



Computational complexity aspects of super domination [☆]

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ABSTRACT

Let G be a graph. A dominating set $D \subseteq V(G)$ is a super dominating set if for every vertex $x \in V(G) \setminus D$ there exists $y \in D$ such that $N_G(y) \cap (V(G) \setminus D) = \{x\}$. The cardinality of a smallest super dominating set of G is the super domination number of G . An exact formula for the super domination number of a tree T is obtained, and it is demonstrated that a smallest super dominating set of T can be computed in linear time. It is proved that it is NP-complete to decide whether the super domination number of a graph G is at most a given integer if G is a bipartite graph of girth at least 8. The super domination number is determined for all k -subdivisions of graphs. Interestingly, in half of the cases the exact value can be efficiently computed from the obtained formulas, while in the other cases the computation is hard. While obtaining these formulas, Π -matching numbers are introduced and proved that they are computationally hard to determine.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. Then $D \subseteq V(G)$ is a *dominating set* if every vertex in $\bar{D} = V(G) \setminus D$ is adjacent to at least one vertex in D . The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . Graph domination theory has been extensively researched so far. To capture the current state of the field, we refer the reader to two recent edited books [10,11].

Many variations of the domination have been introduced, some of which are significant and important (such as total domination and connected domination), while others are of only sporadic importance. In our view, the group of significant domination concepts includes super domination which was introduced in 2015 by Lemańska, Swaminathan, Venkatakrisnan, and Zuazua [18].

It is known [3] that every isolate-free graph G has a minimum dominating set D such that for each $v \in D$ there exists a vertex u (called external private neighbor) in $\bar{D} = V(G) \setminus D$ with $N_G(u) \cap D = \{v\}$. (As usual, $N_G(x)$ stands for the open neighborhood of x in G and $N_G[x]$ for the closed neighborhood of x in G .) By switching the roles of D and \bar{D} in the latter property, we arrive at the following concept. A dominating set D of G is a *super dominating set* of G , if for every vertex $u \in \bar{D}$ there exists $v \in D$ such that $N_G(v) \cap \bar{D} = \{u\}$. The cardinality of a smallest super dominating set of G is the *super*

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domination number $\gamma_{sp}(G)$ of G . A super dominating set of cardinality $\gamma_{sp}(G)$ is briefly called a γ_{sp} -set. The initial study of the concept has been followed by several sequels, of which the reader is referred to [6,8,14,16,22,28].

The *subdivision* of a graph G is the graph obtained from G by replacing each edge with a disjoint path of length 2 and is denoted by $S(G)$. More generally, if $k \geq 1$, then the graph $S_k(G)$ is obtained from G by replacing each edge with a disjoint path of length $k + 1$, that is, subdividing each of its edges k times. Clearly, $S(G) = S_1(G)$. Some authors use the term *complete k -subdivision* for what we call k -subdivision, but since all our subdivisions are complete, we simplify the terminology. The concept of k -subdivisions is ubiquitous in graph theory, here we cite its presence in graph colorings [1,24], spectral graph theory [4], structural graph theory [19,27], and chemical graph theory [2,13].

In this article we discuss various aspects of super domination, which are in one way or another intertwined with the computational complexity of the problem of determining the super domination number. In the next section we recall some definitions and known results, and state a useful characterization of super dominating sets. Super domination has already been considered on trees from several perspectives, see [16,18,22,28]. By now, only sharp upper and lower bounds have been obtained. In Section 3 we fill this gap by providing an exact formula for the super domination number of a tree. Moreover, we demonstrate that if T is a tree, then $\gamma_{sp}(T)$ as well as a γ_{sp} -set of T can be computed in linear time. On the negative side, in Section 4, we prove that it is NP-complete to decide whether $\gamma_{sp}(G) \leq k$ holds if G is a bipartite graph of girth $g(G) \geq 8$ and the positive integer k is part of the input. In our longest part of the paper, Section 5, we consider k -subdivisions of arbitrary graphs. Depending on $k \bmod 4$, four closed formulas for $\gamma_{sp}(S_k(G))$ are proved. When $k \bmod 4 \in \{1, 3\}$, the corresponding formulas depend only of k , the size of G , and a simple condition on the cycles of G . Note that in these two cases $S_k(G)$ is bipartite. On the other hand, if $k \bmod 4 = 0$, then $\gamma_{sp}(S_k(G))$ is a function of $\gamma_{sp}(G)$ also, and if $k \bmod 4 = 2$, then $\gamma_{sp}(S_k(G))$ depends also on the cardinality of a largest matching that admits a partition into two induced matchings. We name such matchings as II-matchings and prove that it is NP-hard to compute the maximum size of such matchings. It follows that for each even k , it is also NP-hard to determine $\gamma_{sp}(S_k(G))$.

2. Preliminaries

The order and the size of a graph G will be denoted by $n(G)$ and $m(G)$, respectively. If D is a super dominating set of G and if for a vertex $x \in \overline{D}$ the vertex $y \in D$ has the property $N_G(y) \cap \overline{D} = \{x\}$, then we will say that x is *super dominated* by y .

Let G be a graph. Then the independence number of G will be denoted by $\alpha(G)$, the matching number of G by $\alpha'(G)$, and the vertex cover number of G by $\beta(G)$. A set $X \subseteq V(G)$ is a *2-packing* of G if $d_G(u, v) \geq 3$ holds for each pair of vertices $u, v \in X$. In other words, each pair of vertices of X has disjoint closed neighborhoods. The cardinality of a smallest 2-packing of G will be denoted by $\rho(G)$.

The path in $S_k(G)$ obtained by k times subdividing an edge $uv \in E(G)$ will be denoted by P_{uv} and addressed to as a *super edge*. The vertices of P_{uv} will be denoted by $u, (uv)_1, \dots, (uv)_k, v$. Note that $n(S_k(G)) = n(G) + k \cdot m(G)$ and $m(S_k(G)) = (k + 1)m(G)$. We say that a graph is a *k -subdivision graph* if it can be obtained as a k -subdivision of some graph.

We next recall a few results on the super domination number needed later on.

Theorem 2.1. [18] *If G is a graph without isolated vertices, then,*

$$1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq n(G) - 1.$$

Theorem 2.2. [14, Theorem 3, Corollary 2] *If G is a graph with $n(G) \geq 2$, then*

$$n(G) - \alpha'(G) \leq \gamma_{sp}(G) \leq n(G) - \rho(G).$$

Since $n(G) = \alpha(G) + \beta(G)$ in general and $\alpha'(G) = \beta(G)$ when G is bipartite, Theorem 2.2 implies that if G is bipartite, then $\gamma_{sp}(G) \geq \alpha(G)$.

Theorem 2.3. [18] *The following exact values are valid.*

- (i) *If $n \geq 2$, then $\gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$.*
- (ii) *If $n \geq 3$, then*

$$\gamma_{sp}(C_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil; & n \equiv 2 \pmod{4}, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise.} \end{cases}$$

- (iii) *If $n \geq 2$, then $\gamma_{sp}(K_{1,n}) = n$.*

Let G be a graph, and let D be a super dominating set of G . For each $u \in \overline{D}$, select an arbitrary vertex $u' \in D$ such that u is the unique neighbor of u' in \overline{D} , that is, $N(u') \cap \overline{D} = \{u\}$. Then we say that the set

$$D^* = \{u' \in D : u \in \overline{D}\}$$

is a core of D . By this definition, there exists a matching between \overline{D} and D^* that covers $\overline{D} \cup D^*$. Moreover, the following result holds which seems of independent interest. Before stating it, we introduce a notation. For two disjoint vertex sets $A, B \subseteq V(G)$ let $E_G[A, B]$ denote the set of all edges between A and B in G .

Lemma 2.4. *Let A and B be two disjoint vertex sets of a graph G . Then $D = \overline{A}$ is a super dominating set and B is a core of D if and only if $E_G[A, B]$ is a matching that covers all vertices in $A \cup B$.*

Proof. Suppose that $D = \overline{A}$ is a super dominating set and let B be its core. By definition of the core, every vertex from B has exactly one neighbor from A , so that $|N_G(x) \cap B| = 1$ if $x \in A$ and $|N_G(y) \cap A| = 1$ whenever $y \in B$. It shows that $E_G[A, B]$ is a matching that covers $A \cup B$.

Now, suppose that A, B are disjoint vertex sets in G such that $E_G[A, B]$ is a matching that covers $A \cup B$. Define $D = \overline{A}$ and observe that every $x \in A$ is super dominated by the vertex y that is the pair of x in the matching $E_G[A, B]$. Indeed, by our assumption, $y \in D$ and $N_G(y) \cap A = \{x\}$. It also follows that B can be considered as a core of D . \square

Note that Lemma 2.4 implies the lower bound of Theorem 2.2. Moreover, it also implies the following.

