Discrete Applied Mathematics 343 (2024) 340–349

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Outerplane bipartite graphs with isomorphic resonance graphs

Simon Brezovnik [a,b], Zhongyuan Che [c], Niko Tratnik [b,d,*], Petra Žigert Pleteršek [d,e]

[a] University of Ljubljana, Faculty of Mechanical Engineering, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[c] Department of Mathematics, Penn State University, Beaver Campus, Monaca, USA
[d] University of Maribor, Faculty of Natural Sciences and Mathematics, Slovenia
[e] University of Maribor, Faculty of Chemistry and Chemical Engineering, Slovenia

ARTICLE INFO

Article history:
Received 13 June 2023
Received in revised form 30 October 2023
Accepted 1 November 2023
Available online 8 November 2023

Keywords:
Distributive lattice
Inner dual
Isomorphic resonance graphs
Isomorphic resonance digraphs
Order ideal
2-connected outerplane bipartite graph

ABSTRACT

We present novel results related to isomorphic resonance graphs of 2-connected outerplane bipartite graphs. As the main result, we provide a structure characterization for 2-connected outerplane bipartite graphs with isomorphic resonance graphs. Three additional characterizations are expressed in terms of resonance digraphs, via local structures of inner duals, as well as using distributive lattices on the set of order ideals of posets defined on inner faces of 2-connected outerplane bipartite graphs.
© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

Resonance graphs reflect interactions between perfect matchings (in chemistry known as Kekulé structures) of plane bipartite graphs. These graphs were independently introduced by chemists (El-Basil [9,10], Gründler [11]) and also by mathematicians (Zhang, Guo, and Chen [16]) under the name Z-transformation graph. Initially, resonance graphs were investigated on hexagonal systems [16]. Later, this concept was generalized to plane bipartite graphs, see [15,19–21].

In recent years, various structural properties of resonance graphs of plane bipartite graphs were obtained [5–8]. The problem of characterizing 2-connected outerplane bipartite graphs with isomorphic resonance graphs is also interesting and nontrivial. There are outerplane bipartite graphs G and G' whose inner duals are isomorphic paths but with non-isomorphic resonance graphs. For example, let G be a linear benzenoid chain (a chain in which every non-terminal hexagon is linear) with n hexagons, and let G' be a fibonaccene (a benzenoid chain in which every non-terminal hexagon is angular, see [12]) with n hexagons, where $n > 2$. Then the inner dual T of graph G is isomorphic to the inner dual T' of graph G', since T and T' are both paths on n vertices. However, their resonance graphs $R(G)$ and $R(G')$ are not isomorphic: $R(G)$ is a path and $R(G')$ is a Fibonacci cube, see Fig. 1.

In [2,3] the problem of finding catacondensed even ring systems (shortly called CERS) with isomorphic resonance graphs was investigated. More precisely, the relation of evenly homeomorphic CERS was introduced and it was proved that

* Corresponding author at: University of Maribor, Faculty of Natural Sciences and Mathematics, Slovenia.
E-mail addresses: simon.brezovnik@fs.uni-lj.si (S. Brezovnik), zxc10@psu.edu (Z. Che), niko.tratnik@um.si (N. Tratnik), petra.zigert@um.si (P. Žigert Pleteršek).

https://doi.org/10.1016/j.dam.2023.11.006
0166-218X/© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

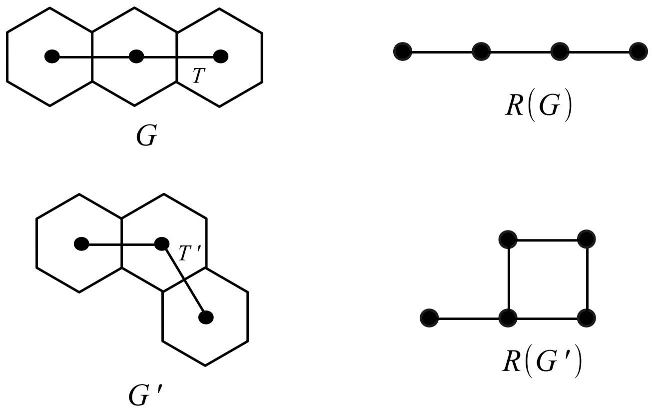

Fig. 1. Resonance graphs of the linear benzenoid chain and fibonaccene with three hexagons.

if two CERS are evenly homeomorphic, then their resonance graphs are isomorphic. Conversely, it is true for catacondensed even ring chains but not for all CERS [3]. Moreover, in [4] it was proved that if two 2-connected outerplane bipartite graphs are evenly homeomorphic, then their resonance graphs are isomorphic. In papers [3,4], the following open problem was stated.

Problem 1. Characterize 2-connected outerplane bipartite graphs with isomorphic resonance graphs.

In this paper we solve the above problem. Firstly, we state all the needed definitions and previous results as preliminaries. The main result, Theorem 3.4, is presented in Section 3. The necessity part of this result is stated as Theorem 3.2. Moreover, in Corollary 3.3 we show that two 2-connected outerplane bipartite graphs have isomorphic resonance graphs if and only if they can be properly two colored so that their resonance digraphs are isomorphic. In addition, by Corollary 3.6, it follows that 2-connected outerplane bipartite graphs G and G' have isomorphic resonance graphs if and only if there exists an isomorphism α between their inner duals T and T' such that for any 3-path xyz of T, the triple (x, y, z) is regular if and only if $(\alpha(x), \alpha(y), \alpha(z))$ is regular. Finally, we provide Corollary 4.2 to connect our results with a result from [18] which showed that the distributive lattice on the set of perfect matchings of G and the distributive lattice on the set of order ideals of the poset defined on all inner faces of G are isomorphic for any 2-connected outerplane bipartite graph G.

2. Preliminaries

We say that two faces of a plane graph G are *adjacent* if they have an edge in common. An *inner face* (also called a *finite face*) adjacent to the *outer face* (also called the *infinite face*) is named a *peripheral face*. In addition, we denote the set of edges lying on some face s of G by $E(s)$. The subgraph induced by the edges in $E(s)$ is the *periphery of* s and denoted by ∂s. The periphery of the outer face is also called the *periphery of* G and denoted by ∂G. Moreover, for a peripheral face s and the outer face s_0, the subgraph induced by the edges in $E(s) \cap E(s_0)$ is called the *common periphery* of s and G, and denoted by $\partial s \cap \partial G$. The vertices of G that belong to the outer face are called *peripheral vertices* and the remaining vertices are *interior vertices*. Furthermore, an *outerplane graph* is a plane graph in which all vertices are peripheral vertices.

A bipartite graph G is *elementary* if and only if it is connected and each edge is contained in some perfect matching of G. Any elementary bipartite graph other than K_2 is 2-connected. Hence, if G is a plane elementary bipartite graph with more than two vertices, then the periphery of each face of G is an even cycle. A peripheral face s of a plane elementary bipartite graph G is called *reducible* if the subgraph H of G obtained by removing all internal vertices (if exist) and edges on the common periphery of s and G is elementary.

The *inner dual* of a plane graph G is a graph whose vertex set is the set of all inner faces of G, and two vertices being adjacent if the corresponding faces are adjacent.

A *perfect matching* M of a graph G is a set of independent edges of G such that every vertex of G is incident with exactly one edge from M. An even cycle C of G is called M-*alternating* if the edges of C appear alternately in M and in $E(G) \setminus M$. Also, a face s of a 2-connected plane bipartite graph is M-*resonant* if ∂s is an M-alternating cycle.

