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On the structure of consistent cycles in cubic symmetric graphs

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Abstract

A cycle in a graph is *consistent* if the automorphism group of the graph admits a one-step rotation of this cycle. A thorough description of consistent cycles of arc-transitive subgroups in the full automorphism groups of finite cubic symmetric graphs is given.

KEYWORDS

1/2-consistent cycle, automorphism, consistent cycle, cubic symmetric graph, shunt, *s*-regular graph

MATHEMATICS SUBJECT CLASSIFICATION

05C25, 05C45

1 | INTRODUCTION

Even though somewhat overlooked, the concept of consistent cycles together with a beautiful result of Conway [4], given in Proposition 2.1, has proved to be quite useful when dealing with symmetric graphs (i.e., graphs whose automorphism groups act transitively on the set of all ordered pairs of adjacent vertices). Understandingly, cubic symmetric graphs have been the most extensively studied subclass of symmetric graphs, resulting in a long list of papers dealing with structural results of these graphs (see, e.g., [6] and the references therein). None of these papers, however, addresses the concept of consistent cycles in an explicit way. Roughly speaking a cycle in a graph is consistent if there exists an automorphism of the graph acting on it as a one-step rotation (see Section 2). A further insight into the structure of symmetric graphs follows from the generalization of the concept of consistent cycles to $1/k$-consistent cycles, introduced in [12]. The aim of this paper is to describe the structure of consistent cycles for arc-transitive subgroups of the full automorphism groups of finite cubic symmetric graphs (throughout the paper all graphs are assumed to be finite). In this respect 1/2-consistent cycles will need to be analyzed in detail, too.

The main results of this paper are given in Theorems 3.6 and 4.4 and in Section 5. Some of the more interesting corollaries, to name just two, are:

(i) A cubic symmetric graph admitting a proper arc-transitive subgroup has at least one orbit of consistent cycles of even length.
(ii) All consistent cycles are of the same length for every 1-arc-regular, 2^2-arc-regular, and 4^2-arc-regular subgroup of automorphisms as well as for every 2^1-arc-regular and 4^1-arc-regular subgroup of automorphisms in a, respectively, 3-arc-regular and 5-arc-regular cubic symmetric graph.

We would like to remark that, at least implicitly, some of these results can be deduced from the results of Djoković and Miller [5]. It is the explicit form of these results, however, that is, the central focus of this paper.

2 | PRELIMINARIES

In this section we recall the definitions and notions pertaining to consistent cycles and their generalization to $1/k$-consistent cycles. We also review the results of Tutte [13, 14] and Djoković and Miller [5] on the automorphism groups of cubic symmetric graphs.

2.1 | Consistent cycles and their generalization

For a formal definition of consistent cycles and their generalization to $(1/k)$-consistent cycles we follow [12], where the latter concept was first introduced. Let X be a graph. A *cyclet* in X is a sequence $\vec{C} = (v_0, v_1, v_2, \ldots, v_{m-1})$ of $m \geq 3$ pairwise distinct vertices of X such that v_i is adjacent to v_{i+1} for all $i \in \mathbb{Z}_m$ (addition computed modulo m). For an $s \in \mathbb{Z}_m$ the *s-shift* of $\vec{C}$ is the cyclet $\vec{C}^{+s} = (v_s, v_{s+1}, v_{s+2}, \ldots, v_{m-1}, v_0, v_1, \ldots, v_{s-1})$, while its *reverse* is $\vec{C}^{*} = (v_0, v_{m-1}, v_{m-2}, \ldots, v_1)$. Note that the cyclet and its reverse have the same root. Introducing the equivalence relation on the set of all cyclets by saying that two cyclets are equivalent if they are of the same length, say m, and one is an s-shift of the other for some $s \in \mathbb{Z}_m$, gives equivalence classes which we call *directed cycles*. A directed cycle will be represented by any one of its member cyclets. Finally, taking the transitive hull of the relation on the set of all cyclets where two cyclets are related whenever one is the reverse or the shift of the other we get an equivalence relation whose equivalence classes are called *cycles*. The cycle containing a cyclet $\vec{C}$ will be denoted by C and called the *underlying cycle* of the cyclet $\vec{C}$ (as well as the underlying cycle of the directed cycle $\vec{C}$).

Let $G \leq \mathrm{Aut}(X)$ and let $\vec{C} = (v_0, v_1, v_2, \ldots, v_{m-1})$ be a cyclet. We say that $\vec{C}$ is *G-consistent* if there is a $g \in G$ such that $v_i^g = v_{i+1}$ for all $i \in \mathbb{Z}_m$ (addition computed modulo m). In other words, $\vec{C}$ is G-consistent if some $g \in G$ maps it to its 1-shift $\vec{C}^{+1}$. In this case g is said to be a *shunt* for the cyclet $\vec{C}$ (as well as for the corresponding directed cycle $\vec{C}$ and the corresponding cycle C). More generally, for an integer $k \geq 1$ we say $\vec{C}$ is $(G, 1/k)$*-consistent* if there is a $g \in G$ such that $v_i^g = v_{i+k}$ for all $i \in \mathbb{Z}_m$, or in other words if g maps $\vec{C}$ to $\vec{C}^{+k}$. The automorphism g will be called a k*-shunt* of $\vec{C}$ (and of C) in this case (we mention that in [12] the term $1/k$-shunt was used instead of k-shunt, but we think the latter is more natural as g corresponds to a k-step rotation of the corresponding cycle). Clearly, a cyclet $\vec{C}$ is $(G, 1/k)$-consistent if and only if each of its shifts and each of their reverses is $(G, 1/k)$-consistent. This is why we can thus speak of $(G, 1/k)$-consistent (directed) cycles. Whenever the group G is clear from the context or $G = \mathrm{Aut}(X)$ we simply speak of $(1/k)$-consistent cyclets, directed cycles, and cycles (or simply consistent cyclets, directed cycles, and cycles when $k = 1$). Note that G acts in a natural way on the set of all cyclets of G. Moreover, if $h \in G$ is a k-shunt of a $(G, 1/k)$-consistent cyclet $\vec{C}$, then for each $g \in G$ the automorphism $g^{-1}hg$ is a k-shunt for the cyclet $\vec{C}^g$, showing that G also acts on the set of all $(G, 1/k)$-consistent cyclets of X. The above-mentioned result of Conway [4] can now be stated as follows.

Proposition 2.1 (Biggs [1] and Conway [4]). *Let X be a graph of valency $d \geq 2$ and assume $G \leq \mathrm{Aut}(X)$ is an arc-transitive subgroup of automorphisms. Let $\mathcal{C}$ be the set of all G-consistent directed cycles of X. Then G has exactly $d - 1$ orbits in its action on $\mathcal{C}$.*

A first written proof of Proposition 2.1 was given by Biggs in [1] (but see also [11, 12]). For cubic symmetric graphs Proposition 2.1 implies that for any arc-transitive group G there are two orbits of G-consistent directed cycles. For example, the well-known Möbius–Kantor graph GP(8, 3) is of girth 6 and has consistent cycles of length 8 and 12 (see, e.g., [9] or Section 6 of this paper). As mentioned above the generalization from consistent (directed) cycles to $(1/k)$-consistent cyclets and (directed) cycles was introduced in [12]. The following result can be deduced from [12, Theorem 4.3].

Proposition 2.2 (Miklavič et al. [12]). *Let G be an arc-transitive subgroup of automorphisms of a d-valent arc-transitive graph X, where $d \geq 2$, let $k \geq 1$ be such that $2k + 1$ does not exceed the girth of X, and let Ω be the set of all $(G, 1/k)$-consistent cyclets of X. Then G has exactly $(d - 1)^k$ orbits in its action on Ω.*

Observe that if $\vec{C}$ is a G-consistent cyclet then each of its shifts is in the G-orbit of the cyclet $\vec{C}$, and so in the case of $k = 1$ Proposition 2.2 does indeed give Proposition 2.1.

