MATCH
Communications in Mathematical and in Computer Chemistry

*MATCH Commun. Math. Comput. Chem.* **90** (2023) 453–468
**ISSN:** 0340–6253
**doi:** 10.46793/match.90-2.453B

# Resonance Graphs and a Binary Coding of Perfect Matchings of Outerplane Bipartite Graphs

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(Received February 13, 2023)

**Abstract**

The aim of this paper is to investigate resonance graphs of 2-connected outerplane bipartite graphs, which include various families of molecular graphs. Firstly, we present an algorithm for a binary coding of perfect matchings of these graphs. Further, 2-connected outerplane bipartite graphs with isomorphic resonance graphs are considered. In particular, it is shown that if two 2-connected outerplane bipartite graphs are evenly homeomorphic, then its resonance graphs are isomorphic. Moreover, we prove that for any 2-connected outerplane bipartite graph $G$ there exists a cata-condensed even ring systems $H$ such that the resonance graphs of $G$ and $H$ are isomorphic. We conclude with the characterization of 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes.

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# 1 Introduction

Kekulé structures of aromatic hydrocarbons reflect the positions of double bonds in a molecule. In graph theory, Kekulé structures are modelled by perfect matchings of the corresponding molecular graph. On the other hand, the interaction between Kekulé structures is reflected in the resonance graph of a given molecule. Resonance graphs were independently introduced by chemists (El-Basil [3, 4], Gründler [9]) and also by mathematicians (Zhang, Guo, and Chen [15]) under the name $Z$-transformation graph.

Initially, various properties of resonance graphs of hexagonal systems were established in [15]. Later, the concept of resonance graphs was generalized to all plane (elementary) bipartite graphs (for example, see [17, 18]).

In [13, 16], a binary coding procedure of vertices of resonance graphs of catacondensed hexagonal systems was developed. Later [1], this binary coding was generalized to catacondensed even ring systems (CERS), which form a subfamily of 2-connected outerplane bipartite graphs (see also [14]). In recent years, various structural properties of resonance graphs of 2-connected (outer)plane bipartite graphs were deduced [5–8]. For example, in [8] all plane bipartite graphs whose resonance graphs can be constructed from an edge by a sequence of peripheral convex expansions are characterized.

The paper is organized as follows. Firstly, in Section 3 we generalize the binary coding procedure of perfect matchings from CERS [1] to all 2-connected outerplane bipartite graphs. Next, in Section 4 we study 2-connected outerplane bipartite graphs with isomorphic resonance graphs. In particular, we prove that if $G$ and $H$ are evenly homeomorphic, then its resonance graphs are isomorphic. Furthermore, in Section 5 we prove that for any 2-connected outerplane bipartite graph $G$ there exists a CERS $H$ such that the resonance graphs of $G$ and $H$ are isomorphic. Finally, we characterize 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes, which extends results from [19] and [2].

# 2 Preliminaries

The *distance* $d_G(u, v)$ between vertices $u$ and $v$ of a graph $G$ is defined as the usual shortest path distance. The distance between two edges $e$ and $f$ of $G$, denoted by $d_G(e, f)$, is defined as the distance between corresponding vertices in the line graph $L(G)$ of $G$.

The *hypercube* $Q_n$ of dimension $n$ is defined in the following way: all vertices of $Q_n$ are presented as $n$-tuples $x_1x_2 \ldots x_n$ where $x_i \in \{0, 1\}$ for each $i \in \{1, \ldots, n\}$, and two vertices of $Q_n$ are adjacent if the corresponding $n$-tuples differ in precisely one position. A subgraph $H$ of a graph $G$ is an *isometric subgraph* if for all $u, v \in V(H)$ it holds $d_H(u, v) = d_G(u, v)$. If a graph is isomorphic to an isometric subgraph of $G$, we say that it can be *isometrically embedded* in $G$. Any isometric subgraph of a hypercube is called a *partial cube* [11].