Let G be a plane elementary bipartite graph and $\mathcal{M}(G)$ be the set of all perfect matchings of G. Assume that s is a reducible face of G. By [14], the common periphery of s and G is an odd length path P. By Proposition 4.1 in [5], P is M-alternating for any perfect matching M of G, and $\mathcal{M}(G) = \mathcal{M}(G; P^-) \cup \mathcal{M}(G; P^+)$, where $\mathcal{M}(G; P^-)$ is the set of perfect matchings M of G such that two end edges of P are not contained in M or P is a single edge and not contained in M;

$\mathcal{M}(G; P^+)$ is the set of perfect matchings M of G such that two end edges of P are contained in M or P is a single edge and contained in M. Furthermore, $\mathcal{M}(G; P^-)$ and $\mathcal{M}(G; P^+)$ can be partitioned as

$$\mathcal{M}(G; P^-) = \mathcal{M}(G; P^-, \partial s) \cup \mathcal{M}(G; P^-, \overline{\partial s})$$
$$\mathcal{M}(G; P^+) = \mathcal{M}(G; P^+, \partial s) \cup \mathcal{M}(G; P^+, \overline{\partial s})$$

where $\mathcal{M}(G; P^-, \partial s)$ (resp., $\mathcal{M}(G; P^-, \overline{\partial s})$) is the set of perfect matchings M in $\mathcal{M}(G; P^-)$ such that s is M-resonant (resp., not M-resonant), and $\mathcal{M}(G; P^+, \partial s)$ (resp., $\mathcal{M}(G; P^+, \overline{\partial s})$) is the set of perfect matchings M in $\mathcal{M}(G; P^+)$ such that s is M-resonant (resp., not M-resonant).

Let G be a plane bipartite graph with a perfect matching. The *resonance graph* (also called *Z-transformation graph*) $R(G)$ of G is the graph whose vertices are the perfect matchings of G, and two perfect matchings M_1, M_2 are adjacent whenever their symmetric difference forms the edge set of exactly one inner face s of G. In this case, we say that the edge M_1M_2 has the *face-label* s.

Let H and K be two graphs with vertex sets $V(H)$ and $V(K)$, respectively. The Cartesian product of H and K is a graph with the vertex set $\{(h, k) \mid h \in V(H), k \in V(K)\}$ such that two vertices (h_1, k_1) and (h_2, k_2) are adjacent if either h_1h_2 is an edge of H and $k_1 = k_2$ in K or k_1k_2 is an edge of K and $h_1 = h_2$ in H. Assume that G is a disjoint union of two plane bipartite graphs G_1 and G_2. Then by definitions, the resonance graph $R(G)$ is the Cartesian product of $R(G_1)$ and $R(G_2)$.

Assume that G is a plane bipartite graph whose vertices are properly colored black and white such that adjacent vertices receive different colors. Let M be a perfect matching of G. An M-alternating cycle C of G is *M-proper* (resp., *M-improper*) if every edge of C belonging to M goes from white to black vertex (resp., from black to white vertex) along the clockwise orientation of C. A plane bipartite graph G with a perfect matching has a unique perfect matching $M_{\hat{0}}$ (resp., $M_{\hat{1}}$) such that G has no proper $M_{\hat{0}}$-alternating cycles (resp., no improper $M_{\hat{1}}$-alternating cycles) [19].

The *resonance digraph*, denoted by $\overrightarrow{R}(G)$, is the digraph obtained from $R(G)$ by adding a direction for each edge so that $\overrightarrow{M_1M_2}$ is a directed edge from M_1 to M_2 if $M_1 \oplus M_2$ is a proper M_1-alternating (or, an improper M_2-alternating) cycle surrounding an inner face of G. Let $\mathcal{M}(G)$ be the set of all perfect matchings of G. Then a partial order $\leq$ can be defined on $\mathcal{M}(G)$ such that $M' \leq M$ if there is a directed path from M to M' in $\overrightarrow{R}(G)$. When G is a plane elementary bipartite graph, $\mathbf{M}(G) := (\mathcal{M}(G), \leq)$ is a finite distributive lattice whose Hasse diagram is isomorphic to $\overrightarrow{R}(G)$ [13]. It is well known that $M_{\hat{0}}$ is the minimum element and $M_{\hat{1}}$ the maximum element of the distributive lattice $\mathbf{M}(G)$ [13,17].

We now present the concept of a reducible face decomposition, see [5,6,20]. Firstly, we introduce the *bipartite ear decomposition* of a plane elementary bipartite graph G with n inner faces. Starting from an edge e of G, we join its two end vertices by a path P_1 of odd length and proceed inductively to build a sequence of bipartite graphs as follows. If $G_{i-1} = e + P_1 + \cdots + P_{i-1}$ has already been constructed, add the ith ear P_i of odd length by joining any two vertices belonging to different bipartition sets of G_{i-1} such that P_i has no internal vertices in common with the vertices of G_{i-1} to obtain G_i. A bipartite ear decomposition of a plane elementary bipartite graph G is called a *reducible face decomposition* (shortly *RFD*) if G_1 is a periphery of an inner face s_1 of G, and the ith ear P_i lies in the exterior of G_{i-1} such that P_i and a part of the periphery of G_{i-1} surround an inner face s_i of G for all $i \in \{2, \ldots, n\}$. For such a decomposition, we use notation $RFD(G_1, G_2, \ldots, G_n)$, where $G_n = G$. It was shown [20] that a plane bipartite graph with more than two vertices is elementary if and only if it has a reducible face decomposition.

Let H be a convex subgraph of a graph G. The *peripheral convex expansion* of G with respect to H, denoted by $pce(G; H)$, is the graph obtained from G by the following procedure:

(i) Replace each vertex v of H by an edge v_1v_2.
(ii) Insert edges between v_1 and the neighbors of v in $V(G) \setminus V(H)$.
(iii) Insert the edges u_1v_1 and u_2v_2 whenever u, v of H are adjacent in G.

Two edges uv and xy of a connected graph G are said to be in *relation Θ* (also known as *Djoković–Winkler relation*), denoted by $uv\Theta xy$, if $d_G(u, x) + d_G(v, y) \neq d_G(u, y) + d_G(v, x)$. It is well known that if G is a plane elementary bipartite graph, then its resonance graph $R(G)$ is a median graph [17] and therefore, the relation Θ is an equivalence relation on the set of edges $E(R(G))$.

Let xy be an edge of a resonance graph $R(G)$ and $F_{xy} = \{e \in E(R(G)) \mid e\Theta xy\}$ be the set of all edges in relation Θ with xy in $R(G)$, where G is a plane elementary bipartite graph. By Proposition 3.2 in [5], all edges in F_{xy} have the same face-label. On the other hand, two edges with the same face-label can be in different Θ-classes of $R(G)$.

We now present several results from previous papers which will be needed later.

Proposition 2.1 (*[14]*). *Let G be a plane elementary bipartite graph other than K_2. Then the outer cycle of G is improper $M_{\hat{0}}$-alternating as well as proper $M_{\hat{1}}$-alternating, where $M_{\hat{0}}$ is the minimum element and $M_{\hat{1}}$ the maximum element in the finite distributive lattice $\mathbf{M}(G)$.*

The induced subgraph of a graph G on $W \subseteq V(G)$ will be denoted as $\langle W \rangle$.