Corollary 2.5. *If D is a super dominating set of a graph G , and D^* is a core of D , then $(D \setminus D^*) \cup \overline{D}$ is also a super dominating set of G . In particular, if D is a γ_{sp} -set, then $(D \setminus D^*) \cup \overline{D}$ is also a γ_{sp} -set.*

Proof. The first assertion follows directly by Lemma 2.4. And as $|(D \setminus D^*) \cup \overline{D}| = |D|$, the second assertion also follows. \square

Corollary 2.6. *If G is a graph and $v \in V(G)$, then there exists a γ_{sp} -set of G that contains v .*

Proof. Let D be an arbitrary γ_{sp} -set of G . If $v \in D$ there is nothing to prove. Otherwise consider a core D^* of G . Then $v \in (D \setminus D^*) \cup \overline{D}$ which is a γ_{sp} -set by Corollary 2.5. \square

It is interesting to compare Corollary 2.6 with [28, Proposition 2.4] which asserts that if v is a leaf of a non-trivial tree T , then there exists a γ_{sp} -set of T which does not contain v .

3. Super domination number of trees

The main result of this section reads as follows.

Theorem 3.1. *If T is a tree, then $\gamma_{sp}(T) = n(T) - \alpha'(T)$. Moreover, a γ_{sp} -set of T can be determined in linear time over the class of trees.*

Proof. Consider a maximum matching M in T . We will show that $V(M)$ can be partitioned into two vertex sets A and B such that $E_T[A, B] = M$. By Lemma 2.4, it will imply that there is a super dominating set of cardinality $n(T) - |A| = n(T) - |M| = n(T) - \alpha'(T)$. Together with the inequality $\gamma_{sp}(G) \geq n(G) - \alpha'(G)$ from Theorem 2.1 we obtain $\gamma_{sp}(T) = n(T) - \alpha'(T)$ for the tree, as stated.

To construct the sets A and B , we first specify a root vertex r such that r is covered by M . We first put r into A and consider the children v_1, \dots, v_k of r . If $rv_i \in M$, we put v_i into B ; if $rv_i \notin M$ but $v_i \in V(M)$, we put v_i into A ; if $v_i \notin V(M)$, then v_i remains outside $A \cup B$. We continue analogously while traversing the tree in preorder. When we decide about the children u_1, \dots, u_ℓ of a vertex u , we have three main cases.

- First, let $u \notin V(M)$. Then, if $u_i \in V(M)$, we put u_i into A ; if $u_i \notin V(M)$, we put it into neither A nor B .
- Suppose that $u \in A$. If $uu_i \in M$, put u_i into B ; if $uu_i \notin M$ and $u_i \in V(M)$, put u_i into A ; if $u_i \notin V(M)$, leave u_i outside $A \cup B$.
- The case when $u \in B$ is analogous to the previous one. If $uu_i \in M$, we put u_i into A ; if $uu_i \notin M$ and $u_i \in V(M)$, we put u_i into B ; if $u_i \notin V(M)$, we leave u_i outside $A \cup B$.

It is clear that for the constructed sets, (A, B) results in a partition of $V(M)$ such that $M = E_T[A, B]$. Therefore, by Lemma 2.4, $D = \overline{A}$ is a super dominating set in T and we may infer $\gamma_{sp}(T) = n(T) - \alpha'(T)$.

Concerning the construction of a γ_{sp} -set of a tree, we remark that a maximum matching of a tree can be obtained in linear time. Once the matching M is in hand, the algorithm described in the proof assigns labels $A, B, \overline{A \cup B}$ to the vertices in preorder, visiting every vertex only once and making a choice according to local properties. Thus, the determination of a γ_{sp} -set of a tree can be done in linear time as stated. \square

Extending the definition of the subdivision of a graph by setting $S_0(G) = G$, the following result can be considered as a generalization of Theorem 2.3(iii).

Corollary 3.2. *If $k \geq 0$ and $n \geq 2$, then*

$$\gamma_{sp}(S_k(K_{1,n})) = \begin{cases} \frac{n(k+2)}{2}; & k \text{ even,} \\ \frac{n(k+1)}{2} + 1; & k \text{ odd.} \end{cases}$$

Proof. By Theorem 2.3(iii), $\gamma_{sp}(S_0(K_{1,n})) = \gamma_{sp}(K_{1,n}) = n$, hence the assertion holds for $k = 0$.

It is straightforward to see that if $k \geq 2$ is even, then $\alpha'(S_k(K_{1,n})) = n\frac{k}{2} + 1$, and if $k \geq 1$ is odd, then $\alpha'(S_k(K_{1,n})) = n\frac{k+1}{2}$. The result now follows by applying Theorem 3.1. \square

4. Bipartite graphs

By Theorem 3.1, the super domination number and a γ_{sp} -set can be determined in linear time for trees. In this section we show that the same problem is NP-hard over the class of bipartite graphs.

Theorem 4.1.

- (a) *It is NP-complete to decide whether $\gamma_{sp}(G) \leq k$ holds if G is a bipartite graph of girth $g(G) \geq 8$ and the positive integer k is part of the input.*
- (b) *It is NP-complete to decide whether $\gamma_{sp}(G) = n(G) - \alpha'(G)$ holds if G is bipartite and $g(G) \geq 8$.*

Proof. As $\gamma_{sp}(G) \geq n(G) - \alpha'(G)$ holds for every graph G , the equality in (b) is equivalent to the inequality $\gamma_{sp}(G) \leq n(G) - \alpha'(G)$. Thus, both decision problems (a) and (b) belong to NP. In order to prove that the decision problems in (a) and (b) are NP-hard, we present a polynomial-time reduction from 3-SAT problem, which is a classical NP-complete problem [7].

Let F be a 3-SAT instance with clauses C_1, \dots, C_ℓ over the Boolean variables x_1, \dots, x_s . We construct a graph G_F such that F is satisfiable if and only if $\gamma_{sp}(G_F) \leq 4s + 3\ell + 1$.

Construction of G_F For each variable x_i , we take eight vertices that form the set $X_i = \{x_i^-, x_i^+, x_i^1, \dots, x_i^6\}$ and add edges such that $x_i^1 x_i^- x_i^2 x_i^4 x_i^5 x_i^+ x_i^6$ is an induced path and $x_i^3 x_i^4$ is a pendant edge in G_F . Each clause C_j , will be represented by a vertex c_j in G_F . If x_i is a literal in C_j , we add an edge $x_i^+ c_j$ and subdivide it by a vertex $y_{j,i}$. Similarly, if $\neg x_i$ is a literal in C_j , we add an edge $x_i^- c_j$ and subdivide it by a vertex $y_{j,i}$. The set of these subdivision vertices will be denoted by Y . To finish the construction, we add two further vertices, namely v and v^* , and the edges vv^* and $v^* c_j$ for each $j \in [\ell]$. (See Fig. 1 for illustration.) It is easy to check that the constructed graph G_F is bipartite with $n(G_F) = 8s + 4\ell + 2$ and, moreover, if G is not a tree,¹ then its girth is at least 8.

We first prove that $\alpha'(G_F) = 4s + \ell + 1$. Let M be a matching in G_F . For every $i \in [s]$, the vertices in X_i may be incident with at most four edges from M . Each clause vertex c_j may be incident with one edge from M . The only edge in G_F that is not covered by the previous vertices is vv^* and it may belong to M only if $c_j v^* \notin M$ holds for all $j \in [\ell]$. This proves $\alpha'(G_F) \leq 4s + \ell + 1$ and it is easy to find a matching of size $4s + \ell + 1$ in G_F . Therefore, $\alpha'(G_F) = 4s + \ell + 1$. Moreover, every maximum matching contains the following edges: vv^* ; $x_i^2 x_i^3$ and $x_i^4 x_i^5$ for every $i \in [s]$; one edge between c_j and Y for every $j \in [\ell]$; one edge between x_i^+ and $Y \cup \{x_i^6\}$ and one edge between x_i^- and $Y \cup \{x_i^1\}$ for every $i \in [s]$.

Now, suppose that $\gamma_{sp}(G_F) \leq 4s + 3\ell + 1$ holds and prove that the 3-SAT formula F is satisfiable. Since $\alpha'(G_F) = 4s + \ell + 1$, by Theorem 2.1 the condition is equivalent with $\gamma_{sp}(G_F) = 4s + 3\ell + 1$. Let D be a minimum γ_{sp} -set in G_F . By Lemma 2.4, there are two disjoint vertices sets $A = \overline{D}$ and $B = D^*$ such that $|A| = |B| = n(G_F) - \gamma_{sp}(G_F) = 4s + \ell + 1$ and $E_{G_F}[A, B]$ is a matching M . Since M is a maximum matching, for every $i \in [s]$, we have $x_i^-, x_i^+ \in V(M)$ and $x_i^2 x_i^3, x_i^4 x_i^5 \in M$. By the condition $E_{G_F}[A, B] = M$, if $x_i^+ \in A$ holds, then $x_i^5 \in A$ and $x_i^4, x_i^2, x_i^- \in B$ follow. Analogously, if $x_i^+ \in B$, we may conclude $x_i^- \in A$. Therefore, we may define a truth function $\varphi: X \rightarrow \{\text{true, false}\}$ in the following way:

$$\varphi(x_i) = \begin{cases} \text{true; } & x_i^+ \in B, \\ \text{false; } & x_i^- \in B. \end{cases}$$

¹ We may suppose, without loss of generality, that G_F is of girth of at least 8. Indeed, if G_F is a tree, we may consider the formula $F' = F \wedge (x_1 \vee \neg x_1 \wedge x_2)$. Clearly, a truth assignment satisfies F if and only if it satisfies F' , and $\gamma_{sp}(G_F) = n(G_F) - \alpha'(G_F)$ if and only if the same is true for $G_{F'}$.

