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Bordering of symmetric matrices and an application to the minimum number of distinct eigenvalues for the join of graphs

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A B S T R A C T

An important facet of the inverse eigenvalue problem for graphs is to determine the minimum number of distinct eigenvalues of a particular graph. We resolve this question for the join of a connected graph with a path. We then focus on bordering a matrix and attempt to control the change in the number of distinct eigenvalues induced by this operation. By applying bordering techniques to the join of graphs, we obtain numerous results on the nature of the minimum number of distinct eigenvalues as vertices are joined to a fixed graph.

1. Introduction

Given a simple graph G on $|G| = n$ vertices, let $S(G)$ denote the set of all $n \times n$ real symmetric matrices $A = (a_{ij})$ such that, for $i \neq j$, $a_{ij} \neq 0$ if and only if i and j are adjacent in G. There are no restrictions on the main diagonal entries of A. The inverse eigenvalue problem for G asks which possible multi-sets of eigenvalues (spectra) occur in the class $S(G)$. This is a very difficult problem for most graphs (which generally remains open, except for some sporadic graphs, including, for example, paths, cycles, complete graphs and some basic families of trees). Considerable work on this important problem has occurred over the past several decades (see the recent book [12]). Our work generally pertains to multiplicity lists associated to the spectra of matrices in $S(G)$.

Suppose A is an $n \times n$ real symmetric matrix and λ is an eigenvalue of A, that is $\lambda \in \sigma(A)$, where $\sigma(A)$ denotes the collection of eigenvalues (spectrum) of the matrix A. We let $m_A(\lambda)$ denote the multiplicity of λ in $\sigma(A)$; if a scalar λ is not an eigenvalue of a matrix A then we define $m_A(\lambda) = 0$. Perhaps one of the most important results on the eigenvalues of real symmetric matrices is Cauchy's interlacing inequalities, from which it immediately follows that if A is an $n \times n$ principal submatrix of an $(n+1) \times (n+1)$ real symmetric matrix B, then $|m_A(\lambda) - m_B(\lambda)| \leq 1$ for any scalar λ. Another way to view the principal submatrix A of B is to consider that B was obtained from A by bordering A with one row and column, and since the spectrum is invariant under permutation similarity, we might as well assume that the new row and column added to A are the first row and column of B. More generally, given a symmetric $n \times n$ matrix A and $r \geq 1$, an *r-bordering* of A is any symmetric $(n + r) \times (n + r)$ matrix B which contains A as a trailing principal $n \times n$ submatrix (that is, A lies in rows and columns indexed by $\{r+1, r+2, \ldots, r+n\}$ of B), and it follows that $|m_A(\lambda) - m_B(\lambda)| \leq r$. For brevity, we will also let $A[S]$ denote the principal submatrix of A lying in rows and columns indexed by $S \subseteq \{1, 2, \ldots, n\}$.

We define the *maximum multiplicity* of a symmetric matrix A to be

$$M(A) = \max\{m_A(\lambda) : \lambda \in \sigma(A)\},$$

and the *maximum multiplicity of a graph* G is

$$M(G) = \max\{M(A) : A \in S(G)\}.$$

Let $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}_0^k$ be a sequence of k nonnegative integers and $q(\mathbf{m}) = |\{i \colon m_i > 0\}|$. We say $\mathbf{m}$ is an ordered multiplicity list for a symmetric matrix A, if A possesses $q(\mathbf{m})$ distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_{q(\mathbf{m})}$ and $m_A(\lambda_i) = m_{j_i}$ for $i = 1, 2, \ldots, q(\mathbf{m})$, where $1 \leq j_1 < j_2 < \cdots < j_{q(\mathbf{m})} \leq k$ are the $q(\mathbf{m})$ indices j with $m_j > 0$. In this case we write $\mathbf{m} = \mathbf{m}(A)$. For any matrix A, we write $q(A) = k$ if A has k distinct eigenvalues. For a given graph G, we define

$$q(G) = \min\{q(A) : A \in S(G)\}.$$

It is easy to observe that for any graph G we have $q(G) \geq \lceil \frac{|G|}{M(G)} \rceil$. In this paper our goal is to investigate the behaviour of $q(\cdot)$ upon appending vertices to a fixed graph G. Here, when a vertex is appended, all possible edges between the existing vertices and the new vertex are inserted.

We let K_n $(n \geq 1)$, P_n $(n \geq 1)$, C_n $(n \geq 3)$ denote the complete graph, the path, and the cycle on n vertices. If G and H are two graphs, then the *join of* G *and* H, denoted by $G \vee H$, is the graph obtained from the union of G and H by adding all edges with one endpoint in G and one endpoint in H. Hence, our goal in this paper is to investigate the behaviour of $q(G \vee H)$ for various graphs G and H.

Given a graph G, let $V(G)$ denote its vertex set. For $v \in V(G)$, we define $\text{jdup}(G, v)$ to be the supergraph of G obtained from G by duplicating v, with an edge connecting v to its duplicate. That is, $V(\text{jdup}(G, v)) = V(G) \cup \{w\}$, where $w \notin V(G)$, and $\{v, w\} \in E(\text{jdup}(G, v))$, and w has the same neighbours as v in $\text{jdup}(G, v)$. As observed in [2, Theorem 3] and [15, Lemma 2.9],

$$q(\text{jdup}(G, v)) \leq q(G). \tag{1}$$

Since $K_{n+1} \vee H = \text{jdup}(K_n \vee H, v)$ for any vertex $v \in K_n$, we see that $q(K_n \vee H)$ is monotone decreasing in n.

One of the first examples considered along these lines was the case of determining $q(K_1 \vee P_n)$. In [5, Example 4.5] it was shown that $q(K_1 \vee P_n) = \lceil \frac{n+1}{2} \rceil$ for $n \geq 2$.

We note here that the lower bound on $q(K_1 \vee P_n)$ follows from Cauchy's interlacing inequalities since $q(P_n) = n$. Another important example is the star, or $S_n = K_1 \vee E_{n-1}$, where E_k represents the empty graph on k vertices. It is straightforward to show that $M(S_n) = n-2$ and that $q(S_n) = 3$. We remark that the star has played a key role in the inverse eigenvalue problem for graphs (mostly in the case of trees), and in many ways was a critical tool used in [7] to establish a converse to Cauchy's interlacing inequalities. This technique has been extended and adapted to broaden the scope of which spectra can be realized by a graph that contains a dominating vertex (see, for example, [3,14,16]).

Merging the concepts of bordering a particular matrix and joining a vertex to a given graph, we are interested in determining the minimum number of distinct eigenvalues of a graph joined by a sequence of vertices, and we develop techniques, based in part of the nature of ordered multiplicity lists and eigenvectors, to aid this computation. We begin, in Section 2, with the necessary background and present a general upper bound (Theorem 2.2) on $q(G \vee H)$ for connected graphs G and H, which reduces to a simple exact formula in the case $G = P_n$. In Section 3 we investigate the borderings of a given symmetric matrix. Theorem 3.1 describes in detail how a 1-bordering can change the spectrum of a symmetric matrix, and in Proposition 3.5 we find a necessary and sufficient condition for the existence of an r-bordering of a symmetric matrix with a given value of q. In Section 4 we make several observations on the patterns of such bordered matrices, and we apply them to estimate $q(K_n \vee H)$ when H is either a hypercube or a cycle. Finally, in Section 5, we pay particular attention to some possible limitations of our methods (Corollary 5.2).

2. General graphs and paths

It is known that if G and H are two connected graphs and $|G| = |H|$, then $q(G \vee H) = 2$ (see [11, Theorem 5.2]). This result was extended in [16,17] where it was shown that $q(G \vee H) = 2$ if G and H are connected graphs with $\big||G| - |H|\big| \leq 2$. Moreover, for trees T_1 and T_2 we have $q(T_1 \vee T_2) = 2$ if and only if $\big||T_1| - |T_2|\big| \leq 2$, so in this case the result is sharp.

