Journal of Computational and Applied Mathematics 437 (2024) 115463

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the exact region determined by Spearman's rho and Spearman's footrule

Damjana Kokol Bukovšek [a,*,1], Nik Stopar [b,1]

[a] *University of Ljubljana, School of Economics and Business, and Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[b] *University of Ljubljana, Faculty of Civil and Geodetic Engineering, and Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*

A R T I C L E   I N F O

Article history:
Received 13 August 2022
Received in revised form 28 May 2023

MSC:
62H20
62H05
60E05

Keywords:
Copula
Dependence
Concordance measure
Spearman's rho
Spearman's footrule

A B S T R A C T

We determine the lower bound for possible values of Spearman's rho of a bivariate copula given that the value of its Spearman's footrule is known and show that this bound is always attained. We also give an estimate for the exact upper bound and prove that the estimate is exact for some but not all values of Spearman's footrule. Nevertheless, we show that the estimate is quite tight.

1. Introduction

Intuitively, two continuous random variables X and Y are in concordance when large values of X occur simultaneously with large values of Y. More precisely, two realisations (x_1, y_1) and (x_2, y_2) of the random vector (X, Y) are concordant when $(x_2-x_1)(y_2-y_1) > 0$ and they are discordant when $(x_2-x_1)(y_2-y_1) < 0$. We can measure the concordance of a pair of random variables (X, Y) in various ways, see [1]. A concordance measure is often a better way to model dependence than Pearson's correlation coefficient since it is invariant with respect to monotone increasing transformations of the random variables. Because of this invariance, the concordance of a random vector (X, Y) is uniquely determined by its copula, which is given by

$$C(u, v) = H(F^{(-1)}(u), G^{(-1)}(v)),$$

where H is a joint distribution function of (X, Y), F, G are univariate distribution functions of random variables X and Y, respectively, and $F^{(-1)}, G^{(-1)}$ are their generalised inverses.

Due to their importance in statistical analysis, which is based on their connection to measures for the degree of association between two random variables, concordance measures have been studied intensively since their introduction. Recent references for bivariate concordance measures include [2–8] and their multivariate generalisations were studied

* Corresponding author.
E-mail addresses: damjana.kokol.bukovsek@ef.uni-lj.si (D. Kokol Bukovšek), nik.stopar@fgg.uni-lj.si (N. Stopar).
1 Both authors acknowledge financial support from the ARIS (Slovenian Research and Innovation Agency, research core funding No. P1-0222).

https://doi.org/10.1016/j.cam.2023.115463
0377-0427/

in [9–12], to name just a few. Given their widespread use in a variety of practical applications, it is natural to compare different concordance measures in terms of the values that they can attain. In particular, if a value of one measure is known, we may ask what are the possible values of the other measures. Here we only give a brief overview of the known results regarding this question, and for explicit formulas we refer the reader to the papers referenced below or to [13], where all the formulas are collected in one place. The investigation of the above question was started by Daniels [14] and Durbin and Stuart [15], who compared Spearman's rho and Kendall's tau and gave some estimates for the values of the two measures. The exact region of all possible pairs of values $(\tau(C), \rho(C))$, $C \in \mathcal{C}$, was only determined recently in [16]. The regions determined by Blomqvist's beta and the other three concordance measures (Spearman's rho, Kendall's tau, and Gini's gamma) are given in [17] as an exercise for the reader, while the region determined by Blomqvist's beta and Spearman's footrule was given in [5]. The region determined by Spearman's footrule and Gini's gamma was given in [13]. The regions determined by Spearman's footrule and Kendall's tau respectively Gini's gamma and Kendall's tau are considered in [18].

In this paper we investigate the relation between Spearman's rho and Spearman's footrule of a bivariate copula. We determine the exact lower bound for the value of Spearman's rho if the value of Spearman's footrule is known. The determination of the exact upper bound seems to be quite difficult, but we are able to give a tight estimate for it.