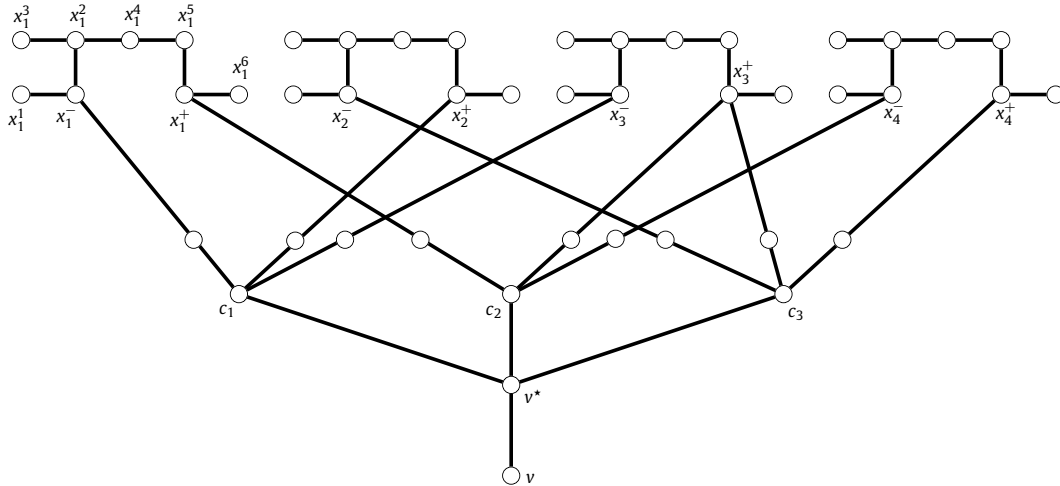


Fig. 1. Graph G_F for the formula $F = (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_3 \vee \neg x_4) \wedge (\neg x_2 \vee x_3 \vee x_4)$ constructed in the proof of Theorem 4.1.

By Corollary 2.5, we may suppose that $v^* \in A$. Consider a clause vertex c_j . As $v^*c_j \notin M$ and $c_j \in V(M)$, the vertex c_j also belongs to A . If $y_{j,i}$ is the vertex from Y such that $c_jy_{j,i} \in M$, then $y_{j,i} \in B$. Suppose first that the other neighbor of $y_{j,i}$ is x_i^+ i.e., the clause C_j contains the positive literal x_i . As $x_i \in V(M)$ and $y_{j,i}$ is already covered by one matching edge, $y_{j,i}x_i^+ \notin M$ and $x_i^+ \in B$ holds. Then, by definition, we have $\varphi(x_i) = \text{true}$ and the positive literal x_i^+ satisfies clause C_j . Similarly, if the other neighbor of $y_{j,i}$ is x_i^- , then C_j contains the literal $\neg x_i$. As $E_{G_F}[A, B] = M$, we may infer $x_i^- \in B$. It implies $\varphi(x_i) = \text{false}$ and hence, the literal $\neg x_i$ satisfies C_j . It is true for all clauses in F and proves the satisfiability of the formula.

To prove the other direction of the statement, we suppose that F is satisfied by a truth assignment $\phi: X \rightarrow \{\text{true}, \text{false}\}$. Let us define

$$D = Y \cup \{v\} \cup \{x_i^1, x_i^3, x_i^5, x_i^+ : i \in [s] \text{ and } \phi(x_i) = \text{true}\} \\ \cup \{x_i^-, x_i^2, x_i^4, x_i^6 : i \in [s] \text{ and } \phi(x_i) = \text{false}\}.$$

It is easy to check that D is a super dominating set and $|D| = 4s + 3\ell + 1$. Indeed, it is enough to consider the following connections:

- v super dominates v^* ;
- if $\phi(x_i) = \text{true}$, then x_i^1 super dominates x_i^- , x_i^3 super dominates x_i^2 , x_i^5 super dominates x_i^4 , and x_i^+ super dominates x_i^6 ;
- if $\phi(x_i) = \text{false}$, then x_i^- super dominates x_i^1 , x_i^2 super dominates x_i^3 , x_i^4 super dominates x_i^5 , and x_i^6 super dominates x_i^+ ;
- if a clause C_j is satisfied by a literal x_i or $\neg x_i$, then the corresponding subdivision vertex $y_{j,i} \in D$ and $N(y_{j,i}) \cap \overline{D} = \{c_j\}$ and thus, $y_{j,i}$ super dominates c_j .

We have proved that the NP-complete problem 3-SAT can be reduced to the problem of deciding whether $\gamma_{\text{sp}}(G_F) \leq n(G_F) - \alpha'(G_F) = 4s + 3\ell + 1$ holds. The reduction can be done in polynomial time and therefore, we may conclude that both problems (a) and (b) are NP-complete. \square

5. Super domination in subdivision graphs

5.1. $(4t + 3)$ -subdivisions

For a graph G , let $\widehat{n}(G)$ be the maximum size of a subset $\widehat{V} \subseteq V(G)$ such that there exists an injective mapping $\phi: \widehat{V} \rightarrow E(G)$ so that $v \in \phi(v)$ holds for every $v \in \widehat{V}$. We will say that a function ϕ with these properties is a *DR-function* in G ; and if $\widehat{n}(G) = n(G)$, we may say that the vertex set of G has a *set of distinct representatives (SDR)*.

Lemma 5.1. *If G is a connected graph that is not a tree, then $\widehat{n}(G) = n(G)$. If G is a tree, then $\widehat{n}(G) = n(G) - 1$.*

Proof. Associate every vertex $v \in V(G)$ with the set $E(v)$ of edges that are incident to v . First, consider a proper subset X of $V(G)$ and the set $E(X) = \bigcup_{v \in X} E(v)$. In the induced subgraph $G[X]$, every component F satisfies $m(F) \geq n(F) - 1$ and,

since G is connected and $F \neq G$, the vertex set of F is incident with at least one edge not contained in the subgraph F . These extra edges are pairwise different for different components of $G[X]$. We therefore conclude $|E(X)| \geq |X|$ for every vertex set $X \subsetneq V(G)$. Notice that it is true for every graph G , no matter G is a tree or not. Consider now the case $X = V(G)$. If G is not a tree, then $n(G) = |X| \leq |E(X)| = m(G)$ and, as Hall's Condition is satisfied for each $X \subseteq V(G)$, there exists a system of distinct representatives for the vertex set of G . That is, $\widehat{n}(G) = n(G)$. If G is a tree, $|V(G)| > |E(G)|$ and there is no SDR for the vertex set. On the other hand, if we consider G as a tree rooted in r and map every non-root vertex v to the edge between v and its parent, the obtained mapping is a DR-function from $V(G) \setminus \{r\}$ to $E(G)$. It proves $\widehat{n}(G) = n(G) - 1$ for every tree G . \square

Theorem 5.2. For every connected graph G and integer $k \equiv 3 \pmod 4$,

$$\gamma_{\text{sp}}(S_k(G)) = \begin{cases} \frac{k+1}{2} m(G) + 1; & G \text{ is a tree,} \\ \frac{k+1}{2} m(G); & \text{otherwise.} \end{cases}$$

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and $k = 4t + 3$. Observe that $n(S_k(G)) = n + km(G)$. We first show that

$$\alpha'(S_k(G)) \leq \frac{k-1}{2} m(G) + \widehat{n}(G). \tag{1}$$

Let M be a maximum matching in $S_k(G)$. For every super edge $P_{v_i v_j}$, we have two possibilities:

- (a) M contains at most $\frac{k-1}{2}$ edges from $P_{v_i v_j}$;
- (b) M contains exactly $\frac{k+1}{2}$ edges from $P_{v_i v_j}$ and at least one of v_i and v_j is covered by a matching edge belonging to $P_{v_i v_j}$.

As each $v_i \in V(G)$ is covered by at most one edge from M , the number of super edges satisfying (b) is at most n . Moreover, if (b) is valid for a super edge $P_{v_i v_j}$, then $v_i(v_i v_j)_1 \in M$ or $(v_i v_j)_k v_j \in M$. In the first case, we define $\phi(v_i) = v_i v_j$, while we set $\phi(v_j) = v_i v_j$ in the latter case. (If both edges $v_i(v_i v_j)_1$ and $(v_i v_j)_k v_j$ belong to M , then to keep ϕ injective, we set just $\phi(v_i) = v_i v_j$ for the smaller index i .) As ϕ is a DR-function, the number of super edges with property (b) is at most $\widehat{n}(G)$. This proves the inequality (1), and together with Theorem 2.2 we conclude

$$\gamma_{\text{sp}}(S_k(G)) \geq n(S_k(G)) - \alpha'(S_k(G)) \geq \frac{k+1}{2} m(G) + (n(G) - \widehat{n}(G)) \tag{2}$$

where, according to Lemma 5.1, the last term is 1 if G is a tree and 0 if G contains a cycle.