If $G$ is a plane graph, then an edge $e$ of $G$ that belongs to two inner faces of $G$ will be called an *inner edge*. We say that two faces of $G$ are *adjacent* if they have an edge in common. An inner face adjacent to the outer face is called a *peripheral face*. In addition, we denote the edges lying on some face $s$ of $G$ by $E(s)$. The subgraph induced by the edges in $E(s)$ is the *periphery of* $s$ and the periphery of the outer face is also called the *periphery of* $G$. Moreover, for a peripheral face $s$ and the outer face $s_0$, the subgraph induced by the edges in $E(s) \cap E(s_0)$ is called the *common periphery* of $s$ and $G$. The vertices of $G$ that belong to the outer face are called *peripheral vertices* and the remaining vertices are *interior vertices*. Furthermore, an *outerplane graph* is a plane graph in which all vertices are peripheral vertices.

The following definitions can be found, for example, in [5]. A bipartite graph $G$ is *elementary* if and only if it is connected and each edge is contained in some perfect matching of $G$. A peripheral face $s$ of a plane elementary bipartite graph $G$ is called *reducible* if the subgraph $H$ of $G$ obtained by removing all internal vertices (if exist) and edges on the common periphery of $s$ and $G$ is elementary.

An *even ring system* is a 2-connected plane bipartite graph with all interior vertices of degree 3 and all peripheral vertices of degree 2 or 3.

Moreover, an outerplane even ring system is called *catacondensed even ring system* or shortly CERS [14].

A *1-factor* of a graph $G$ is a spanning subgraph of $G$ such that every vertex has degree one. The edge set of a 1-factor is called a *perfect matching* of $G$, which is a set of independent edges covering all vertices of $G$. In chemical literature, perfect matchings are known as Kekulé structures (see [10] for more details).

Let $G$ be a 2-connected plane bipartite graph. The *resonance graph* $R(G)$ of $G$ is the graph whose vertices are the perfect matchings of $G$, and two perfect matchings $M_1, M_2$ are adjacent whenever their symmetric difference forms the edge set of exactly one inner face $s$ of $G$, i.e. $M_1 \oplus M_2 = E(s)$.

Next, we state the definition of a reducible face decomposition, see [17] and [5, 6]. Firstly, we introduce the *bipartite ear decomposition* of a plane elementary bipartite graph $G$ with $n$ inner faces. Starting from an edge $e$ of $G$, join its two end-vertices by a path $P_1$ of odd length and proceed inductively to build a sequence of bipartite graphs as follows. If $G_{i-1} = e + P_1 + \cdots + P_{i-1}$ has already been constructed, add the $i$th ear $P_i$ of odd length by joining any two vertices belonging to different bipartition sets of $G_{i-1}$ such that $P_i$ has no internal vertices in common with the vertices of $G_{i-1}$. A bipartite ear decomposition of a plane elementary bipartite graph $G$ is called a *reducible face decomposition* (shortly RFD) if $G_1$ is a periphery of a finite face $s_1$ of $G$, and the $i$th ear $P_i$ lies in the exterior of $G_{i-1}$ such that $P_i$ and a part of the periphery of $G_{i-1}$ surround a finite face $s_i$ of $G$ for all $i \in \{2, \ldots, n\}$. For such a decomposition, we use notation $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$.

Furthermore, if $G$ is a graph and $X \subseteq V(G)$, then the notation $G[X]$ is used to denote the subgraph of $G$ induced by the set $X$.

# 3 Binary coding of perfect matchings

In this section, we develop an algorithm for constructing binary codes of perfect matchings of 2-connected outerplane bipartite graphs. This represents a generalization of the result from [1]. For this purpose, we firstly need several auxiliary results.

Let $G$ be a 2-connected outerplane bipartite graph and $H_1$, $H_2$ two induced subgraphs of $G$ such that $V(H_1) \cup V(H_2) = V(G)$ and $|E(H_1) \cap E(H_2)| = 1$. If $e \in E(H_1) \cap E(H_2)$, we say that $H_1$ and $H_2$ are *e-subgraphs* of graph $G$. Moreover, let $M$ be a perfect matching of $G$. We say that a vertex $x \in V(H_i)$ is *$M$-covered* in $H_i$, $i \in \{1, 2\}$, if there exists a vertex $y \in V(H_i)$ such that $xy \in M$. Furthermore, an edge $f$ is $M$-covered in $H_i$ if its end-vertices are both $M$-covered in $H_i$.