The paper is structured as follows. In Section 2 we give some basic definitions that will be used throughout the paper. In Section 3 we define doubly symmetric shuffles of M and prove that any doubly symmetric copula can be approximated by a doubly symmetric shuffle of M. Sections 4 and 5 are devoted to determining the exact lower bound for Spearman's rho in terms of Spearman's footrule and the corresponding upper bound is considered in Section 6. We give an estimate for the exact upper bound and prove that the estimate is exact for some but not all values of Spearman's footrule. Nevertheless, we show that the estimate is quite tight. In Section 7 we estimate the similarity measure between Spearman's footrule and Spearman's rho. In Appendix we give the proofs of some technical lemmas and propositions.

2. Preliminaries on concordance measures

Denote the unit interval by $\mathbb{I} = [0, 1]$ and the set of all bivariate copulas by $\mathcal{C}$. For any copula $C \in \mathcal{C}$ let $V_C(B)$ be the C-volume of rectangle B. Let us introduce some standard transformations that are naturally defined on $\mathcal{C}$ and are induced by reflections of the unit square $\mathbb{I}^2$. We denote by C^t the transpose of the copula C, i.e., $C^t(u, v) = C(v, u)$, which is induced by the reflection over the main diagonal. A copula C that satisfies the condition $C = C^t$ is called *symmetric*. The two reflections $\sigma_1 : (u, v) \mapsto (1 - u, v)$ and $\sigma_2 : (u, v) \mapsto (u, 1 - v)$ induce reflections C^{σ_1} and C^{σ_2} of the copula C, which are defined by $C^{\sigma_1}(u, v) = v - C(1 - u, v)$ and $C^{\sigma_2}(u, v) = u - C(u, 1 - v)$ (see [19, §1.7.3]), and are again copulas. If we apply both reflections to C, we obtain the survival copula of C, which we denote by $\widehat{C} = (C^{\sigma_1})^{\sigma_2} = (C^{\sigma_2})^{\sigma_1}$. It is induced by the reflection $(u, v) \mapsto (1 - u, 1 - v)$ and is given by $\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$.

Given two copulas C and D we denote $C \le D$ if $C(u, v) \leqslant D(u, v)$ for all $(u, v) \in \mathbb{I}^2$. This is the so-called *pointwise order* of copulas. The set $\mathcal{C}$ equipped with the pointwise order is a partially ordered set, but not a lattice [20, Theorem 2.1]. For any copula C we have $W \le C \le M$, where $W(u, v) = \max\{0, u + v - 1\}$ and $M(u, v) = \min\{u, v\}$ are the lower and upper *Fréchet–Hoeffding bounds* for the set of all copulas.

Formal axioms for a concordance measure were introduced by Scarsini [1] (see also [21,22] for multivariate versions). A mapping $\kappa : \mathcal{C} \to [-1, 1]$ is called a *concordance measure* if it satisfies the following properties (see [19, Definition 2.4.7]):

(C1) $\kappa(C) = \kappa(C^t)$ for every $C \in \mathcal{C}$.
(C2) $\kappa(C) \leqslant \kappa(D)$ when $C \leqslant D$.
(C3) $\kappa(M) = 1$.
(C4) $\kappa(C^{\sigma_1}) = \kappa(C^{\sigma_2}) = -\kappa(C)$.
(C5) If a sequence of copulas C_n, $n \in \mathbb{N}$, converges uniformly to $C \in \mathcal{C}$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

If a sequence of copulas converges pointwise to a function C, then C is a copula and the sequence converges uniformly to C (see [19] for details). Hence, in axiom (C5) we may replace the uniform convergence requirement with the pointwise convergence condition without loss of generality. Any concordance measure automatically satisfies also the following additional properties, which are sometimes stated as part of the definition, but are actually consequences of conditions (C1)–(C5) (see [23, §3] for more details):

(C6) $\kappa(\Pi) = 0$, where Π is the independence copula $\Pi(u, v) = uv$.
(C7) $\kappa(W) = -1$.
(C8) $\kappa(C) = \kappa(\widehat{C})$ for every $C \in \mathcal{C}$.

The four most commonly used concordance measures of a copula C are Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist's beta. The *Spearman's rho* is defined by

$$\rho(C) = 12 \int_{\mathbb{I}^2} C(u, v)\,du\,dv - 3, \tag{1}$$

Table 1
(κ_1, κ_2)-similarity measure between pairs of (weak) concordance measures.