An important notion used in [16] is generic realizability. Recall that a matrix (vector) is said to be *nowhere zero* if none of its entries is zero. Suppose G is a graph with $|G| = n$ vertices and σ is a collection of realizable eigenvalues in $S(G)$ (with multiplicities), i.e., $\sigma = \sigma(A)$ for some $A \in S(G)$. The collection σ is said to be *generically realizable* in $S(G)$ if, for any finite set $\mathcal{Y}$ of nonzero vectors in $\mathbb{R}^n$, there is an orthogonal matrix U such that Uy is nowhere zero for all $y \in \mathcal{Y}$, and $UDU^T \in S(G)$, where D is a diagonal matrix with eigenvalues equal to σ (see [16] for more details). Observe that if we form an $n \times |\mathcal{Y}|$ matrix Y from the columns of $\mathcal{Y}$ then this property ensures that there is an orthogonal matrix U so that $UDU^T \in S(G)$ and UY is nowhere zero, whenever Y has no zero column (that is, no column of Y is the zero vector in $\mathbb{R}^n$). We will use the following result.

Theorem 2.1. *[16, Theorem 2.5] Suppose G is a connected graph. Then any σ with $|G|$ distinct elements is generically realizable in $S(G)$.*

Theorem 2.1 allows us to construct matrices in $S(G \vee H)$ with some desired spectral properties, using matrices $A \in S(G)$ and $B \in S(H)$ with distinct eigenvalues. In particular, in the next result we explore this idea of constructing matrices in $S(G \vee H)$ with bounded number of distinct eigenvalues.

Theorem 2.2. *Suppose G and H are two connected graphs. If k is a positive integer and $|G| \le |H| \le k|G| + k + 1$, then*

$$q(G \vee H) \le k + 1.$$

In particular, for any connected graphs G and H with $\max\{|G|, |H|\} \neq 1$ *we have:*

$$q(G \vee H) \le \left\lceil \frac{|G| + |H|}{\min\{|G|, |H|\} + 1} \right\rceil .$$

Proof. Suppose $|G| = n$, $|H| = m$ and k is a positive integer with $n \le m \le kn + k + 1$. To prove the first claim, we will construct a matrix in $S(G \vee H)$ with distinct eigenvalues contained in $\mathcal{S} := \{\lambda_j\}_{j=1}^{k+1}$ for any chosen set $\mathcal{S}$ of $k+1$ distinct numbers. To this end, choose real numbers $\lambda_1 < \cdots < \lambda_{k+1}$, and integers k_i with $1 \le k_i \le k$ for $i = 1, \dots, n$, satisfying:

$$0 \le k' := m - \sum_{i=1}^{n} k_i \le k + 1.$$

Now select n sets of real numbers $\mathcal{M}_i := \{\mu_{i,1}, \dots, \mu_{i,k_i}\}$, $i = 1, \dots, n$, where we assume

$$\mu_{i,1} < \cdots < \mu_{i,k_i}.$$

Furthermore, we assume that $\mathcal{M}_i$ strictly interlaces $\{\lambda_1, \dots, \lambda_{k_i+1}\}$, that the numbers $\mu_{i,j}$ for $j = 1, \dots, k_i$ and $i = 1, \dots, n$ are all distinct, and finally we demand that numbers $a_i := \left(\sum_{j=1}^{k_i+1} \lambda_j\right) - \left(\sum_{j=1}^{k_i} \mu_{i,j}\right)$, $i = 1, \dots, n$, are all distinct. Writing $\operatorname{diag}(x_1, \dots, x_m)$ for the diagonal matrix with main diagonal $(x_1, \dots, x_m)$, we define

$$\begin{aligned} \Lambda &:= \operatorname{diag}(\lambda_1, \dots, \lambda_{k'}), \\ D_i &:= \operatorname{diag}(\mu_{i,1}, \dots, \mu_{i,k_i}), \ i = 1, \dots, n, \\ D_a &:= \operatorname{diag}(a_1, \dots, a_n), \\ D_\mu &:= D_1 \oplus \cdots \oplus D_n. \end{aligned}$$

By [10, Theorem 4.2] and our strict eigenvalue interlacing requirement, for $i = 1, \dots, n$ there exist matrices:

$$M_i := \begin{pmatrix} a_i & \mathbf{b}_i^T \\ \mathbf{b}_i & D_i \end{pmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_{k_i+1}$, where $\mathbf{b}_i$ is a nowhere zero vector. Clearly, the distinct eigenvalues of $M := M_1 \oplus \cdots \oplus M_n \oplus \Lambda$ are contained in $\{\lambda_1, \ldots, \lambda_{k+1}\}$, and in particular, $q(M) \leq k+1$. The same is true for the matrix:

$$M' := \begin{pmatrix} D_a & B^T & 0 \\ B & D_\mu & 0 \\ 0 & 0 & \Lambda \end{pmatrix},$$

where $B = \bigoplus_{i=1}^n \mathbf{b}_i$, since M' is permutationally similar to M.

Observe that the $n \times n$ matrix D_a and the $m \times m$ matrix $D_\mu \oplus \Lambda$ are both diagonal matrices with distinct eigenvalues. By Theorem 2.1, their spectra are generically realizable for G and H, respectively. Since the $m \times n$ matrix $Y := \binom{B}{0}$ has no zero column, by generic realizability for H there is an orthogonal matrix V so that $V(D_\mu \oplus \Lambda)V^T \in S(H)$ and VY is nowhere zero. Since $(VY)^T = Y^T V^T$ is nowhere zero and so has no zero column, by generic realizability for G there is an orthogonal matrix U so that $UD_aU^T \in S(G)$ and UY^TV^T is nowhere zero. Now

$$(U \oplus V)M'(U^T \oplus V^T) = \begin{pmatrix} UD_aU^T & UY^TV^T \\ VYU^T & V(D_\mu \oplus \Lambda)V^T \end{pmatrix} \in S(G \vee H),$$

so $q(G \vee H) \leq k+1$ as required.

To see that the second claim follows from the first, observe that if $\max\{|G|, |H|\} > 1$, then $|G|+|H| > \min\{|G|, |H|\}+1$, so $k := \left\lceil \frac{|G|+|H|}{\min\{|G|,|H|\}+1} \right\rceil - 1$ is a positive integer, and if $|G| \leq |H|$, then $|G| \leq |H| \leq k|G| + k + 1$, so $q(G \vee H) \leq k+1 = \left\lceil \frac{|G|+|H|}{\min\{|G|,|H|\}+1} \right\rceil$. By symmetry, the same holds if $|H| \leq |G|$. □

We remark that the hypothesis $|G| \leq |H| \leq k|G| + k + 1$ in Theorem 2.2 cannot be relaxed in general, since if we take $k = 1$ and G and H are trees with $|H| > k|G|+k+1 = |G|+2$, then $q(G \vee H) > 2 = k+1$ by [16, Example 3.5].

The upper bound of Theorem 2.2 is sharp when H is a path, as shown below.

Corollary 2.3. *If $m > 1$ and G is a connected graph with $|G| = n \leq m$, then*

$$q(G \vee P_m) = \left\lceil \frac{n+m}{n+1} \right\rceil.$$

Proof. Let X be a matrix in $S(G \vee P_m)$. Since X has an $m \times m$ principal submatrix corresponding to P_m, this submatrix must have distinct eigenvalues. By eigenvalue interlacing, the matrix X can have maximum eigenvalue multiplicity at most $n+1$. Hence

$$q(X) \geq \left\lceil \frac{|G \vee P_m|}{M(X)} \right\rceil \geq \left\lceil \frac{n+m}{n+1} \right\rceil.$$

The opposite inequality was established in Theorem 2.2. □

Remark 2.4. In the case $G = P_n$ where $2 \leq n \leq m$, the formula of Corollary 2.3 improves on the upper bound $q(P_n \vee P_m) \leq \lceil \frac{n+m}{2} \rceil$ which follows from [4, Corollary 49], since $P_n \vee P_m$ contains a Hamiltonian cycle.

We conclude this section with a theorem which resolves a question from [16, Remark 3.13].

Corollary 2.5. *If $m, n \geq 2$, then*

$$q(K_n \vee P_m) = \left\lceil \frac{n+m}{n+1} \right\rceil .$$

Proof. For $n \leq m$, this is a special case of Corollary 2.3. For $n \geq m$, note that $\lceil \frac{m+n}{n+1} \rceil = 2$. We know from Theorem 5.2 in [11] that $q(K_n \vee P_n) = 2$, and for $n > m$, it follows that $q(K_n \vee P_m) = 2$ by applying the notion of join duplication (jdup) and the inequality presented in (1). □

3. Bordering

Recall from the introduction that an *r-bordering* of a symmetric $n \times n$ matrix A is any symmetric $(n+r) \times (n+r)$ matrix B which contains A as its $n \times n$ trailing principal submatrix of B. Building upon the classical results derived from Cauchy's interlacing inequalities that characterize all possible eigenvalues of a 1-bordering of A, we aim to understand the fewest number of distinct eigenvalues possible for an r-bordering of A.

