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The core of a vertex-transitive complementary prism of a lexicographic product

Marko Orel*

University of Primorska, FAMNIT, Glagoljaška 8, Koper, Slovenia and
University of Primorska, IAM, Muzejski trg 2, Koper, Slovenia and
IMFM, Jadranska 19, Ljubljana, Slovenia

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Abstract

The complementary prism of a graph Γ is the graph $\Gamma\bar{\Gamma}$, which is formed from the union of Γ and its complement $\bar{\Gamma}$ by adding an edge between each pair of identical vertices in Γ and $\bar{\Gamma}$. Vertex-transitive self-complementary graphs provide vertex-transitive complementary prisms. It was recently proved by the author that $\Gamma\bar{\Gamma}$ is a core, i.e. all its endomorphisms are automorphisms, whenever Γ is vertex-transitive, self-complementary, and either Γ is a core or its core is a complete graph. In this paper the same conclusion is obtained for some other classes of vertex-transitive self-complementary graphs that can be decomposed as a lexicographic product $\Gamma = \Gamma_1[\Gamma_2]$. In the process some new results about the homomorphisms of a lexicographic product are obtained.

Keywords: Graph homomorphism, core, complementary prism, self-complementary graph, vertex-transitive graph, lexicographic product.

Math. Subj. Class.: 05C60, 05C76

1 Introduction

Given a graph it is often difficult to decide if it is a core or not. In the case of some graphs with high degree of symmetry and nice combinatorial properties, the decision is equivalent to some of the longstanding open problems in finite geometry [4, 23]. A well known core is the Petersen graph, which has many generalizations. One such family is given by the Kneser graphs. Another is constructed from the invertible hermitian matrices over the field

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E-mail address: marko.orel@upr.si (Marko Orel)

with four elements [22]. Both of these two families consist of cores if we exclude Kneser graphs with trivial parameters (see [10, Theorem 7.9.1] and [22, Theorem 7]). Generalized Petersen graphs $G(n, k)$ that are cores were very recently studied in [8]. The complementary prism $\Gamma\bar{\Gamma}$, which was introduced in [14], is also a generalization of the Petersen graph. In fact, the Petersen graph is isomorphic to $C_5\overline{C_5}$, where C_5 is the 5-cycle. Recall that C_5 is strongly regular vertex-transitive self-complementary graph. In [25] it was shown that $\Gamma\bar{\Gamma}$ is a core whenever Γ is strongly regular and self-complementary (see also the arXiv version [24, Theorem 5.7]). The same conclusion was obtained in [26] (see also [24, Theorem 5.10]) for all vertex-transitive self-complementary graphs Γ, provided that Γ is core-complete, i.e. Γ is either a core or it has an endomorphism that maps Γ onto a maximum clique. In general, the existence of a vertex-transitive self-complementary graph Γ, such that $\Gamma\bar{\Gamma}$ is not a core, was stated as an open problem ([26], [24, Open Problem 5.12]). Here, it should be emphasized that despite the possible orders of vertex-transitive self-complementary graphs are fully determined [21], the graphs themselves are poorly understood. In particular, the first non-Cayley vertex-transitive self-complementary graph was constructed only in 2001 [19]. Moreover, graphs with high degree of symmetry or nice combinatorial properties are often core-complete [4, 11, 28]. In this paper the focus is on the only vertex-transitive self-complementary graphs, the author is aware of, which are not core-complete (see Remarks 4.2, 4.6 and Example 4.9). They are in the form of a lexicographic product $\Gamma = \Gamma_1[\Gamma_2]$ for specially chosen graphs Γ_1 and Γ_2. It turns out that also in these cases the complementary prism $\Gamma\bar{\Gamma}$ is a core (see Corollaries 4.3 and 4.7), which means that the problem stated in [26] remains open.

The rest of the paper is organized as follows. In Section 2 we recall the necessary definitions and auxiliary lemmas, together with a result from [25] (and [24]) that we rely on in the proofs. Several results about the homomorphisms of a lexicographic product are recalled and developed in Section 3. The main results are presented in Section 4.

2 Preliminaries

All graphs in this paper are finite and simple. The vertex set and the edge set of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A subset of pairwise adjacent vertices in $V(\Gamma)$ is a *clique*, while a set of pairwise nonadjacent vertices in $V(\Gamma)$ is an *independent set*. If a clique or an independent set has the largest possible order, then it is referred to as the *maximum clique* or *maximum independent set*, respectively. The corresponding orders are the *clique number* $\omega(\Gamma)$ and the *independence number* $\alpha(\Gamma)$ of Γ, respectively. In particular, $\alpha(\Gamma) = \omega(\bar{\Gamma})$, where $\bar{\Gamma}$ is the complement of Γ. If $\{u, v\} \in E(\Gamma)$, then we write $u \sim_\Gamma v$ or simply $u \sim v$ if it is clear from the context which graph is meant. The set $N_\Gamma(u) = \{v \in V(\Gamma) : u \sim v\}$ is the *neighborhood* of $u \in V(\Gamma)$, and the set $N_\Gamma[u] = N_\Gamma(u) \cup \{u\}$ is the *closed neighborhood* of $u \in V(\Gamma)$.