κsm	ρ	τ	γ	ϕ	β
ρ	1	0.7114			0.3750
τ	0.7114	1	0.7500	0.7500	0.3333
γ		0.7500	1	0.8125	0.5000
ϕ		0.7500	0.8125	1	0.5000
β	0.3750	0.3333	0.5000	0.5000	1

Kendall's tau by

$$\tau(C) = 4\int_{\mathbb{I}^2} C(u, v)dC(u, v) - 1, \tag{2}$$

Gini's gamma by

$$\gamma(C) = 4\int_0^1 C(u, u)du + 4\int_0^1 C(u, 1-u)du - 2, \tag{3}$$

and *Blomqvist's beta* by

$$\beta(C) = 4\,C\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \tag{4}$$

We refer the reader to [19, §2.4], [24], and [17, §5] for more details on these measures.

In 2014 Liebscher [6] considered measures that are slightly more general than concordance measures, in particular, if we replace property (C4) with property (C6) in the definition of a concordance measure, we get what Liebscher calls *weak concordance measure*. The most important example of a weak concordance measure is the *Spearman's footrule* defined by

$$\phi(C) = 6\int_0^1 C(u, u)du - 2. \tag{5}$$

While the range of any concordance measure is the interval $[-1, 1]$, the range of a weak concordance measure may be different. For example, the range of Spearman's footrule is the interval $\left[-\frac{1}{2}, 1\right]$ (see [12, §4]). Note that Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta are concordance measures, hence, they satisfy conditions (C1)–(C8). On the other hand, Spearman's footrule only satisfies conditions (C1)–(C3), (C5)–(C6) and (C8).

All five (weak) concordance measures mentioned above are well established in statistical literature and their statistical meaning has been investigated in detail. See e.g. [25–28] for Spearman's rho, [25,28–30] for Kendall's tau, [31–33] for Gini's gamma, [12] for Blomqvist's beta and [12,32,34,35] for Spearman's footrule. More general families of concordance measures were also considered in the literature, these include some of the above examples, see [36–40]. Multivariate versions of concordance measures were investigated in [11,12,21,22,41].

In [41] the author compares the efficacies of various (multivariate) concordance measures, giving examples where one measure might be preferable over the other. The question which concordance measure is better to use is often debated in the literature, as indicated by a vast number of the references above. We consider a slightly different approach to this question. We compare different concordance measures in terms of the information they give to practitioners about a pair of random variables. To this end, the authors of paper [13] introduce the (κ_1, κ_2)-*similarity measure* between (week) concordance measures κ_1 and κ_2 as

$$\kappa sm(\kappa_1, \kappa_2) = 1 - \frac{A(\kappa_1, \kappa_2)}{(1-\kappa_1(W))(1-\kappa_2(W))},$$

where $A(\kappa_1, \kappa_2)$ is the area of the exact region determined by κ_1 and κ_2.

The (κ_1, κ_2)-similarity measure between two (weak) concordance measures κ_1 and κ_2 is defined in such a way that a value close to 1 indicates the measures are very similar, i.e. given the value of κ_1, we have a lot of information about the value of κ_2 on average. In practice this means that there is little difference between the two concordance measures, so a practitioner could use either one. A value of similarity measure close to 0 suggests that given the value of κ_1, we have very little information about the value of κ_2 on average. This means that computing both makes sense to get more information about a pair of random variables.

Table 1 gives κ-similarity measures for pairs of (weak) concordance measures for which the exact region determined by them is known.

The table suggests that knowing the value of Blomqvist's beta gives us on average very little information about possible values of other (weak) concordance measures. This is not surprising since Blomqvist's beta itself contains very little information. On the other hand, due to the similar structure of Gini's gamma and Spearman's footrule, their similarity measure is relatively big compared to that of Spearman's rho and Kendall's tau. The amount of information that Kendall's tau gives is on average almost the same for Spearman's rho, Gini's gamma and Spearman's footrule.

3. Doubly symmetric shuffles

It is well known that any copula can be approximated arbitrarily well in the sup norm by a shuffle of M. In this section we investigate approximations of doubly symmetric copulas with doubly symmetric shuffles of M, defined below.

Definition 1. A copula C is called *doubly symmetric* if $C = C^t = \widehat{C}$.