First we have a look at 1-borderings, noting that any r-bordering of A can be obtained by repeated 1-bordering.

Theorem 3.1. *Let A be an $n \times n$ symmetric matrix and A' a 1-bordering of A. The following statements are equivalent:*

1. *$\mathcal{N}$ is the set of distinct eigenvalues λ of A' that satisfy $m_{A'}(\lambda) = m_A(\lambda) + 1$, and $\mathcal{R}_0$ is the set of distinct eigenvalues λ of A that satisfy $m_{A'}(\lambda) = m_A(\lambda) - 1$.*
2. *$A' = \begin{pmatrix} \alpha & \mathbf{b}^T U_0^T \\ U_0 \mathbf{b} & A \end{pmatrix}$ where $k := |\mathcal{R}_0|$, U_0 is an $n \times k$ matrix with $U_0^T U_0 = I_k$, and $U_0^T A U_0$ is a $k \times k$ diagonal matrix D_0 with distinct eigenvalues equal to $\mathcal{R}_0$. Further, $\mathbf{b} \in \mathbb{R}^k$ is a nowhere zero vector so that the matrix*

$$B = \begin{pmatrix} \alpha & \mathbf{b}^T \\ \mathbf{b} & D_0 \end{pmatrix}$$

has eigenvalues $\mathcal{N}$.

If the above hold, then A' is similar to a matrix of the form $D_{\mathcal{N}} \oplus D_1$ for some diagonal matrix D_1 via an orthogonal similarity using

$$W = \begin{pmatrix} \mathbf{v}^T & 0 \\ U_0 V_0 & U_1 \end{pmatrix}, \tag{2}$$

where $V = \begin{pmatrix} \mathbf{v}^T \\ V_0 \end{pmatrix} \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{N}|}$ is an orthogonal matrix that satisfies $V^T B V = D_{\mathcal{N}}$, and $U = \begin{pmatrix} U_0 & U_1 \end{pmatrix}$ is an orthogonal matrix that satisfies $U^T A U = D_0 \oplus D_1$.

Proof. $(1 \Rightarrow 2)$ Let $\lambda_1, \ldots, \lambda_q$ be the distinct eigenvalues of A with multiplicities $m_i := m_A(\lambda_i)$, $i = 1, \ldots, q$, and let U' be an orthogonal matrix that diagonalizes A, that is, ${U'}^T A U' = \oplus_{j=1}^q \lambda_j I_{m_j}$. Then for some $\alpha \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$, we have

$$A_1' := (1 \oplus {U'}^T) A' (1 \oplus U') = \begin{pmatrix} \alpha & \mathbf{a}^T \\ \mathbf{a} & \oplus_{j=1}^q \lambda_j I_{m_j} \end{pmatrix}.$$

Write $\mathbf{a}^T = \begin{pmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_q^T \end{pmatrix}$, where $\mathbf{a}_i \in \mathbb{R}^{m_i}$. Choose orthogonal matrices $Z_i \in \mathbb{R}^{m_i \times m_i}$ that satisfy $Z_i \mathbf{a}_i = b_i \mathbf{e}_1$, where $b_i \in \mathbb{R}$ and $\mathbf{e}_1$ denotes the basic unit vector in $\mathbb{R}^{m_i}$ whose first element is equal to 1. Note that $b_i \neq 0$ if and only if $\lambda_i \in \mathcal{R}_0$ (see for example [18, Lemma 5.1] for the nontrivial implication). Applying the orthogonal similarity $1 \oplus (\oplus_{i=1}^q Z_i)$ to A_1', followed by a permutation similarity $1 \oplus P$, we see that A' is orthogonally similar to $B \oplus D_1$, where D_1 is a diagonal $(n-k) \times (n-k)$ matrix,

$$B = \begin{pmatrix} \alpha & \mathbf{b}^T \\ \mathbf{b} & D_0 \end{pmatrix}$$

and $\mathbf{b} \in \mathbb{R}^k$ is a nowhere zero vector. In summary, $U := U'(\oplus_{i=1}^k Z_i^T) P$ satisfies $U^T A U = D_0 \oplus D_1$ and $(1 \oplus U)^T A' (1 \oplus U) = B \oplus D_1$. Writing $U = \begin{pmatrix} U_0 & U_1 \end{pmatrix}$ where $U_0 \in \mathbb{R}^{n \times k}$ and computing $A' = (1 \oplus U)(B \oplus D_1)(1 \oplus U^T)$ gives the form for A' as in item 2.

$(2 \Rightarrow 1)$ Let A' and U_0 be as in item 2, and $U_1 \in \mathbb{R}^{n \times (n-k)}$ be such that $U := \begin{pmatrix} U_0 & U_1 \end{pmatrix}$ is orthogonal and $U^T A U$ is a diagonal matrix $D_0 \oplus D_1$. From

$$(1 \oplus U^T) A' (1 \oplus U) = B \oplus D_1$$

we conclude that A' has eigenvalues as stated in item 1.

To prove the final claim we note that:

$$W := (1 \oplus U)(V \oplus I_{n-k}) = \begin{pmatrix} \mathbf{v}^T & 0 \\ U_0 V_0 & U_1 \end{pmatrix}$$

and $W^T A' W = (V^T \oplus I)(B \oplus D_1)(V \oplus I) = D_{\mathcal{N}} \oplus D_1$, as claimed. □

Theorem 3.1 provides a construction of a 1-bordering of a symmetric matrix, subject to quite general eigenvalue constraints. Our first application of this theorem produces a known result [13, Theorem 4.3.10]. We include it here mostly to establish notation that we will depend on in the rest of this section.

Corollary 3.2. *Let A be an $n \times n$ symmetric matrix, $\mathcal{R}$ the set of distinct eigenvalues of A, and $\mathcal{R}_0 \subseteq \mathcal{R}$. If $\mathcal{N}$ is any set of $|\mathcal{R}_0|+1$ distinct real numbers which strictly interlaces $\mathcal{R}_0$, then there is a 1-bordering A' of A so that for $\lambda \in \mathbb{R}$,*

$$m_{A'}(\lambda) = \begin{cases} m_A(\lambda) - 1 & \text{if } \lambda \in \mathcal{R}_0, \\ m_A(\lambda) + 1 & \text{if } \lambda \in \mathcal{N}, \\ m_A(\lambda) & \text{otherwise}, \end{cases}$$

where $m_{A'}(\lambda) = 0$ means that λ is not an eigenvalue of A'.

Proof. Let D_0 be a diagonal matrix with distinct diagonal elements equal to the elements in $\mathcal{R}_0$. By [6], since $\mathcal{N}$ strictly interlaces $\mathcal{R}_0$, there exist $a \in \mathbb{R}$ and a (nowhere zero) vector $\mathbf{b} \in \mathbb{R}^{|\mathcal{R}_0|}$ so that the matrix

$$B = \begin{pmatrix} a & \mathbf{b}^T \\ \mathbf{b} & D_0 \end{pmatrix}$$

has the set of eigenvalues equal to $\mathcal{N}$. The result now follows from Theorem 3.1. □

Starting with the eigenvalues of A, we will reduce the number of distinct eigenvalues of an r-bordering of A by removing all eigenvalues from different intervals. Along these lines, we let $m_A(\alpha, \beta)$ denote the sum of the multiplicities of all eigenvalues λ of A that are contained in the open interval (α, β), where $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$ with $\alpha < \beta$.

The following straightforward consequence of eigenvalue interlacing produces a lower bound on r for an r-bordering to have no eigenvalues in a given interval.