A *graph homomorphism* between graphs Γ_1, Γ_2 is a map $\varphi\colon V(\Gamma_1) \to V(\Gamma_2)$ such that $\varphi(N_{\Gamma_1}(u)) \subseteq N_{\Gamma_2}\big(\varphi(u)\big)$ for all $u \in V(\Gamma_1)$. In other words, $\varphi(u) \sim_{\Gamma_2} \varphi(v)$ whenever $u \sim_{\Gamma_1} v$. If in addition φ is bijective and $\varphi(u) \sim_{\Gamma_2} \varphi(v)$ if and only if $u \sim_{\Gamma_1} v$, for all $u, v \in V(\Gamma_1)$, then φ is a *graph isomorphism* and graphs Γ_1, Γ_2 are *isomorphic*, which we denote by $\Gamma_1 \cong \Gamma_2$. The sets of all graph homomorphisms/isomorphisms from Γ_1 to Γ_2 are denoted by $\mathrm{Hom}(\Gamma_1, \Gamma_2)$ and $\mathrm{Iso}(\Gamma_1, \Gamma_2)$, respectively. Similarly, we use $\mathrm{End}(\Gamma) = \mathrm{Hom}(\Gamma, \Gamma)$ and $\mathrm{Aut}(\Gamma) = \mathrm{Iso}(\Gamma, \Gamma)$ to denote the sets of all *graph endomorphisms* and *automorphisms*, respectively. The elements in the set $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{Iso}(\Gamma, \bar{\Gamma})$ are *antimorphisms* (or *complementing permutations*) of Γ. If $\overline{\mathrm{Aut}(\Gamma)}$ is nonempty, then Γ is a *self-complementary* graph. Observe that $\overline{\mathrm{Aut}(\bar{\Gamma})} = \overline{\mathrm{Aut}(\Gamma)}$. A graph Γ is *vertex-transitive* if for each pair of vertices $u, v \in V(\Gamma)$ there exists an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ such that $u = \varphi(v)$. If $N_\Gamma(u)$ has constant number of elements for each $u \in V(\Gamma)$, then Γ is a *regular* graph. If this number equals k, then we say that Γ is k*-regular*. Clearly, each vertex-transitive graph is regular. A k-regular graph on n vertices is *strongly regular* with parameters (n, k, λ, μ) if $|N_\Gamma(u) \cap N_\Gamma(v)| = \lambda$ whenever $\{u, v\} \in E(\Gamma)$ and $|N_\Gamma(u) \cap N_\Gamma(v)| = \mu$ whenever $\{u, v\} \notin E(\Gamma)$, for all distinct vertices $u, v \in V(\Gamma)$. We use K_n to denote the complete graph on n vertices. The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the minimal value n such that the set $\mathrm{Hom}(\Gamma, K_n)$ is nonempty. It is well known that $\chi(\Gamma) \geq \omega(\Gamma)$ and $\chi(\Gamma) \geq \frac{n}{\alpha(\Gamma)}$, where $n = |V(\Gamma)|$ (cf. [1]). From our definition of the chromatic number we immediately deduce the following claim (cf. [12, 13]).

Lemma 2.1. *If Hom(Γ_1, Γ_2) is nonempty for graphs Γ_1, Γ_2, then $\chi(\Gamma_1) \leq \chi(\Gamma_2)$.*

Lemma 2.2 for vertex-transitive graphs and strongly regular graphs can be found in [9, Corollaries 2.1.2, 2.1.3] and [9, Theorem 3.8.4 and Corollary 3.8.6], respectively, where it is stated in a more general settings.

Lemma 2.2. *If a graph Γ is vertex-transitive or strongly regular, then $\alpha(\Gamma)\omega(\Gamma) \leq |V(\Gamma)|$. If the equality holds, then $|C \cap I| = 1$ for each maximum clique C and each maximum independent set I.*

In Corollary 2.3, the claim for vertex-transitive graphs follows directly from the proof of [10, Theorem 6.13.2]. We provide a short proof that works also for strongly regular graphs.

Corollary 2.3. *If a graph Γ is vertex-transitive or strongly regular, and $\varphi \in$ Hom$(\Gamma, K_{\omega(\Gamma)})$, then $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$ and $|\varphi^{-1}(v)| = \alpha(\Gamma)$ for each vertex v in $K_{\omega(\Gamma)}$.*

Proof. By Lemma 2.1, $\chi(\Gamma) \leq \chi(K_{\omega(\Gamma)}) = \omega(\Gamma)$. Hence, $\omega(\Gamma) = \chi(\Gamma) \geq \frac{|V(\Gamma)|}{\alpha(\Gamma)}$. By Lemma 2.2, it follows that $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$. Since the preimages $\varphi^{-1}(v)$, with $v \in V(K_{\omega(\Gamma)})$, are independent sets in Γ that partition $V(\Gamma)$, we have

$$|V(\Gamma)| = \sum_{v \in V(K_{\omega(\Gamma)})} |\varphi^{-1}(v)| \leq \alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|.$$

Consequently, $|\varphi^{-1}(v)| = \alpha(\Gamma)$ for all v. □

It is obvious that each regular self-complementary graph on n vertices is $(\frac{n-1}{2})$-regular. Consequently, the hand-shaking lemma implies that $n = 4m + 1$ for some nonnegative integer m, and Lemma 2.4 follows from the results in [27] or [30] (see also [7, page 12] and [26]).