Theorem 2.2 (*[5]*). *Assume that G is a plane elementary bipartite graph and s is a reducible face of G. Let P be the common periphery of s and G. Let H be the subgraph of G obtained by removing all internal vertices and edges of P. Assume that $R(G)$*

and $R(H)$ are resonance graphs of G and H respectively. Let F be the set of all edges in $R(G)$ with the face-label s. Then F is a Θ-class of $R(G)$ and $R(G) - F$ has exactly two components $\langle \mathcal{M}(G; P^-) \rangle$ and $\langle \mathcal{M}(G; P^+) \rangle$. Furthermore,

(i) F is a matching defining an isomorphism between $\langle \mathcal{M}(G; P^-, \partial s) \rangle$ and $\langle \mathcal{M}(G; P^+, \partial s) \rangle$;
(ii) $\langle \mathcal{M}(G; P^-, \partial s) \rangle$ is convex in $\langle \mathcal{M}(G; P^-) \rangle$, $\langle \mathcal{M}(G; P^+, \partial s) \rangle$ is convex in $\langle \mathcal{M}(G; P^+) \rangle$;
(iii) $\langle \mathcal{M}(G; P^-) \rangle$ and $\langle \mathcal{M}(G; P^+) \rangle$ are median graphs, where $\langle \mathcal{M}(G; P^-) \rangle \cong R(H)$.

In particular, $R(G)$ can be obtained from $R(H)$ by a peripheral convex expansion if and only if $\mathcal{M}(G; P^+) = \mathcal{M}(G; P^+, \partial s)$.

Proposition 2.3 ([6]). *Let G be a 2-connected outerplane bipartite graph. Assume that s is a reducible face of G. Then s is adjacent to exactly one inner face of G.*

For any 2-connected outerplane bipartite graph G and a reducible face s of G, we know from [14] that the common periphery of s and G is an odd length path P. By Proposition 2.3, s is adjacent to exactly one inner face s' of G. It is clear that the common edge of s and s' is a single edge e. Therefore, $E(s) = e \cup E(P)$ and the odd length path P must have at least three edges.

Theorem 2.4 ([6]). *Let G be a 2-connected outerplane bipartite graph. Assume that s is a reducible face of G and P is the common periphery of s and G. Let H be the subgraph of G obtained by removing all internal vertices and edges of P. Then $R(G)$ can be obtained from $R(H)$ by a peripheral convex expansion, that is, $R(G) = pce(R(H); T)$ where the set of all edges between $R(H)$ and T is a Θ-class of $R(G)$ with the face-label s. Moreover,*

(i) $R(G)$ has exactly one more Θ-class than $R(H)$ and it has the face-label s, and
(ii) each of other Θ-classes of $R(G)$ can be obtained from the corresponding Θ-class of $R(H)$ with the same face-label (adding more edges if needed).

Theorem 2.5 ([6]). *Let G be a 2-connected outerplane bipartite graph and $R(G)$ be its resonance graph. Assume that G has a reducible face decomposition $G_i (1 \leq i \leq n)$ where $G_n = G$ associated with a sequence of inner faces $s_i (1 \leq i \leq n)$ and a sequence of odd length ears $P_i (2 \leq i \leq n)$. Then $R(G)$ can be obtained from the one edge graph by a sequence of peripheral convex expansions with respect to the above reducible face decomposition of G. Furthermore, $R(G_1) = K_2$ where the edge has the face-label s_1; for $2 \leq i \leq n$, $R(G_i) = pce(R(G_{i-1}); T_{i-1})$ where the set of all edges between $R(G_{i-1})$ and T_{i-1} is a Θ-class in $R(G_i)$ with the face-label s_i, $R(G_i)$ has exactly one more Θ-class than $R(G_{i-1})$ and it has the face-label s_i, each of other Θ-classes of $R(G_i)$ can be obtained from the corresponding Θ-class of $R(G_{i-1})$ with the same face-label (adding more edges if needed).*

The *induced graph $\Theta(R(G))$* on the Θ-classes of $R(G)$ is a graph whose vertex set is the set of Θ-classes, and two vertices E and F of $\Theta(R(G))$ are adjacent if $R(G)$ has two incident edges $e \in E$ and $f \in F$ such that e and f are not contained in a common 4-cycle of $R(G)$. It is well-known that if s and t are two face labels of incident edges of a 4-cycle of $R(G)$, then s and t are vertex disjoint in G and M-resonant for a perfect matching M of G; if s and t are two face labels of incident edges not contained in a common 4-cycle of $R(G)$, then s and t are adjacent in G and M-resonant for a perfect matching M of G.

Theorem 2.6 ([6]). *Let G be a 2-connected outerplane bipartite graph and $R(G)$ be its resonance graph. Then the graph $\Theta(R(G))$ induced by the Θ-classes of $R(G)$ is a tree and isomorphic to the inner dual of G.*

3. Main results

In this section, we characterize 2-connected outerplane bipartite graphs with isomorphic resonance graphs. We start with the following lemma, which is a more detailed version of Theorem 2.4 [6] and Lemma 1 [8] for 2-connected outerplane bipartite graphs. We use $\mathcal{M}(G; e)$ to denote the set of perfect matchings of a graph G containing the edge e of G.

Lemma 3.1. *Let G be a 2-connected outerplane bipartite graph. Assume that s is a reducible face of G, P is the common periphery of s and G and $e \in E(s)$ is a unique edge that does not belong to P. Let H be the subgraph of G obtained by removing all internal vertices and edges of P.*

Further, assume that H has more than two vertices. Let $M_{\hat{0}}$ be the minimum element and $M_{\hat{1}}$ be the maximum element in the distributive lattice $\mathbf{M}(H)$. Then e is contained in exactly one of $M_{\hat{0}}$ and $M_{\hat{1}}$.

(i) Suppose that $M_{\hat{0}} \notin \mathcal{M}(H; e)$. Let $\widehat{M_{\hat{0}}}$ be the perfect matching of G such that $M_{\hat{0}} \subseteq \widehat{M_{\hat{0}}}$ and $\widehat{M_{\hat{1}}}$ be the perfect matching of G such that $M_{\hat{1}} \setminus \{e\} \subseteq \widehat{M_{\hat{1}}}$. Then $\widehat{M_{\hat{0}}} \in \mathcal{M}(G; P^-, \overline{\partial s})$ is the minimum element, and $\widehat{M_{\hat{1}}} \in \mathcal{M}(G; P^+, \partial s)$ is the maximum element of the finite distributive lattice $\mathbf{M}(G)$.
(ii) Suppose that $M_{\hat{0}} \in \mathcal{M}(H; e)$. Let $\widehat{M_{\hat{0}}}$ be the perfect matching of G such that $M_{\hat{0}} \setminus \{e\} \subseteq \widehat{M_{\hat{0}}}$ and $\widehat{M_{\hat{1}}}$ be the perfect matching of G such that $M_{\hat{1}} \subseteq \widehat{M_{\hat{1}}}$. Then $\widehat{M_{\hat{0}}} \in \mathcal{M}(G; P^+, \partial s)$ is the minimum element, and $\widehat{M_{\hat{1}}} \in \mathcal{M}(G; P^-, \overline{\partial s})$ is the maximum element of the finite distributive lattice $\mathbf{M}(G)$.

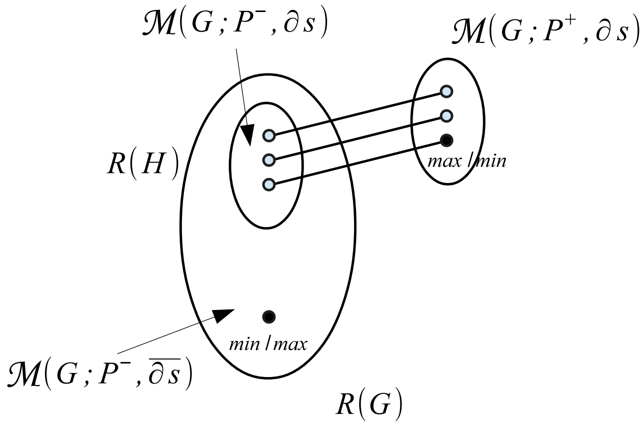

Fig. 2. A peripheral convex expansion of the resonance graph $R(G)$.