Note that a copula is doubly symmetric if and only if the distribution of its mass in the unit square is symmetric with respect to both the main and the opposite diagonal. The reflection with respect to the main diagonal is given by $(u, v) \mapsto (v, u)$ while the reflection with respect to the opposite diagonal is given by $(u, v) \mapsto (1 - v, 1 - u)$.

A shuffle of M

$$C = M(n, J, \pi, \omega)$$

is determined by a positive integer n, a partition $J = \{J_1, J_2, \ldots, J_n\}$ of the interval $\mathbb{I}$ into n pieces, where $J_i = [u_{i-1}, u_i]$ and $0 = u_0 \le u_1 \le u_2 \le \cdots \le u_{n-1} \le u_n = 1$, shortly written as $(n-1)$-tuple of splitting points $J = (u_1, u_2, \ldots, u_{n-1})$, a permutation $\pi \in S_n$, written as n-tuple of images $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, and a mapping $\omega : \{1, 2, \ldots, n\} \to \{-1, 1\}$, written as n-tuple of images $\omega = (\omega(1), \omega(2), \ldots, \omega(n))$. The mass of C is concentrated on squares $J_i \times [v_{\pi(i)-1} \times v_{\pi(i)}]$ where $0 = v_0 \le v_1 \le v_2 \le \ldots \le v_{n-1} \le v_n = 1$. For more details see [17, §3.2.3]. Notice that we allow some of the intervals in the partition J to be singletons. We can now define doubly symmetric shuffles of M.

Definition 2. We will say that a shuffle $C = M(n, J, \pi, \omega)$ of M with $J = (u_1, u_2, \ldots, u_{n-1})$, $u_0 = 0$, $u_n = 1$, is a *doubly symmetric shuffle* if the following properties hold

(S1) n is even,
(S2) $\pi^2 = \mathrm{id}$ and $\pi(n - i + 1) = n - \pi(i) + 1$ for all $i = 1, 2, \ldots, n$,
(S3) $\omega(i) = \omega(\pi(i)) = \omega(n - i + 1)$ for all $i = 1, 2, \ldots, n$,
(S4) $u_i - u_{i-1} = u_{\pi(i)} - u_{\pi(i)-1} = u_{n-i+1} - u_{n-i}$ for all $i = 1, 2, \ldots, n$.

In the following lemma we give some properties of doubly symmetric shuffles. The proof is given in Appendix.

Lemma 3. *If C is a doubly symmetric shuffle of M then*

(a) C is a doubly symmetric copula,
(b) $u_{n-i} = 1 - u_i$ for all $i = 1, 2, \ldots, n$, in particular, $u_{\frac{n}{2}} = \frac{1}{2}$,
(c) all the mass of C is concentrated on the squares $J_i \times J_{\pi(i)} = [u_{i-1}, u_i] \times [u_{\pi(i)-1}, u_{\pi(i)}]$, $i = 1, 2, \ldots, n$, and

$$\begin{aligned} V_C(J_i \times J_{\pi(i)}) &= V_C(J_{\pi(i)} \times J_i) = V_C(J_{n-i+1} \times J_{\pi(n-i+1)}) \\ &= V_C(J_{\pi(n-i+1)} \times J_{n-i+1}) = u_i - u_{i-1}. \end{aligned}$$

Note that a doubly symmetric copula C which is also a shuffle $M(n, J, \pi, \omega)$ is not necessarily a doubly symmetric shuffle. But there exists a doubly symmetric shuffle $M(n', J', \pi', \omega')$ such that $C = M(n', J', \pi', \omega')$. For example, if $\frac{1}{2}$ is not one of the splitting points, we can add it and adjust π and ω accordingly.

Next lemma shows that doubly symmetric copulas can be approximated by doubly symmetric shuffles of M. The proof is given in Appendix. In fact, we show that the original construction by Mikusiński et al. [42] (c.f. also [17, §3.2.3]) of a shuffle of M that approximates a copula C produces a doubly symmetric shuffle of M whenever C is a doubly symmetric copula. Recall that a shuffle of M is called *straight* if the mapping ω has all values equal to 1.