**Proposition 1.** *Let $G$ be a 2-connected outerplane bipartite graph and $H_1$, $H_2$ two induced subgraphs of $G$ such that $V(H_1) \cup V(H_2) = V(G)$ and $e = uv$ is the only edge in the set $E(H_1) \cap E(H_2)$. Moreover, let $M$ be a perfect matching of $G$. Then $e$ is $M$-covered in $H_1$ or $H_2$.*

*Proof.* Suppose that $u$ is $M$-covered in $H_1$ but not in $H_2$ and $v$ is $M$-covered in $H_2$ but not in $H_1$. It is easy to see that $H_1$ is again a 2-connected outerplane bipartite graph and therefore, it has an even number of vertices. Let $H_1' = H_1 - v$. Then, $M \setminus E(H_2)$ is a perfect matching of the graph $H_1'$. However, graph $H_1'$ has an odd number of vertices, which is a contradiction with the existence of a perfect matching. ∎

**Proposition 2.** *Let $G$ be a 2-connected outerplane bipartite graph and $e = uv \in E(G)$ an edge belonging to two inner faces $s$ and $s'$ of $G$. Also, let $H$ be the $e$-subgraph of $G$ containing $s'$, $f \neq e$ an edge of face $s$, and $H_f$ the $f$-subgraph of $G$ not containing $e$. Suppose that $M$ is a perfect matching of $G$ such that $e$ is $M$-covered in $H$. Then $d_G(e, f)$ is even if and only if $f$ is $M$-covered in $H_f$.*

*Proof.* Let $P = (e, f_1, f_2, \ldots, f_k = f)$ be a shortest path in $G$ between $e$ and $f$. Moreover, let $H_i$ be the $f_i$-subgraph of $G$ that does not contain $s$. Since $e$ is $M$-covered in $H$, by Proposition 1 it follows that $f_1$ is not $M$-covered in $H_1$. Using the same argument, the edge $f_2$ must be $M$-covered in $H_2$. Inductively, we obtain that $f_i$ is $M$-covered in $H_i$ if and only if $i$ is even. As a consequence, $f$ is $M$-covered in $H_f$ if and only if $k = d_G(e, f)$ is even. ∎

**Remark.** *Let $s$ be an inner face of a 2-connected outerplane bipartite graph $G$. Then $s$ is reducible if and only if it is adjacent to exactly one inner face of $G$ [6].*

Let $G$ be a 2-connected outerplane bipartite graph with $n$ inner faces, $s$ a reducible face of $G$, and $e = uv$ the edge of $s$ that belongs to exactly two inner faces of $G$. Moreover, let $G'$ be the graph obtained from $G$ by removing face $s$. In addition, we denote by $H$ the subgraph of $G$ induced on the vertices of $s$. We partition the perfect matchings of $G$ into the sets $\mathcal{M}_e(G)$, $\mathcal{M}_{\overline{e}}^{G'}(G)$, and $\mathcal{M}_{\overline{e}}^{H}(G)$. More precisely, $\mathcal{M}_e(G)$ is the set of all perfect matchings of $G$ that contain edge $e$ and $\mathcal{M}_{\overline{e}}^{G'}(G)$ is the set of all perfect matchings $M$ of $G$ such that $e \notin M$ and $e$ is $M$-covered in $G'$. Similarly, $\mathcal{M}_{\overline{e}}^{H}(G)$ is the set of all perfect matchings $M$ of $G$ such that $e \notin M$ and $e$ is $M$-covered in $H$.

It is straightforward to see that the subgraph of $R(G)$ induced by the vertices from $\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)$, denoted as $R(G)[\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)]$, is isomorphic to $R(G')$. Moreover, $R(G)[\mathcal{M}_e(G)]$ and $R(G)[\mathcal{M}_{\overline{e}}^{H}(G)]$ are also isomorphic. By using these facts, we can obtain binary codes of length $n$ of perfect matchings of $G$, where $n$ is the number of inner faces of $G$. Then, $G'$ contains $n-1$ inner faces and suppose that we have already obtained binary codes of perfect matchings of $G'$.

Every perfect matching $M'$ of $G'$ with binary code $b(M')$ can be in the unique way extended to a perfect matching $M$ of $G$, see Figure 1 (a). Binary code $b(M)$ is obtained by concatenation of 0 to $b(M')$. In this way, we obtain the binary codes for perfect matchings in the set $\mathcal{M}_e(G) \cup \mathcal{M}_{\overline{e}}^{G'}(G)$.