Proof. By Theorem 2.4, $R(G) = pce(R(H), \langle\mathcal{M}(H; e)\rangle)$, where the edges between $R(H)$ and $\langle\mathcal{M}(H; e)\rangle$ is a Θ-class of $R(G)$ with the face-label s. Moreover, $R(H) \cong \langle\mathcal{M}(G; P^-)\rangle$ and $\langle\mathcal{M}(H; e)\rangle \cong \langle\mathcal{M}(G; P^-, \partial s)\rangle \cong \langle\mathcal{M}(G; P^+, \partial s)\rangle$. See Fig. 2.

Any 2-connected outerplane bipartite graph has two perfect matchings whose edges form alternating edges on the outer cycle of the graph. By Proposition 2.1, one is the maximum element and the other is the minimum element in the finite distributive lattice on the set of perfect matchings of the graph.

Let $M_{\hat{0}}$ be the minimum element and $M_{\hat{1}}$ be the maximum element in the finite distributive lattice $\mathbf{M}(H)$. Then the outer cycle of H is both improper $M_{\hat{0}}$-alternating and proper $M_{\hat{1}}$-alternating. Note that e is an edge of the outer cycle of H. Then e is contained in exactly one of $M_{\hat{0}}$ and $M_{\hat{1}}$.

We will show only part (i), since the proof of (ii) is analogous. Suppose that $M_{\hat{0}}$ does not contain the edge e. Recall that the outer cycle of H is improper $M_{\hat{0}}$-alternating. By the definition of $\widehat{M_{\hat{0}}}$, the outer cycle of G is improper $\widehat{M_{\hat{0}}}$-alternating. Therefore, $\widehat{M_{\hat{0}}}$ is the minimum element of the distributive lattice $\mathbf{M}(G)$ since G is an outerplane bipartite graph. Note that three consecutive edges on the periphery of s, namely e and two end edges of P, are not contained in $\widehat{M_{\hat{0}}}$. Then s is not $\widehat{M_{\hat{0}}}$-resonant. So, $\widehat{M_{\hat{0}}} \in \mathcal{M}(G; P^-, \overline{\partial s})$.

Note that $M_{\hat{1}}$ contains the edge e since $M_{\hat{0}}$ does not contain e by our assumption for part (i). Recall that the outer cycle of H is proper $M_{\hat{1}}$-alternating. By the definition of $\widehat{M_{\hat{1}}}$, $\widehat{M_{\hat{1}}} \in \mathcal{M}(G; P^+)$ and the outer cycle of G is again proper $\widehat{M_{\hat{1}}}$-alternating. It follows that s is $\widehat{M_{\hat{1}}}$-resonant. Consequently, $\widehat{M_{\hat{1}}} \in \mathcal{M}(G; P^+, \partial s)$ is the maximum element of the finite distributive lattice $\mathbf{M}(G)$. $\square$

To state the next theorem, we need the following notation. Let G and G' be 2-connected outerplane bipartite graphs. Suppose that ϕ is an isomorphism between resonance graphs $R(G)$ and $R(G')$. By Theorem 2.6, the isomorphism ϕ induces an isomorphism between inner duals of G and G', which we denote by $\widehat{\phi}$.

Theorem 3.2. *Let G and G' be 2-connected outerplane bipartite graphs. If there exists an isomorphism ϕ between resonance graphs $R(G)$ and $R(G')$, then G has a reducible face decomposition $G_i(1 \le i \le n)$ where $G_n = G$ associated with the face sequence $s_i(1 \le i \le n)$ and the odd length path sequence $P_i(2 \le i \le n)$; and G' has a reducible face decomposition $G'_i(1 \le i \le n)$ where $G'_n = G'$ associated with the face sequence $s'_i(1 \le i \le n)$ and the odd length path sequence $P'_i(2 \le i \le n)$ satisfying three properties:*

(i) the isomorphism $\widehat{\phi}$ between the inner duals of G and G' maps s_i to s'_i for $1 \le i \le n$;
(ii) G and G' can be properly two colored so that odd length paths P_i and P'_i either both start from a black vertex and end with a white vertex, or both start from a white vertex and end with a black vertex in clockwise orientation along the peripheries of G_i and G'_i for $2 \le i \le n$;
(iii) ϕ is an isomorphism between resonance digraphs $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ with respect to the colorings from property (ii).

Proof. Let $\phi : R(G) \longrightarrow R(G')$ be an isomorphism between $R(G)$ and $R(G')$. By Theorem 2.6, the graph $\Theta(R(G))$ induced by the Θ-classes of $R(G)$ is a tree and isomorphic to the inner dual of G, and the graph $\Theta(R(G'))$ induced by the Θ-classes of $R(G')$ is a tree and isomorphic to the inner dual of G'. By the peripheral convex expansions with respect to a reducible face decomposition of a 2-connected outerplane bipartite graph given by Theorem 2.5, we can see that ϕ induces an isomorphism $\widehat{\phi}$ between the inner duals of G and G'. So, G and G' have the same number of inner faces.

Suppose that G and G' have n inner faces. Obviously, all three properties hold if $n = 1$ or $n = 2$. Let $n \ge 3$. We proceed by induction on n and therefore assume that all three properties hold for any 2-connected outerplane bipartite graphs with less than n inner faces.

Let s_n be a reducible face of G, P_n be the common periphery of s_n and G, and E be the Θ-class in $R(G)$ corresponding to s_n. Moreover, we denote by E' the Θ-class in $R(G')$ obtained from E by the isomorphism ϕ, and s'_n the corresponding reducible face of G'. Then $s'_n = \widehat{\phi}(s_n)$. Also, we denote by P'_n the common periphery of s'_n and G'.

By Theorem 2.4, the graph $R(G)$ is obtained from $R(G_{n-1})$ by a peripheral convex expansion with respect to the Θ-class E. Similarly, the graph $R(G')$ is obtained from $R(G'_{n-1})$ by a peripheral convex expansion with respect to the Θ-class E'. Since ϕ is an isomorphism between $R(G)$ and $R(G')$ such that ϕ maps E to E', it follows that $R(G_{n-1})$ and $R(G'_{n-1})$ are isomorphic and the restriction of ϕ on $R(G_{n-1})$ is an isomorphism ϕ_{n-1} between $R(G_{n-1})$ and $R(G'_{n-1})$. Let $\widehat{\phi}_{n-1}$ be the induced isomorphism between the inner duals of G_{n-1} and G'_{n-1}. Then $\widehat{\phi}_{n-1}$ is the restriction of $\widehat{\phi}$ on the inner dual of G_{n-1}.

Since G_{n-1} and G'_{n-1} have $n-1$ inner faces, by the induction hypothesis G_{n-1} has a reducible face decomposition $G_i (1 \le i \le n-1)$ associated with the face sequence $s_i (1 \le i \le n-1)$ and the odd length path sequence $P_i (2 \le i \le n-1)$; and G' has a reducible face decomposition $G'_i (1 \le i \le n-1)$ associated with the face sequence $s'_i (1 \le i \le n-1)$ and the odd length path sequence $P'_i (2 \le i \le n-1)$ satisfying properties (i) $s'_i = \widehat{\phi}_{n-1}(s_i)$ for $1 \le i \le n-1$, (ii) G_{n-1} and G'_{n-1} can be properly two colored so that odd length paths P_i and P'_i either both start from a black vertex and end with a white vertex, or both start from a white vertex and end with a black vertex in clockwise orientation along the peripheries of G_i and G'_i for $2 \le i \le n-1$, and (iii) ϕ_{n-1} is an isomorphism between resonance digraphs $\overrightarrow{R}(G_{n-1})$ and $\overrightarrow{R}(G'_{n-1})$ with respect to the colorings from property (ii).