Lemma 2.4. *If Γ is a regular self-complementary graph and $\sigma \in \overline{Aut(\Gamma)}$, then there exists a unique vertex $v \in V(\Gamma)$ such that $\sigma(v) = v$.*

A graph Γ is a *core* if $\mathrm{End}(\Gamma) = \mathrm{Aut}(\Gamma)$. Given a graph Γ, we use $\mathrm{core}(\Gamma)$ to denote any subgraph in Γ that is a core and such that the set $\mathrm{Hom}\big(\Gamma, \mathrm{core}(\Gamma)\big)$ is nonempty. The graph $\mathrm{core}(\Gamma)$ is referred to as the *core of* Γ. It is always an induced subgraph and unique up to isomorphism [10, Lemma 6.2.2]. Clearly, a graph Γ is a core if and only if $\Gamma = \mathrm{core}(\Gamma)$. On the other hand, $\mathrm{core}(\Gamma)$ is a complete graph if and only if $\chi(\Gamma) = \omega(\Gamma)$. It is well known that there always exists a *retraction* $\psi : \Gamma \to \mathrm{core}(\Gamma)$, i.e. a graph homomorphism that fixes each vertex in $\mathrm{core}(\Gamma)$. In fact, if $\varphi \in \mathrm{Hom}\big(\Gamma, \mathrm{core}(\Gamma)\big)$ is arbitrary, then the restriction $\varphi|_{V(\mathrm{core}(\Gamma))}$ is invertible and the composition $(\varphi|_{V(\mathrm{core}(\Gamma))})^{-1} \circ \varphi$ is the required retraction.

Let Γ be a graph with the vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$. The *complementary prism* of Γ is the graph $\Gamma\bar{\Gamma}$, which is constructed from the disjoint union of Γ and its complement $\bar{\Gamma}$ if we add an edge between each vertex in Γ and its copy in $\bar{\Gamma}$. More precisely, the vertex set $V(\Gamma\bar{\Gamma})$ equals $W_1 \cup W_2$, where

$$W_1 = W_1(\Gamma\bar{\Gamma}) = \{(v_1, 1), \ldots, (v_n, 1)\} \text{ and } W_2 = W_2(\Gamma\bar{\Gamma}) = \{(v_1, 2), \ldots, (v_n, 2)\},$$

while the edge set $E(\Gamma\bar{\Gamma})$ is the union of the sets

$$\begin{gathered}\big\{\{(u,1),(v,1)\} : \{u,v\} \in E(\Gamma)\big\},\\ \big\{\{(u,2),(v,2)\} : \{u,v\} \in E(\bar{\Gamma})\big\},\\ \big\{\{(u,1),(u,2)\} : u \in V(\Gamma)\big\}.\end{gathered}$$

Obviously, $\Gamma\bar{\Gamma}$ is regular if and only if Γ is $\left(\frac{n-1}{2}\right)$-regular (cf. [5, Theorem 3.6]). For a general graph Γ the core of its complementary prism $\Gamma\bar{\Gamma}$ was recently studied in [25] (see also the arXiv version [24]). In particular, the following result was proved for a regular graph $\Gamma\bar{\Gamma}$.

Lemma 2.5 ([25, Corollary 3.4]). *Let Γ be any graph on n vertices that is $\left(\frac{n-1}{2}\right)$-regular. If core$(\Gamma\bar{\Gamma})$ is any core of $\Gamma\bar{\Gamma}$, then one of the following three possibilities is true.*

(i) *$\Gamma\bar{\Gamma}$ is a core.*

(ii) *All vertices of core$(\Gamma\bar{\Gamma})$ are contained in W_1, in which case*

$$\mathit{core}(\Gamma\bar{\Gamma}) \cong \mathit{core}(\Gamma).$$

(iii) *All vertices of core$(\Gamma\bar{\Gamma})$ are contained in W_2, in which case*

$$\mathit{core}(\Gamma\bar{\Gamma}) \cong \mathit{core}(\bar{\Gamma}).$$

3 Homomorphisms of the lexicographic product

In this section we recall and develop some properties of the homomorphisms of the lexicographic product of graphs. The *lexicographic product* of graphs Γ_1 and Γ_2 is the graph $\Gamma_1[\Gamma_2]$ with the vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where $(u_1, u_2) \sim (v_1, v_2)$ if and only if either $u_1 \sim_{\Gamma_1} v_1$ or $u_1 = v_1$ and $u_2 \sim_{\Gamma_2} v_2$. Here we follow the notation in [12]. The same product appears in the literature also as $\Gamma_1 \circ \Gamma_2$ (see [2, 3, 13, 29, 31, 32]) and as $\Gamma_1 \wr \Gamma_2$ in which case it is referred to as the *wreath product* [6, 20]. Observe that $\overline{\Gamma_1[\Gamma_2]} = \overline{\Gamma_1}[\overline{\Gamma_2}]$. In particular, $\Gamma_1[\Gamma_2]$ is self-complementary whenever Γ_1 and Γ_2 are self-complementary. The converse is also true. Namely, if $\Gamma_1[\Gamma_2]$ is self-complementary, then $\overline{\Gamma_1}[\overline{\Gamma_2}] \cong \Gamma_1[\Gamma_2]$ and [13, Theorem 10.8] implies that $\Gamma_1 \cong \overline{\Gamma_1}$ and $\Gamma_2 \cong \overline{\Gamma_2}$. Similarly, $\Gamma_1[\Gamma_2]$ is vertex-transitive if and only if Γ_1 and Γ_2 are vertex-transitive [13, Theorem 10.14]. Analogous claim for regularity can be proved straightforward.