To prove the other direction, we construct a γ_{sp} -set D for G . Let ϕ be a DR-function of G with domain $V(G)$ if G is not a tree, and with domain $V(G) \setminus \{v_n\}$ otherwise.

- (i) If $\phi(v_i) = v_i v_j$, let D contain the following vertices from the super edge $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t+1}, (v_i v_j)_{4t+2}.$$

- (ii) If $v_i v_j$ does not belong to the image set of ϕ and $i < j$, let D contain the following vertices from the super edge $P_{v_i v_j}$:

$$(v_i v_j)_2, (v_i v_j)_3, (v_i v_j)_6, (v_i v_j)_7, \dots, (v_i v_j)_{4t+2}, (v_i v_j)_{4t+3}.$$

- (iii) If G is a tree and v_n does not have a representative edge in ϕ , then v_n also belongs to D . Note that the other vertices of G belong to \overline{D} .

If an internal subdivision vertex, $(v_i v_j)_s$ with $3 \leq s \leq 4t + 1$, does not belong to D , it is easy to identify a neighbor that super dominates it. A vertex v_i with $\phi(v_i) = v_i v_j$ is always super dominated by $(v_i v_j)_1$. The subdivision vertices $(v_i v_j)_2$ and $(v_i v_j)_{4t+2}$ always belong to D ; if $(v_i v_j)_1 \notin D$, it is super dominated by $(v_i v_j)_2$; if $(v_i v_j)_{4t+3} \notin D$, it is super dominated by $(v_i v_j)_{4t+2}$.

No matter whether (i) or (ii) was applied when we specified the vertices in $V(P_{v_i v_j}) \cap D$, we added exactly $2t + 2 = \frac{k+1}{2}$ subdivision vertices to D in each step. Thus, D contains $\frac{k+1}{2} m(G)$ subdivision vertices and also contains v_n if G is a tree. This proves the upper bound

$$\gamma_{\text{sp}}(S_k(G)) \leq \frac{k+1}{2} m(G) + (n(G) - \widehat{n}(G)).$$

We infer that the equality $\gamma_{\text{sp}}(S_k(G)) = \frac{k+1}{2} m(G) + (n(G) - \widehat{n}(G))$ holds for every graph G as stated. \square

For connected graphs, Theorem 5.2 and inequality (2) in its proof together imply the following statement. Since $\gamma_{sp}(F)$ and $\alpha'(F)$ are additive under disjoint union of graphs, we may state:

Proposition 5.3. *For every graph G and integer $k \equiv 3 \pmod 4$, it holds that*

$$\gamma_{sp}(S_k(G)) = n(S_k(G)) - \alpha'(S_k(G)).$$

As the number of tree components in G can be computed in linear time, and $\gamma_{sp}(F)$ is additive under taking disjoint union of graphs, we conclude the subsection with the following consequence of Theorem 5.2.

Theorem 5.4. *If k is a positive integer with $k \equiv 3 \pmod 4$, then the super domination number can be computed in linear time over the class of k -subdivision graphs.*

5.2. $(4t + 1)$ -subdivisions

Theorem 5.5. *For every connected graph G and integer $k \equiv 1 \pmod 4$,*

$$\gamma_{sp}(S_k(G)) = \begin{cases} \frac{k+1}{2} m(G); & G \text{ contains an even cycle;} \\ \frac{k+1}{2} m(G) + 1; & \text{otherwise.} \end{cases}$$

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and $k = 4t + 1$. Suppose first that D is a minimum super dominating set of $S_k(G)$ and consider $A = \overline{D}$ and a core B of D . By Lemma 2.4, the edges $E[A, B]$ form a matching M in $S_k(G)$. If M is fixed, we have three possibilities for a super edge $P_{v_i v_j}$.

- (a) M contains at most $\frac{k-1}{2}$ edges from $P_{v_i v_j}$. The set of the corresponding edges $v_i v_j \in E(G)$ will be denoted by E_0 .
- (b) M contains exactly $\frac{k+1}{2}$ edges from $P_{v_i v_j}$ and exactly one of v_i and v_j is covered by an edge from $M \cap E(P_{v_i v_j})$. If this vertex, say v_i , is contained in A , we set $v_i v_j \in E_A$. Similarly, if $v_i \in B$, $v_i(v_i v_j)_1 \in M$, and $(v_i v_j)_k v_j \notin M$, then the edge $v_i v_j$ belongs to E_B .
- (c) M contains exactly $\frac{k+1}{2}$ edges from $P_{v_i v_j}$ and both $v_i(v_i v_j)_1$ and $(v_i v_j)_k v_j$ belong to M . In this case, we set $v_i v_j \in E_2$.

As E_0, E_A, E_B, E_2 is a partition of $E(G)$, we may estimate the size of M as follows:

$$|M| \leq |E_0| \frac{k-1}{2} + (|E_A| + |E_B| + |E_2|) \frac{k+1}{2} = \frac{k-1}{2} m(G) + |E_A| + |E_B| + |E_2|. \tag{3}$$

By definition, if $v_i v_j \in E_A \cup E_B$, then only one of $v_i(v_i v_j)_1$ and $(v_i v_j)_k v_j$ belongs to M . If $v_i v_j \in E_2$, then both $v_i(v_i v_j)_1$ and $(v_i v_j)_k v_j$ are contained in M . Since each vertex $v_i \in V(G)$ is covered by at most one M -edge, we infer $|E_A| + |E_B| + 2|E_2| \leq n$ and, in turn, we get from (3) that

$$|M| \leq \frac{k-1}{2} m(G) + n - |E_2|. \tag{4}$$

As $|M| = |A| = |\overline{D}| = n(S_k(G)) - \gamma_{sp}(S_k(G))$ and $n(S_k(G)) = n + km(G)$, inequality (4) implies

$$\gamma_{sp}(S_k(G)) = n(S_k(G)) - |M| \geq \frac{k+1}{2} m(G) + |E_2|. \tag{5}$$

If $E_2 \neq \emptyset$ or G contains an even cycle, (5) itself proves the required lower bound. From now on, we assume that there is no even cycle in G and that $E_2 = \emptyset$.

Consider an edge $v_i v_j \in E_A$ with $v_i(v_i v_j)_1 \in M$. As M contains $\frac{k+1}{2}$ edges from P_{v_i, v_j} that includes $v_i(v_i v_j)_1$ but not $(v_i v_j)_k v_j$, M contains the following edges from the super edge:

$$v_i(v_i v_j)_1, (v_i v_j)_2(v_i v_j)_3, \dots, (v_i v_j)_{4t}(v_i v_j)_{4t+1}.$$

By Lemma 2.4, $v_i \in A$ implies $(v_i v_j)_1 \in B$; the latter implies $(v_i v_j)_2 \in B$. Since $(v_i v_j)_2(v_i v_j)_3 \in M$, we infer $(v_i v_j)_3 \in A$; and so on. We obtain that $(v_i v_j)_s \in B$ if and only if $s \pmod 4 \in \{1, 2\}$; otherwise, $(v_i v_j)_s \in A$. In the last step, $v_j \in B$ also follows. It can be proved analogously that $v_i v_j \in E_B$ and $v_i \in B$ implies $v_j \in A$.

Let

$$V' = \{v_i : \exists j \in [n] \text{ s.t. } v_i(v_i v_j)_1 \in M\}, \quad V'' = \{v_j : \exists i \in [n] \text{ s.t. } v_i(v_i v_j)_1 \in M\},$$

and define a DR-function $\phi : V' \rightarrow E_A \cup E_B$ such that $\phi(v_i) = v_i v_j$ if $v_i(v_i v_j)_1 \in M$. Since $E_2 = \emptyset$, it is an injective function. Moreover, if the edge $v_i v_j$ is in the image of ϕ , then one from v_i and v_j belongs to A and the other one to B . Thus, ϕ remains a DR-function on V' , if we consider the following bipartite subgraph F instead of G . We first take the induced subgraph $G[V_F]$, where $V_F = V' \cup V''$, and then delete the edges inside $V_F \cap A$ and $V_F \cap B$. We may also say that this graph F is defined by the edge set $E_G[V_F \cap A, V_F \cap B]$. By supposition, G contains no even cycle. Therefore, the bipartite subgraph F contains no cycle at all. By Lemma 5.1, $\widehat{n}(F) \leq n(F) - 1 \leq n - 1$. Consequently, no more than $n - 1$ vertices of G are covered by an edge from M in $S_k(G)$. This implies $|E_A| + |E_B| \leq n - 1$ and, by (3), we infer $|M| \leq \frac{k-1}{2} m(G) + n - (n - 1)$ that yields

$$\gamma_{sp}(S_k(G)) \geq \frac{k+1}{2} m(G) + 1,$$

if G contains no even cycle.