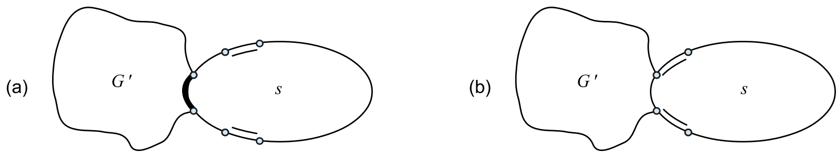


**Figure 1.** Two possibilities for the perfect matchings of $G$. The common edge of $G'$ and $s$ is bold iff it is $M$-covered in $G'$.

On the other hand, let $M'$ be a perfect matching of $G'$ such that $e \in M'$. We define $M$ as the unique perfect matching of $G$ such that $M' \setminus \{e\} \subseteq M$

and $e \notin M$, see Figure 1 (b). Binary code $b(M)$ is then obtained by concatenation of 1 to $b(M')$. Here, we obtain the binary codes for perfect matchings in the set $\mathcal{M}_{\overline{e}}^{H}(G)$.

The obtained binary coding procedure of perfect matchings of a 2-connected outerplane bipartite graph $G$ follows a peripheral convex expansion described in [6]. Therefore, by Theorem 3.2 [6] our procedure gives an isometric embedding of the resonance graph $R(G)$ into the hypercube of dimension $n$, where $n$ is the number of inner faces of $G$. Consequently, two perfect matchings $M_1$ and $M_2$ of $G$ are adjacent in $R(G)$ if and only if their binary codes differ in exactly one position.

In [1, 13] the algorithms for binary coding of perfect matchings of benzenoid graphs and CERS were presented. The mentioned algorithms are here generalized to 2-connected outerplane bipartite graphs. We first extend the following definition from [1] to a larger family of graphs.

**Definition 1.** *Let $s$, $s'$, $s''$ be three inner faces of a 2-connected outerplane bipartite graph such that $s$ and $s'$ have common edge $e$ and $s', s''$ have common edge $f$. The triple $(s, s', s'')$ is called an **adjacent triple of inner faces**. Moreover, $(s, s', s'')$ is **regular** if the distance $d_G(e, f)$ is an even number and **irregular** otherwise.*

To show an example, consider the triple $(s_i, s_j, s_{r+1})$ from Figure 2. The mentioned triple is regular in case (a) and irregular in case (b).

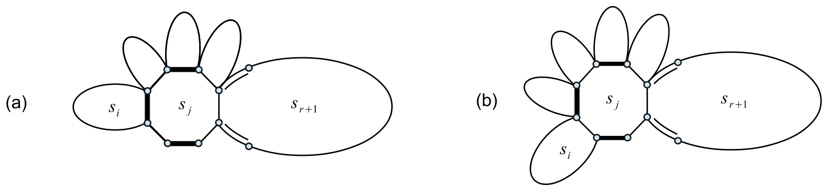


**Figure 2.** Perfect matchings of $G_{r+1}$ with respect to the regularity of triple $(s_i, s_j, s_{r+1})$. An edge $f$ is bold iff it is $M$-covered in the $f$-subgraph that does not contain face $s_j$.

Suppose $G$ is a 2-connected outerplane bipartite graph. Moreover, let $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$, be a reducible face decomposition associated with a sequence of inner faces $s_1, s_2, \ldots, s_n$. The set of all binary codes for the perfect matchings of $G_r$ will be denoted as $B_r$ for every $r \in \{1, \ldots, n\}$.

If $G$ has only two faces, $s_1$ and $s_2$, we define the binary codes $B_2 = \{00, 01, 10\}$ in the following way: code 00 represents the perfect matching that contains the common edge of $s_1$ and $s_2$. Further, let 01 be the perfect matching obtained from 00 by rotating the edges in $s_2$, and 10 the remaining perfect matching, see Figure 3.