Lemma 3.3. *If M is an r-bordering of a symmetric matrix A and $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$ with $\alpha < \beta$, then*

$$|m_A(\alpha, \beta) - m_M(\alpha, \beta)| \leq r.$$

Proof. The eigenvalues of A and any 1-bordering of A must interlace by Cauchy's interlacing inequalities, which establishes the case $r = 1$. In general, M is obtained by r successive 1-borderings of A, and the statement follows immediately. □

Table 1
The list of possible values of p_2 and the corresponding parameters inside the formula for $C(\mathbf{m}, t)$ for $\mathbf{m} = (1, 2, 5, 5, 3, 1)$ and $t = 3$. Note that in each case, we have $p_1 = 1$ and $p_3 = 6$, whereas p_2 can vary. The list of multiplicities in each of the two gaps derived from each value of p_2 and the corresponding maximum gap multiplicities are given. Taking the minimum of the final column, we obtain $C(\mathbf{m}, 3) = 7$.

p_2	$g_{\mathbf{m}}(p_1, p_2)$	$g_{\mathbf{m}}(p_2, p_3)$	maximum gap multiplicity
1	0	$m_2 + m_3 + m_4 + m_5 = 2 + 5 + 5 + 3$	15
2	0	$m_3 + m_4 + m_5 = 5 + 5 + 3$	13
3	$m_2 = 2$	$m_4 + m_5 = 5 + 3$	8
4	$m_2 + m_3 = 2 + 5$	$m_5 = 3$	7
5	$m_2 + m_3 + m_4 = 2 + 5 + 5$	0	12
6	$m_2 + m_3 + m_4 + m_5 = 2 + 5 + 5 + 3$	0	15

Let $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{N}_0^k$ be an ordered multiplicity list of a symmetric matrix. For $2 \le t \le k$, we define

$$C(\mathbf{m}, t) = \min_{1=p_1 \le p_2 \le \cdots \le p_t = k} \left(\max_{1 \le i \le t-1} g_{\mathbf{m}}(p_i, p_{i+1}) \right) \tag{3}$$

where

$$g_{\mathbf{m}}(p_i, p_{i+1}) := \sum_{j=p_i+1}^{p_{i+1}-1} m_j.$$

In colloquial terms, $C(\mathbf{m}, t)$ is the solution to the problem of minimizing the largest "gap multiplicity" $g_{\mathbf{m}}$ of $\mathbf{m}$, over the gaps given by the various choices of t "gap boundaries" $1 = p_1 \le p_2 \le \cdots \le p_t = k$.

Example 3.4. To illustrate the preceding definition, we demonstrate how $C(\mathbf{m}, t)$ is computed for the case when $\mathbf{m} = (1, 2, 5, 5, 3, 1)$ and $t = 3$, by listing the possible values of the gap multiplicities for the various choices of gap boundary p_2 in Table 1. The minimum of the maximum gap multiplicities is 7, so $C(\mathbf{m}, 3) = 7$. One can also determine that $C(\mathbf{m}, 4) = 3$, $C(\mathbf{m}, 5) = 2$, and $C(\mathbf{m}, 2) = 15$.

Note that $q(\mathbf{m}) \le t$ if and only if $C(\mathbf{m}, t) = 0$. Indeed, if $q(\mathbf{m}) \le t$, then we may assume that $\mathbf{m} \in \mathbb{N}^k$ where $k = q(\mathbf{m}) \le t$, and then choosing $p_i = \min\{i, k\}$ in (3) shows that $C(\mathbf{m}, t) = 0$; conversely, if $C(\mathbf{m}, t) = 0$ is attained for some particular $1 = p_1 \le p_2 \le \cdots \le p_t = k$, then $m_i = 0$ for all $i \in [k] \setminus \{p_1, \dots, p_t\}$, hence $q(\mathbf{m}) \le t$. Hence, we can view $C(\mathbf{m}, t)$ as a measure of how far the multiplicity list $\mathbf{m}$ is from having $q(\mathbf{m}) = t$. This will be made more precise in the next proposition.

Proposition 3.5. *Let A be a symmetric matrix with $k \ge 2$ distinct eigenvalues and ordered multiplicity list $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{N}^k$, and let $2 \le t \le k$. For $r \in \mathbb{N}_0$ the following statements are equivalent:*

1. *there is an r-bordering M of A with $q(M) \leq t$;*
2. $C(\mathbf{m}, t) \leq r$.

Proof. Let $\lambda_1 < \cdots < \lambda_k$ be the distinct eigenvalues of A, with $m_A(\lambda_i) = m_i$ for $i = 1, \ldots, k$.

(1 ⇒ 2) Suppose $\mu_1 < \cdots < \mu_\tau$ are the distinct eigenvalues of some r-bordering M of A, where $\tau \leq t$. By eigenvalue interlacing, we have $\lambda_j \in [\mu_1, \mu_\tau]$ for every j. Hence, there is a unique i_0 with $\lambda_1 \in [\mu_{i_0}, \mu_{i_0+1}) = [\nu_1, \nu_2)$, where $\nu_i := \mu_{i_0-1+i}$. For $1 \leq i \leq \tau - i_0$, define

$$p_i := \min\{j \colon 1 \leq j \leq k, \lambda_j \in [\nu_i, \nu_{i+1})\}$$

and let $p_i := k$ for $i > \tau - i_0$. Then $1 = p_1 \leq p_2 \leq \cdots \leq p_t = k$. Moreover, if $p_i < j < p_{i+1}$, then $\lambda_j \in (\nu_i, \nu_{i+1})$, so

$$g_{\mathbf{m}}(p_i, p_{i+1}) = \sum_{j: p_i < j < p_{i+1}} m_A(\lambda_j) \leq m_A(\nu_i, \nu_{i+1}) \leq r,$$

where the final inequality follows from Lemma 3.3, since $m_M(\nu_i, \nu_{i+1}) = 0$. Hence,

$$C(\mathbf{m}, t) \leq \max_{1 \leq i \leq t-1} g_{\mathbf{m}}(p_i, p_{i+1}) \leq r,$$

as required.

(2 ⇒ 1) If $C(\mathbf{m}, t) = 0$, then $q(A) \leq t$ and we can take $r = 0$. From now on we assume $C(\mathbf{m}, t) > 0$. Since $r \geq C(\mathbf{m}, t)$, there exist $p_1 = 1 < p_2 < \cdots < p_{t'} = k$ where $t' \leq t$ so that

$$m_A(\lambda_{p_i}, \lambda_{p_{i+1}}) = g_{\mathbf{m}}(p_i, p_{i+1}) \leq r, \quad 1 \leq i < t'.$$

It suffices to find a 1-bordering M_1 of A so that $\sigma(M_1) \subseteq [\lambda_1, \lambda_k]$ and

$$m_{M_1}(\lambda_{p_i}, \lambda_{p_{i+1}}) \leq \max\{r - 1, 0\}, \quad 1 \leq i < t',$$

since we can then continue inductively to find $A = M_0, M_1, \ldots, M_r =: M$, where $M_{\ell+1}$ is a 1-bordering of M_ℓ, so that $m_{M_r}(\lambda_{p_i}, \lambda_{p_{i+1}}) = 0$ for $1 \leq i < t'$ and every eigenvalue of M_r is in $[\lambda_1, \lambda_k]$, hence M_r has only the t' distinct eigenvalues $\{\lambda_{p_1}, \ldots, \lambda_{p_{t'}}\}$.

To show that such a matrix M_1 exists, first enumerate the open intervals $L_i := (\lambda_{p_i}, \lambda_{p_{i+1}})$ which contain at least one eigenvalue of A as $L_{i_1}, \ldots, L_{i_s}$, where $1 \leq i_1 < \cdots < i_s < t'$, and choose $\mu_j \in \sigma(A) \cap L_{i_j}$ for $1 \leq j \leq s$. (The assumption $C(\mathbf{m}, t) > 0$ guarantees that at least one such interval exists.) Let $\mathcal{R}_0 = \{\mu_1, \ldots, \mu_s\}$, and choose any set $\mathcal{N} \subseteq \{\lambda_{p_1}, \ldots, \lambda_{p_{t'}}\}$ of size $s + 1$ which strictly interlaces $\mathcal{R}_0$. The matrix constructed in Corollary 3.2 then has the desired properties. □

Given an $n \times n$ symmetric matrix A with $\sigma(A) = \{\lambda_1^{(m_1)}, \dots, \lambda_k^{(m_k)}\}$, $\sum_{i=1}^k m_i = n$, the general procedure to find an r-bordering matrix M of A with $q(M) \leq t$ is shown in Algorithm 3.1 below. Note that we may have some freedom in how we choose the sets $\mathcal{R}_0$ and $\mathcal{N}$ in each step. One possible choice is given in the proof of Proposition 3.5, and we show all possible choices for a particular case in Example 4.4.

Algorithm 3.1 Find an r-bordering matrix M of A with $q(M) \leq t$.

1. Choose an integer t, $2 \leq t \leq q(A)$. Define $M_0 := A$, $r := C(\mathbf{m}, t)$.
2. For $\ell = 1, \dots, r$, use Corollary 3.2 to construct an $(n+\ell) \times (n+\ell)$ matrix M_ℓ such that

$$C(\mathbf{m}(M_\ell), t) = C(\mathbf{m}(M_{\ell-1}), t) - 1.$$

   Note that we may have some freedom in how we choose the sets $\mathcal{R}_0$ and $\mathcal{N}$ in each step.
3. The resulting $(n+r) \times (n+r)$ matrix $M := M_r$ has $q(M) \leq t$.