To prove the reverse inequalities, we construct a γ_{sp} -set D for G . First we define an appropriate DR-function ϕ .

- If G contains an even cycle C , take a unicyclic spanning subgraph H of G such that C is the only cycle in H . Then H is bipartite, not a tree, and therefore, by Lemma 5.1, $\widehat{n}(H) = n$. Let A_H and B_H be the partite classes of H . By Lemma 5.1, there is a DR-function ϕ which assigns a representative edge from $E(H)$ to each vertex from $V(G)$.
- If G contains no even cycle, choose a spanning tree H in G . Again, H is bipartite, but now we have $\widehat{n}(H) = n - 1$. Let A_H and B_H be the partite classes of H . By Lemma 5.1, we can define a DR-function ϕ that assigns a representative edge from $E(H)$ to each vertex from $V(G) \setminus \{v_n\}$.

Having a DR-function ϕ in hand, we define a super dominating set D in $S_k(G)$ with a size that matches the required upper bound.

- (i) If $v_i \in B_H$ or $v_i \notin A_H \cup B_H$, we set $v_i \in D$.
- (ii) If $\phi(v_i) = v_i v_j$ and $v_i \in A_H$, let D contain the following vertices from the super edge $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t-3}, (v_i v_j)_{4t-2}, (v_i v_j)_{4t+1}.$$

- (iii) If $\phi(v_i) = v_i v_j$ and $v_i \in B_H$, let D contain the following vertices from $P_{v_i v_j}$:

$$(v_i v_j)_3, (v_i v_j)_4, (v_i v_j)_7, (v_i v_j)_8, \dots, (v_i v_j)_{4t-1}, (v_i v_j)_{4t}.$$

- (iv) If $v_i v_j$ does not belong to the image set of ϕ , $v_i \in A_H$, and $i < j$, then let D contain the following vertices from $P_{v_i v_j}$:

$$(v_i v_j)_2, (v_i v_j)_3, (v_i v_j)_6, (v_i v_j)_7, \dots, (v_i v_j)_{4t-2}, (v_i v_j)_{4t-1}, (v_i v_j)_{4t+1}.$$

- (v) If $v_i v_j$ does not belong to the image set of ϕ , $v_i \in B_H$, and $i < j$, let D contain the following vertices from $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_4, (v_i v_j)_5, (v_i v_j)_8, (v_i v_j)_9, \dots, (v_i v_j)_{4t}, (v_i v_j)_{4t+1}.$$

If an internal subdivision vertex, $(v_i v_j)_s$ with $3 \leq s \leq 4t - 1$, does not belong to D , it is easy to see that a neighbor super dominates it. A vertex $v_i \notin D$ with $\phi(v_i) = v_i v_j$ is always super dominated by $(v_i v_j)_1$. The subdivision vertex $u = (v_i v_j)_1$ does not belong to D , if (iii) or (iv) was applied. In the latter case, $(v_i v_j)_2$ super dominates u . In the first case, $v_i \in B_H$ and $(N[v_i] \setminus \{u\}) \subseteq D$ holds by the determination of D . The subdivision vertex $w = (v_i v_j)_2$ is missing from D , if (iii) or (v) was applied. In the first case, $(v_i v_j)_3$ super dominates w . For the second case, the condition in (v) ensures that $v_i \in D$. Hence, w is the only neighbor of $(v_i v_j)_1$ which is outside D . In case (ii), vertex $(v_i v_j)_{4t}$ is super dominated by $(v_i v_j)_{4t+1}$ as in this case $v_i \in A_H$ implies $v_j \in B_H$ and therefore, we have $v_j \in D$. In case (iv), vertex $(v_i v_j)_{4t}$ is super dominated by $(v_i v_j)_{4t-1}$. A vertex $(v_i v_j)_{4t+1}$ is outside D , only if (iii) was applied. In this case, $(v_i v_j)_{4t}$ super dominates it.

Finally, we determine the size of D . Case (i) puts $n - |A_H|$ vertices into D . When the subdivision vertices are considered, we put $\frac{k+1}{2}$ vertices from each $P_{i,j}$ into D , except when case (iii) is applied. There we deal with $|B_H|$ super edges putting $\frac{k-1}{2}$ internal vertices into D from each. This gives

$$\gamma_{sp}(S_k(G)) \leq |D| = n - |A_H| + m(G) \frac{k+1}{2} - |B_H|.$$

By the determination of the DR-function ϕ , $|A_H| + |B_H| = n$ if G contains an even cycle; and $|A_H| + |B_H| = n - 1$ if every cycle in G is of odd order. Substituting these values in the inequality, we get the required upper bounds on $\gamma_{sp}(S_k(G))$. This finishes the proof of the theorem. \square

Let $oc(G)$ denote the number of components in G that contain no even cycles. Then Theorem 5.5 directly implies:

Proposition 5.6. For every graph G and integer $k \equiv 1 \pmod 4$, it holds that

$$\gamma_{\text{sp}}(S_k(G)) = \frac{k+1}{2}m(G) + oc(G).$$

A shortest even cycle in a graph can be found in polynomial (actually quadratic) time [26], hence the number of even-cycle-free components is easy to determine. Thus we may deduce the following result:

Theorem 5.7. If k is a positive integer with $k \equiv 1 \pmod 4$, then the super domination number can be computed in polynomial time over the class of k -subdivision graphs.

5.3. $4t$ -subdivisions

Theorem 5.8. For every graph G and integer $k \equiv 0 \pmod 4$,

$$\gamma_{\text{sp}}(S_k(G)) = \frac{k}{2}m(G) + \gamma_{\text{sp}}(G).$$

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ and $k = 4t$. Suppose that D is a γ_{sp} -set of $S_k(G)$ and consider $A = \overline{D}$ and a core B of D . By Lemma 2.4, the edges $E[A, B]$ form a matching M in $S_k(G)$. We have three possibilities for a super edge $P_{v_i v_j}$.

- (a) M contains at most $\frac{k}{2} - 1$ edges from $P_{v_i v_j}$. Let E_0 denote the set of the edges $v_i v_j \in E(G)$ with this property.
- (b) M contains exactly $\frac{k}{2}$ edges from $P_{v_i v_j}$. The set of the corresponding edges $v_i v_j$ in G is denoted by E_1 .
- (c) M contains exactly $\frac{k}{2} + 1$ edges from $P_{v_i v_j}$. In this case, both $(v_i v_j)_1$ and $(v_i v_j)_k$ belong to M , and we set $v_i v_j \in E_2$.

By definitions given for E_0, E_1, E_2 , the following is true:

$$|M| \leq \frac{k}{2}m(G) + |E_2| - |E_0|. \tag{6}$$

To prove that $|E_2| - |E_0| \leq n - \gamma_{\text{sp}}(G)$, we first consider a super edge $P_{v_i v_j}$ so that $v_i v_j \in E_2$. The edges

$$v_i(v_i v_j)_1, (v_i v_j)_2(v_i v_j)_3, \dots, (v_i v_j)_{4t-2}(v_i v_j)_{4t-1}, (v_i v_j)_{4t} v_j$$

are all included in M . If $v_i \in A$ then, by Lemma 2.4, $E[A, B]$ is a matching and the vertices

$$(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t-2}, v_j$$

are from B ; while the remaining subdivision vertices belong to A . Therefore, $v_i \in A$ implies $v_j \in B$ and, similarly, $v_i \in B$ implies $v_j \in A$.

If $v_i v_j \in E_2$, $v_p v_q \in E_2$ with $v_i \in A, v_q \in B$, and G contains an edge $v_i v_q$, then we say that $v_i v_q$ is a *critical edge*. As v_i and v_q are already covered by M -edges, $v_i v_q \notin E_2$. If $v_i v_q \in E_1$, then the $k/2$ edges in $M \cap E(P_{v_i v_q})$ have to be

$$(v_i v_q)_1(v_i v_q)_2, (v_i v_q)_3(v_i v_q)_4, \dots, (v_i v_q)_{k-1}(v_i v_q)_k.$$

By Lemma 2.4, $v_i \in A$ implies $(v_i v_q)_1 \in A, (v_i v_q)_2, (v_i v_q)_3 \in B, (v_i v_q)_4, (v_i v_q)_5 \in A, \dots, (v_i v_q)_k \in A$. Finally, we infer $v_q \in A$ that contradicts the assumption $v_q \in B$. It implies that $v_i v_q \in E_0$. By symmetry, the same is true if $v_i \in B$ and $v_q \in A$ and therefore, every critical edge belongs to E_0 .