If graphs Γ_i and Γ_i' are isomorphic for $i=1,2$, then we define

$$\mathrm{Iso}(\Gamma_1,\Gamma_1')\wr\mathrm{Iso}(\Gamma_2,\Gamma_2') \tag{3.1}$$

as the set of all maps $(\varphi,\beta)\colon \Gamma_1[\Gamma_2]\to\Gamma_1'[\Gamma_2']$ that are of the form

$$(\varphi,\beta)(v_1,v_2):=\big(\varphi(v_1),\beta(v_1)(v_2)\big) \tag{3.2}$$

for all $v_1\in V(\Gamma_1), v_2\in V(\Gamma_2)$, where $\varphi\in\mathrm{Iso}(\Gamma_1,\Gamma_1')$ and $\beta: V(\Gamma_1)\to\mathrm{Iso}(\Gamma_2,\Gamma_2')$ is a map. Clearly, the map (3.2) is an isomorphism and (3.1) is a subset in $\mathrm{Iso}(\Gamma_1[\Gamma_2],\Gamma_1'[\Gamma_2'])$. If $\Gamma_i'=\Gamma_i$ for $i=1,2$ or $\Gamma_i'=\overline{\Gamma_i}$ for $i=1,2$, then we write $\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$ or $\overline{\mathrm{Aut}(\Gamma_1)}\wr\overline{\mathrm{Aut}(\Gamma_2)}$ instead of (3.1), respectively. Here we emphasize that $\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$ is known as the *wreath product of groups* $\mathrm{Aut}(\Gamma_1)$ and $\mathrm{Aut}(\Gamma_2)$ (see [6, Section 4.2]).

Given a graph Γ let $R_\Gamma=\{(u,v)\in V(\Gamma)\times V(\Gamma): N_\Gamma(u)=N_\Gamma(v)\}$, $S_\Gamma=\{(u,v)\in V(\Gamma)\times V(\Gamma): N_\Gamma[u]=N_\Gamma[v]\}$, and $\triangle_\Gamma=\{(u,u)\in V(\Gamma)\times V(\Gamma): u\in V(\Gamma)\}$. Sabidussi [29] proved the following result (see also [13, Theorem 10.13]).

Lemma 3.1. *For any graphs Γ_1,Γ_2 we have $Aut\big(\Gamma_1[\Gamma_2]\big)=Aut(\Gamma_1)\wr Aut(\Gamma_2)$ if and only if the following two assertions hold:*

(i) *If $R_{\Gamma_1}\neq\triangle_{\Gamma_1}$, then Γ_2 is connected.*

(ii) *If $S_{\Gamma_1}\neq\triangle_{\Gamma_1}$, then $\overline{\Gamma_2}$ is connected.*

Corollary 3.2 follows easily from Lemma 3.1. A short proof is provided for the reader's convenience (for more elaborate results of this kind see [15]).

Corollary 3.2. *Suppose that $\Gamma_i\cong\Gamma_i'$ for $i=1,2$. If Γ_2 and $\overline{\Gamma_2}$ are both connected, then $Iso\big(\Gamma_1[\Gamma_2],\Gamma_1'[\Gamma_2']\big)=Iso(\Gamma_1,\Gamma_1')\wr Iso(\Gamma_2,\Gamma_2')$.*

Proof. Let $\Phi\in\mathrm{Iso}\big(\Gamma_1[\Gamma_2],\Gamma_1'[\Gamma_2']\big)$. Pick any $\psi_1\in\mathrm{Iso}(\Gamma_1,\Gamma_1')$, $\psi_2\in\mathrm{Iso}(\Gamma_2,\Gamma_2')$, and define $\Psi\in\mathrm{Iso}\big(\Gamma_1[\Gamma_2],\Gamma_1'[\Gamma_2']\big)$ by $\Psi(v_1,v_2)=(\psi_1(v_1),\psi_2(v_2))$ for all $v_1\in V(\Gamma_1)$ and $v_2\in V(\Gamma_2)$. Then $\Psi^{-1}\circ\Phi\in\mathrm{Aut}\big(\Gamma_1[\Gamma_2]\big)$. By Lemma 3.1, $\Psi^{-1}\circ\Phi=(\varphi,\beta)$ for some $\varphi\in\mathrm{Aut}(\Gamma_1)$ and a map $\beta\colon V(\Gamma_1)\to\mathrm{Aut}(\Gamma_2)$. Consequently, $\Phi=(\psi_1\circ\varphi,\gamma)$, where $\gamma\colon V(\Gamma_1)\to\mathrm{Iso}(\Gamma_2,\Gamma_2')$ is defined by $\gamma(v_1)=\psi_2\circ\beta(v_1)$. Hence, $\Phi\in\mathrm{Iso}(\Gamma_1,\Gamma_1')\wr\mathrm{Iso}(\Gamma_2,\Gamma_2')$. $\square$

Since self-complementary graphs are connected (cf. [7]), we deduce the following.

Corollary 3.3. *Let Γ_1 and Γ_2 be self-complementary graphs. Then*

(i) $Aut\big(\Gamma_1[\Gamma_2]\big)=Aut(\Gamma_1)\wr Aut(\Gamma_2)$,

(ii) $\overline{Aut\big(\Gamma_1[\Gamma_2]\big)}=\overline{Aut(\Gamma_1)}\wr\overline{Aut(\Gamma_2)}$.

Similarly as above, for given graphs $\Gamma_1,\Gamma_1',\Gamma_2,\Gamma_2'$ with nonempty sets $\mathrm{Hom}(\Gamma_1,\Gamma_1')$ and $\mathrm{Hom}(\Gamma_2,\Gamma_2')$, we define

$$\mathrm{Hom}(\Gamma_1,\Gamma_1')\wr\mathrm{Hom}(\Gamma_2,\Gamma_2') \tag{3.3}$$

as the set of all homomorphisms $(\varphi,\beta)\colon\Gamma_1[\Gamma_2]\to\Gamma_1'[\Gamma_2']$ that are of the form (3.2), where $\varphi\in\mathrm{Hom}(\Gamma_1,\Gamma_1')$ and $\beta\colon V(\Gamma_1)\to\mathrm{Hom}(\Gamma_2,\Gamma_2')$ is a map. If $\Gamma_i'=\Gamma_i$ for $i=1,2$, we write $\mathrm{End}(\Gamma_1)\wr\mathrm{End}(\Gamma_2)$ instead of (3.3) (cf. [16, 17, 18]).

