0	1	0	0	1	$\dots$

We now take a look at the columns i_1, i_2, i_3 and i_4. Let v_1 be a vertex having zeroes at all positions except at i_1 and i_3 and let v_2 be a vertex having zeroes at all positions except at i_2 and i_4. Then, $d(v_1, s) = d(v_2, s)$, for any vertex $s \in S$, a contradiction. Therefore, at least one of the sets S'_i has to be a mixed metric generator, and therefore, the equality $\operatorname{mdim}(Q_d) = \dim(Q_d)$ follows since any mixed metric basis is also a metric basis. $\square$

In view of the asymptotical result for the metric dimension of hypercubes from [2], Theorems 2.4 and 2.7 give us the following consequences.

Corollary 2.8. *Let $d \geq 3$. Then*

$$\dim(Q_d) - 1 \leq \operatorname{edim}(Q_d) \leq \dim(Q_d) = \operatorname{mdim}(Q_d).$$

Corollary 2.9. *Let $d \geq 2$. Then*

$$\operatorname{mdim}(Q_d) \sim \operatorname{edim}(Q_d) \sim \dim(Q_d) \sim \frac{2d}{\log_2 d}.$$

We conclude this short paper with the following conjecture.

Conjecture 2.10. *If d is large enough, then*

$$\operatorname{edim}(Q_d) = \dim(Q_d).$$

As the above conjecture does not hold for $d = 4$, d must be at least 5.

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