We now prove that the maximum for $|E_2| - |E_0|$ can be attained without the presence of critical edges. Indeed, if $v_i v_q$ is a critical edge such that $v_i v_j, v_p v_q \in E_2$ and $v_i \in A, v_q \in B$, then we may perform the following changes in M :

- Remove the edges $M \cap E(P_{v_i v_j})$ from M and replace them by the complement edge set $E(P_{v_i v_j}) \setminus M$. By this change, $v_i v_j$ is moved to E_1 , and v_i, v_j become uncovered by M . So, this step decreases $|E_2|$ by 1. After this change $v_i \notin A \cup B$ and we can replace the (at most) $\frac{k}{2} - 1$ M -edges on $P_{v_i v_q}$ with the following $\frac{k}{2}$ edges:

$$(v_i v_q)_1(v_i v_q)_2, (v_i v_q)_3(v_i v_q)_4, \dots, (v_i v_q)_{4t-1}(v_i v_q)_{4t}$$

such that we put $(v_i v_q)_1, (v_i v_q)_4, (v_i v_q)_5, \dots, (v_i v_q)_{4t}$ into B and the remaining subdivision vertices into A . Since $v_q \in B$, this step keeps the property $E[A, B] = M$ and, by Lemma 2.4, $D = \overline{A}$ is a super dominating set in $S_k(G)$. Note that this modification removes $v_i v_q$ from E_0 .

After applying the described changes, $|E_2| - |E_0|$ remains the same and we have less critical edges than before. Thus, performing the steps iteratively while there is a critical edge, we obtain a matching M' and sets A', B' without critical edges such that $|E_2| - |E_0|$ remains unchanged. As there are no critical edges, the E_2 -edges now form a matching M^* in G such that $E[A^*, B^*] = M^*$ for the sets $A^* = A' \cap V(G)$ and $B^* = B' \cap V(G)$. Applying Lemma 2.4 again, we conclude that $D^* = \overline{A^*}$ is a super dominating set in G . This yields

$$|E_2| = |M^*| = |A^*| = n - |D^*| \leq n - \gamma_{sp}(G).$$

Now inequality chain in (6) can be continued and we obtain

$$|M| \leq \frac{k}{2}m(G) + |E_2| - |E_0| \leq \frac{k}{2}m(G) + n - \gamma_{sp}(G)$$

which, in turn, proves

$$\begin{aligned} \gamma_{sp}(S_k(G)) &= n(S_k(G)) - |M| \\ &\geq (n + km(G)) - \left(\frac{k}{2}m(G) + n - \gamma_{sp}(G)\right) \\ &= \frac{k}{2}m(G) + \gamma_{sp}(G). \end{aligned}$$

In the second part of the proof we construct a super dominating set D of size $\frac{k}{2}m(G) + \gamma_{sp}(G)$ in $S_k(G)$. Let D^* be a γ_{sp} -set in G with the corresponding sets A^*, B^* and matching $M^* = E_G[A^*, B^*]$.

- (i) If $v_i \in D^*$, we set $v_i \in D$.
- (ii) If $v_i v_j \in M^*$ with $v_i \in A^*$ and $v_j \in B^*$, let D contain the following vertices from the super edge $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t-3}, (v_i v_j)_{4t-2}.$$

- (iii) If $v_i v_j \notin M^*$ and $v_i \in A^*$ then, as $E_G[A^*, B^*] = M^*$, we have $v_j \in V(G) \setminus B^*$. Let us put into D the following subdivision vertices from $P_{v_i v_j}$:

$$(v_i v_j)_2, (v_i v_j)_3, (v_i v_j)_6, (v_i v_j)_7, \dots, (v_i v_j)_{4t-2}, (v_i v_j)_{4t-1}.$$

- (iv) If $v_i v_j \notin M^*$ and $v_i \in B^*$ hold and also if both v_i and v_j are outside $A^* \cup B^*$, we put into D the following subdivision vertices from $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_4, (v_i v_j)_5, \dots, (v_i v_j)_{4t-4}, (v_i v_j)_{4t-3}, (v_i v_j)_{4t}.$$

In step (i), we put $|D^*| = \gamma_{sp}(G)$ non-subdivision vertices into D . Then, for each super edge considered in (ii) – (iv), we put exactly $k/2$ subdivision vertices into D . As there are no edge $v_i v_j$ in G with $v_i \in A^*, v_j \in B^*$ and $v_i v_j \notin M^*$, we treated each super edge of $S_k(G)$ in the steps (ii) – (iv). These sum up $|D| = \frac{k}{2}m(G) + \gamma_{sp}(G)$.

To check that D is a super dominating set is mainly automatic. We note that if $v_i \notin D$, then $v_i \in A^*$ and there is an edge $v_i v_j \in M^*$ which is considered in (ii). Then, $(v_i v_j)_1$ super dominates v_i . We also remark that in step (ii), the vertex $(v_i v_j)_{4t}$ is super dominated by v_j as all the other super edges $P_{v_j v_p}$ being incident to v_j were considered in step (iv). There, subdivision neighbors $(v_j v_p)_1 = (v_p v_j)_{4t}$ were put into D . For a super edge $P_{v_i v_j}$ that was treated in (iv), the condition implies $v_i, v_j \in D$. Therefore, $(v_i v_j)_1$ and $(v_i v_j)_{4t}$ super dominate $(v_i v_j)_2$ and $(v_i v_j)_{4t-1}$, respectively. It shows $\gamma_{sp}(S_k(G)) \leq |D| = \frac{k}{2}m(G) + \gamma_{sp}(G)$, and together with the first part of the proof give the equality $\gamma_{sp}(S_k(G)) = \frac{k}{2}m(G) + \gamma_{sp}(G)$ as required. \square

The problem of deciding whether $\gamma_{sp}(F) \leq \ell$ holds, clearly belongs to NP. Let ℓ be part of the input of the problem and k be a fixed integer with $k \equiv 0 \pmod 4$. By Theorem 4.1, it is NP-hard to decide whether $\gamma_{sp}(G) \leq \ell$ holds over the class of all graphs. By Theorem 5.8, $\gamma_{sp}(G) \leq \ell$ holds if and only if $\gamma_{sp}(S_k(G)) \leq \frac{k}{2}m(G) + \ell$. Thus, we may conclude the following:

Theorem 5.9. *Over the class of k -subdivision graphs, it is NP-complete to decide whether $\gamma_{sp}(F) \leq \ell$ holds, if ℓ is part of the input and k is a fixed integer with $k \equiv 0 \pmod 4$.*

5.4. II-matchings

Before continuing our study with the last case for subdivision graphs, we introduce a graph invariant and prove an additional complexity result.

In a graph G , an *induced matching* is a matching $M \subseteq E(G)$ such that the induced subgraph $G[V(M)]$ contains only the edges from M . We denote by $i(G)$ the maximum size of an induced matching in G . Induced matchings are applicable in network flow problems, secure communication, VLSI design, and elsewhere, cf. [9,21]. To decide whether $i(G) \geq \ell$ holds is known to be NP-hard in many classes of graphs, say in planar bipartite graphs [23] and in claw-free graphs [17]. For exact algorithms for maximum induced matchings see [20,25], and for the complexity aspects of the maximum-weight induced matchings and dominating induced matchings see [15,5], respectively, and references therein.

We further say that a matching M is an *II-matching* if M can be partitioned into two induced matchings M_1 and M_2 . The *II-matching number* $ii(G)$ of G is the maximum size of an II-matching in G . We prove that the II-matching number is hard to determine.

Proposition 5.10. *It is NP-complete to decide whether $ii(G) \geq \ell$ holds, if ℓ is part of the input.*

Proof. To decide whether the independence number $\alpha(F)$ of a graph F is at least k is a classical NP-complete problem [12]. We show a polynomial-time reduction from the decision problem of $\alpha(F) \geq k$ to the problem of $ii(G) \geq 2k$.

Construction For every graph F , let G_F be the graph constructed on the vertex set $V(G) \times V(K_4)$, where $V(K_4) = [4]$, by making two different vertices (x, i) and (y, j) adjacent in G_F if either $xy \in E(F)$ or $x = y$. (We note in passing that G_F is isomorphic to the lexicographic product $F \circ K_4$.) Let $V(F) = \{v_1, \dots, v_n\}$ and let V_i denote the vertex set $\{v_i\} \times [4]$ in G_F .

Reduction We show that $ii(G_F) = 2\alpha(F)$ holds for every graph F and therefore, deciding whether $\alpha(F) \geq k$ is equivalent to the problem of whether $ii(G_F) \geq 2k$ holds.

First, consider a maximum independent set S in F and define the edge sets

$$M_1 = \{(x, 1)(x, 2) : x \in S\} \quad \text{and} \quad M_2 = \{(x, 3)(x, 4) : x \in S\}.$$

By definition, $|M_1| = |M_2| = \alpha(F)$, $M_1 \cup M_2$ is a matching in G_F and, since S is an independent set in F , both M_1 and M_2 are induced matchings. It follows that $ii(G_F) \geq 2\alpha(F)$.