Lemma 4. *For any doubly symmetric copula C and any $\varepsilon > 0$ there exists a straight doubly symmetric shuffle of M, which we denote by C', such that*

$$\sup_{(u,v)\in\mathbb{I}^2} |C(u, v) - C'(u, v)| < \varepsilon.$$

4. Lower bound - special case

In the following two sections we prove a lower bound for the value of Spearman's rho for any copula with a given value of Spearman's footrule. In this section we consider copulas which have all the mass concentrated on the main and opposite diagonal. In the next section we will reduce the general case to this special case.

Copulas which have all mass concentrated on the two diagonals correspond to uniformly distributed random variables U and V on interval $\mathbb{I}$ with the property $P(U = V) + P(U = 1 - V) = 1$. As we will see, any such copula is completely determined by its diagonal $\delta_C(u) = C(u, u)$ and it is doubly symmetric, so it is easier to tackle. The proof of next proposition is given in Appendix.

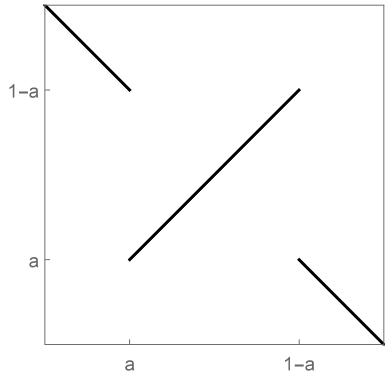

Fig. 1. The scatterplot of copula C_a from Example 6.

Proposition 5. *Let U and V be uniformly distributed random variables on interval $\mathbb{I}$ with the property $P(U = V) + P(U = 1 - V) = 1$ and let $C \in \mathcal{C}$ be their copula. Then*

$$\rho(C) \geq \frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} - 1.$$

It turns out that the copula C from Proposition 5 is the Bertino copula, determined by diagonal section δ_C, which is the smallest copula with this diagonal section, see [43]. This is because the function $\widehat{\delta}(u) = u - \delta_C(u)$ is increasing on the interval $[0, \frac{1}{2}]$, decreasing on the interval $[\frac{1}{2}, 1]$, and it is symmetric with respect to $u = \frac{1}{2}$, as it can be seen from the proof.

In the following example, we note that all points on the curve $r = \frac{2}{9}\sqrt{3}(1 + 2p)^{3/2} - 1$ can be attained by shuffles of M.

Example 6. Let $a \in [0, \frac{1}{2}]$ and let C_a be a shuffle of M

$$C_a = M(3, (a, 1 - a), (3, 2, 1), (-1, 1, -1)).$$

Notice that $C_0 = M$ and $C_{\frac{1}{2}} = W$. We have

$$\delta_{C_a}(u) = \begin{cases} 0; & u \leq a, \\ u - a; & a \leq u \leq 1 - a, \\ 2u - 1; & 1 - a \leq u \leq 1, \end{cases}$$

and

$$C_a(u, v) = \begin{cases} 0; & 0 \leq u \leq a, u \leq v \leq 1 - u, \\ 0; & 0 \leq v \leq a, v \leq u \leq 1 - v, \\ u + v - 1; & 1 - a \leq u \leq 1, 1 - u \leq v \leq u, \\ u + v - 1; & 1 - a \leq v \leq 1, 1 - v \leq u \leq v, \\ u - a; & a \leq u \leq 1 - a, u \leq v \leq 1 - a, \\ v - a; & a \leq u \leq 1 - a, a \leq v \leq u, \end{cases}$$

It follows that

$$\phi(C_a) = 6a^2 - 6a + 1 = \tfrac{3}{2}(1 - 2a)^2 - \tfrac{1}{2}$$

and

$$\rho(C_a) = -16a^3 + 24a^2 - 12a + 1 = 2(1 - 2a)^3 - 1,$$

so that $\rho(C_a) = \frac{2}{9}\sqrt{3}(1 + 2\phi(C_a))^{3/2} - 1$, the point $(\phi(C_a), \rho(C_a))$ lies on the curve $r = \frac{2}{9}\sqrt{3}(1 + 2p)^{3/2} - 1$ and every point on this curve for $p \in [-\frac{1}{2}, 1]$ is attained. The scatterplot of copula C_a is shown in Fig. 1.