4. Joins with complete graphs

In this section we consider the join of two graphs and develop a technique for determining, under certain conditions, the minimum number of distinct eigenvalues for the join of a graph with a complete graph.

4.1. Patterns and eigenvectors

If we want a 1-bordering of the matrix $A \in S(G)$ to produce a matrix $A' \in S(K_1 \vee G)$, then we need $U_0\mathbf{b}$ to have no zero entries in Theorem 3.1 above. This will happen for most choices of $\mathbf{b}$, unless U_0 contains a zero row, or equivalently, unless eigenvectors corresponding to the eigenvalues in $\mathcal{R}_0$ all have a zero entry in the same position. The next results consider the case $|\mathcal{R}_0| = 1$. We call an eigenvalue of a symmetric matrix *extreme* if it is the smallest or the largest eigenvalue of that matrix.

Corollary 4.1. *Suppose G is a non-empty graph and there exists an $A \in S(G)$ with a nowhere zero eigenvector associated with some eigenvalue λ of A. Then there exists a 1-bordering A' of A in $S(K_1 \vee G)$ so that:*

- $q(A') = q(A) + 1$ *if λ is an extreme eigenvalue,*
- $q(A') = q(A)$ *if λ is not an extreme eigenvalue,*
- $q(A') = q(A) - 1$ *if λ is simple and not an extreme eigenvalue.*

Proof. In Theorem 3.1 we choose $\mathcal{R}_0 = \{\lambda\}$, $U_0 \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ a nowhere zero eigenvector of A with eigenvalue λ, and B with eigenvalues μ_1, μ_2, satisfying $\mu_1 < \lambda < \mu_2$, so that either μ_1 or μ_2 agrees with an eigenvalue of A, if λ is an extreme eigenvalue, and so that both μ_1 and μ_2 are eigenvalues of A, if λ is not an extreme eigenvalue of A. Since U_0 is a single column with no zero entries we get $A' \in S(K_1 \vee G)$, and since the spectrum of A' can be obtained from the spectrum of A by removing one multiple of λ and increasing the multiplicity of μ_1 and μ_2 by 1, the result follows. □

In Theorem 3.1 we have seen that after 1-bordering, some eigenvectors will necessarily have a zero entry, and this has an interesting consequence for the patterns of 2-borderings.

Corollary 4.2. *Let A be a symmetric matrix, A' a 1-bordering of A, and A'' a 1-bordering of A'. If $(A'')_{1,2} \neq 0$, then there is an eigenvalue λ of A' so that $m_{A''}(\lambda) = m_{A'}(\lambda) - 1 = m_A(\lambda)$.*

Proof. Adopting the notation and definitions from Theorem 3.1, recall that $W = (W_{\mathcal{N}}\ W_1)$ where

$$W_{\mathcal{N}} = \begin{pmatrix} \mathbf{v}^T \\ U_0 V_0 \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} 0 \\ U_1 \end{pmatrix},$$

and $W^T A' W = D_{\mathcal{N}} \oplus D_1$. Hence, $A'W = W(D_{\mathcal{N}} \oplus D_1)$, i.e., $(A'W_{\mathcal{N}}\ \ A'W_1) = (W_{\mathcal{N}} D_{\mathcal{N}}\ \ W_1 D_1)$, so the columns of the matrices $W_{\mathcal{N}}$ and W_1 are eigenvectors of A' corresponding to the eigenvalues of $D_{\mathcal{N}}$ and D_1, respectively. If λ is an eigenvalue of A' which is not in $\mathcal{N}$, then the λ-eigenspace of A' is contained in the column space of W_1, so every vector in this eigenspace has first entry equal to zero. It follows that any eigenvector of A' with nonzero first entry must have its corresponding eigenvalue λ in $\mathcal{N}$.

Consider now the 1-bordering A'' of A'. Let us define $\mathcal{R}_0'$, D_0', U_0' and $\mathbf{b}'$ for this 1-bordering, analogously as was done above for the 1-bordering A' of A. If $(A'')_{1,2} \neq 0$, then $(U_0'\mathbf{b}')_1 \neq 0$ by the above, so the first row of U_0' cannot be a zero row. Since ${U_0'}^T A' U_0' = D_0'$, this implies that there is some eigenvector of A', with eigenvalue $\lambda \in \mathcal{R}_0'$, which has a nonzero first entry. Hence, by the previous paragraph, $\lambda \in \mathcal{N} \cap \mathcal{R}_0'$, and thus $m_{A''}(\lambda) = m_{A'}(\lambda) - 1 = m_A(\lambda)$. □

Remark 4.3. Suppose $r \geq 2$ and $A_0, A_1, \ldots, A_r$ are successive 1-borderings of a matrix $A_0 \in S(G)$. If $A_r \in S(K_r \vee G)$, then by Corollary 4.2, it is necessarily the case that for $0 \leq s \leq r-2$, there is a real number λ_s so that $m_{A_{s+2}}(\lambda_s) = m_{A_{s+1}}(\lambda_s) - 1 = m_{A_s}(\lambda_s)$.

In the following example we illustrate how Algorithm 3.1 may be used to border a matrix achieving a small q value in 3-bordering in different ways. We also identify cases when Remark 4.3 implies that the resulting 3-bordering cannot be in $S(K_3 \vee G)$.

Example 4.4. Let A be a 9×9 symmetric matrix with ordered multiplicity list $\mathbf{m} = (1, 3, 3, 1, 1)$ and spectrum $\{1, 2^{(3)}, 3^{(3)}, 4, 5\}$. The goal is to find the spectra of all 3-borderings of A that have three distinct eigenvalues. Observe that $C(\mathbf{m}, 3) = 3$, and to achieve this goal the value of C must decrease by one every time we border. Table 2

Table 2
Red eigenvalues are the ones that are forced to have reduced multiplicity in the next bordering, the blue ones satisfy the conclusion of Corollary 4.2 for the 2-bordering of A, and the green ones satisfy the same condition when we consider instead the 3-bordering of A. Moreover, $\lambda, \lambda' \in [3, 4]$, $\nu \in [4, 5]$, $\mu, \mu', \mu'' \geq 5$ and $\rho \in (3, \mu)$, are arbitrary. (For interpretation of the colours in the table, the reader is referred to the web version of this article.)

A	$\{1, 2^{(3)}, 3^{(3)}, 4, 5\}$		
1-bordering	$\{1^{(2)}, 2^{(2)}, 3^{(4)}, 5, \mu\}$	$\{1^{(2)}, 2^{(2)}, 3^{(3)}, \lambda, \mu, \nu\}$	$\{1^{(2)}, 2^{(2)}, 3^{(3)}, \lambda, 5^{(2)}\}$
2-bordering	$\{1^{(3)}, 2, 3^{(5)}, \mu', \mu''\}$	$\{1^{(3)}, 2, 3^{(4)}, \rho, \mu^{(2)}\}$	$\{1^{(3)}, 2, 3^{(3)}, \lambda', 5^{(3)}\}$
3-bordering	$\{1^{(4)}, 3^{(6)}, \mu''^{(2)}\}$	$\{1^{(4)}, 3^{(5)}, \mu^{(3)}\}$	$\{1^{(4)}, 3^{(4)}, 5^{(4)}\}$

shows all possible eigenvalues we can obtain in this way. We produced this table by exhaustive search.

We note that the construction in the proof of Proposition 3.5 produces only the spectrum $\{1^{(4)}, 3^{(6)}, 5^{(2)}\}$, which we obtain after 1-bordering with spectrum $\{1^{(2)}, 2^{(2)}, 3^{(4)}, 4, 5\}$ and 2-bordering with spectrum $\{1^{(3)}, 2, 3^{(5)}, 4, 5\}$. This example shows that there may be several options of choosing appropriate sets $\mathcal{N}$ and $\mathcal{R}_0$ in each step as we develop an r-bordering with the desired number of distinct eigenvalues.

In all three situations (corresponding to three columns of Table 2), if $A \in S(G)$, then by appropriately choosing the free parameters, it is possible to satisfy the necessary conditions of Remark 4.3 for the 3-bordering of A to be in $S(K_3 \vee G)$. However, if, for example, we choose $\lambda = 4$ or $\lambda' = \lambda$ in the last column, then the conditions of the remark do not hold.