Assume now, that we have a maximum II-matching $M = M_1 \cup M_2$ in G_F . If the induced matching M_1 contains a *cross edge* xy , that is an edge xy with $x \in V_i$, $y \in V_j$ such that $i \neq j$, then M_1 cannot cover any other vertex from the neighborhood of x . Equivalently, if $v_i v_{i'}$ is an edge in F , then $V_{i'} \cap V(M_1)$ cannot contain a vertex different from y . Thus, if the cross edge xy is replaced in M_1 with an arbitrary edge inside V_i , the set M_1 remains an induced matching. As M_2 is also an induced matching, it covers at most two vertices from V_i . We therefore have a vertex $x' \in V_i$ such that $(M \setminus \{xy\}) \cup \{xx'\}$ is an II-matching and $|M| = |M'|$. Repeating this procedure for all cross edges in $M_1 \cup M_2$, we obtain an II-matching without cross edges. Again, we may refer to the property that if V_i contains an edge from M_p , for $p \in [2]$, then $V(M_p) \cap V_{i'} = \emptyset$ whenever $v_i v_{i'} \in E(F)$. We may conclude that $|M_p| \leq \alpha(F)$ and hence, $|M| \leq 2\alpha(F)$. This finishes the proof for $ii(G_F) = 2\alpha(F)$.

Conclusion As G_F is obtained by a polynomial-time construction from F , and the NP-complete problem of deciding whether $\alpha(F) \geq k$ holds can be reduced to the problem of deciding about $ii(G_F) \geq 2k$, the latter problem is also NP-hard. It is also clear that the decision problem of $ii(G) \geq \ell$ belongs to NP. Thus, we may infer that the problem is NP-complete over the class of all graphs. \square

5.5. $(4t + 2)$ -subdivisions

Theorem 5.11. *For every graph G and integer $k \equiv 2 \pmod 4$,*

$$\gamma_{\text{sp}}(S_k(G)) = \frac{k}{2} m(G) + n(G) - ii(G).$$

Proof. Let $V(G) = \{v_1, \dots, v_n\}$, and $k = 4t + 2$. Choose a γ_{sp} -set D of $S_k(G)$. Let $A = \overline{D}$ and B an arbitrary core of D . By Lemma 2.4, the edges $E[A, B]$ form a matching M in $S_k(G)$. Again, we have three possibilities for a super edge $P_{v_i v_j}$.

- (a) M contains at most $\frac{k}{2} - 1$ edges from $P_{v_i v_j}$. Let E_0 denote the set of the edges $v_i v_j \in E(G)$ with this property.
- (b) M contains exactly $\frac{k}{2}$ edges from $P_{v_i v_j}$. The set of the corresponding edges $v_i v_j$ in G is denoted by E_1 .
- (c) M contains exactly $\frac{k}{2} + 1$ edges from $P_{v_i v_j}$. In this case, both $v_i(v_i v_j)_1$ and $(v_i v_j)_k v_j$ belong to M , and we set $v_i v_j \in E_2$.

By definitions, the following inequality holds:

$$|M| \leq \frac{k}{2} m(G) + |E_2| - |E_0|. \tag{7}$$

For every edge $v_i v_j \in E_2$ of G , the super edge $P_{v_i v_j}$ must contain the edges $v_i(v_i v_j)_1, (v_i v_j)_2(v_i v_j)_3, \dots, (v_i v_j)_{4t} v_j$ from M . Moreover, if $v_i \in A$ in $S_k(G)$ then, by Lemma 2.4, $E[A, B]$ is a matching in $S_k(G)$ and

$$B \cap V(P_{v_i v_j}) = \{(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t+1}, (v_i v_j)_{4t+2}\}.$$

The remaining subdivision vertices and v_j then belong to A . Therefore, $v_i \in A$ implies $v_j \in A$ and, similarly, $v_i \in B$ implies $v_j \in B$ if $v_i v_j \in E_2$. We may therefore partition E_2 into

$$E_A = \{v_i v_j : v_i v_j \in E_2 \text{ and } v_i, v_j \in A\} \quad \text{and} \quad E_B = \{v_i v_j : v_i v_j \in E_2 \text{ and } v_i, v_j \in B\}.$$

Suppose now that E_A contains two edges $v_i v_j$ and $v_p v_q$ and there exists an edge $v_i v_q \in E(G)$. We will say that $v_i v_q$ is an A -critical edge. As v_i and v_q are already covered by M -edges, $v_i v_q \notin E_2$. By the same reason, if $v_i v_q \in E_1$, then the $k/2$ edges in $M \cap E(P_{v_i v_q})$ are $(v_i v_q)_1(v_i v_q)_2, \dots, (v_i v_q)_{4t+1}(v_i v_q)_{4t+2}$. Referring to Lemma 2.4 again, $v_i \in A$ implies $(v_i v_q)_1 \in A, (v_i v_q)_2, (v_i v_q)_3 \in B, \dots, (v_i v_q)_{4t+2} \in B$, and also that $v_q \in B$. As $v_q \in A$ was supposed, it is a contradiction. We may infer that $v_i v_q \in E_0$ holds for every A -critical edge. The same is true for the set B ; that is if $v_i v_j, v_p v_q \in E_B$ and a B -critical edge $v_i v_q$ is present in G , then $v_i v_q \in E_0$.

We now prove that the maximum for $|E_2| - |E_0|$ can be attained without the presence of A - and B -critical edges. Indeed, let $v_i v_q$ be an A -critical edge such that $v_i v_j, v_p v_q \in E_A$. We may perform the following alteration in M without decreasing $|E_2| - |E_0|$. First we remove the edges $M \cap E(P_{v_i v_j})$ from M and replace them by the complement edge set $E(P_{v_i v_j}) \setminus M$. Then, we also remove the at most $\frac{k}{2} - 1$ edges $M \cap E(P_{v_i v_q})$ and replace them by the $\frac{k}{2}$ edges $(v_i v_q)_1(v_i v_q)_2, \dots, (v_i v_q)_{4t+1}(v_i v_q)_{4t+2}$. After these changes we update the sets A and B along $P_{v_i v_j}$ and $P_{v_i v_q}$ such that, for $x \in \{j, q\}$ and $s \in [4t + 2]$, a subdivision vertex $(v_i v_x)_s$ belongs to B if $s \bmod 4 \in \{0, 1\}$, otherwise it is put into A . The vertices v_i, v_j are not in A anymore, but we still have $v_q \in A$. It can be checked that $E[A, B]$ remains a matching and, by Lemma 2.4, $D = \bar{A}$ is a super dominating of $S_k(G)$. By this change, both $v_i v_j$ and $v_i v_q$ are moved to E_1 , the number of A -critical edges is decreased, while $|E_2| - |E_0|$ remains the same. By the symmetry of the roles of the sets A and B , if a B -critical edge exists, we may do the analogous changes.

Repeating these changes while there are critical edges, we obtain a matching M and a super dominating set D without decreasing $|E_2| - |E_0|$. The edges in E_2 still form a matching as every $u \in V(G)$ is covered only one edge from M ; and the absence of A - and B -critical edges means that both E_A and E_B are induced matchings in G . It implies that the obtained M is an II-matching and hence, $|E_2| \leq \text{ii}(G)$. From (7), we now obtain

$$|M| \leq \frac{k}{2} m(G) + \text{ii}(G),$$

and we may conclude

$$\gamma_{\text{sp}}(S_k(G)) = n(S_k(G)) - |M| \geq \frac{k}{2} m(G) + n(G) - \text{ii}(G). \tag{8}$$

To complete the proof, we show that there exists a super dominating set D of the required cardinality in $S_k(G)$. Let $M^* = M_1^* \cup M_2^*$ be a maximum II-matching in G . The set D is constructed by the following five rules.