For $k > 1$ the G-orbits of cyclets and directed cycles do not necessarily "coincide". To be able to describe this phenomenon we introduce an additional notion. Let $\vec{C}$ be a $(G, 1/k)$-consistent cyclet for some positive integer k. If k_0 is the smallest positive integer such that $\vec{C}$ is $(G, 1/k_0)$-consistent then we say that $\vec{C}$ (and at the same time the corresponding directed cycle $\vec{C}$ and the cycle C) is a *proper* $(G, 1/k_0)$-consistent cyclet (or directed cycle or cycle). For instance, the proper $(G, 1/2)$-consistent directed cycles are precisely the $(G, 1/2)$-consistent directed cycles that are not G-consistent. Note that if $\vec{C}$ is $(G, 1/k)$-consistent, it is $(G, 1/k')$-consistent for each positive multiple k'

of k, and moreover, that if it is $(G, 1/k_1)$- and $(G, 1/k_2)$-consistent for some positive integers k_1 and k_2, then it is also $(G, 1/\gcd(k_1, k_2))$-consistent.

To be able to give further insight into the nature of $1/k$-consistent cyclets, directed cycles, and cycles we introduce three more concepts. Let $\vec{C}$ be a $(G, 1/k)$-consistent cyclet. If there exists a $g \in G$ such that for some nonnegative integer s the cyclet $\vec{C}$ is mapped by g to $(\vec{C}^*)^{+s}$, then the corresponding cycle C is said to be *G-reflexible*. Otherwise it is said to be *G-chiral*. Suppose C is G-reflexible and let $g \in G$ be a corresponding automorphism mapping the cyclet $\vec{C}$ to $(\vec{C}^*)^{+s}$ for some $s \geq 0$. If s or $m - s$ (where m is the length of $\vec{C}$) is even, then the restriction of the action of g to C (as a subgraph) is a reflection with respect to one of its vertices (in particular, with respect to v_{m-s_0} if $s = 2s_0$ is even, and with respect to v_{s_0} if $m - s = 2s_0$ is even). If such a g and $s \geq 0$ exist we say that the cycle C is *G-vertex-reflexible*. Note that in the case of $k = 1$ each G-reflexible cycle is also G-vertex-reflexible. Observe also that if a $(G, 1/k)$-consistent cycle is G-reflexible but is not G-vertex-reflexible then its length is necessarily even. We mention that in [12] the G-orbits of (directed) cycles which we call reflexible were called symmetric. Since we want to be able to give this finer distinction between the two different types of reflexibility of such cycles, we decided to use this different term.

2.2 | Cubic symmetric graphs

An *s-arc* in a graph Γ is a sequence $(v_0, v_1, ..., v_s)$ of $s + 1$ vertices, such that v_i is adjacent to v_{i+1} for all i with $0 \leq i < s$ and $v_{i-1} \neq v_{i+1}$ for all i with $1 \leq i < s$. A subgroup $G \leq \mathrm{Aut}(X)$ acts *s-arc-transitively* on Γ if it acts transitively on the set of all s-arcs of Γ. In this case Γ is said to be *(G, s)-arc-transitive*. If this action is regular, then Γ is *(G, s)-arc-regular*. When $G = \mathrm{Aut}(\Gamma)$ we simply speak of s-arc-transitivity and s-arc-regularity of Γ.

In [13, 14] Tutte proved that every finite cubic symmetric graph is s-arc-regular for some $s \leq 5$. A further deeper insight into the structure of cubic symmetric graphs is due to Djoković and Miller [5]. It turns out that a vertex stabilizer in an s-arc-regular subgroup of automorphisms of a cubic symmetric graph is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2$, S_4, or $S_4 \times \mathbb{Z}_2$, depending on whether s is 1, 2, 3, 4, or 5, respectively. Moreover, for $s \in \{1, 3, 5\}$ there is just one possibility for edge stabilizers, while there are two possibilities for each of $s \in \{2, 4\}$, see Table 1. In particular, for $s = 2$ the edge stabilizer is either isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, and for $s = 4$ the edge stabilizer is either isomorphic to the dihedral group D_{16} of order 16 or to the quasi-dihedral group QD_{16} of order 16 (see p. 71 of [7]).

TABLE 1 List of all possible pairs of vertex and edge stabilizers in cubic s-arc-regular graphs.

s	$\mathrm{Aut}(X)_v$	$\mathrm{Aut}(X)_e$
1	$\mathbb{Z}_3$	$\mathbb{Z}_2$
2	S_3	$\mathbb{Z}_2^2$ or $\mathbb{Z}_4$
3	$S_3 \times \mathbb{Z}_2$	D_8
4	S_4	D_{16} or QD_{16}
5	$S_4 \times \mathbb{Z}_2$	$(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$

The results of [5, 8] have enabled Conder and Lorimer [2] to give presentations of the following seven infinite groups, such that each arc-transitive subgroup of automorphisms of a finite cubic symmetric graph is a quotient group of one of these seven groups by a normal torsion-free subgroup:

$$G_1 = \langle h, a | h^3 = a^2 = 1 \rangle, \tag{1}$$

$$G_2^1 = \langle h, a, p | h^3 = a^2 = p^2 = 1, apa = p, php = h^{-1} \rangle, \tag{2}$$

$$G_2^2 = \langle h, a, p | h^3 = p^2 = 1, a^2 = p, php = h^{-1} \rangle, \tag{3}$$

$$G_3 = \langle h, a, p, q | h^3 = a^2 = p^2 = q^2 = 1, apa = q, qp = pq, ph = hp, qhq = h^{-1} \rangle, \tag{4}$$

$$\begin{aligned} G_4^1 = \langle h, a, p, q, r | & h^3 = a^2 = p^2 = q^2 = r^2 = 1, apa = p, aqa = r, h^{-1}ph = q, \\ & h^{-1}qh = pq, rhr = h^{-1}, pq = qp, pr = rp, rq = pqr \rangle, \end{aligned} \tag{5}$$

$$\begin{aligned} G_4^2 = \langle h, a, p, q, r | & h^3 = p^2 = q^2 = r^2 = 1, a^2 = p, a^{-1}qa = r, h^{-1}ph = q, \\ & h^{-1}qh = pq, rhr = h^{-1}, pq = qp, pr = rp, rq = pqr \rangle, \end{aligned} \tag{6}$$

$$\begin{aligned} G_5 = \langle h, a, p, q, r, s | & h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1, apa = q, ara = s, \\ & h^{-1}ph = p, h^{-1}qh = r, h^{-1}rh = pqr, shs = h^{-1}, pq = qp, pr = rp, ps = sp, \\ & qr = rq, qs = sq, sr = pqrs \rangle. \end{aligned} \tag{7}$$

In particular, an s-arc-regular subgroup of automorphisms of a finite cubic symmetric graph X is a quotient group of a group G_s if $s \in \{1, 3, 5\}$, and of a group G_s^i, $i \in \{1, 2\}$, if $s \in \{2, 4\}$. Moreover, as was shown in [2], the graph X can be reconstructed as a double-coset graph as follows. Let G be the corresponding quotient group (in which we denote the elements corresponding to the generators $h, a, \ldots$ of G_s or G_s^i by the same symbols) and let H be the subgroup generated by all the generators except a (so, e.g., $H = \langle h, p \rangle$ in the case that G is a quotient of G_2^1 or G_2^2). The vertex set of the graph is then the coset space $\{Hg : g \in G\}$ of all right cosets of H in G and two cosets Hx and Hy are adjacent whenever $xy^{-1} \in HaH$. In other words, a coset Hx is adjacent to the distinct cosets Hax, $Hahx$, and Hah^2x (since G is a quotient with respect to a torsion-free normal subgroup $aha^{-1} \notin H$). In this way, G has a natural faithful action on the constructed graph by right multiplication.

Agreement 2.3. *We make an agreement that throughout this paper whenever we speak of a finite cubic symmetric graph X and a corresponding arc-transitive subgroup G of* $\mathrm{Aut}(X)$ *we may think of G being a quotient group of one of the groups G_s, $s \in \{1, 3, 5\}$, or G_s^i, $s \in \{2, 4\}$, $i \in \{1, 2\}$, and we let H be the subgroup of G generated by the images of the generators of G_s or G_s^i, other than a (we use the same symbols for the generators of G_s or G_s^i and G). Finally, we may think of X being the double-coset graph as described in the previous paragraph, and so G has a natural faithful action on X via multiplication on the right.*

In [3] a complete characterization of admissible types of cubic symmetric graphs according to the structure of arc-transitive subgroups of the automorphism group was obtained. For example, a cubic symmetric graph X is said to be of type $\{1, 2^1, 2^2, 3\}$ if its automorphism group

TABLE 2 All possible types of cubic symmetric graphs.

s	Type	Bipartite?	s	Type	Bipartite?	s	Type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

is 3-arc-regular but admits a 2-arc-regular subgroup, which is a quotient of the group G_2^1, a 2-arc-regular subgroup, which is a quotient of the group G_2^2, and also a 1-arc-regular subgroup. All possible types are summarized in Table 2 (for details see [3]).