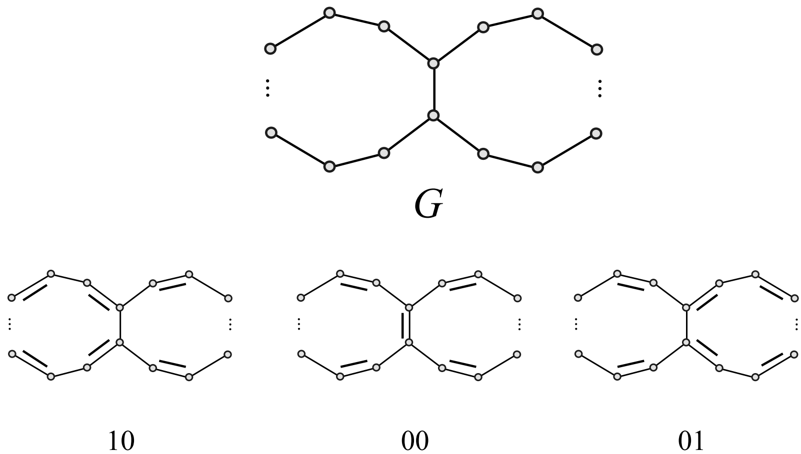


**Figure 3.** Binary coding of perfect matchings of a graph $G$ with two inner faces.

Assume that $B_r$ is the set of all the binary codes for perfect matchings of the graph $G_r$, which is composed of faces $s_1, \ldots, s_r$. Graph $G_{r+1}$ is then obtained from $G_r$ by adding a new face $s_{r+1}$. Let $s_j$, $j \in \{1, \ldots, r\}$, be the unique face adjacent to $s_{r+1}$. Moreover, let $s_i$ be the inner face adjacent to $s_j$ with the smallest index $i \in \{1, \ldots, r\}$.

The set $B_{r+1}$ of all binary codes for perfect matchings of the graph $G_{r+1}$ then contains all the strings that are obtained by concatenating 0 to every $x = x_1 x_2 \ldots x_r \in B_r$. Moreover, the set $B_{r+1}$ also contains additional codes, which are due to Proposition 2 obtained in one of the following ways:

(a) If $(s_i, s_j, s_{r+1})$ is regular, then $B_{r+1}$ also contains all the strings that are obtained by concatenating 1 to every $x = x_1 x_2 \ldots x_r \in B_r$ with

$x_j = 0$, see Figure 2 (a).

(b) If $(s_i, s_j, s_{r+1})$ is irregular, then $B_{r+1}$ also contains all the strings that are obtained by concatenating 1 to every $x = x_1x_2\dots x_r \in B_r$ with $x_j = 1$, see Figure 2 (b).

Finally, we present the procedure for binary coding of perfect matchings for a 2-connected outerplane bipartite graph, see Algorithm 1. In the algorithm, we denote $B := B_r$ and $B' := B_{r+1}$.

**Algorithm 1:** Binary coding of perfect matchings of a 2-connected outerplane bipartite graph.

**Input:** $RFD(G_1, G_2, \dots, G_n)$ of a graph $G$ associated with a sequence $s_1, \dots, s_n$.

**Output:** Binary codes for all perfect matchings of $G$.

```
B := {00, 01, 10}
for r = 2, ..., n − 1 do
    B' := ∅
    set j ∈ {1, ..., r} such that s_j is adjacent to s_{r+1}
    i = min{l | s_l is adjacent to s_j}
    if (s_i, s_j, s_{r+1}) is regular then
        for each x ∈ B do
            B' := B' ∪ {x0}
            if x_j = 0 then
                B' := B' ∪ {x1}
            end
        end
    else
        for each x ∈ B do
            B' := B' ∪ {x0}
            if x_j = 1 then
                B' := B' ∪ {x1}
            end
        end
    end
    B := B'
end
```

We now apply Algorithm 1 on graph $G$ from Figure 4. Its faces are denoted as $s_1, \dots, s_4$. As usual, by $G_k$ we denote the subgraph of $G$ that contains faces $s_1, \ldots, s_k$, where $k \in \{2,3,4\}$, and therefore $G_4 = G$. The resonance graphs obtained by Algorithm 1 are shown in Figure 4.