Obviously, since $s'_n = \widehat{\phi}(s_n)$ and $s'_i = \widehat{\phi}_{n-1}(s_i) = \widehat{\phi}(s_i)$ for $1 \le i \le n-1$, property (i) holds for the above reducible face decompositions of G and G'. It remains to show that the these reducible face decompositions satisfy property (ii) when $i = n$, and ϕ is an isomorphism between resonance digraphs $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ with respect to the colorings from property (ii), that is, property (iii) holds.

By Proposition 2.3, s_n is adjacent to exactly one inner face of G since s_n is a reducible face of G. Suppose that the unique inner face adjacent to s_n is s_j. Since $\widehat{\phi}$ is an isomorphism between the inner duals of G and G', the unique inner face adjacent to s'_n is s'_j. By Lemma 3.1, $\partial s_n \cap \partial s_j$ is an edge uv on ∂G_{n-1}, and $\partial s'_n \cap \partial s'_j$ is an edge $u'v'$ on $\partial G'_{n-1}$. It is clear that u and v (resp., u' and v') are two end vertices of P_n (resp., P'_n). Moreover, $R(G) = pce(R(G_{n-1}), \langle \mathcal{M}(G_{n-1}; uv)\rangle)$ where $R(G_{n-1}) \cong \langle \mathcal{M}(G; P_n^-)\rangle$ and $\langle \mathcal{M}(G_{n-1}; uv)\rangle \cong \langle \mathcal{M}(G; P_n^-, \partial s_n)\rangle \cong \langle \mathcal{M}(G; P_n^+, \partial s_n)\rangle$, and $R(G') = pce(R(G'_{n-1}), \langle \mathcal{M}(G'_{n-1}; u'v')\rangle)$ where $R(G'_{n-1}) \cong \langle \mathcal{M}(G'; P'^-_n)\rangle$ and $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle \cong \langle \mathcal{M}(G'; P'^-_n, \partial s'_n)\rangle \cong \langle \mathcal{M}(G'; P'^+_n, \partial s'_n)\rangle$.

Recall that ϕ is an isomorphism between $R(G) = pce(R(G_{n-1}), \langle \mathcal{M}(G_{n-1}; uv)\rangle)$ and $R(G') = pce(R(G'_{n-1}), \langle \mathcal{M}(G'_{n-1}; u'v')\rangle)$. We also have that ϕ_{n-1} is an isomorphism between resonance digraphs $\overrightarrow{R}(G_{n-1})$ and $\overrightarrow{R}(G'_{n-1})$, where ϕ_{n-1} is the restriction of ϕ on $R(G_{n-1})$. Hence, ϕ_{n-1} maps $\langle \mathcal{M}(G_{n-1}; uv)\rangle$ to $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$ such that if an edge xy of $\langle \mathcal{M}(G_{n-1}; uv)\rangle$ is directed from x to y in $\overrightarrow{R}(G_{n-1})$, then $\phi_{n-1}(x)\phi_{n-1}(y)$ is an edge of $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$ directed from $\phi_{n-1}(x)$ to $\phi_{n-1}(y)$ in $\overrightarrow{R}(G'_{n-1})$.

Claim 1. *Each edge of $\langle \mathcal{M}(G; P_n^+, \partial s_n)\rangle$ resulted from the peripheral convex expansion of an edge x_1y_1 in $\langle \mathcal{M}(G_{n-1}; uv)\rangle$ has the same orientation as the edge of $\langle \mathcal{M}(G'; P'^+_n, \partial s'_n)\rangle$ resulted from the peripheral convex expansion of $\phi_{n-1}(x_1)\phi_{n-1}(y_1)$ in $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$.*

Proof of Claim 1. Let x_1y_1 be an edge in $\langle \mathcal{M}(G_{n-1}; uv)\rangle$. Then $\phi_{n-1}(x_1)\phi_{n-1}(y_1)$ is its corresponding edge under ϕ_{n-1} in $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$.

Assume that x_1x_2 and y_1y_2 are two edges between $R(G_{n-1})$ and $\langle \mathcal{M}(G; P_n^+, \partial s_n)\rangle$, where both edges have face-label s_n. Then x_2y_2 is an edge of $\langle \mathcal{M}(G; P_n^+, \partial s_n)\rangle$ resulted from the peripheral convex expansion of the edge x_1y_1. Note that $\phi_{n-1}(x_1)\phi(x_2)$ and $\phi_{n-1}(y_1)\phi(y_2)$ are two edges between $R(G'_{n-1})$ and $\langle \mathcal{M}(G'; P'^+_n, \partial s'_n)\rangle$, where both edges have face-label $s'_n = \widehat{\phi}(s_n)$. Hence, $\phi(x_2)\phi(y_2)$ is an edge of $\langle \mathcal{M}(G'; P'^+_n, \partial s'_n)\rangle$ resulted from the peripheral convex expansion of the edge $\phi_{n-1}(x_1)\phi_{n-1}(y_1)$.

Without loss of generality, we show that if x_2y_2 is directed from x_2 to y_2 in $\overrightarrow{R}(G)$, then $\phi(x_2)\phi(y_2)$ is directed from $\phi(x_2)$ to $\phi(y_2)$ in $\overrightarrow{R'}(G)$.

Recall both edges x_1x_2 and y_1y_2 of $R(G)$ have face-label s_n. Then $x_1 = x_2 \oplus \partial s_n$ and $y_1 = y_2 \oplus \partial s_n$. By the peripheral convex expansion structure of $R(G)$ from $R(G_{n-1})$, vertices x_1, y_1, y_2, x_2 form a 4-cycle C in $R(G)$. It is well known [5] that two antipodal edges of a 4-cycle in $R(G)$ have the same face-label and two face-labels of adjacent edges of a 4-cycle in $R(G)$ are vertex disjoint faces of G. Assume that two antipodal edges x_1y_1 and x_2y_2 of C in $R(G)$ have the face-label s_k for some $1 \le k \le n-1$. Then $x_1 \oplus y_1 = x_2 \oplus y_2 = \partial s_k$ where s_k is vertex disjoint from s_n. By our assumption that x_2y_2 is directed from x_2 to y_2 in $\overrightarrow{R}(G)$, it follows that x_1y_1 is directed from x_1 to y_1 in $\overrightarrow{R}(G_{n-1}) \subset \overrightarrow{R}(G)$.