3 | THE CONSISTENT CYCLES

Let X be a cubic symmetric graph and G an s-arc-regular subgroup of $\mathrm{Aut}(X)$. Our aim in this section is to determine in what way s relates to whether the G-consistent cycles of X are G-chiral or G-reflexible. We also provide shunts giving rise to the representatives of the G-orbits of G-consistent cycles. We point out that in fact the information on when the G-consistent cycles are G-chiral and when G-reflexible can be extracted from the results of [5]. Nevertheless, for self-completeness and since we are also interested in the corresponding shunts we provide a proof of these facts. Before doing this we record a straightforward but very useful observation that follows from the fact that the group G is s-arc-regular for some $s \geq 1$, implying that no nontrivial element of G can fix a cycle pointwise (but see also [10] where the structure of subgraphs induced on fixed points of automorphisms of cubic symmetric graphs was studied in great detail).

Lemma 3.1. *Let G be an arc-transitive subgroup of $\mathrm{Aut}(X)$ for a cubic symmetric graph X and let $\vec{C}$ be a $(G, 1/k)$-consistent cyclet of X for some $k \geq 1$. Then $\vec{C}$ has a unique k-shunt in G, and thus also in any subgroup K with $G \leq K \leq \mathrm{Aut}(X)$.*

Recall our Agreement 2.3 and consider the action of $ha \in G$ on X. It maps the vertex H to its neighbor Ha and maps Ha to $Haha$. Since $aha^{-1} \notin H$ and $a^2 \in H$, we also have that $aha \notin H$, which thus implies that ha is a shunt of a G-consistent cyclet containing the vertex H. Since $(ha)^m$, where m is the length of this cyclet, clearly fixes each of its vertices, the above comment shows that ha is of order m. We thus have the following result.

Proposition 3.2. *Let X be a cubic symmetric graph and let $G \leq \mathrm{Aut}(X)$ be arc-transitive on X. Then with the notation from Agreement 2.3 $g = ha \in G$ is a shunt of a G-consistent cyclet of X containing the vertex H. Consequently, this cyclet is of the form $\vec{C} = (H, Ha, Haha, \ldots)$ and its length coincides with the order of g.*

We call the G-consistent cyclet $\vec{C}$ from Proposition 3.2 (as well as the corresponding G-consistent directed cycle and the underlying cycle C) the *canonical* G-consistent cyclet (or directed cycle or cycle) of X.

The following straightforward but very useful fact is well known (see, e.g., [12]). By Proposition 2.1, X has two G-orbits of G-consistent directed cycles. If any of the corresponding cycles is G-chiral, then the reverse of any G-consistent cyclet must be in the other G-orbit of G-consistent cyclets, and so in this case both G-orbits of G-consistent directed cycles give rise to a single G-orbit of G-chiral consistent cycles. The only other possibility is thus that the two G-orbits of G-consistent directed cycles both give rise to G-vertex-reflexible cycles, in which case the G-orbits of directed G-consistent cycles "coincide" with those of G-consistent cycles.

We make another useful observation (see also [12]). Let $\vec{C} = (v_0, v_1, \ldots, v_{m-1})$ be a G-consistent cyclet of X (recall that G is assumed to be s-arc-regular, and so $m > s$) and let g be the shunt of $\vec{C}$. If $s = 1$ then let $k \in G$ be the unique element fixing v_0 and mapping v_{m-1} to v_1 and note that then k maps v_1 to the unique neighbor of v_0, different from v_1 and v_{m-1}. If however $s > 1$, let $k \in G$ be the unique nontrivial element fixing each of v_i, $0 \le i < s$, which thus moves v_s. Then kg maps each of v_i, $0 \le i < s$, to v_{i+1} but maps v_s to the unique neighbor of v_s, different from v_{s-1} and v_{s+1}, say u_{s+1}. Therefore, kg is the shunt of the G-consistent cyclet starting with $(v_0, v_1, \ldots, v_s, u_{s+1})$, which cannot be in the G-orbit of $\vec{C}$ (as G is s-arc-regular), and is thus a representative of the other G-orbit of G-consistent cyclets. For future reference we record these two observations in the following proposition.

Proposition 3.3. *Let X be a cubic symmetric graph and let $G \le \mathrm{Aut}(X)$ be s-arc-regular for some s with $1 \le s \le 5$. Let $\vec{C} = (v_0, v_1, \ldots, v_{m-1})$ be a G-consistent cyclet of X and let $g \in G$ be its shunt. In the case that $s = 1$ let $k \in G$ be the unique element fixing v_0 and mapping v_{m-1} to v_1, while if $s > 1$ let $k \in G$ be the unique nontrivial element fixing the first s vertices of $\vec{C}$. Then kg is the shunt of the G-consistent cyclet $\vec{C}'$ of X which shares the first $s + 1$ vertices with $\vec{C}$ but is not in the same G-orbit as $\vec{C}$. Moreover, either all G-consistent cycles are G-chiral in which case there is a single G-orbit of G-consistent cycles, or all G-consistent cycles are G-vertex-reflexible and the underlying cycles of $\vec{C}$ and $\vec{C}'$ are representatives of the two G-orbits of G-consistent cycles.*

We thus only need to determine for which groups G_s and G_s^i the canonical G-consistent cycle is G-chiral and then for the ones for which it is G-reflexible determine the corresponding element $k \in G$ to obtain the shunt of a representative from the second G-orbit of G-consistent cycles. The following result will be of use (in Section 5 we study the situations where we have more than one arc-transitive subgroup of the automorphism group, which is why we state this result in a more general form).

Proposition 3.4. *Let X be a cubic symmetric graph and let $G \le \mathrm{Aut}(X)$ be arc-transitive on X. Let $\vec{C}$ be a G-consistent directed cycle of X and let m be its length. Then for any subgroup $K \le \mathrm{Aut}(X)$ with $G \le K$ the stabilizer of C in K is isomorphic to the cyclic group of order m or the dihedral group of order $2m$, depending on whether C is K-chiral or K-reflexible, respectively. Consequently, for any such K the cycle C is K-reflexible if and only if for some, and thus each, vertex of C (and for some, and thus each, edge of C) there is an* *involution in K reflecting C with respect to this vertex (this edge). Finally, letting $g \in G$ be the shunt of $\vec{C}$ the cycle C is K-reflexible if and only if for some (and thus each) vertex of C there exists an involution $k \in K$ fixing this vertex and such that $kgk = g^{-1}$, which occurs if and only if for some (and thus each) edge of C there exists an involution $k' \in K$ reflecting this edge such that $k'gk' = g^{-1}$.*

Proof. Let $K \leq \mathrm{Aut}(X)$ with $G \leq K$. Since $\vec{C}$ is G-consistent, G (and thus also K) contains a cyclic subgroup of order m fixing C (as a subgraph). Since the automorphism group of a cycle of length m is the dihedral group of order $2m$, the remark preceding Lemma 3.1 implies that the stabilizer of C in K is either the full dihedral group of order $2m$ or its index 2 cyclic subgroup. That C is K-reflexible if and only if the former holds is now clear. This also proves the claim regarding the existence of certain involutions in K reflecting C.

For the last part of the proposition, let $\vec{C} = (v_0, v_1, ..., v_{m-1})$ and let $g \in G$ be the shunt of $\vec{C}$. Suppose first that C is K-reflexible. The first part of this proposition then implies that for each i there exists an involution $k \in K$ reflecting C with respect to v_i. But then $(kg)^2$ fixes C pointwise, implying that $kgk = g^{-1}$. Conversely, suppose $kgk = g^{-1}$ for some involution $k \in K$ fixing v_i. A straightforward calculation then shows that $v_j^k = v_{2i-j}$ for all $j \in \mathbb{Z}_m$, and so k reflects C (with respect to v_i), showing that C is K-reflexible. Taking $k' = kg$ we get the part concerning the reflection with respect to an edge. □

We now show that the types $\{1\}, \{2^2\}$, and $\{4^2\}$ give rise to chiral consistent cycles.