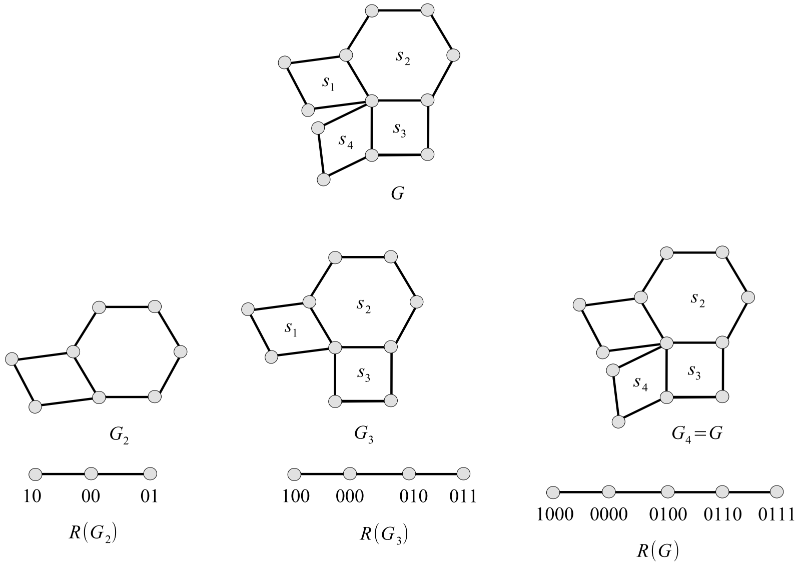


**Figure 4.** Binary coding procedure of perfect matchings of a graph $G$ together with resonance graphs.

# 4 Evenly homeomorphic 2-connected outerplane bipartite graphs

In this section, we consider 2-connected outerplane bipartite graphs with isomorphic resonance graphs. The main result of the section represents a generalization of a result from [1]. Firstly, we need to define two transformations. As usual, for a graph $G$ we denote by $\deg u$ the degree of a vertex $u \in V(G)$.

**Transformation 1.** *Let $G$ be a 2-connected outerplane bipartite graph and $P = (x, y, z)$ a path on three vertices in $G$ such that $\deg y = 2$ and the face containing $P$ is not a 4-cycle. The graph $G'$ is obtained from $G$ by deleting $y$ and identifying vertices $x$ and $z$, see Figure 5.*

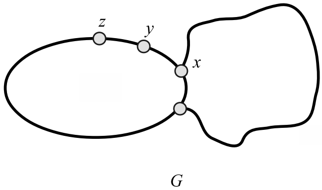


**Figure 5.** Transformation 1.

**Transformation 2.** *Let $G$ be a 2-connected outerplane bipartite graph and $v \in V(G)$ such that $\deg v = k$. Then $v$ belongs to exactly $k-1$ inner faces of $G$. Moreover, let $u_1, \ldots, u_k$ be the neighbours of $v$ ordered in the clockwise direction such that $vu_1$ and $vu_k$ belong to the outer face.*

*(i) If $k \geq 3$, then the graph $G'$ is obtained from $G$ be deleting vertex $v$, adding the path $(v_1, v_2, v_3)$ and inserting edges $v_1u_1$, $v_1u_2$, and $v_3u_i$ for any $i \in \{3, \ldots, k\}$, see Figure 6 (i).*

*(ii) If $k = 2$, then the graph $G'$ is obtained from $G$ be deleting vertex $v$, adding the path $(v_1, v_2, v_3)$ and inserting edges $v_1u_1$ and $v_3u_2$, see Figure 6 (ii).*

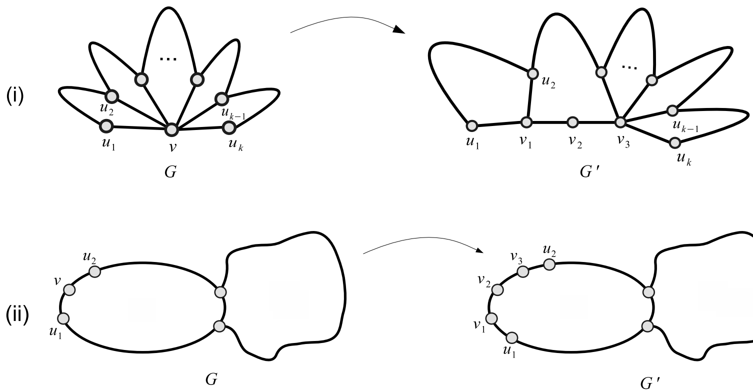


**Figure 6.** Transformation 2.

Note that if $\deg(v) = k \geq 4$, then after applying Transformation 2 the maximum degree of $v_1$ and $v_3$ in graph $G'$ is $k-1$. It is also obvious that the graph $G'$ obtained by Transformation 1 or Transformation 2 is again a 2-connected outerplane bipartite graph.