Since ϕ_{n-1} is an isomorphism between resonance digraphs $\overrightarrow{R}(G_{n-1})$ and $\overrightarrow{R'}(G_{n-1})$, we have that $\phi_{n-1}(x_1)\phi_{n-1}(y_1)$ is directed from $\phi_{n-1}(x_1)$ to $\phi_{n-1}(y_1)$ in $\overrightarrow{R'}(G_{n-1})$. Similarly to the above argument, vertices $\phi_{n-1}(x_1), \phi_{n-1}(y_1), \phi(y_2), \phi(x_2)$ form a 4-cycle in $R(G')$, where two antipodal edges $\phi_{n-1}(x_1)\phi_{n-1}(y_1)$ and $\phi(x_2)\phi(y_2)$ of C' in $R(G')$ have the face-label $s'_k = \widehat{\phi}_{n-1}(s_k)$, where s'_k and s'_n are vertex disjoint faces of G'. Recall both edges $\phi_{n-1}(x_1)\phi(x_2)$ and $\phi_{n-1}(y_1)\phi(y_2)$ have face-label $s'_n = \widehat{\phi}(s_n)$. Then $\phi(x_2) = \phi_{n-1}(x_1) \oplus \partial s'_n$ and $\phi(y_2) = \phi_{n-1}(y_1) \oplus \partial s'_n$. It follows that $\phi(x_2)\phi(y_2)$ is directed from $\phi(x_2)$ to $\phi(y_2)$ in $\overrightarrow{R'}(G)$. Therefore, Claim 1 holds. □

Claim 2. *The edges between $\mathcal{M}(G; P_n^-, \partial s_n)$ and $\mathcal{M}(G; P_n^+, \partial s_n)$ in $\overrightarrow{R}(G)$ have the same orientation as the edges between $\mathcal{M}(G'; P_n'^-, \partial s_n')$ and $\mathcal{M}(G'; P_n'^+, \partial s_n')$ in $\overrightarrow{R}(G')$.*

Proof of Claim 2. By definitions of $\mathcal{M}(G; P_n^-, \partial s_n)$, $\mathcal{M}(G; P_n^+, \partial s_n)$, and directed edges in $\overrightarrow{R}(G)$, we can see that all edges between $\mathcal{M}(G; P_n^-, \partial s_n)$ and $\mathcal{M}(G; P_n^+, \partial s_n)$ are directed from one set to the other. Similarly, all edges between $\mathcal{M}(G'; P_n'^-, \partial s_n')$ and $\mathcal{M}(G'; P_n'^+, \partial s_n')$ are directed from one set to the other.

Let $M_{\hat{0}}$ be the minimum element and $M_{\hat{1}}$ the maximum element in the distributive lattice $\mathbf{M}(G_{n-1})$. By Lemma 3.1, exactly one of these two perfect matchings contains the edge uv. Without loss of generality, let $M_{\hat{0}} \in \mathcal{M}(G_{n-1}; uv)$ where $\langle\mathcal{M}(G_{n-1}; uv)\rangle \cong \langle\mathcal{M}(G; P_n^-, \partial s_n)\rangle \cong \langle\mathcal{M}(G; P_n^+, \partial s_n)\rangle$. Let $\widehat{M_{\hat{0}}}$ be the perfect matching of G such that $M_{\hat{0}} \setminus \{uv\} \subseteq \widehat{M_{\hat{0}}}$. Then $\widehat{M_{\hat{0}}} \in \mathcal{M}(G; P^+, \partial s)$ is the minimum element of the distributive lattice $\mathbf{M}(G)$.

Let $M'_{\hat{0}} = \phi_{n-1}(M_{\hat{0}})$. Then $M'_{\hat{0}}$ is the minimum element of the distributive lattice $\mathbf{M}(G'_{n-1})$, and $M'_{\hat{0}} \in \mathcal{M}(G'_{n-1}; u'v')$ where $\langle\mathcal{M}(G'_{n-1}; u'v')\rangle \cong \langle\mathcal{M}(G'; P'^-_n, \partial s'_n)\rangle \cong \langle\mathcal{M}(G'; P'^+_n, \partial s'_n)\rangle$. As before, define $\widehat{M'_{\hat{0}}}$ as the perfect matching of G' such that $M'_{\hat{0}} \setminus \{u'v'\} \subseteq \widehat{M'_{\hat{0}}}$. By Lemma 3.1, $\widehat{M'_{\hat{0}}} \in \mathcal{M}(G'; P'^+, \partial s)$ is the minimum element of the distributive lattice $\mathbf{M}(G')$. This implies that Claim 2 holds. □

Consequently, ϕ is also an isomorphism between resonance digraphs $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$, which means that property (iii) holds.

Suppose that $P_n = \partial s_n - uv$ starts with u and ends with v along the clockwise orientation of the periphery of G, and $P_n' = \partial s_n' - u'v'$ starts with u' and ends with v' along the clockwise orientation of the periphery of G'. Since the resonance digraphs $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ are isomorphic, it follows that u and u' have the same color and v and v' have the same color. So, the above reducible face decompositions of G and G' also satisfy property (ii) when $i = n$. Therefore, property (ii) holds. □

The following corollary follows directly from Theorem 3.2.

Corollary 3.3. *Let G and G' be 2-connected outerplane bipartite graphs. Then their resonance graphs $R(G)$ and $R(G')$ are isomorphic if and only if G and G' can be properly two colored so that $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ are isomorphic.*

We are now ready to state the following main result of the paper.

Theorem 3.4. *Let G and G' be 2-connected outerplane bipartite graphs. Then their resonance graphs $R(G)$ and $R(G')$ are isomorphic if and only if G has a reducible face decomposition $G_i (1 \le i \le n)$ associated with the face sequence $s_i (1 \le i \le n)$ and the odd length path sequence $P_i (2 \le i \le n)$; and G' has a reducible face decomposition $G_i' (1 \le i \le n)$ associated with the face sequence $s_i' (1 \le i \le n)$ and the odd length path sequence $P_i' (2 \le i \le n)$ satisfying two properties:*

- *(i) the map sending s_i to s_i' induces an isomorphism between the inner dual of G and inner dual of G' for $1 \le i \le n$; and*
- *(ii) G and G' can be properly two colored so that odd length paths P_i and P_i' either both start from a black vertex and end with a white vertex, or both start from a white vertex and end with a black vertex in clockwise orientation along the peripheries of G_i and G_i' for $2 \le i \le n$.*

Proof. *Necessity.* This implication follows by Theorem 3.2.

Sufficiency. Let G and G' be 2-connected outerplane bipartite graphs each with n inner faces. Use induction on n. The result holds when $n = 1$ or 2. Assume that $n \ge 3$ and the result holds for any two 2-connected outerplane bipartite graphs each with less than n inner faces. By Theorem 2.5, $R(G)$ can be obtained from an edge by a peripheral convex expansions with respect to a reducible face decomposition $G_i (1 \le i \le n)$ associated with the face sequence $s_i (1 \le i \le n)$ and the odd length path sequence $P_i (2 \le i \le n)$; and $R(G')$ can be obtained from an edge by a peripheral convex expansions with respect to a reducible face decomposition $G_i' (1 \le i \le n)$ associated with the face sequence $s_i' (1 \le i \le n)$ and the odd length path sequence $P_i' (2 \le i \le n)$.

Assume that properties (i) and (ii) hold for the above reducible face decompositions of G and G'. By induction hypothesis, $R(G_{n-1})$ and $R'(G_{n-1})$ are isomorphic.

Similarly to the argument in Theorem 3.2, we can see that s_n is adjacent to exactly one inner face s_j of G such that $\partial s_n \cap \partial s_j$ is an edge uv on ∂G_{n-1}; and s_n' is adjacent to exactly one inner face s_j' of G' such that $\partial s_n' \cap \partial s_j'$ is an edge $u'v'$ on $\partial G'_{n-1}$. It is clear that u and v (resp., u' and v') are two end vertices of P_n (resp., P_n'). Moreover, $R(G) = pce(R(G_{n-1}), \langle\mathcal{M}(G_{n-1}; uv)\rangle)$, and $R(G') = pce(R(G'_{n-1}), \langle\mathcal{M}(G'_{n-1}; u'v')\rangle)$. To show that $R(G)$ and $R(G')$ are isomorphic, it remains to prove that $\langle\mathcal{M}(G_{n-1}; uv)\rangle$ and $\langle\mathcal{M}(G'_{n-1}; u'v')\rangle$ are isomorphic.