It is obvious that Γ_1 and Γ_2 are cores whenever $\Gamma_1[\Gamma_2]$ is a core (see [18, Proposition 3.10]). The converse is not true in general. The following result is proved in [18, Theorem 3.11].

Lemma 3.4. *Let n be a positive integer and Γ_2 a graph. Then $K_n[\Gamma_2]$ is a core if and only if Γ_2 is a core.*

Lemma 3.5 is proved in [17, Theorem 14].

Lemma 3.5. *Let Γ_1 and Γ_2 be graphs, where Γ_1 is a core, and let core$(\Gamma_1[\Gamma_2])$ be any core of $\Gamma_1[\Gamma_2]$. Then End$\big(\Gamma_1[\Gamma_2]\big) =$ End$(\Gamma_1) \wr$ End(Γ_2) if and only if the following assertions are true:*

(i) *core$(\Gamma_1[\Gamma_2]) = \Gamma_1[$core$(\Gamma_2)]$, where core$(\Gamma_2)$ is some core of Γ_2.*

(ii) *$S_{\Gamma_1} = \triangle_{\Gamma_1}$ or $\overline{\text{core}(\Gamma_2)}$ is connected.*

Corollary 3.6. *Let Γ_1 and Γ_2 be graphs, where $\Gamma_1[$core$(\Gamma_2)]$ is a core. If Γ_1 is self-complementary, then End$\big(\Gamma_1[\Gamma_2]\big) =$ End$(\Gamma_1) \wr$ End(Γ_2).*

Proof. Obviously there exist a homomorphism from $\Gamma_1[\Gamma_2]$ onto $\Gamma_1[\text{core}(\Gamma_2)]$. Hence, (i) from Lemma 3.5 is satisfied. Since $\Gamma_1[\text{core}(\Gamma_2)]$ is a core, the same is true for Γ_1. Consequently, to end the proof it suffices to prove that $S_{\Gamma_1} = \triangle_{\Gamma_1}$. Suppose on the contrary that there are distinct $u, v \in V(\Gamma_1)$ such that $N_{\Gamma_1}[u] = N_{\Gamma_1}[v]$. Clearly, u and v are adjacent in Γ_1. Consequently, $N_{\overline{\Gamma_1}}(u) = N_{\overline{\Gamma_1}}(v)$ and u, v are nonadjacent in $\overline{\Gamma_1}$. The map φ on $V(\overline{\Gamma_1})$ that maps u to v and fixes all other vertices is a nonbijective endomorphism of $\overline{\Gamma_1}$. Since $\overline{\Gamma_1}$ is a core by self-complementarity, we get a contradiction. $\square$

Lemma 3.7. *Let Γ_1, Γ_2 be graphs, where Γ_1 is vertex-transitive, while $\Gamma_2, \overline{\Gamma_2}$ are both connected. If core(Γ_1) is any core of Γ_1 and core$(\Gamma_1)[\Gamma_2]$ is a core, then*

$$\mathit{Hom}\big(\Gamma_1[\Gamma_2], \mathit{core}(\Gamma_1)[\Gamma_2]\big) = \mathit{Hom}\big(\Gamma_1, \mathit{core}(\Gamma_1)\big) \wr \mathit{Hom}(\Gamma_2, \Gamma_2).$$

Proof. Let $\Phi \in \text{Hom}\big(\Gamma_1[\Gamma_2], \text{core}(\Gamma_1)[\Gamma_2]\big)$ and $v_1 \in V(\Gamma_1)$. Since Γ_1 is vertex-transitive, there exists a subgraph Γ in Γ_1, which is isomorphic to $\text{core}(\Gamma_1)$ and such that $v_1 \in V(\Gamma)$. The restriction $\Phi|_{V(\Gamma[\Gamma_2])}$ is a homomorphism from $\Gamma[\Gamma_2]$ to $\text{core}(\Gamma_1)[\Gamma_2]$. Since $\text{core}(\Gamma_1)[\Gamma_2]$ is a core, we deduce that $\Phi|_{V(\Gamma[\Gamma_2])}$ is an isomorphism. By Corollary 3.2, $\Phi|_{V(\Gamma[\Gamma_2])} = (\varphi, \beta)$ for some $\varphi \in \text{Iso}(\Gamma, \text{core}(\Gamma_1))$ and a map $\beta\colon V(\Gamma) \to \text{Iso}(\Gamma_2, \Gamma_2) = \text{Aut}(\Gamma_2)$. Since $v_1 \in V(\Gamma_1)$ is arbitrary, we deduce in particular that

$$\Phi(v_1, v_2) = (\psi(v_1), \beta_{v_1}(v_2))$$

for all $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$, where $\psi\colon V(\Gamma_1) \to V\big(\text{core}(\Gamma_1)\big)$ is some map and $\beta_{v_1} \in \text{Aut}(\Gamma_2)$.