- (i) A vertex $v_i \in V(G)$ belongs to D if and only if $v_i \notin V(M_1^*)$.
- (ii) If $v_i v_j \in M_1^*$, then D contains the following vertices from the super edge $P_{v_i v_j}$:

$$(v_i v_j)_1, (v_i v_j)_2, (v_i v_j)_5, (v_i v_j)_6, \dots, (v_i v_j)_{4t+1}, (v_i v_j)_{4t+2}.$$

- (iii) If $v_i v_j \in M_2^*$, then D contains the following vertices from $P_{v_i v_j}$:

$$(v_i v_j)_3, (v_i v_j)_4, (v_i v_j)_7, (v_i v_j)_8, \dots, (v_i v_j)_{4t-1}, (v_i v_j)_{4t}.$$

- (iv) If $v_i v_j \notin M^*$ and $v_i \in V(M_1^*)$, then $v_j \notin V(M_1^*)$. In this case, we put the following subdivision vertices into D :

$$(v_i v_j)_2, (v_i v_j)_3, (v_i v_j)_6, (v_i v_j)_7, \dots, (v_i v_j)_{4t-2}, (v_i v_j)_{4t-1}, (v_i v_j)_{4t+2}.$$

- (v) If $v_i v_j \notin M^*$ and $v_i \in V(M_2^*), v_j \notin V(M^*)$, and also if $v_i, v_j \notin V(M^*)$, we put the following subdivision vertices into D :

$$(v_i v_j)_1, (v_i v_j)_4, (v_i v_j)_5, \dots, (v_i v_j)_{4t}, (v_i v_j)_{4t+1}.$$

In step (i) we put $n(G) - 2|M_1^*|$ non-subdivision vertices into D ; in step (ii) we consider $|M_1^*|$ super edges and put $\frac{k+2}{2}$ subdivision vertices from each into D ; in (iii) $|M_2^*|$ super edges are considered and put $|M_2^*| \frac{k+2}{2}$ vertices into D . For the remaining edges $v_i v_j$ of G step (iv) or (v) is applied. In either case, D contains exactly $\frac{k}{2}$ subdivision vertices from $P_{v_i v_j}$. The size of D is therefore

$$\begin{aligned} |D| &= (n(G) - 2|M_1^*|) + |M_1^*| \left(\frac{k}{2} + 1\right) + |M_2^*| \left(\frac{k}{2} - 1\right) + (m(G) - |M_1^*| - |M_2^*|) \frac{k}{2} \\ &= \frac{k}{2} m(G) + n(G) - (|M_1^*| + |M_2^*|) \\ &= \frac{k}{2} m(G) + n(G) - \text{ii}(G). \end{aligned}$$

It is straightforward to check that D is a super dominating set in $S_k(G)$. We notice that if a vertex $v_i \in V(G)$ does not belong to D , then there exists an edge $v_i v_j \in M_1^*$ and, by (ii), the subdivision vertex $(v_i v_j)_1$ super dominates v_i . If $v_i \in V(M_2^*)$ such that $v_i v_j \in M_2^*$, then $v_i \in D$ and $(v_i v_j)_1$ is v_i 's only neighbor which is not in D . Then, v_i super dominates $(v_i v_j)_1$ (that is the same as $(v_j v_i)_{4t+2}$). For a super edge $P_{v_i v_j}$ considered in (v), both ends v_i and v_j belong to D and hence, $(v_i v_j)_1$ super dominates $(v_i v_j)_2$.

Since the constructed set D is a super dominating set, we may conclude

$$\gamma_{\text{sp}}(S_k(G)) \leq |D| = \frac{k}{2} m(G) + n(G) - \text{ii}(G)$$

which, together with (8), complete the proof of the theorem. \square

As a consequence of Proposition 5.10 and Theorem 5.11, we obtain the following result.

Theorem 5.12. *Over the class of k -subdivision graphs, it is NP-complete to decide whether $\gamma_{\text{sp}}(F) \leq \ell$ holds, if ℓ is part of the input and k is a fixed integer with $k \equiv 2 \pmod 4$.*

6. Conclusions

We conclude the paper by summarizing our main results on the computational complexity of the following problem.

SUPER DOMINATION (S-DOM) PROBLEM

Instance: A simple undirected graph G and an integer ℓ .
Question: Does $\gamma_{\text{sp}}(G) \leq \ell$ hold?

By Theorems 3.1, 4.1, 5.4, 5.7, 5.9, and 5.12, we can conclude the following summary.

- The S-DOM problem is NP-complete over the following graph classes:
 - (A) Bipartite graphs of girth at least 8;
 - (B) Class of k -subdivision graphs if k is a fixed even integer.
- The S-DOM problem can be solved in polynomial time over the following graph classes:
 - (C) Trees;
 - (D) Class of k -subdivision graphs if k is an odd integer.

Classes (A), (C), and (D) are subclasses of bipartite graphs. However, to get a better picture of the computational complexity of the S-DOM problem on the class of bipartite graphs, we propose the following problems.

Problem 6.1. Find a subclass \mathcal{C} of (A) so that the S-DOM problem remains NP-complete on \mathcal{C} .

Problem 6.2. Find further subclasses of bipartite graphs over which the S-DOM problem can be solved in polynomial time.

The ultimate goal we set is a complete characterization:

Problem 6.3. Characterize the subclasses of bipartite graphs over which the S-DOM problem remains NP-complete.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

- [1] S. Alikhani, N. Ghanbari, S. Soltani, Total dominator chromatic number of k -subdivision of graphs, *Art Discrete Appl. Math.* 6 (2023) 1.10.
- [2] M. Arockiaraj, S. Klavžar, S. Mushtaq, K. Balasubramanian, Topological characterization of the full k -subdivision of a family of partial cubes and their applications to α -types of novel graphyne and graphdiyne materials, *Polycycl. Aromat. Compd.* 41 (2021) 1902–1924.
- [3] B. Bollobás, E.J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance, *J. Graph Theory* 3 (1979) 241–249.
- [4] W. Barrett, S. Butler, M. Catral, S.M. Fallat, M. Shaun, H.T. Hall, L. Hogben, M. Young, The maximum nullity of a complete subdivision graph is equal to its zero forcing number, *Electron. J. Linear Algebra* 27 (2014) 444–457.
- [5] A. Brandstädt, F. Mosca, Finding dominating induced matchings in P_9 -free graphs in polynomial time, *Discuss. Math., Graph Theory* 42 (2022) 1139–1162.
- [6] M. Dettlaff, M. Lemańska, J.A. Rodríguez-Velázquez, R. Zuazua, On the super domination number of lexicographic product graphs, *Discrete Appl. Math.* 263 (2019) 118–129.
- [7] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, WH Freeman and Co., New York, 1979.
- [8] N. Ghanbari, Some results on the super domination number of a graph, *Discrete Math. Algorithms Appl.* (2023), <https://doi.org/10.1142/S1793830923500441>.
- [9] M.C. Golumbic, M. Lewenstein, New results on induced matchings, *Discrete Appl. Math.* 101 (2000) 157–165.
- [10] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (Eds.), *Topics in Domination in Graphs*, Developments in Mathematics, vol. 64, Springer, Cham, 2020.
- [11] T.W. Haynes, S.T. Hedetniemi, M.A. Henning (Eds.), *Structures of Domination in Graphs*, Developments in Mathematics, vol. 66, Springer, Cham, 2021.
- [12] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher, J.D. Bohlinger (Eds.), *Complexity of Computer Computations*, Plenum, New York, 1972.
- [13] S. Klavžar, K. Knauer, T. Marc, On the Djoković-Winkler relation and its closure in subdivisions of fullerenes, triangulations, and chordal graphs, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 327–342.
- [14] D.J. Klein, J.A. Rodríguez-Velázquez, E. Yi, On the super domination number of graphs, *Commun. Comb. Optim.* 5 (2020) 83–96.
- [15] B. Klemz, G. Rote, Linear-time algorithms for maximum-weight induced matchings and minimum chain covers in convex bipartite graphs, *Algorithmica* 84 (2022) 1064–1080.
- [16] B. Krishnakumari, Y.B. Venkatakrishnan, Double domination and super domination in trees, *Discrete Math. Algorithms Appl.* 08 (2016) 1650067.
- [17] D. Kobler, U. Rotics, Finding maximum induced matchings in subclasses of claw-free and P_5 -free graphs, and in graphs with matching and induced matching of equal maximum size, *Algorithmica* 37 (2003) 327–346.
- [18] M. Lemańska, V. Swaminathan, Y.B. Venkatakrishnan, R. Zuazua, Super dominating sets in graphs, *Proc. Natl. Acad. Sci. India Sect. A* 85 (2015) 353–357.
- [19] V. Lozin, I. Razgon, Tree-width dichotomy, *Eur. J. Comb.* 103 (2022) 103517.
- [20] V.D. Nguyen, B.T. Pham, P.T. Do, Efficient algorithms for maximum induced matching problem in permutation and trapezoid graphs, *Fundam. Inform.* 182 (2021) 257–283.
- [21] S. Saeedi Madani, D. Kiani, Induced matchings in strongly biconvex graphs and some algebraic applications, *Math. Nachr.* 294 (2021) 1160–1174.
- [22] B. Senthilkumar, Y.B. Venkatakrishnan, H.N. Kumar, Super domination in trees, *Discrete Math. Algorithms Appl.* 14 (2022) 2150137.
- [23] L.J. Stockmeyer, V.V. Vazirani, NP-completeness of some generalizations of the maximum matching problem, *Inf. Process. Lett.* 15 (1982) 14–19.
- [24] F. Wang, X. Liu, Coloring 3-power of 3-subdivision of subcubic graph, *Discrete Math. Algorithms Appl.* 10 (2018) 1850041.
- [25] M. Xiao, H. Tan, Exact algorithms for maximum induced matching, *Inf. Comput.* 256 (2017) 196–211.
- [26] R. Yuster, U. Zwick, Finding even cycles even faster, *SIAM J. Discrete Math.* 10 (1997) 209–222.
- [27] J. Zhou, L. Sun, C. Bu, Resistance characterizations of equiarboreal graphs, *Discrete Math.* 340 (2017) 2864–2870.
- [28] W. Zhuang, Super domination in trees, *Graphs Comb.* 38 (2022) 21.