Let H_{n-1} be the subgraph of G_{n-1} obtained by removing two end vertices of the edge uv, and repeatedly removing end vertices of resulted pendant edges during the process, and H'_{n-1} be the subgraph of G'_{n-1} obtained by removing two end vertices of the edge $u'v'$, and repeatedly removing end vertices of resulted pendant edges during the process. Note that all vertices of G_{n-1} (resp., G'_{n-1}) are on the outer cycle of G_{n-1} (resp., G'_{n-1}) since G_{n-1} (resp., G'_{n-1}) is an outerplane graph. Then all resulted pendant edges during the process of obtaining H_{n-1} (resp., H'_{n-1}) from G_{n-1} (resp., G'_{n-1}) are the edges on the outer cycle of G_{n-1} (resp., G'_{n-1}). It follows either both H_{n-1} and H'_{n-1} are empty, or H_{n-1} and H'_{n-1} are connected subgraphs

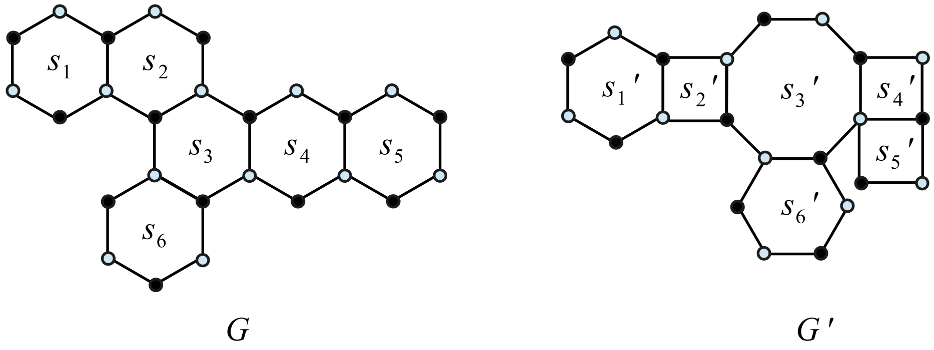

Fig. 3. Two outerplane bipartite graphs with isomorphic resonance graphs.

of G_{n-1} and G'_{n-1}, respectively. Moreover, if both H_{n-1} and H'_{n-1} are empty, then $\mathcal{M}(G_{n-1}; uv)$ contains a unique perfect matching of G_{n-1} and $\mathcal{M}(G'_{n-1}; u'v')$ contains a unique perfect matching of G'_{n-1}. So, $\langle \mathcal{M}(G_{n-1}; uv)\rangle$ and $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$ are isomorphic as single vertices.

We now assume that H_{n-1} and H'_{n-1} are connected subgraphs of G_{n-1} and G'_{n-1}, respectively. It is clear that all resulted pendant edges during the process of obtaining H_{n-1} from G are the edges of each perfect matching in $\mathcal{M}(G_{n-1}; uv)$, and so each perfect matching of H_{n-1} can be extended uniquely to a perfect matching in $\mathcal{M}(G_{n-1}; uv)$. Hence, there is a 1-1 correspondence between the set of perfect matchings of H_{n-1} and the set of perfect matchings in $\mathcal{M}(G_{n-1}; uv)$. Two perfect matchings of H_{n-1} are adjacent in $R(H_{n-1})$ if and only if the corresponding two perfect matchings in $\mathcal{M}(G_{n-1}; uv)$ are adjacent in $\langle \mathcal{M}(G_{n-1}; uv)\rangle$. Therefore, $R(H_{n-1})$ is isomorphic to $\langle \mathcal{M}(G_{n-1}; uv)\rangle$. Similarly, $R(H'_{n-1})$ is isomorphic to $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$.

Next, we show that $R(H_{n-1})$ and $R(H'_{n-1})$ are isomorphic. Note that G_{n-1} and G'_{n-1} have reducible face decompositions satisfying properties (i) and (ii). By the constructions of H_{n-1} and H'_{n-1}, we can distinguish two cases based on whether H_{n-1} and H'_{n-1} are 2-connected or not.

Case 1. H_{n-1} and H'_{n-1} are 2-connected. Then by their constructions, H_{n-1} and H'_{n-1} have reducible face decompositions satisfying properties (i) and (ii). Then $R(H_{n-1})$ and $R(H'_{n-1})$ are isomorphic by induction hypothesis.

Case 2. Each of H_{n-1} and H'_{n-1} has more than one 2-connected component. Note that any 2-connected component of H_{n-1} and H'_{n-1} is a 2-connected outerplane bipartite graph. This implies that any bridge of H_{n-1} (resp., H'_{n-1}) cannot belong to any perfect matching of H_{n-1} (resp., H'_{n-1}). Hence, any perfect matching of H_{n-1} (resp., H'_{n-1}) is the perfect matching of the union of its 2-connected components. It follows that $R(H_{n-1})$ (resp., $R(H'_{n-1})$) is the Cartesian product of resonance graphs of its 2-connected components. By the constructions of H_{n-1} and H'_{n-1}, there is a 1-1 correspondence between the set of 2-connected components of H_{n-1} and the set of 2-connected components of H'_{n-1} such that each 2-connected component of H_{n-1} and its corresponding 2-connected component of H'_{n-1} have reducible face decompositions satisfying properties (i) and (ii). Then $R(H_{n-1})$ and $R(H'_{n-1})$ are isomorphic by induction hypothesis.

We have shown that $R(H_{n-1})$ is isomorphic to $\langle \mathcal{M}(G_{n-1}; uv)\rangle$, and $R(H'_{n-1})$ is isomorphic to $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$. Therefore, $\langle \mathcal{M}(G_{n-1}; uv)\rangle$ and $\langle \mathcal{M}(G'_{n-1}; u'v')\rangle$ are isomorphic. It follows that $R(G)$ and $R(G')$ are isomorphic. □

Example. Let G and G' be 2-connected outerplane bipartite graphs shown in Fig. 3. It is easy to check that these two graphs satisfy the conditions of Theorem 3.4. So, they have isomorphic resonance graphs $R(G)$ and $R(G')$.

Finally, we can formulate the main result using local structures of given graphs. Let e and f be two edges of a graph G. Let $d_{L(G)}(e, f)$ denote the distance between corresponding vertices in the line graph $L(G)$ of G. The following concepts introduced in [4] will be also needed for that purpose.

Definition 3.5 (*[4]*)**.** Let G be a 2-connected outerplane bipartite graph and s, s', s'' be three inner faces of G. Then the triple (s, s', s'') is called an *adjacent triple of inner faces* if s and s' have the common edge e and s', s'' have the common edge f. The adjacent triple of inner faces (s, s', s'') is *regular* if the distance $d_{L(G)}(e, f)$ is an even number, and *irregular* otherwise.

It is easy to see that 2-connected outerplane bipartite graphs G and G' have reducible face decompositions satisfying properties (i) and (ii) if and only if there exists an isomorphism between the inner dual of G and G' that preserves the (ir)regularity of adjacent triples of inner faces. Therefore, the next result follows directly from Theorem 3.4.

Corollary 3.6. *Let G and G' be two 2-connected outerplane bipartite graphs with inner duals T and T', respectively. Then G and G' have isomorphic resonance graphs if and only if there exists an isomorphism $\alpha : V(T) \to V(T')$ such that for any 3-path xyz of T: the adjacent triple (x, y, z) of inner faces of G is regular if and only if the adjacent triple $(\alpha(x), \alpha(y), \alpha(z))$ of inner faces of G' is regular.*

4. Remarks

One referee pointed out that our main results are closely related to Theorem 5.4 in [18]. The following terminologies used in [18] are needed to explain their relations.