Proposition 3.5. *Let X be a cubic symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of automorphisms of X. If G is of type $\{1\}, \{2^2\}$, or $\{4^2\}$, then the G-consistent cycles of X are all G-chiral. Consequently, the canonical cycle is a representative of the unique G-orbit of G-consistent cycles of X.*

Proof. Suppose to the contrary that the underlying cycle C of the canonical cyclet $\vec{C}$ is G-reflexible. By Proposition 3.4 we then must have that for each vertex and each edge of C there exist involutions $k, k' \in G$ such that k reflects C with respect to this vertex and k' reflects C with respect to this edge.

If G is of type $\{1\}$, then no nontrivial element of G can fix a vertex and one of its neighbors, contradicting the existence of the above k. Similarly, if G is of type $\{2^2\}$, then (3) implies that the stabilizer of the edge $\{H, Ha\}$ in G is $\langle a \rangle \cong \mathbb{Z}_4$, and so the unique involution in this stabilizer fixes the end-vertices of the edge, contradicting the existence of the above k'.

We are left with the possibility that G is of type $\{4^2\}$. By (6) we have that $H = \langle h, p, q, r \rangle$ and that the stabilizer of the edge $\{H, Ha\}$ is $\langle a, p, q, r \rangle = \langle a, q \rangle$ with each of p, q, r fixing H. The above-mentioned k' thus must belong to the coset $a\langle p, q, r \rangle$. However, by (6) four of the elements of this coset (namely, $a, ap, aqr, apqr$) are of order 4, while the remaining four are of order 8, showing that k' does not exist. □

To complete the analysis of consistent cycles for different types of groups we now prove that the remaining four types give rise to reflexible consistent cycles and in each case provide a shunt giving rise to the second G-orbit of G-consistent cycles by exhibiting an appropriate k from Proposition 3.3.

Theorem 3.6. *Let X be a cubic symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of automorphisms of X. Then the G-consistent cycles of X are G-chiral if and only if G is of one of the types $\{1\}$, $\{2^2\}$, and $\{4^2\}$. In this case the canonical cycle is a representative of the unique G-orbit of G-consistent cycles of X. In the case that G is of one of the remaining four types, with the notation from Agreement 2.3, a shunt g' of a representative through the vertex H of the G-orbit of G-consistent cycles not containing the canonical cycle is of the following form:*

- If G is of type $\{2^1\}$ then we can take $g' = pha$.
- If G is of type $\{3\}$ then we can take $g' = qha$.
- If G is of type $\{4^1\}$ then we can take $g' = rha$.
- If G is of type $\{5\}$ then we can take $g' = sha$.

Proof. By Proposition 3.5 it suffices to assume that G is of one of the types $\{2^1\}$, $\{3\}$, $\{4^1\}$, and $\{5\}$, show that the G-consistent cycles are G-reflexible, and verify that the stated g' are appropriate. For the first part Proposition 3.4 implies that it suffices to exhibit an involution $k' \in G$ which interchanges the vertices H and Ha and conjugates the shunt $g = ha$ of the canonical G-consistent cycle to its inverse. We claim that for types $\{2^1\}$, $\{3\}$, $\{4^1\}$, and $\{5\}$, respectively, we can take k' to be pa, qpa, qra, and rsa, respectively. Observe first that (2), (4), (5), and (7) imply that in the respective groups we have $pa = ap$, $qpa = apq$, $qra = arq$, and $rsa = asr$, respectively. This clearly shows that in each of the four cases the chosen k' is an involution and that $Hk' = Ha$ and $Hak' = H$. We thus only need to verify that $k'gk' = g^{-1}$. The above observation regarding k' makes this easy:

$$\begin{aligned}
&\text{type } \{2^1\}: k'gk' = (pa)ha(pa) = a(php) = ah^{-1} = g^{-1},\\
&\text{type } \{3\}: k'gk' = (qpa)ha(qpa) = ap(qh)(pq) = a(ph^{-1})qqp = ah^{-1}pp = ah^{-1} = g^{-1},\\
&\text{type } \{4^1\}: k'gk' = (qra)ha(qra) = ar(qh)rq = a(rh)(pqr)q = ah^{-1}rrqq = ah^{-1} = g^{-1},\\
&\text{type } \{5\}: k'gk' = (rsa)ha(rsa) = as(rh)sr = a(sh)(pqrs)r = ah^{-1}ssrr = ah^{-1} = g^{-1}.
\end{aligned}$$

For the second part Proposition 3.3 implies that it suffices to verify that p, q, r, and s, respectively, satisfy the requirements for k from that proposition when G is of type $\{2^1\}$, $\{3\}$, $\{4^1\}$, and $\{5\}$, respectively. This can easily be verified using (2), (4), (5), and (7). For instance, for type $\{3\}$ we have that $Hq = H, Haq = Hpa = Ha$, and

$$Hah(aq) = Ha(hp)a = H(ap)ha = Hqaha = Haha.$$

Similarly, for type $\{5\}$ we have that $Hs = H, Has = Hra = Ha, Hah(as) = Ha(hr)a = H(aq)ha = Hpaha = Haha$,

$$Hah(ahas) = Ha(hp)aha = H(ap)haha = Hqahaha = Hahaha,$$

and as $(ap)qr = q(aq)r = qp(ar) = qpsa$ also

$$\begin{aligned} Hah(ahahas) &= Ha(hq)ahaha = H(apqr)hahaha = Hqpsahahaha \\ &= Hahahaha. \end{aligned}$$

We leave the verification for types $\{2^1\}$ and $\{4^1\}$ to the reader. □

4 | THE 1/2-CONSISTENT CYCLES

Let X be a cubic symmetric graph and G an s-arc-regular subgroup of $\mathrm{Aut}(X)$, where $1 \leq s \leq 5$. In this section we analyze the proper $(G, 1/2)$-consistent cycles of X. To this end we assume in addition that X is of girth at least 5 (note that this only excludes three graphs—the complete graph K_4, the complete bipartite graph $K_{3,3}$ and the cube graph Q_3), so that Proposition 2.2 implies that X has four G-orbits of $(G, 1/2)$-consistent cyclets. Of course, two of these are the two G-orbits of G-consistent cyclets, while the remaining two contain proper $(G, 1/2)$-consistent cyclets.

Suppose $\vec{C}$ is a proper $(G, 1/2)$-consistent cyclet. Then its shift $\vec{C}^{+1}$ cannot be in the same G-orbit (otherwise $\vec{C}$ would be G-consistent), and so $\vec{C}^{+1}$ must be in the second G-orbit of proper $(G, 1/2)$-consistent cyclets. Thus, X has just one G-orbit of proper $(G, 1/2)$-consistent directed cycles which are necessarily of even length. The cyclets $\vec{C}^{*}$ and $(\vec{C}^{+1})^{*}$ are of course also proper $(G, 1/2)$-consistent cyclets and are in different G-orbits. We thus have two essentially different possibilities. The first is that the G-orbit of $\vec{C}$ coincides with that of $\vec{C}^{*}$ (and thus the G-orbit of $\vec{C}^{+1}$ coincides with that of $(\vec{C}^{+1})^{*}$), in which case the cycle C is of course G-vertex-reflexible. The other possibility is that the G-orbit of $\vec{C}$ coincides with that of $(\vec{C}^{+1})^{*}$ (and the G-orbit of $\vec{C}^{+1}$ coincides with that of $\vec{C}^{*}$), in which case the cycle C is G-reflexible, but is not G-vertex-reflexible. Observe that in each of these possibilities the restriction G_C^C of the action of the stabilizer G_C of C (as a subgraph) to C is the dihedral group of order m, where m is the length of C. However, in the first possibility this action is transitive on the set of edges of C (and not on its vertex set), while in the second possibility the action is transitive on the vertex set of C (but not on the set of its edges). We summarize all of these remarks in the following proposition.

Proposition 4.1. *Let X be a cubic symmetric graph of girth at least 5 and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of automorphisms of G. Then X has two G-orbits of proper $(G, 1/2)$-consistent cyclets and the union of these two G-orbits corresponds to the unique G-orbit of proper $(G, 1/2)$-consistent directed cycles, which are thus all of the same even length, say m. Moreover, the underlying cycles of all proper $(G, 1/2)$-consistent directed cycles are G-reflexible and letting C be any such cycle the restriction G_C^C is the dihedral group of order m. Furthermore, either all proper $(G, 1/2)$-consistent cycles are G-vertex-reflexible,* *in which case G_C^C acts regularly on the set of the edges of C but has two orbits on the set of its vertices, or no proper $(G, 1/2)$-consistent cycle is G-vertex-reflexible, in which case G_C^C acts regularly on the set of the vertices of C but has two orbits on the set of its edges.*

The next lemma gives an easy way of obtaining representatives of proper $(G, 1/2)$-consistent cycles and is the key ingredient in the proof of Proposition 4.3.