4.2. Hypercubes

In this section we explore the minimum number of distinct eigenvalues for joins of complete graphs with a hypercube graph. Recall that for $t \geq 1$, the vertices of the hypercube graph Q_t are the 2^t binary strings of length t, and its edges are the pairs of vertices with Hamming distance one. It was shown in [1, Corollary 6.9] that if Q_t is the hypercube graph with $t \geq 2$, then $q(Q_t) = 2$. In the following, we use the matrix construction from [1] to demonstrate that Q_t has a realization A having $q(A) = 2$ and a nowhere zero eigenvector.

Theorem 4.5. *For any two positive integers s and t,*

$$q(K_s \vee Q_t) \leq 3.$$

Moreover, if $s \leq t$, then

$$q(K_s \vee Q_{2t+2}) = 3.$$

Proof. We will demonstrate that Q_t has a realization A having $q(A) = 2$ and a nowhere zero eigenvector. Corollary 4.1 will then imply that $q(K_1 \vee Q_t) \leq 3$ and so the result follows from the inequality (1). As observed in [1], for any nonzero α and β with $\alpha^2+\beta^2 = 1$, Q_t has a realization

$$B = \begin{pmatrix} \alpha A & \beta I \\ \beta I & -\alpha A \end{pmatrix}$$

such that $A^2 = I$ and $q(B) = 2$. The vector

$$\begin{pmatrix} (I + \alpha A)\mathbf{1} \\ \beta\mathbf{1} \end{pmatrix}$$

with $\mathbf{1}$ representing the all ones vector, will be a nowhere zero eigenvector of B with eigenvalue 1 for any α sufficiently small.

The second part of the statement is a generalization of [5, Proposition 5.1]. It uses [5, Theorem 1.9], which is a small correction of [1, Theorem 4.4]. For $i = 1, 2, \ldots, t+1$, consider the vertices of the hypercube Q_t given by the binary strings $v_i = 00\cdots01100\cdots0$, with the two ones in positions $2i-1$ and $2i$. Then $\{v_1, \ldots, v_{t+1}\}$ is a set of $t+1$ independent vertices in $K_s \vee Q_{2t+2}$, and $N(v_i) \cap N(v_j) = V(K_s)$ for $i \neq j$. Therefore

$$\left| \bigcup_{i \neq j} N(v_i) \cap N(v_j) \right| = s < t+1,$$

hence $q(K_s \vee Q_{2t+2}) \geq 3$ by [1, Theorem 4.4]. □

By Theorem 2.2, if s is chosen sufficiently large, then $q(K_s \vee Q_t) = 2$. Thus, in light of Theorem 4.5, and the fact that $q(K_s \vee Q_t)$ is a non-increasing function of s as per Equation (1), it is natural to ask the following question: What is the minimum s for which $q(K_s \vee Q_t) = 2$?

4.3. Cycles

Given $A \in S(H)$ and a graph G, let $S(G \vee A)$ be the set of all matrices $X \in S(G \vee H)$ so that $X[H] = A$, and let $q(G \vee A)$ be the minimum $q(X)$ over all such matrices X. Note that $q(G \vee A) \geq q(G \vee H)$. Suppose A has ordered multiplicity list $\mathbf{m} = \mathbf{m}(A)$. Given a number $t \geq 2$ and a graph G, we want to determine whether or not $q(G \vee A) \leq t$. By Proposition 3.5, a necessary condition is that

$$C(\mathbf{m}, t) \leq |G|.$$

In Section 5 we will show that this condition is not sufficient in general, since it may happen that none of the $|G|$-borderings guaranteed by Proposition 3.5 has the correct graph, $G \vee H$, where for any $n \times n$ symmetric matrix $A = (a_{ij})$, $H = G(A)$ is defined as the graph on n vertices with edges $\{i, j\}$ whenever $i \neq j$ and $a_{ij} \neq 0$. In fact, it is not generally sufficient even in the case that G is a complete graph. Despite this, we provide examples when the procedure from Section 3 is applied successfully.

Note that the necessary condition above may be written as

$$q(G \vee A) \geq \min\{t \geq 2 : C(\mathbf{m}(A), t) \leq |G|\}. \tag{4}$$

Turning to cycles, it is known by [16, Theorem 3.4] that $q(K_{2k-2} \vee C_{2k}) = 2$. Next, we use the following result on the inverse eigenvalue problem for cycles to determine the minimum number of eigenvalues allowed for joins of complete graphs with even cycles.

Proposition 4.6. *(IEPG for cycles [9]). Nonincreasing real numbers $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of some $A \in S(C_n)$ if and only if either*

$$\lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \lambda_5 \geq \cdots$$

or

$$\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \cdots .$$

Hence, if $k \geq 2$, $q(C_{2k}) = k$ and $M(C_{2k}) = 2$.

Observe that if λ is a multiple eigenvalue of $A \in S(C_n)$, then the multiplicity of λ is two and there exists a nowhere zero eigenvector for λ associated with A. If the latter did not hold then every eigenvector $\mathbf{x}$ for λ would satisfy $\mathbf{x}_i = 0$ for some $i = 1, 2, \ldots, n$. In this case λ is a multiple eigenvalue for the principal submatrix of A obtained by deleting row and column i. However, this submatrix lies in $S(P_{n-1})$, and can only possess simple eigenvalues.

Theorem 4.7. *If $k \geq 2$ then $q(K_1 \vee C_{2k}) = k$.*

Proof. To obtain the upper bound $q(K_1 \vee C_{2k}) \leq k$, use Proposition 4.6 to choose a matrix $A \in S(C_{2k})$ with multiplicity list $(2, 2, \ldots, 2)$, choose a non-extreme eigenvalue of A and a nowhere zero eigenvector and apply Corollary 4.1.

To show the lower bound, assume that $M \in S(K_1 \vee C_{2k})$ has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2k+1}$ and that A is the submatrix corresponding to C_{2k} and has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{2k}$. By Proposition 4.6, we have that the maximum multiplicity of an eigenvalue λ_i is 2 and furthermore, if there are eigenvalues λ_i and λ_j with multiplicity 2 then $m_A(\lambda_i, \lambda_j)$ must be even. By eigenvalue interlacing we have that the maximum multiplicity of any eigenvalue of M is 3. We claim that if μ_i and μ_j each have multiplicity 3, then there must be an eigenvalue of multiplicity 1 between them, and the lower bound follows once we show this.

By way of contradiction, assume that there is some pair of eigenvalues with multiplicity 3 and j distinct eigenvalues between them, each with multiplicity 2 (with the possibility that j is 0). That is, we have

$$\mu_i = \mu_{i+1} = \mu_{i+2} < \cdots < \mu_{i+2+2j+1} = \mu_{i+2+2j+2} = \mu_{i+2+2j+3}.$$

From eigenvalue interlacing we must have $\lambda_i = \lambda_{i+1}$ and $\lambda_{i+2j+3} = \lambda_{i+2j+4}$. Hence it follows that both λ_{i+1} and λ_{i+2j+3} have multiplicity 2 and $m_A(\lambda_{i+1}, \lambda_{i+2j+3}) = 2j+1$ is odd, a contradiction. □

As we saw in Example 4.4, we have to be careful about the choice of 1-bordering of A in order to ensure that a subsequent 2-bordering of A has the desired pattern. As another illustration of this issue, observe that if an eigenvalue λ of $A \in S(C_6)$ has multiplicity 2 for A, and multiplicity 1 for a 1-bordering A' of A, then eigenvectors of λ for A' will not be nowhere zero—since the interlacing is not strict, an entry of the eigenvector for A' is 0 (see [13, Theorem 4.3.17]). This shows that if we apply Algorithm 3.1 starting with $A \in S(C_6)$ with multiplicity list $(2,2,2)$ to produce a 2-bordering A'' with $q(A'') = 2$, then $A'' \notin S(K_2 \vee C_6)$. In the next example we show that starting with a matrix $A \in S(C_6)$ with a different multiplicity list, it is possible that $q(A'') = 2$ can still be reached for some $A'' \in S(K_2 \vee C_6)$.