We claim that ψ is a graph homomorphism. Let $v_1 \sim_{\Gamma_1} v_1'$. Then

$$(\psi(v_1), \beta_{v_1}(v_2)) = \Phi(v_1, v_2) \sim \Phi(v_1', v_2') = (\psi(v_1'), \beta_{v_1'}(v_2')) \tag{3.4}$$

for all $v_2, v_2' \in V(\Gamma_2)$. Since $\beta_{v_1}, \beta_{v_1'}$ are automorphisms, we can find v_2, v_2' such that $\beta_{v_1}(v_2) = \beta_{v_1'}(v_2')$. It follows from (3.4) that $\psi(v_1) \sim_{\Gamma_1} \psi(v_1')$ and ψ is a graph homomorphism. Hence, $\Phi = (\psi, \widehat{\beta})$, where the map $\widehat{\beta}\colon V(\Gamma_1) \to \text{Aut}(\Gamma_2)$ is defined by $\widehat{\beta}(v_1) := \beta_{v_1}$. Therefore, $\Phi \in \text{Hom}\big(\Gamma_1, \text{core}(\Gamma_1)\big) \wr \text{Hom}(\Gamma_2, \Gamma_2)$. $\square$

We remark that since $\text{core}(\Gamma_1)[\Gamma_2]$ is a core in Lemma 3.7, then Γ_2 is also a core. Consequently, $\text{Hom}(\Gamma_2, \Gamma_2) = \text{Aut}(\Gamma_2)$.

4 Main results

In [24, Corollary 3.8] it was recently proved that the complementary prism $\Gamma\bar{\Gamma}$ is vertex-transitive if and only if Γ is vertex-transitive and self-complementary. In [26] it was proved that a vertex-transitive complementary prism $\Gamma\bar{\Gamma}$ is a core whenever Γ is a core or its core is complete. In Corollaries 4.3 and 4.7 we consider the only vertex-transitive self-complementary graphs the author is aware of, which are neither cores nor their cores are complete. We prove that $\Gamma\bar{\Gamma}$ is a core also for such graphs. Theorems 4.1 and 4.4 generalize Corollaries 4.3 and 4.7. They simultaneously generalize also a result from [26] (see Remark 4.8).

Theorem 4.1. *Let Γ_1, Γ_2 be self-complementary graphs, where Γ_1 is vertex-transitive and Γ_2 is regular. If core(Γ_1) is complete, Γ_2 is a core, and $\Gamma = \Gamma_1[\Gamma_2]$, then $\Gamma\bar{\Gamma}$ is a core.*

Proof. Recall from the beginning of Section 3 that the lexicographic product Γ is regular and self-complementary. Consequently, only the possibilities (i), (ii), (iii) in Lemma 2.5 may occur. Suppose that (iii) is correct, that is, $V(\text{core}(\Gamma\bar{\Gamma})) \subseteq W_2$ and $\text{core}(\Gamma\bar{\Gamma}) \cong \text{core}(\bar{\Gamma})$. Let $\phi\colon \overline{\Gamma_1} \to K_m$ be any homomorphism onto a complete core of $\overline{\Gamma_1}$. Then the map $\bar{\Gamma} = \overline{\Gamma_1}[\overline{\Gamma_2}] \to K_m[\overline{\Gamma_2}]$, defined by $(v_1, v_2) \mapsto (\phi(v_1), v_2)$ for all $v_1 \in V(\overline{\Gamma_1}), v_2 \in V(\overline{\Gamma_2})$, is a graph homomorphism. By Lemma 3.4 it follows that

$$\text{core}(\Gamma\bar{\Gamma}) \cong \text{core}(\bar{\Gamma}) = K_m[\overline{\Gamma_2}].$$

Let $\psi\colon \text{core}(\Gamma\bar{\Gamma}) \to K_m[\overline{\Gamma_2}]$ be any isomorphism and let $\Phi\colon \Gamma\bar{\Gamma} \to \text{core}(\Gamma\bar{\Gamma})$ be any homomorphism. The map ψ_1, defined by $\psi_1(v) = (v, 1)$ for all $v \in V(\Gamma)$, is the canonical isomorphism between Γ and the subgraph in $\Gamma\bar{\Gamma}$, which is induced by the set W_1. Similarly, $\psi_2(v) = (v, 2)$, where $v \in V(\Gamma)$, is the canonical isomorphism between $\bar{\Gamma}$ and the subgraph induced by W_2. Then $f_2 := \psi \circ (\Phi|_{W_2}) \circ \psi_2 \in \text{Hom}(\overline{\Gamma_1}[\overline{\Gamma_2}], K_m[\overline{\Gamma_2}])$. Similarly, if $\sigma\colon \Gamma \to \bar{\Gamma}$ is any antimorphism and $f_1 := \psi \circ (\Phi|_{W_1}) \circ \psi_1$, then $f_1 \circ \sigma^{-1} \in \text{Hom}(\overline{\Gamma_1}[\overline{\Gamma_2}], K_m[\overline{\Gamma_2}])$. By Lemma 3.7 and Corollary 3.3 we deduce that

$$f_2 = (\varphi_2, \beta_2) \quad \text{and} \quad f_1 = (\varphi_1, \beta_1) \circ (\sigma_1, \gamma),$$

where

$$\begin{aligned}
&\varphi_1, \varphi_2 \in \text{Hom}(\overline{\Gamma_1}, K_m),\\
&\beta_1, \beta_2\colon V(\overline{\Gamma_1}) \to \text{Hom}(\overline{\Gamma_2}, \overline{\Gamma_2}) = \text{Aut}(\overline{\Gamma_2}),\\
&\sigma_1 \in \overline{\text{Aut}(\Gamma_1)},\\
&\gamma\colon V(\Gamma_1) \to \overline{\text{Aut}(\Gamma_2)}.
\end{aligned}$$