Lemma 4.2. *Let X be a cubic symmetric graph of girth at least 5 and let $G \leq \mathrm{Aut}(X)$ be s-arc-regular for some s with $1 \leq s \leq 5$. Let $A_s = (v_0, v_1, ..., v_s)$ be an s-arc of X and let $g_1, g_2 \in G$ be the shunts of the two G-consistent cyclets, say $\overrightarrow{C}_1$ and $\overrightarrow{C}_2$, respectively, starting with A_s. Then there are precisely two proper $(G, 1/2)$-consistent cyclets starting with A_s, they are in different G-orbits and their 2-shunts are g_1g_2 and g_2g_1. Moreover, through every path of length $s + 1$ there is precisely one proper $(G, 1/2)$-consistent cycle.*

Proof. Since G is s-arc-regular there is precisely on cyclet $\overrightarrow{C}_i$, $1 \leq i \leq 4$, starting with A_s, from each of the four G-orbits of $(G, 1/2)$-consistent cyclets. Moreover, if we let $\overrightarrow{C}_1 = (v_0, v_1, ..., v_{-1})$ be one of the G-consistent ones and $\overrightarrow{C}_2$ be the other G-consistent one, it is clear that $\overrightarrow{C}_2$ is of the form $\overrightarrow{C}_2 = (v_0, v_1, ..., v_s, u_{s+1}, u_{s+2}, ..., u_{-1})$, where $u_{s+1} \neq v_{s+1}$ and $u_{-1} \neq v_{-1}$ (otherwise s-arc-regularity would imply that they have the same shunt and are thus equal—see also Figure 1). Let $g_1, g_2 \in G$ be their respective shunts. It is easy to see that then g_1g_2 is the 2-shunt of a $(G, 1/2)$-consistent cyclet starting with $(v_0, v_1, ..., v_s, u_{s+1}, x_{s+2})$, where $x_{s+2} \neq u_{s+2}$, which thus implies that this cyclet, denote it by $\overrightarrow{C}_3$, is not in the G-orbit of $\overrightarrow{C}_1$ or $\overrightarrow{C}_2$, and thus must be a proper $(G, 1/2)$-consistent cyclet. Similarly, g_2g_1 is the 2-shunt of a proper $(G, 1/2)$-consistent cyclet starting with $(v_0, v_1, ..., v_s, v_{s+1}, y_{s+2})$, where $y_{s+2} \neq v_{s+2}$, and so this cyclet, say $\overrightarrow{C}_4$ is in the fourth G-orbit of $(G, 1/2)$-consistent cyclets. The last part now follows from the fact that there is a unique proper $(G, 1/2)$-consistent cyclet through each of $(v_0, v_1, ..., v_s, v_{s+1})$ and $(v_0, v_1, ..., v_s, u_{s+1})$, that $\overrightarrow{C}_3$ and $\overrightarrow{C}_3^{+1}$ are representatives of the two G-orbits of proper $(G, 1/2)$-consistent cyclets but have the same underlying cycle, and that $\overrightarrow{C}_3^{*}$ is in the same G-orbit as one of $\overrightarrow{C}_3$ and $\overrightarrow{C}_3^{+1}$. □

Proposition 4.3. *Let X be a cubic arc-transitive graph of girth at least 5 and let $G \leq \mathrm{Aut}(X)$ be s-arc-regular for some s with $1 \leq s \leq 5$. Then the proper $(G, 1/2)$-consistent cycles are G-reflexible. Moreover, the following holds:*

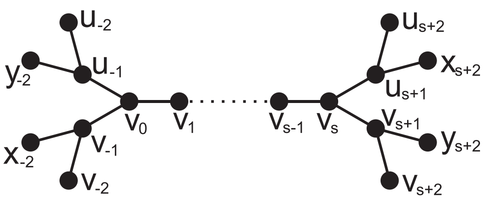


FIGURE 1 The $(G, 1/2)$-consistent cyclets starting with $(v_0, v_1, ..., v_s)$.

- *If the G-consistent cycles are G-chiral, then the proper $(G, 1/2)$-consistent cycles are G-vertex-reflexible if and only if s is even.*
- *If the G-consistent cycles are G-reflexible, then the proper $(G, 1/2)$-consistent cycles are G-vertex-reflexible if and only if s is odd.*

Proof. Fix an s-arc $A_s = (v_0, v_1, \ldots, v_s)$ and let $\vec{C_i}$, $1 \le i \le 4$, and $g_1, g_2 \in G$ be as in the proof of Lemma 4.2. Then $v_1^{(g_1g_2)^{-1}} = v_0^{g_1^{-1}} = v_{-1}$ but $v_0^{(g_1g_2)^{-1}} = u_{-1}^{g_1^{-1}} \neq v_{-2}$, and so $\vec{C_3}$ is of the form

$$\vec{C_3} = (v_0, v_1, \ldots, v_s, u_{s+1}, x_{s+2}, \ldots, x_{-2}, v_{-1}),$$

where $x_{-2} \neq v_{-2}$. Similarly,

$$\vec{C_4} = (v_0, v_1, \ldots, v_s, v_{s+1}, y_{s+2}, \ldots, y_{-2}, u_{-1}),$$

where $y_{-2} \neq u_{-2}$ (see Figure 1).

Suppose now that the underlying cycle C_1 is G-chiral and let $g \in G$ be the unique element mapping A_s to $(v_s, v_{s-1}, \ldots, v_0)$. Since the directed cycle $\vec{C_1}$ is not in the same G-orbit as the directed cycle corresponding to $\vec{C}^*_1$, the directed cycle $\vec{C_1}$ is mapped to the one corresponding to $\vec{C}^*_2$. Consequently, the $(s+1)$-arc $(v_0, v_1, \ldots, v_{s+1})$ is mapped by g to $(v_s, v_{s-1}, \ldots, v_0, u_{-1})$. Then Lemma 4.2 implies that g reflects the underlying cycle C_4 of $\vec{C_4}$. If $s = 2s_0$ is even, then g fixes v_{s_0}, and so C_4 is G-vertex-reflexible. If however $s = 2s_0 + 1$ is odd, then g interchanges v_{s_0} with v_{s_0+1}, and so Proposition 4.1 implies that C_4 is not G-vertex-reflexible.

Suppose finally that C_1 is G-reflexible and let this time $g' \in G$ be the unique element mapping A_s to $(v_{s-1}, v_{s-2}, \ldots, v_0, v_{-1})$. Since $\vec{C_1}$ is the unique G-consistent directed cycle through $(v_{-1}, v_0, v_1, \ldots, v_{s-1})$ and C_1 is G-reflexible, we must have that $v_{-1}^{g'} = v_s$. But then the $(s+1)$-arc $(v_{-1}, v_0, v_1, \ldots, v_s)$ is reflected by g', and so Lemma 4.2 implies that g' reflects the underlying cycle C_3 of $\vec{C_3}$. We now proceed as in the previous paragraph. □

Combining the results of this and the previous section we obtain the following theorem.

Theorem 4.4. *Let X be a cubic symmetric graph of girth at least 5 and let $G \le \mathrm{Aut}(X)$ be an arc-transitive subgroup of automorphisms of X. Then there exists a single G-orbit of proper $(G, 1/2)$-consistent cycles which are all G-reflexible. Moreover, they are G-vertex-reflexible if and only if G is of one of the types $\{2^2\}$, $\{3\}$, $\{4^2\}$, and $\{5\}$. Furthermore, with the notation from Agreement 2.3, for each of the seven possible types of G a 2-shunt of a proper $(G, 1/2)$-consistent cycle containing the edge $\{H, Ha\}$ can be obtained taking the element given in the column labeled "2-shunt" from Table 3.*

Proof. The first part of the theorem follows from Theorem 3.6, Proposition 4.1, and Proposition 4.3. It thus remains to verify that the stated elements of G in Table 3 are indeed 2-shunts of proper $(G, 1/2)$-consistent cycles. In view of Proposition 3.2, Proposition 3.3, and Lemma 4.2 we only need to see that the part of the stated

TABLE 3 G-consistent and proper $(G, 1/2)$-consistent cycles in cubic symmetric graphs.