In the following definition, we generalize the concept of evenly homeomorphic CERS [2] to all 2-connected outerplane bipartite graphs.

**Definition 2.** *Let $G$ and $H$ be two 2-connected outerplane bipartite graphs. Then $G$ is* ***evenly homeomorphic*** *to $H$ if it is possible to successively apply Transformation 1 or 2 on $G$ and $H$ to obtain graphs $G'$ and $H'$, respectively, such that $G'$ and $H'$ are isomorphic. In such a case we write $G \stackrel{R}{\sim} H$.*

It is obvious that the relation $\stackrel{R}{\sim}$ is an equivalence relation on the set of all 2-connected outerplane bipartite graphs. Moreover, if $G$ and $H$ are evenly homeomorphic, then both graphs have the same number of inner faces.

The following two lemmas are also needed.

**Lemma 1.** *Let $G$ and $G'$ be 2-connected outerplane bipartite graphs such that $G'$ is obtained from $G$ by applying Transformation 1 or Transformation 2. Then any two inner edges $e, f \in E(G)$ are also in $E(G')$ and it holds $d_{G'}(e, f) - d_G(e, f) \in \{-2, 0, 2\}$.*

*Proof.* Obviously, if we apply Transformation 1, then the distance between two inner edges $e$ and $f$ remains the same or decreases by 2. On the other hand, after using Transformation 2 the distance between $e$ and $f$ remains the same or increases by 2. ■

**Lemma 2.** *Let $G$ and $H$ be evenly homeomorphic 2-connected outerplane bipartite graphs and let $(s_1, s_2, s_3)$ be an adjacent triple of inner faces in $G$. If $(s_1', s_2', s_3')$ denotes the corresponding adjacent triple of inner faces in $H$, then the triple $(s_1, s_2, s_3)$ is regular if and only if the triple $(s_1', s_2', s_3')$ is regular.*

*Proof.* Let $e \in E(s_1) \cap E(s_2)$, $f \in E(s_2) \cap E(s_3)$, $e' \in E(s_1') \cap E(s_2')$, and $f' \in E(s_2') \cap E(s_3')$. By the definitions of Transformations 1, 2 and Lemma 1, it holds that $d_G(e, f)$ is even if and only if $d_H(e', f')$ is even. Therefore, the triple $(s_1, s_2, s_3)$ is regular if and only if $(s_1', s_2', s_3')$ is regular. ■

Finally, we can state the main result of this section.

**Theorem 1.** *Let $G$ and $H$ be 2-connected outerplane bipartite graphs. If $G$ and $H$ are evenly homeomorphic, then the resonance graph $R(G)$ is isomorphic to the resonance graph $R(H)$.*

*Proof.* Suppose $G$ is a 2-connected outerplane bipartite graph. Moreover, let $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$, be a reducible face decomposition associated with the sequence of inner faces $s_1, s_2, \ldots, s_n$. Also, denote by $s'_i$, $i \in \{1, \ldots, n\}$, the corresponding inner faces of graph $H$, which give the reducible face decomposition $RFD(H_1, H_2, \ldots, H_n)$ such that $H_n = H$.

We show that for any $r \in \{2, \ldots, n\}$, the set of binary codes $B_r$ of the graph $G_r$ obtained by Algorithm 1 coincides with the set of binary codes $B'_r$ of the graph $H_r$. Consequently, the resonance graphs $R(G_r)$ and $R(H_r)$ are isomorphic for all $r \in \{2, \ldots, n\}$, which implies that $R(G)$ and $R(H)$ are isomorphic. We proceed by induction on the number of inner faces.

Obviously, the sets of binary codes $B_2$ and $B'_2$ are equal. Next, assume that for some $r \geq 2$ the sets of codes $B_r$ and $B'_r$ coincide. Let $s_j$ be the face of $G_{r+1}$ from the set $\{s_1, \ldots, s_r\}$ that is adjacent to $s_{r+1}$. In addition, define $s_i$ as the face with the smallest index among all the adjacent inner faces of $s_j$. Analogously, we also define $s'_j$ and $s'_i$ in the graph $H_{r+1}$. By Lemma 2 we obtain that the adjacent triple of inner faces $(s_i, s_j, s_{r+1})$ is regular if and only if $(s'_i, s'_j, s'_{r+1})$ is regular. Hence, by Algorithm 1 we obtain $B_{r+1} = B'_{r+1}$. ■

We conclude the section with the following open problem.