That is,

$$f_2(v_1, v_2) = \big(\varphi_2(v_1), \beta_2(v_1)(v_2)\big), \tag{4.1}$$

$$f_1(v_1, v_2) = \Big((\varphi_1 \circ \sigma_1)(v_1), \big(\beta_1(\sigma_1(v_1)) \circ \gamma(v_1)\big)(v_2)\Big) \tag{4.2}$$

for all $v_1 \in V(\Gamma_1), v_2 \in V(\Gamma_2)$. Pick any $v \in V(K_m)$. By Corollary 2.3, $\varphi_2^{-1}(v)$ is an independent set in $\overline{\Gamma_1}$ of order $\alpha(\overline{\Gamma_1})$, $(\varphi_1 \circ \sigma_1)^{-1}(v)$ is a clique in $\overline{\Gamma_1}$ of order $\alpha(\Gamma_1) = \omega(\overline{\Gamma_1})$, and $\alpha(\overline{\Gamma_1})\omega(\overline{\Gamma_1}) = |V(\Gamma_1)|$. By Lemma 2.2, there exists

$$v_1 \in \varphi_2^{-1}(v) \cap (\varphi_1 \circ \sigma_1)^{-1}(v).$$

Since $\big(\beta_1(\sigma_1(v_1)) \circ \gamma(v_1)\big)^{-1} \circ \beta_2(v_1) \in \overline{\mathrm{Aut}(\overline{\Gamma_2})} = \overline{\mathrm{Aut}(\Gamma_2)}$, Lemma 2.4 yields a vertex $v_2 \in V(\Gamma_2)$ such that

$$\big(\beta_1(\sigma_1(v_1)) \circ \gamma(v_1)\big)(v_2) = \beta_2(v_1)(v_2).$$

Consequently, $f_1(v_1, v_2) = f_2(v_1, v_2)$ by (4.1) – (4.2). Therefore

$$\begin{aligned}\Phi\big((v_1, v_2), 1\big) &= (\psi^{-1} \circ f_1 \circ \psi_1^{-1})\big((v_1, v_2), 1\big) \\ &= (\psi^{-1} \circ f_2 \circ \psi_2^{-1})\big((v_1, v_2), 2\big) = \Phi\big((v_1, v_2), 2\big),\end{aligned}$$

which is a contradiction, since $\{\big((v_1, v_2), 1\big), \big((v_1, v_2), 2\big)\}$ is an edge in $\Gamma\bar{\Gamma}$.

In the same way we see that (ii) in Lemma 2.5 is not possible, so (i) is true. □

Remark 4.2. It follows from Lemma 3.4 that the core of Γ in Theorem 4.1 is isomorphic to $\mathrm{core}(\Gamma_1)[\Gamma_2]$. Hence, Γ is neither a core nor its core is complete whenever Γ_1 and Γ_2 have more than one vertex.

Recall from Section 3 that $\Gamma_1[\Gamma_2]$ is vertex-transitive and self-complementary if and only if Γ_1 and Γ_2 both have these properties. Consequently, Corollary 4.3 follows directly from Theorem 4.1.

Corollary 4.3. *Let $\Gamma = \Gamma_1[\Gamma_2]$ be vertex-transitive and self-complementary. If core(Γ_1) is complete and Γ_2 is a core, then $\Gamma\bar{\Gamma}$ is a core.*

If we swap the assumptions regarding Γ_1 and Γ_2 in Theorem 4.1, then we are able to deduce the same conclusion under the additional condition that $\Gamma_1[\mathrm{core}(\Gamma_2)]$ is a core.

Theorem 4.4. *Let Γ_1, Γ_2 be self-complementary graphs, where Γ_1 is regular and Γ_2 is vertex-transitive. If core(Γ_2) is complete, $\Gamma_1[\mathrm{core}(\Gamma_2)]$ is a core, and $\Gamma = \Gamma_1[\Gamma_2]$, then $\Gamma\bar{\Gamma}$ is a core.*

Remark 4.5. The claim and the proof of Theorem 4.4 remains valid if we replace vertex-transitivity of Γ_2 by strong regularity. However, at this point in time the author is not aware of any strongly regular self-complementary graph that is not vertex-transitive (cf. [7, page 88]).

Since Theorems 4.1 and 4.4 have similar proofs, we sketch only the main differences.

Sketch of the proof. Denote $\mathrm{core}(\overline{\Gamma_2}) =: K_m$. Similarly as in the proof of Theorem 4.1 we deduce that the condition (iii) in Lemma 2.5 would yield

$$\mathrm{core}(\Gamma\bar{\Gamma}) \cong \mathrm{core}(\bar{\Gamma}) = \overline{\Gamma_1}[K_m].$$

Here, the only difference is the application of Lemma 3.4, which is replaced by the assumption that $\overline{\Gamma_1}[K_m]$ is a core. Let $\psi\colon \mathrm{core}(\Gamma\bar{\Gamma}) \to \overline{\Gamma_1}[K_m]$ be any isomorphism, and define $\Phi, \psi_1, \psi_2, f_1, f_2, \sigma$ as in the proof of Theorem 4.1. Then $f_2, f_1 \circ \sigma^{-1} \in \mathrm{Hom}(\overline{\Gamma_1}[\overline{\Gamma_2}], \overline{\Gamma_1}[K_m])$. Hence, these maps may be interpreted as members of $\mathrm{End}(\overline{\Gamma_1}[\overline{\Gamma_2}])$, which equals $\mathrm{End}(\overline{\Gamma_1}) \wr \mathrm{End}(\overline{\Gamma_2})$ by Corollary 3.6. Note that since $\Gamma_1[\mathrm{core}(\Gamma_2)]$ is a core, the graph $\overline{\Gamma_1} \cong \Gamma_1$ is a core as well. Therefore $f_2 = (\varphi_2, \beta_2)$ and $f_1 = (\varphi_1, \beta_1) \circ (\sigma_1, \gamma)$, where