Type	Shunt(s)	ref/chi	2-Shunt	v-ref/ref
$\{1\}$	ha	Chiral	h^2aha	Reflexible
$\{2^1\}$	ha, pha	Reflexible	$phaha$	Reflexible
$\{2^2\}$	ha	Chiral	$phaha$	Vertex-reflexible
$\{3\}$	ha, qha	Reflexible	$qhaha$	Vertex-reflexible
$\{4^1\}$	ha, rha	Reflexible	$rhaha$	Reflexible
$\{4^2\}$	ha	Chiral	$rhaha$	Vertex-reflexible
$\{5\}$	ha, sha	Reflexible	$shaha$	Vertex-reflexible

elements preceding the $haha$ at the end is the unique k from Proposition 3.3 for the canonical cyclet. That this holds for type $\{1\}$ is clear, while the fact that this holds for types $\{2^1\}$, $\{3\}$, $\{4^1\}$, and $\{5\}$ was established in the proof of Theorem 3.6. For type $\{2^2\}$ note that (3) implies that $Hp = H$ and $Hap = Hpa = Ha$. Finally, for type $\{4^2\}$ (6) implies that $Hr = H, Har = Hqa = Ha$,

$$Hah(ar) = Ha(hq)a = H(ap)ha = Hpaha = Haha,$$

and

$$\begin{aligned} Hah(ahar) &= Ha(hp)aha = H(ap)qhaha = H(a^3q)haha = Hrpahaha \\ &= Hahaha. \end{aligned}$$

□

We summarize the results of Theorems 3.6 and 4.4 in Table 3. For each possible type of an arc-transitive subgroup $G \leq \mathrm{Aut}(X)$ of a cubic symmetric graph of girth at least 5 we give the information on whether the G-consistent cycles are G-reflexible or G-chiral and whether the proper $(G, 1/2)$-consistent cycles are G-vertex-reflexible or not. Moreover, using the notation from Agreement 2.3 we give shunts of representatives through the vertex H of the two G-orbits of G-consistent cycles in the case that they are G-reflexible (and give a shunt of a representative if they are G-chiral), and give a 2-shunt of a representative through the edge $\{H, Ha\}$ of the unique G-orbit of proper $(G, 1/2)$-consistent cycles.

5 | GRAPHS WITH MORE THAN ONE ARC-TRANSITIVE GROUP

In this section we study consistent cycles in cubic symmetric graphs whose automorphism group is s-arc-regular for some s with $2 \leq s \leq 5$, but has at least one proper arc-transitive subgroup. In view of the comment preceding Proposition 2.1 the following observation is clear.

Lemma 5.1. *Let X be a graph and let $H, G \leq \mathrm{Aut}(X)$ be arc-transitive subgroups such that H is normal in G. Then for any $k \geq 1$ the natural action of G on the set of all cyclets of X preserves the set of all $(H, 1/k)$-consistent cyclets.*

Proposition 5.2. *Let X be a cubic symmetric graph and let $H, G \leq \mathrm{Aut}(X)$ be arc-transitive subgroups such that H is an index 2 subgroup of G. Then one G-orbit of G-consistent cycles consists of all H-consistent cycles, which are thus all of the same length, while the other G-orbit consists of all proper $(H, 1/2)$-consistent cycles. As a consequence, the G-consistent cycles are G-reflexible.*

Proof. Let s, where $1 \leq s \leq 4$, be such that H is s-arc-regular (and so G is $(s+1)$-arc-regular). Just like in the proofs of Lemma 4.2 and Proposition 4.3 let $A_s = (v_0, v_1, \ldots, v_s)$ be a chosen s-arc of X and let $\vec{C_i}$, $1 \leq i \leq 4$, be the four $(H, 1/2)$-consistent cyclets starting with A_s, one from each H-orbit, where $\vec{C_1}$ and $\vec{C_2}$ are H-consistent and the successors of v_s on $\vec{C_1}$ and $\vec{C_2}$ are v_{s+1} and u_{s+1}, respectively (recall that $v_{s+1} \neq u_{s+1}$). Let $g \in G$ be the unique element fixing each vertex of A_s but mapping v_{s+1} to u_{s+1}. Since $\vec{C_2}$ is the unique H-consistent cyclet starting with $(v_0, v_1, \ldots, v_s, u_{s+1})$, Lemma 5.1 implies that g maps $\vec{C_1}$ to $\vec{C_2}$, showing that one G-orbit of G-consistent cyclets coincides with the union of the two H-orbits of H-consistent cyclets. Similarly, the proper $(H, 1/2)$-consistent cyclets $\vec{C_3}$ and $\vec{C_4}$ are interchanged by g, and so they are in the same G-orbit. But since the elements of the H-orbit of $\vec{C_4}$ are precisely the 1-shifts of the elements of the H-orbit of $\vec{C_3}$, this shows that $\vec{C_3}$ (and $\vec{C_4}$) must be G-consistent. Since they are H-reflexible, they are of course also G-reflexible. Since H is normal in G, the H-orbits of cyclets are blocks of imprimitivity for the action of G on the set of all cyclets, and so the G-orbit of $\vec{C_3}$ is disjoint from that of $\vec{C_1}$, thus proving that the second G-orbit of G-consistent cyclets coincides with the union of the two H-orbits of proper $(H, 1/2)$-consistent cyclets. □

We remark that, in view of Theorem 3.6, the above proposition gives an alternative proof of the well-known fact that for a cubic symmetric graph a subgroup of type $\{2^2\}$ (respectively, $\{4^2\}$) contains no arc-regular subgroups (respectively, subgroups of type $\{3\}$).

Before stating and proving the next lemma we point out that the consequent Corollaries 5.4 and 5.6 can be extracted from [5], where the interplay of possible arc-transitive subgroups of automorphisms of cubic symmetric graphs was studied using the particular structures of these groups, but here we provide a rather elementary proof of these facts.

Lemma 5.3. *Let X be a cubic symmetric graph and suppose that for some s, $1 \leq s \leq 4$, there are subgroups $H_1, H_2, G \leq \mathrm{Aut}\ (X)$ such that H_1 and H_2 are two different s-arc-regular index 2 subgroups of G. Then no cyclet of X is at the same time H_1-consistent and H_2-consistent.*

Proof. Note first that since H_1 and H_2 are both of index 2 in G, the intersection $K = H_1 \cap H_2$ is of index 2 in each of H_1, H_2 and is normal in G. By way of contradiction suppose there is a cyclet $\vec{C} = (v_0, v_1, \ldots, v_s, v_{s+1}, \ldots, v_{-1})$ which is both H_1- and H_2-consistent.

Proposition 5.2 then implies that all H_1-consistent cyclets are H_2-consistent and vice versa. Let u_{s+1} be the unique neighbor of v_s, different from v_{s-1} and v_{s+1}. Since H_1 is s-arc-regular there is an H_1-consistent cyclet starting with $(v_0, v_1, ..., v_s, u_{s+1})$, say $\overrightarrow{C}'$. Let $g' \in H_1$ be its shunt and let $g \in H_1$ be the shunt of $\overrightarrow{C}$. Then Lemma 3.1 implies that g and g' are the unique shunts of $\overrightarrow{C}$ and $\overrightarrow{C}'$, respectively, in the group G. But since all H_1-consistent cyclets are also H_2-consistent, this in fact implies that $g, g' \in K$, and so $k = g'g^{-1} \in K$. Since k fixes each of v_i, $0 \leq i < s$ and $k \neq 1$, the group K cannot be $(s-1)$-arc-regular. But since any element of order 3 of H_1 is contained in K (as K is of index 2 in H_1), this in fact shows that K is not vertex-transitive. It thus has precisely two orbits and since H_1 acts arc-transitively and K is normal in H_1, the graph X is bipartite with the orbits of K being the sets of bipartition. But since $g \in K$ maps the vertex v_0 to its neighbor v_1, this contradicts the fact that K is not vertex-transitive. □

Corollary 5.4. *Let X be a cubic symmetric graph and suppose $G \leq \mathrm{Aut}(X)$ is s-arc-regular for some s with $2 \leq s \leq 5$. Then G can have at most two $(s-1)$-arc-regular subgroups. Moreover, if G has an $(s-1)$-arc-regular subgroup H then G has precisely two $(s-1)$-arc-regular subgroups if and only if X is bipartite, while H is the unique $(s-1)$-arc-regular subgroup of G otherwise.*

Proof. Suppose H_1 and H_2 are two different $(s-1)$-arc-regular subgroups of G and let $A_s = (v_0, v_1, ..., v_s)$ be an s-arc of X. Since H_1 and H_2 are $(s-1)$-arc-regular, there is a unique H_1-consistent cyclet, say $\overrightarrow{C_1}$, starting with A_s and a unique H_2-consistent cyclet, say $\overrightarrow{C_2}$, starting with A_s. By Lemma 5.3 these two cyclets are different, and so Lemma 3.1 implies (recall that G is s-arc-regular) that the successor of v_s on $\overrightarrow{C_1}$ is different than on $\overrightarrow{C_2}$. Since X is cubic it is now clear that there can be no other $(s-1)$-arc-regular subgroups of G.