Example 4.8. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \in S(C_6), \quad U_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\sigma(A) = \{(-2)^{(2)}, -1, 1, 2^{(2)}\}$. Let $\mathcal{R}_0 = \{-1, 1\}$, and observe that $U_0^T A U_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, corresponding to the setup of Theorem 3.1. Choose any $t \in (-1,1)$ and let $\mathcal{N} = \{-2, t, 2\}$. Following Corollary 3.2 and [6, equation (2.4)], we find

$$B = \begin{pmatrix} t & \sqrt{3}u & \sqrt{3}v \\ \sqrt{3}u & 1 & 0 \\ \sqrt{3}v & 0 & -1 \end{pmatrix}$$

where

$$u = \sqrt{(1-t)/2} \quad \text{and} \quad v = \sqrt{(1+t)/2},$$

and

$$A' = \begin{pmatrix} t & u & v & -u & -v & u & v \\ u & 1 & 1 & 0 & 0 & 0 & -1 \\ v & 1 & -1 & 1 & 0 & 0 & 0 \\ -u & 0 & 1 & 1 & 1 & 0 & 0 \\ -v & 0 & 0 & 1 & -1 & 1 & 0 \\ u & 0 & 0 & 0 & 1 & 1 & 1 \\ v & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \in S(K_1 \vee C_6)$$

with $\sigma(A') = \{(-2)^{(3)}, t, 2^{(2)}\}$.

Repeating the construction, choosing $\mathcal{R}'_0 = \{t\}$ and $\mathcal{N}' = \{-2, 2\}$, we obtain

$$A'' = \begin{pmatrix} -t & \sqrt{1-t^2} & -v & u & v & -u & -v & u \\ \sqrt{1-t^2} & t & u & v & -u & -v & u & v \\ -v & u & 1 & 1 & 0 & 0 & 0 & -1 \\ u & v & 1 & -1 & 1 & 0 & 0 & 0 \\ v & -u & 0 & 1 & 1 & 1 & 0 & 0 \\ -u & -v & 0 & 0 & 1 & -1 & 1 & 0 \\ -v & u & 0 & 0 & 0 & 1 & 1 & 1 \\ u & v & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \in S(K_2 \vee C_6)$$

with $\sigma(A'') = \{(-2)^{(4)}, 2^{(4)}\}$ and thus $q(K_2 \vee C_6) = 2$.

Example 4.9. Using the Jacobi-Ferguson algorithm [8], we can construct numerical matrices $A \in S(C_{10})$ with spectrum $\{(-6)^{(2)}, -4, -2, 0^{(2)}, 4, 2, 6^{(2)}\}$, and hence find numerical matrices $A'' \in S(K_2 \vee C_{10})$ with spectrum $\{(-6)^{(4)}, 0^{(4)}, 6^{(4)}\}$ and thus $q(K_2 \vee C_{10}) = 3$. One such numerical matrix is:

$$A'' = \begin{pmatrix} 0 & -1.9720 & -0.11321 & -0.40399 & -2.4521 & -1.3819 & 0.0061884 & 0.00028437 & -0.0036489 & 2.1264 & -2.9646 & 1.3043 \\ -1.9720 & 0 & -2.2195 & -2.0495 & 2.2752 & -0.83772 & 0.0080390 & 0.005574 & -0.018511 & -1.9731 & -1.7971 & 1.6944 \\ -0.11321 & -2.2195 & 0 & 3.2468 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.6901 \\ -0.40399 & -2.0495 & 3.2468 & 0 & 3.6175 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.4521 & 2.2752 & 0 & 3.6175 & 0 & 1.5399 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.3819 & -0.83772 & 0 & 0 & 1.5399 & 0 & 0.010306 & 0 & 0 & 0 & 0 & 0 \\ 0.0061884 & 0.0080390 & 0 & 0 & 0 & 0.010306 & 0 & 5.4891 & 0 & 0 & 0 & 0 \\ 0.00028437 & 0.005574 & 0 & 0 & 0 & 0 & 5.4891 & 0 & 2.4227 & 0 & 0 & 0 \\ -0.0036489 & -0.018511 & 0 & 0 & 0 & 0 & 0 & 2.4227 & 0 & 0.013409 & 0 & 0 \\ 2.1264 & -1.9731 & 0 & 0 & 0 & 0 & 0 & 0 & 0.013409 & 0 & 2.9999 & 0 \\ -2.9646 & -1.7971 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.9999 & 0 & -2.7171 \\ 1.3043 & 1.6944 & 3.6901 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.7171 & 0 \end{pmatrix}.$$

5. Limitations of Algorithm 3.1 for graph joins

In this section we show that the condition $C(\mathbf{m}(A), t) \leq r$ from Proposition 3.5 is not generally sufficient in the case $G = K_r$ for the existence of a matrix with at most t eigenvalues in $S(G \vee A)$.

Proposition 5.1. *Suppose $t \geq 2$ and $A_1, \dots, A_k$ are successive 1-borderings of a symmetric matrix $A = A_0$, and $\mathbf{m}(A) = (m_1, k, m_2, k, \dots, k, m_t)$ where $m_j \geq k \geq t$ for each j, and $q(A_k) = t$. Then*

$$\mathbf{m}(A_j) = (m_1 + j, k - j, m_2 + j, k - j, \dots, k - j, m_t + j), \quad j = 0, 1, \dots, k.$$

Proof. Let $\mu_1 < \cdots < \mu_{2t-1}$ be the distinct eigenvalues of A_0 and $\lambda_1 < \cdots < \lambda_t$ the distinct eigenvalues of A_k. By eigenvalue interlacing, every eigenvalue of A_j is in the closed interval $[\lambda_1, \lambda_t]$. Moreover, by Lemma 3.3, for $i = 1, \dots, t-1$ and $j = 0, \dots, k$, we have

$$m_{A_j}(\lambda_i, \lambda_{i+1}) \geq m_{A_0}(\lambda_i, \lambda_{i+1}) - j.$$

In particular, $0 = m_{A_k}(\lambda_i, \lambda_{i+1}) \geq m_{A_0}(\lambda_i, \lambda_{i+1}) - k$, so

$$m_{A_0}(\lambda_i, \lambda_{i+1}) \leq k.$$

Let $S = \{\mu_1, \dots, \mu_{2t-1}\} \setminus \{\lambda_1, \dots, \lambda_t\}$. Then $|S| \geq 2t - 1 - t = t - 1$, and each eigenvalue in S has multiplicity at least k in A_0 by hypothesis, so

$$k(t-1) \leq k|S| \leq \sum_{i=1}^{t-1} m_{A_0}(\lambda_i, \lambda_{i+1}) \leq k(t-1).$$

Hence, $|S| = t - 1$, so $\{\lambda_1, \dots, \lambda_t\} \subseteq \{\mu_1, \dots, \mu_{2t-1}\}$. Since $\mu_1, \mu_{2t-1} \in [\lambda_1, \lambda_t]$, this forces $\mu_1 = \lambda_1$ and $\mu_{2t-1} = \lambda_t$. If $\lambda_i = \mu_j$ and $\lambda_{i+1} = \mu_l$ where $l > j + 2$, then $m_{A_0}(\lambda_i, \lambda_{i+1}) \geq 2k$, a contradiction. It follows that $\lambda_i = \mu_{2i-1}$ for $1 \leq i \leq t$.

Hence, $m_{A_0}(\lambda_i, \lambda_{i+1}) = k$ for each i, and the bound we observed above becomes

$$k - j \leq m_{A_j}(\lambda_i, \lambda_{i+1}).$$

Since A_k is a $(k-j)$-bordering of A_j, by Lemma 3.3 we also have

$$m_{A_j}(\lambda_i, \lambda_{i+1}) \leq m_{A_k}(\lambda_i, \lambda_{i+1}) + k - j = k - j,$$

so $m_{A_j}(\lambda_i, \lambda_{i+1}) = k - j$. Moreover, by eigenvalue interlacing, $k - j \leq m_{A_j}(\mu_{2i}) \leq m_{A_j}(\lambda_i, \lambda_{i+1}) = k - j$, so we have equality. Hence, the multiplicity of μ_{2i} as an eigenvalue of A_j is $k - j$, and no other real number in $(\lambda_i, \lambda_{i+1})$ is an eigenvalue of A_j. It follows that every eigenvalue of A_j other than $\mu_2, \dots, \mu_{2(t-1)}$ is in the set $\{\lambda_1, \dots, \lambda_t\}$. Observe that A_j is an $(j + N) \times (j + N)$ matrix, where $N = (t-1)k + \sum_{i=1}^{t} m_i$ is the number of rows and columns of A. Hence,

$$\sum_{i=1}^{t} m_{A_j}(\lambda_i) = j + N - \sum_{i=1}^{t-1} m_{A_j}(\mu_{2i})$$

$$= j - (t-1)(k-j) + (t-1)k + \sum_{i=1}^{t} m_i = \sum_{i=1}^{t}(m_i + j).$$

Since the total multiplicity of the eigenvalues $\lambda_1, \dots, \lambda_t$ in A_j is $\sum_{i=1}^{t}(m_i + j)$, and by eigenvalue interlacing, the multiplicity in A_j of $\lambda_i = \mu_{2i-1}$ is bounded above by $m_i + j$, this must be precisely its multiplicity. □