$$\begin{aligned}
&\varphi_1, \varphi_2 \in \mathrm{End}(\overline{\Gamma_1}) = \mathrm{Aut}(\overline{\Gamma_1}),\\
&\beta_1, \beta_2 \colon V(\overline{\Gamma_1}) \to \mathrm{End}(\overline{\Gamma_2}),\\
&\sigma_1 \in \overline{\mathrm{Aut}(\Gamma_1)},\\
&\gamma\colon V(\Gamma_1) \to \overline{\mathrm{Aut}(\Gamma_2)},
\end{aligned}$$

and the image of $\beta_i(v_1)$ is in $V(K_m)$ for all $v_1 \in V(\overline{\Gamma_1})$ and $i = 1, 2$. Clearly, (4.1) – (4.2) is still true. Since $\varphi_2^{-1} \circ \varphi_1 \circ \sigma_1 \in \overline{\mathrm{Aut}(\Gamma_1)}$, Lemma 2.4 yields a vertex $v_1 \in V(\Gamma_1)$ such that

$$(\varphi_1 \circ \sigma_1)(v_1) = \varphi_2(v_1).$$

Let $v \in V(K_m)$ be any vertex. By Corollary 2.3, $(\beta_2(v_1))^{-1}(v)$ is an independent set in $\overline{\Gamma_2}$ of order $\alpha(\overline{\Gamma_2})$, $\big(\beta_1(\sigma_1(v_1)) \circ \gamma(v_1)\big)^{-1}(v)$ is a clique in $\overline{\Gamma_2}$ of order $\alpha(\Gamma_2) = \omega(\overline{\Gamma_2})$, and $\alpha(\overline{\Gamma_2})\omega(\overline{\Gamma_2}) = |V(\overline{\Gamma_2})|$. By Lemma 2.2 there exists

$$v_2 \in (\beta_2(v_1))^{-1}(v) \cap \big(\beta_1(\sigma_1(v_1)) \circ \gamma(v_1)\big)^{-1}(v).$$

Consequently, (4.1) – (4.2) imply that $f_1(v_1, v_2) = f_2(v_1, v_2)$ and we deduce the same contradiction as in the proof of Theorem 4.1. $\square$

Remark 4.6. It follows from the assumptions that the core of Γ in Theorem 4.4 is isomorphic to $\Gamma_1[\mathrm{core}(\Gamma_2)]$. Hence, Γ is neither a core nor its core is complete whenever Γ_1 and Γ_2 have more than one vertex.

The following claim is deduced analogously as Corollary 4.3.

Corollary 4.7. *Let $\Gamma = \Gamma_1[\Gamma_2]$ be vertex-transitive and self-complementary. If core(Γ_2) is complete and $\Gamma_1[$core$(\Gamma_2)]$ is a core, then $\Gamma\bar{\Gamma}$ is a core.*

Remark 4.8. In [26, Proposition 3.1] it was proved that $\Gamma\bar{\Gamma}$ is a core whenever Γ is regular, self-complementary, and a core. Theorems 4.1 and 4.4 both generalize this result. In fact, K_1 is vertex-transitive and self-complementary. Hence, we can consider $K_1[\Gamma_2] \cong \Gamma_2$ in Theorem 4.1 and $\Gamma_1[K_1] \cong \Gamma_1$ in Theorem 4.4.

Example 4.9. Let q be a power of a prime such that $q \equiv 1 \pmod 4$. The *Paley graph* $P(q)$ has the finite field $\mathbb{F}_q$ as its vertex set, and two of its elements form an edge if and only if their difference is a nonzero square element in $\mathbb{F}_q$. It is well known that each Paley graph is vertex-transitive and self-complementary (cf. [9, page 105]). Moreover, $P(q)$ is a core if q is not a square, while its core is complete if q is a square [4, Proposition 3.3]. Hence, $\Gamma_1 = P(q_1)$ and $\Gamma_2 = P(q_2)$ satisfy the assumptions in Corollary 4.3 whenever q_1 is a square and q_2 is not. In the reversed order, graphs $\Gamma_1 = P(q_2)$ and $\Gamma_2 = P(q_1)$ satisfy the assumptions in Corollary 4.7 at least for $q_2 = 5$. In fact, $P(5)$ is the 5-cycle C_5 and $\Gamma_1[\mathrm{core}(\Gamma_2)] \cong C_5[K_{\sqrt{q_1}}]$ is a core by [18, Theorem 3.11]. $\square$

Clearly, if vertex-transitive self-complementary graphs Γ_1 and Γ_2 both have complete cores, then the same is true for $\Gamma = \Gamma_1[\Gamma_2]$, and therefore $\Gamma\bar{\Gamma}$ is a core by [26, Theorem 3.3]. In view of the open problem in [26], which asks if there exists a vertex-transitive self-complementary graph Γ such that $\Gamma\bar{\Gamma}$ is not a core, the following three combinations in the lexicographic product $\Gamma_1[\Gamma_2]$ of vertex-transitive self-complementary graphs should be also addressed:

- Γ_1 is a core, $\text{core}(\Gamma_2) = K_m$, and $\Gamma_1[K_m]$ is neither a core nor its core is complete (if such graphs exist);
- Γ_1 and Γ_2 are both cores, and $\Gamma_1[\Gamma_2]$ is neither a core nor its core is complete (if such graphs exist);
- lexicographic products with more than two factors.

However, the task seems quite challenging. In fact, in general we only know that the core of $\Gamma_1[\Gamma_2]$ is of the form $\Gamma_1'[\text{core}(\Gamma_2)]$, where Γ_1' is a subgraph in Γ_1 (see [12]).

ORCID iDs

Marko Orel https://orcid.org/0000-0002-2544-2986

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