To prove the second part of the corollary suppose $H \leq G$ is $(s-1)$-arc-regular. Before continuing note first that for any group G having two index 2 subgroups H_1 and H_2 there is always a third index 2 subgroup, namely, $(H_1 \cap H_2) \cup (G \backslash (H_1 \cup H_2))$. Suppose first that G has an $(s-1)$-arc-regular subgroup H_2, different from H and set $H_1 = H$. Then H_1 and H_2 are two index 2 subgroups of G, and so we can form a third index 2 subgroup K of G. By the first part of this proof K is not $(s-1)$-arc-regular and is thus not vertex-transitive. But then X is bipartite (see the proof of Lemma 5.3). Conversely, suppose X is bipartite and let K be the subgroup of G consisting of all the elements preserving the bipartition. Then K is of index 2 in G, and so $H_2 = (H \cap K) \cup (G \backslash (H \cup K))$ is also of index 2 in G. Since H is vertex-transitive and $H \cap K$ is of index 2 in H, K and $H \cap K$ have the same two orbits, and so H_2 is vertex-transitive. But then (as H_2 is of index 2 in G) it is in fact $(s-1)$-arc-regular, and so G has two $(s-1)$-arc-regular subgroups. □

The following immediate corollary of Proposition 5.2 and Lemma 5.3 is also interesting.

Corollary 5.5. *Let X be a cubic symmetric graph and let $H_1, H_2, G \leq \mathrm{Aut}(X)$ be arc-transitive subgroups such that H_1 and H_2 are both of index 2 in G. Then one G-orbit of G-consistent cycles is the set of all H_1-consistent cycles, which coincides with the set of all* *proper $(H_2, 1/2)$-consistent cycles, while the other G-orbit of G-consistent cycles is the set of all H_2-consistent cycles, which coincides with the set of all proper $(H_1, 1/2)$-consistent cycles.*

Corollary 5.6. *Let X be a cubic symmetric graph and let $G \leq \mathrm{Aut}(X)$ be s-arc-regular for some $s \in \{3, 5\}$. If G has two arc-transitive index 2 subgroups, then they are of different types.*

Proof. Suppose $H_1, H_2 \leq G$ are two arc-transitive index 2 subgroups of G and recall that by Corollary 5.4 this implies that X is bipartite with the bipartition sets coinciding with the orbits of the normal subgroup $K = (H_1 \cap H_2) \cup (G \backslash (H_1 \cup H_2))$ of G. Let $A_s = (v_0, v_1, \ldots, v_s)$ be an s-arc of X and let $\vec{C}_1$ and $\vec{C}_2$ be the two G-consistent cyclets starting with A_s with notation just like in the proof of Lemma 4.2 (see Figure 1). By Corollary 5.5 we can assume that $\vec{C}_1$ is H_1-consistent (but not H_2-consistent) and $\vec{C}_2$ is H_2-consistent (but not H_1-consistent). Let $g \in G$ be the unique element mapping A_s to $(v_s, v_{s-1}, \ldots, v_0)$. Since s is odd, say $s = 2s_0 + 1$, g swaps the pair of adjacent vertices v_{s_0} and v_{s_0+1}, and so $g \notin K$. It follows that g is contained in precisely one of H_1 and H_2, say $g \in H_1 \backslash H_2$. For each of $i \in \{1, 2\}$ the unique H_i-consistent directed cycle through $(v_s, v_{s-1}, \ldots, v_0)$ (recall that H_i is $(s-1)$-arc-regular) is $\vec{C}^*_i$. Since $g \in H_1$, this implies that C_1 is H_1-reflexible. On the other hand, since $g \notin H_2$ and H_2 is an index 2 subgroup of G, g swaps the two H_2-orbits of H_2-consistent directed cycles, implying that $\vec{C}^*_2$ is in a different H_2-orbit than $\vec{C}_2$, and so the H_2-consistent cycles are H_2-chiral. By Theorem 3.6 the groups H_1 and H_2 are of different types. □

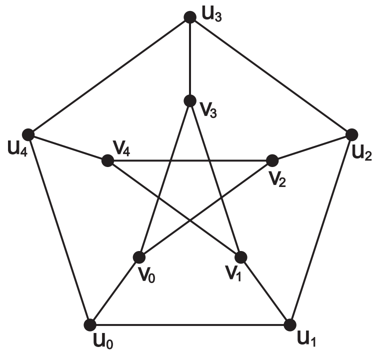
We conclude this section by explaining how the two claims from the Introduction can be deduced from our results. To see that claim (i) holds note that by Table 2 whenever a cubic arc-transitive graph X admits a proper arc-transitive subgroup of $\mathrm{Aut}(X)$ it in fact admits an arc-transitive index 2 subgroup G of $\mathrm{Aut}(X)$, except in the case of type $\{1, 4^1\}$ graphs. The latter are bipartite, while in all of the other cases Proposition 5.2 implies that one $\mathrm{Aut}(X)$-orbit of consistent cycles consists of the proper $(G, 1/2)$-consistent cycles which are all of even length. For claim (ii) its first part follows from Theorem 3.6 (if the consistent cycles are chiral, there is just one orbit of them), while the second part follows from Proposition 5.2.

6 | EXAMPLES

We conclude the paper by three examples illustrating the results of the previous sections and showing how they can be used to infer certain properties of a studied cubic symmetric graph.

Example 6.1. Let $X = \mathrm{GP}(5, 2)$ be the Petersen graph. It is of course well known that its automorphism group is 3-arc-regular and is isomorphic to S_5, from which it is easy to see that X is of type $\{2^1, 3\}$ and has consistent 5-cycles and consistent 6-cycles, the former of which are also A_5-consistent while the latter of which is proper $(A_5, 1/2)$-consistent. But let us see what one can infer about the graph X just by observing some of its rather obvious symmetries and using our results.

The two representations of the graph X from Figure 2 clearly show that X has a consistent 5-cycle with the shunt $g_1 = (u_0\ u_1\ u_2\ u_3\ u_4)(v_0\ v_1\ v_2\ v_3\ v_4)$ and a consistent



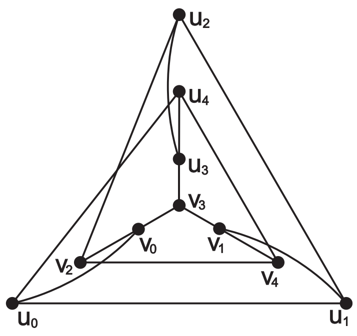


FIGURE 2 Two representations of the Petersen graph.

6-cycle with the shunt $g_2 = (u_0\ u_1\ u_2\ v_2\ v_4\ u_4)(u_3\ v_0\ v_1)$. Since g_2 maps u_2 from one orbit of $\langle g_1 \rangle$ to v_2 from the other orbit of $\langle g_1 \rangle$, and at the same time fixes a vertex while cyclically permuting its three neighbors, the group $G = \langle g_1, g_2 \rangle$ is arc-transitive. Since g_2^3 interchanges the two 3-arcs starting with (u_3, v_3, v_0), G is in fact 3-arc-transitive. As X has girth 5 this shows that $G = \mathrm{Aut}(X)$ and that G is 3-arc-regular. Since g_2 is an odd permutation, there is an index 2 subgroup H of $\mathrm{Aut}(X)$ consisting of all even permutations from G, and since $g_1, g_2^2 \in H$ this group is vertex-transitive and is thus 2-arc-regular on X. As X is not bipartite, Proposition 5.2 and Corollary 5.4 imply that H is the unique 2-arc-regular subgroup of G, that the H-consistent cycles are all of length 5 (implying that H has no 1-arc-regular subgroup) and that the G-consistent 6-cycles are the proper $(H, 1/2)$-consistent cycles. Since $k = (u_1\ u_4)(u_2\ u_3)(v_1\ v_4)(v_2\ v_3) \in H$ fixes the vertex u_0 and inverts g_1, Proposition 3.4 implies that the 5-cycle corresponding to the shunt g_1 is H-reflexible, and so Theorem 3.6 shows that X is of type $\{2^1, 3\}$.