**Problem.** *Characterize 2-connected outerplane bipartite graphs with isomorphic resonance graphs.*

# 5 Resonance graphs of 2-connected outerplane bipartite graphs and CERS

In this final section, we firstly show that the set of all resonance graphs of 2-connected outerplane bipartite graphs coincides with the set of all resonance graphs of CERS. Next, we consider 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes.

**Theorem 2.** *For any* 2*-connected outerplane bipartite graph* $G$ *there exists a CERS* $H$ *such that* $G$ *and* $H$ *are evenly homeomorphic. Consequently, the resonance graphs* $R(G)$ *and* $R(H)$ *are isomorphic.*

*Proof.* Let $G$ be a 2-connected outerplane bipartite graph such that $G$ is not a CERS. Then there exists a vertex $v \in V(G)$ for which $\deg v = k \geq 4$. After applying Transformation 2 on $v$, we obtain a 2-connected outerplane bipartite graph $G_1$ with three new vertices $v_1$, $v_2$, $v_3$, see Figure 6 (i). It is easy to see that $\deg v_1 = 3$, $\deg v_2 = 2$, and $\deg v_3 = k-1$. Note that $G$ and $G_1$ are evenly homeomorphic and by Theorem 1 the resonance graphs $R(G)$ and $R(G_1)$ are isomorphic. Then, we repeat the same procedure until every vertex of the transformed graph has degree at most 3. Consequently, we obtain a sequence of graphs $G_1, G_2, \ldots, G_m$, where $G$ and $G_m$ are evenly homeomorphic and the resonance graphs $R(G)$ and $R(G_m)$ are isomorphic. Let $H = G_m$. Since $H$ is a 2-connected outerplane bipartite graph with the degree of every vertex at most 3, it is a CERS. ■

Next, we characterize 2-connected outerplane bipartite graphs whose resonance graphs are daisy cubes [12]. Therefore, some additional definitions are needed.

Let $B = \{0, 1\}$ and $B^n$ the set of all binary strings of length $n$. Moreover, let $\leq$ be a partial order on $B^n$ defined with $(u_1, \ldots, u_n) \leq (v_1, \ldots, v_n)$ if and only if $u_i \leq v_i$ holds for all $i \in \{1, \ldots, n\}$. For $X \subseteq B^n$, we define the graph $Q_n(X)$ as the subgraph of $Q_n$ as $Q_n(X) = Q_n\left[\{u \in B^n \mid u \leq x \text{ for some } x \in X\}\right]$ and say that $Q_n(X)$ is a *daisy cube* (generated by $X$).

Furthermore, we generalize the concept of regular CERS from [2] to all 2-connected outerplane bipartite graphs.

**Definition 3.** *If a* 2*-connected outerplane bipartite graph* $G$ *has at most two inner faces or if every adjacent triple of inner faces of* $G$ *is regular, then* $G$ *is called* ***regular****.*

The following result was proved in [2].

**Theorem 3.** *[2] If $G$ is a CERS, then $G$ is regular if and only if the resonance graph $R(G)$ is a daisy cube.*

Finally, we generalize the above result to all 2-connected outerplane bipartite graphs.

**Theorem 4.** *If $G$ is a 2-connected outerplane bipartite graph, then $G$ is regular if and only if the resonance graph $R(G)$ is a daisy cube.*

*Proof.* Let $G$ be a 2-connected outerplane bipartite graph. By Theorem 2, there exists a CERS $H$ such that $G$ and $H$ are evenly homeomorphic and the resonance graphs $R(G)$ and $R(H)$ are isomorphic. By Lemma 2, $G$ is regular if and only if $H$ is regular. Also, by Theorem 3, $H$ is regular if and only if $R(H)$ is a daisy cube. Therefore, $G$ is regular if and only if the resonance graph $R(H)$ is a daisy cube and this is further equivalent to $R(G)$ being a daisy cube. ■

***Acknowledgment*:** Niko Tratnik acknowledges the financial support from the Slovenian Research Agency (research programme No. P1-0297 and project No. N1-0285). Petra Žigert Pleteršek acknowledges the financial support from the Slovenian Research Agency (research programme No. P1-0297 and project No. NK-0001).

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