Let $\mathbf{P}$ be a finite poset with a partial order $\leq$. An *order ideal* $\mathbf{I}$ of $\mathbf{P}$ is a subset of $\mathbf{P}$ such that for every $x \in \mathbf{I}$, $y \leq x$ implies $y \in \mathbf{I}$. The set $J(\mathbf{P})$ of order ideals of $\mathbf{P}$, ordered by the set-inclusion, forms a poset $\mathbf{J}(\mathbf{P})$. It is well known that $\mathbf{J}(\mathbf{P})$ is a distributive lattice, see [1].

Assume that G is a 2-connected outerplane bipartite graph whose vertices are properly colored black and white such that adjacent vertices receive different colors and $G^{\#}$ is the inner dual graph of G. Let e^* be the edge of $G^{\#}$ corresponding to the edge e of G, and f^* be the vertex of $G^{\#}$ corresponding to the inner face f of G. An orientation $\overrightarrow{G}^{\#}$ of the inner dual graph $G^{\#}$ can be given such that an edge e^* is oriented as an arc from f_2^* to f_1^* whenever the following condition holds true: if one goes along e^* from f_2^* to f_1^*, the white end-vertex of e is located on the right side. A partial order $\leqslant$ can be defined on the set $\mathcal{F}(G)$ of all inner faces of G such that for $f_1, f_2 \in \mathcal{F}(G)$, $f_1 \leq f_2$ if $\overrightarrow{G}^{\#}$ contains a directed path from f_2^* to f_1^*. It follows that $\mathbf{F}(G) := (\mathcal{F}(G), \leq)$ is a poset since $\overrightarrow{G}^{\#}$ does not contain directed cycles. Let $\mathbf{J}(\mathbf{F}(G))$ be the distributive lattice on the set of order ideals of the poset $\mathbf{F}(G)$. Then Theorem 5.4 in [18] can be stated as follows.

Theorem 4.1 (*[18]*)**.** *Let G be a 2-connected outerplane bipartite graph. Assume that $\mathbf{M}(G)$ is the distributive lattice on the set of perfect matchings of G and $\mathbf{J}(\mathbf{F}(G))$ is the distributive lattice on the set of order ideals of the poset $\mathbf{F}(G)$. Then $\mathbf{M}(G)$ is isomorphic to $\mathbf{J}(\mathbf{F}(G))$.*

Our main results and Theorem 5.4 in [18] can be related by the following corollary.

Corollary 4.2. *Let G and G' be 2-connected outerplane bipartite graphs. Then $R(G)$ and $R(G')$ are isomorphic if and only if G and G' can be properly two colored so that $\mathbf{J}(\mathbf{F}(G))$ and $\mathbf{J}(\mathbf{F}(G'))$ are isomorphic.*

Proof. By Corollary 3.3, $R(G)$ and $R(G')$ are isomorphic if and only if G and G' can be properly two colored so that $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ are isomorphic. Note that $\overrightarrow{R}(G)$ and $\overrightarrow{R}(G')$ are isomorphic if and only if $\mathbf{M}(G)$ and $\mathbf{M}(G')$ are isomorphic since the Hasse diagram of $\mathbf{M}(G)$ is isomorphic to $\overrightarrow{R}(G)$, and the Hasse diagram of $\mathbf{M}(G')$ is isomorphic to $\overrightarrow{R}(G')$. The conclusion follows by Theorem 4.1. $\square$

We conclude the paper with the following open problem.

Problem 2. Characterize plane (elementary) bipartite graphs with isomorphic resonance graphs.

Data availability

No data was used for the research described in the article.

Acknowledgments

Simon Brezovnik, Niko Tratnik, and Petra Žigert Pleteršek acknowledge the financial support from the Slovenian Research and Innovation Agency: research programme No. P1-0297 (Simon Brezovnik, Niko Tratnik, Petra Žigert Pleteršek), projects No. J1-4031, J2-2512 (Simon Brezovnik), N1-0285 (Niko Tratnik), and L7-4494 (Petra Žigert Pleteršek). All four authors thank the Slovenian Research and Innovation Agency for financing our bilateral project between Slovenia and the USA (title: *Structural properties of resonance graphs and related concepts*, project No. BI-US/22-24-158).

The authors would like to thank the referees for their helpful comments.

References

[1] G. Birkhoff, Lattice theory, in: Amer. Math. Soc. Colloq. Publ, Vol. 25, third ed., 1973.
[2] S. Brezovnik, N. Tratnik, P. Žigert Pleteršek, Resonantly equivalent catacondensed even ring systems, MATCH Commun. Math. Comput. Chem. 82 (2019) 625–638.
[3] S. Brezovnik, N. Tratnik, P. Žigert Pleteršek, Resonance graphs of catacondensed even ring systems, Appl. Math. Comput. 374 (2020) 125064.
[4] S. Brezovnik, N. Tratnik, P. Žigert Pleteršek, Resonance graphs and a binary coding of perfect matchings of outerplane bipartite graphs, MATCH Commun. Math. Comput. Chem. 90 (2023) 453–468.
[5] Z. Che, Structural properties of resonance graphs of plane elementary bipartite graphs, Discrete Appl. Math. 247 (2018) 102–110.
[6] Z. Che, Characterizations of the resonance graph of an outerplane bipartite graph, Discrete Appl. Math. 258 (2019) 264–268.
[7] Z. Che, Cube-free resonance graphs, Discrete Appl. Math. 284 (2020) 262–268.
[8] Z. Che, Peripheral convex expansions of resonance graphs, Order 38 (2021) 365–376.
[9] S. El-Basil, Generation of lattice graphs. An equivalence relation on Kekulé counts of catacondensed benzenoid hydrocarbons, J. Mol. Struct. (Theochem) 288 (1993) 67–84.
[10] S. El-Basil, Kekulé structures as graph generators, J. Math. Chem. 14 (1993) 305–318.
[11] W. Gründler, Signifikante Elektronenstrukturen fur benzenoide Kohlenwasserstoffe, Wiss. Z. Univ. Halle 31 (1982) 97–116.
[12] S. Klavžar, P. Žigert Pleteršek, Fibonacci cubes are the resonance graphs of fibonaccenes, Fibonacci Q. 43 (2005) 269–276.

[13] P.C.B. Lam, H. Zhang, A distributive lattice on the set of perfect matchings of a plane bipartite graph, Order 20 (2003) 13–29.
[14] A. Taranenko, A. Vesel, 1-Factors and characterization of reducible faces of plane elementary bipartite graphs, Discuss. Math. Graph Theory 32 (2012) 289–297.
[15] H. Zhang, Z-Transformation graphs of perfect matchings of plane bipartite graphs: a survey, MATCH Commun. Math. Comput. Chem. 56 (2006) 457–476.
[16] F. Zhang, X. Guo, R. Chen, Z-Transformation graph of perfect matching of hexagonal systems, Discrete Math. 72 (1988) 405–415.
[17] H. Zhang, P.C.B. Lam, W.C. Shiu, Resonance graphs and a binary coding for the 1-factors of benzenoid systems, SIAM J. Discrete Math. 22 (2008) 971–984.
[18] H. Zhang, D. Yang, H. Yao, Decomposition theorem on matchable distributive lattices, Discrete Appl. Math. 166 (2014) 239–248.
[19] H. Zhang, F. Zhang, The rotation graphs of perfect matchings of plane bipartite graphs, Discrete Appl. Math. 73 (1997) 5–12.
[20] H. Zhang, F. Zhang, Plane elementary bipartite graphs, Discrete Appl. Math. 105 (2000) 291–311.
[21] H. Zhang, F. Zhang, H. Yao, Z-Transformation graphs of perfect matchings of plane bipartite graphs, Discrete Math. 276 (2004) 393–404.