Example 6.2. Let $X = \mathrm{GP}(8, 3)$ be the Möbius–Kantor graph. Again, its automorphism group is well known, but let us look at the two representations of X in Figure 3. They reveal that X has a consistent 8-cycle with the shunt $g_1 = (u_0\ u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7)(v_0\ v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_7)$ and a consistent 12-cycle with the shunt $g_2 = (u_0\ u_1\ u_2\ v_2\ v_7\ v_4\ u_4\ u_5\ u_6\ v_6\ v_3\ v_0)(u_3\ v_5\ u_7\ v_1)$. As in the previous example it is clear that $G = \langle g_1, g_2 \rangle$ is vertex-transitive. Since g_2^4 fixes the vertex u_3 and cyclically permutes its three neighbors, while $g_1 g_2^{-1}$ swaps the two 2-arcs starting with (u_0, u_1), the group G is 2-arc-transitive. It is not difficult to see that the 3-arc (u_0, u_1, u_2, u_3) lies on a unique 6-cycle while (u_0, u_1, u_2, v_2) lies on two, and so X is 2-arc-regular and $G = \mathrm{Aut}(X)$. Since we have G-consistent cycles of two different lengths (or because the consistent 8-cycles are clearly reflexible), Theorem 3.6 implies that G is of type $\{2^1\}$. It is clear that the subgroup $H_1 = \langle g_1^2, g_2 \rangle$ is arc-transitive. In the corresponding action of G on the edge set of X the element g_1 is an odd while g_2 is an even permutation, and so H_1 is a proper subgroup of G, showing that it is in fact 1-arc-regular and that X is of type $\{1, 2^1\}$. Since X is bipartite, Lemma 3.1, Corollary 5.4, and Corollary 5.5 imply that $H_2 = \langle g_1, g_2^2 \rangle$ is the other 1-arc-regular subgroup of G, that the G-consistent 12-cycles are H_1-consistent and are proper $(H_2, 1/2)$-consistent cycles, while the G-consistent 8-cycles are H_2-consistent and are proper $(H_1, 1/2)$-consistent cycles.

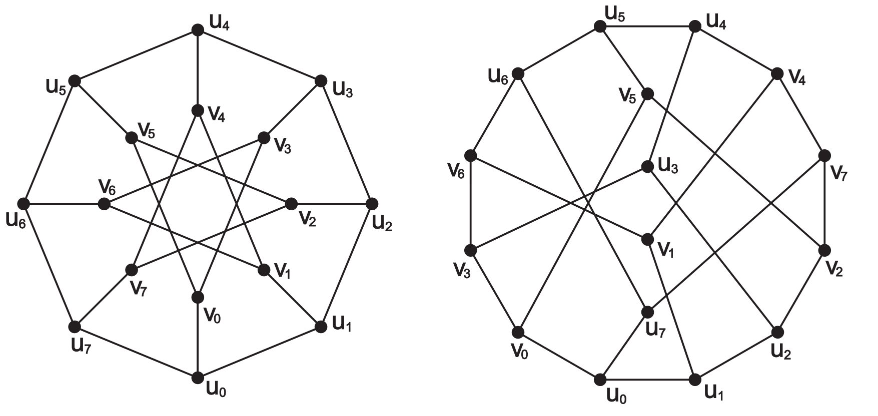


FIGURE 3 Two representations of the Möbius–Kantor graph.

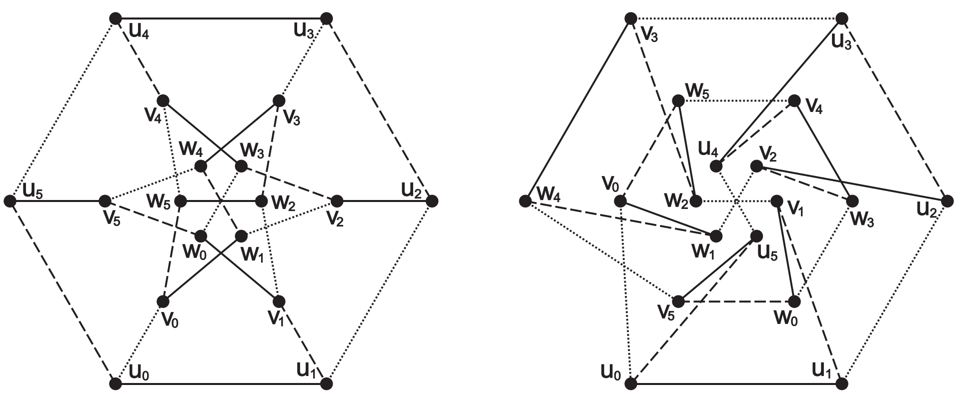


FIGURE 4 Two representations of the Pappus graph.

Example 6.3. Let X be the Pappus graph, the unique cubic symmetric graph of order 18. The two presentations of X in Figure 4 reveal that X has a consistent 6-cycle with the shunt $g_1 = (u_0\ u_1\ u_2\ u_3\ u_4\ u_5)(v_0\ v_1\ v_2\ v_3\ v_4\ v_5)(w_0\ w_1\ w_2\ w_3\ w_4\ w_5)$ and a consistent 12-cycle with the shunt $g_2 = (u_0\ u_1\ u_2\ u_3\ v_3\ w_4\ v_5\ w_0\ w_3\ v_4\ w_5\ v_0)(u_4\ w_2\ w_1\ u_5\ v_1\ v_2)$. The group $G = \langle g_1, g_2 \rangle$ is clearly vertex-transitive and since g_2^6 and $g_1 g_2^{-1}$ both fix v_1 but one swaps w_0 with u_1 while the other swaps w_0 with w_2, G is also arc-transitive. Since $g_1 g_2^{-1}$ swaps the two 3-arcs starting with (u_0, u_1, u_2), the group G is in fact 3-arc-transitive. Since there clearly exist 4-arcs which do not lie on a 6-cycle, $G = \mathrm{Aut}(X)$ and X is 3-arc-regular. The fact that g_1 is an odd while g_2 is an even permutation implies that G contains an index 2 subgroup H_2 consisting of all even permutations, which thus contains g_2 and g_1^2. This shows that H_2 is vertex-transitive and is therefore 2-arc-regular. Suppose H_2 is of type 2^1. Then Proposition 3.4 and Theorem 3.6 imply that there exists an involution in H_2 fixing u_0 and reflecting the consistent 12-cycle corresponding to g_2. But it is easy to see that such an involution is an odd permutation (as it fixes u_0, v_5, u_4, and u_5, while exchanging other vertices in seven pairs) and thus cannot be in H_2. Therefore, H_2 is of type 2^2, the G-consistent 12-cycles are H_2-chiral H_2-consistent cycles, while the G-consistent 6-cycles are proper $(H_2, 1/2)$-consistent cycles.

Since X is bipartite, Corollary 5.4 implies that there is another 2-arc-regular subgroup, say H_1, which by Corollary 5.6 must be of type 2^1. So the G-consistent 6-cycles are H_1-consistent, while the G-consistent 12-cycles are proper $(H_1, 1/2)$-consistent cycles. Note that $g_1 \in H_1$ by Lemma 3.1. Finally, set $k = g_2^2 g_1 g_2^2$ and $K = \langle g_1, k \rangle$. Since k maps u_0 to w_4 and $k g_1^{-1}$ fixes u_4 while permuting its three neighbors, K is arc-transitive. However, observe that g_1 maps the solid edges to the dotted ones, the dotted ones to the dashed ones, and the dashed ones back to the solid ones, while g_2^2 preserves the set of the dotted edges and interchanges the set of the solid ones with the set of the dashed ones. This shows that each of k and g_1^{-1} acts as the same 3-cycle on the set of the three types of edges (solid to dashed, dashed to dotted, and dotted to solid), showing that K contains no element fixing a vertex and swapping two of its neighbors. Therefore, K is 1-regular, and so Corollary 5.6 implies that X is of type $\{1, 2^1, 2^2, 3\}$. Moreover, since H_1 is of index 2 in G, g_2^2 and thus also k are both in H_1, showing that $K \leq H_1$ and therefore that all K-consistent and proper $(K, 1/2)$-consistent cycles are 6-cycles.

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