Corollary 5.2. *If A is a symmetric matrix with $\mathbf{m}(A) = (m_1, k, m_2, k, \dots, k, m_t)$ where $m_i \geq k \geq t \geq 2$ for each i, then $C(\mathbf{m}(A), t) = k$ yet $q(G \vee A) > t$ for all non-empty graphs G with $|G| = k$. Hence, the inequality (4) is strict in this case.*

Proof. We have $C(\mathbf{m}(A), t) = k$, so $q(B) \geq t$ for all k-borderings B of A by Proposition 3.5. Consider a sequence of successive 1-borderings taking us from A to some k-bordering B with $q(B) = t$. By Proposition 5.1, the successive eigenvalue multiplicities of any given $\lambda \in \mathbb{R}$ in this sequence of matrices is monotone. Hence, by Corollary 4.2, the superdiagonal of the leading principal $k \times k$ submatrix of B is zero.

Now let P be a $k \times k$ permutation matrix, and consider $B_P = (P \oplus I_r)B(P^T \oplus I_r)$, where $k + r = |G|$. By the previous paragraph, the superdiagonal of the leading principal $k \times k$ submatrix of B_P is zero, for every such permutation matrix P. Hence, every off-diagonal entry of B is zero, so B has an empty graph. □

This shows a limitation of Algorithm 3.1. However, we show in the following proposition that this limitation is very specific, and that if the multiplicity list is perturbed only slightly we may have success using this procedure.

Proposition 5.3. *Suppose $t \geq 2$ and A is a symmetric matrix with eigenvalues*

$$\lambda_1^{(m_1)} < \beta < \gamma < \lambda_2^{(m_2)} < \mu_2^{(2)} < \lambda_3^{(m_3)} < \mu_3^{(2)} < \cdots < \mu_{t-1}^{(2)} < \lambda_t^{(m_t)}.$$

If A has an eigenbasis such that for each vertex u there is at least one eigenvector corresponding to an eigenvalue in $\{\mu_i\}$ which is nonzero in the entry corresponding to u, then there exists a matrix $B \in S(K_2 \vee A)$ such that B has eigenvalues $\lambda_1^{(m_1+2)}, \dots, \lambda_t^{(m_t+2)}$. In particular, $q(K_2 \vee G(A)) \leq t$.

Proof. By [6] we know that there are 1-borderings C_β and C_γ of the matrices $\mathrm{diag}(\beta, \mu_2, \dots, \mu_t)$ and $\mathrm{diag}(\gamma, \mu_2, \dots, \mu_t)$ respectively which each have eigenvalues $\{\lambda_1, \dots, \lambda_t\}$. Furthermore, we know that these borderings can have no zeros in the first row or column, and by computing traces we see that the $(1,1)$ entries are $k - \beta$ and $k - \gamma$ respectively, where $k = \lambda_1 + \cdots + \lambda_t - (\mu_2 + \cdots + \mu_t)$. Let the first row of C_β have entries $k - \beta, b_1, \dots, b_{t-1}$ and the first row of C_γ have entries $k - \gamma, c_1, \dots, c_{t-1}$. Define $B_0 = [v_\beta, v_\gamma]$ where $v_\beta = (k - \beta, 0, b_1, 0, b_2, 0, \dots)^T$, and $v_\gamma = (0, k - \gamma, 0, c_1, 0, c_2, \dots)^T$. That is, we are making vectors with the first rows of the borderings in the even or odd positions. Now define matrices $D_1 = \mathrm{diag}(k-\beta, k-\gamma)$, $D_2 = \mathrm{diag}(\beta, \gamma, \mu_2^{(2)}, \ldots, \mu_{t-1}^{(2)})$ and $D_0 = \mathrm{diag}(\lambda_1^{(m_1)}, \ldots, \lambda_t^{(m_t)})$, and finally define

$$M = \begin{pmatrix} k-\beta & & b_1 & & b_2 & & \cdots & b_{t-1} & \\ & k-\gamma & & c_1 & & c_2 & \cdots & & c_{t-1} \\ b_1 & & \beta & & & & & & \\ & c_1 & & \gamma & & & & & \\ b_2 & & & & \mu_2 & & & & \\ & c_2 & & & & \mu_2 & & & \\ & & & & & & \ddots & & \\ b_{t-1} & & & & & & & \mu_{t-1} & \\ & c_{t-1} & & & & & & & \mu_{t-1} \end{pmatrix} \oplus D_0$$

where the blank entries denotes 0s. Since M is permutationally similar to the block diagonal matrix with blocks C_β, C_γ, and D_0, the eigenvalues of M are $\lambda_1^{(m_1+2)}, \ldots, \lambda_t^{(m_t+2)}$.

By the assumption, we may choose V to be a matrix which diagonalizes the matrix A such that for any row u, there is a column j corresponding to an eigenvector of some μ_ℓ such that $V_{uj} \neq 0$. Without loss of generality assume that

$$V^T A V = \mathrm{diag}(\beta, \gamma, \mu_2^{(2)}, \ldots, \mu_{t-1}^{(2)}, \lambda_1^{(m_1)}, \ldots, \lambda_t^{(m_t)}) = D_2 \oplus D_0.$$

Define $W' = I_2 \oplus W_2 \oplus \cdots \oplus W_{t-1}$ where the W_i are any orthogonal 2×2 matrices, and define $W = W' \oplus I_{m_1+\cdots+m_t}$. Then, as W' commutes with D_2, we have that

$$W^T V^T A V W = V^T A V = D_2 \oplus D_0.$$

Let V' be the first $2t-2$ columns of V, so that it has columns that are the eigenvectors corresponding to the eigenvalues of D_2. Notate these columns by $v_1, \ldots, v_{2t-2}$. Let U be any orthogonal 2×2 matrix. Then

$$\begin{aligned}(U \oplus VW)M(U^T \oplus W^T V^T) &= (U \oplus VW) \begin{pmatrix} D_1 & B_0 \\ B_0^T & D_2 \oplus D_0 \end{pmatrix} (U^T \oplus W^T V^T) \\ &= \begin{pmatrix} U D_1 U^T & U B_0 W^T V^T \\ V W B_0^T U^T & A \end{pmatrix}.\end{aligned}$$

This matrix is in $S(K_2 \vee G(A))$ if UD_1U^T has nonzero off-diagonal entries and the following matrix has no zero entry:

$$U B_0 W^T V^T = U B_0' (W')^T (V')^T, \quad \text{where} \quad B_0' = \begin{pmatrix} b_1 & & b_2 & & \cdots & b_{t-1} & \\ & c_1 & & c_2 & \cdots & & c_{t-1} \end{pmatrix}.$$

Let $\theta_2, \ldots, \theta_{t-1}$ be uniformly and independently chosen angles and let W_i be the 2×2 rotation matrix by angle θ_i. Then the ij'th entry of $B_0'(W')^T(V')^T$ is

$$b_1 v_1(j) + \sum_{k=2}^{t-1} b_k v_{2k-1}(j) \cos\theta_k - b_k v_{2k}(j) \sin\theta_k$$

if $i = 1$ and

$$c_1 v_2(j) + \sum_{k=2}^{t-1} c_k v_{2k-1}(j) \sin\theta_k + c_k v_{2k}(j) \cos\theta_k$$

if $i = 2$. Since the b_i and c_i are nonzero, and by the choice of V there is at least one u with $3 \leq u \leq 2t - 2$ with $v_u(j) \neq 0$, we have that the ij'th entry of $B_0'(W')^T(V')^T$ is nonzero with probability 1. So we can choose W' for which $B_0'(W')^T(V')^T$ has no zero entries. Moreover, since $\beta \neq \gamma$, D_1 is not a zero matrix. It is now easy to choose U such that UD_1U^T and $UB_0W^TV^T$ have all nonzero entries. □

In this paper we continued the study of the behaviour of $q(G \vee H)$. For a general graph H, we obtained results for the case when G is either a path or a complete graph, and we explored the potential impact of eigenvector patterns on $q(G \vee H)$, for various families of graphs H.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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