5. Lower bound - general case

Let U and V be uniformly distributed random variables on interval $\mathbb{I}$ and let $C \in \mathcal{C}$ be their copula. In this section we are going to prove the lower bound

$$\frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} - 1 \leq \rho(C)$$

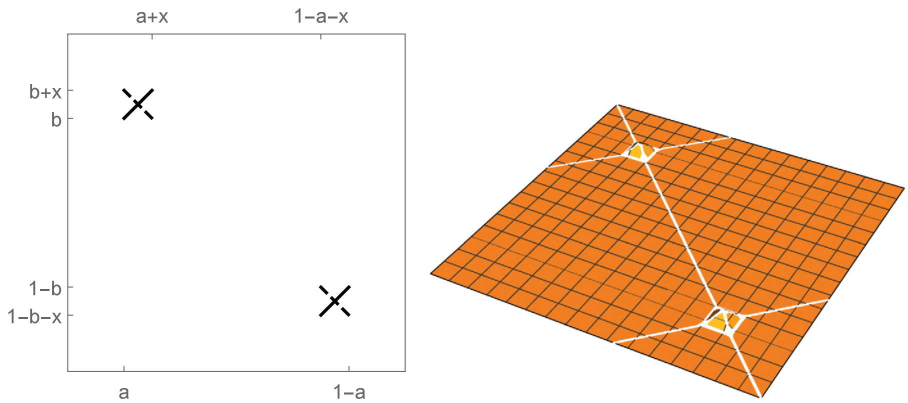

Fig. 2. The mass distribution (left) and the graph (right) of the difference of copulas considered in the proof of Lemma 7 in *Case I*.

holds for arbitrary copula C. Define two function depending on copula $C \in \mathcal{C}$ by

$$f(C) = \rho(C) - \frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} + 1$$

and

$$q(C) = P(U = V) + P(U = 1 - V).$$

Note that in order to prove the above bound we need to prove that $f(C) \geq 0$ for any copula C. We will reduce the general case to the special case considered in Section 4 by redistribution of the mass of the copula. We will first approximate the copula C by a doubly symmetric shuffle C' of M. Then we will transform C' to a copula C'' by shifting the mass of C' in such a way that copula C'' will have all the mass concentrated on both diagonals. By doing this appropriately we will achieve that $f(C'') \leq f(C') \approx f(C)$. This will allow us to use Proposition 5 to finish the proof. The shifting of mass will be done step by step. In each step the value of $f(C)$ will decrease while the value of $q(C)$ will increase, until $q(C)$ becomes 1. We believe that this method of mass shifting may also be useful in other considerations. Next lemma describes a single step of this method.

Lemma 7. *Let $C \in \mathcal{C}$ be a doubly symmetric shuffle of M, $C = M(n, J, \pi, \omega)$. Write $J = \{J_1, \ldots, J_n\}$. Suppose that for some $i \in \{1, 2, \ldots, n\}$ we have $J_i \times J_{\pi(i)} = [a, a + x] \times [b, b + x]$ with $a < b < 1 - a$, $\omega(i) = 1$, and $x > 0$. Then there exists doubly symmetric shuffle of M, $C' = M(n, J, \pi', \omega')$ such that $f(C') < f(C)$ and $q(C') = q(C) + 2x$ in the case $\pi(i) = n + 1 - i$ or $q(C') = q(C) + 4x$ otherwise.*

Proof. We have

$$V_C(J_i \times J_{\pi(i)}) = V_C([a, a + x] \times [b, b + x]) = x$$

and corresponding reflected squares are $J_{\pi(i)} \times J_i = [b, b+x] \times [a, a+x]$, $J_{n+1-i} \times J_{\pi(n+1-i)} = [1-a-x, 1-a] \times [1-b-x, 1-b]$, and $J_{\pi(n+1-i)} \times J_{n+1-i} = [1 - b - x, 1 - b] \times [1 - a - x, 1 - x]$. Furthermore, $\omega(\pi(i)) = \omega(n + 1 - i) = \omega(\pi(n + 1 - i)) = 1$. Since $a < b < 1 - a$, we have $a < \frac{1}{2}$, $i \leq \frac{n}{2}$ and $i < \pi(i) \leq n + 1 - i$. We will consider several cases.

Case I: Suppose that $\pi(i) = n + 1 - i$, so $b = 1 - a - x$ and in C we have two segments crossing the opposite diagonal. Define

$$\pi' = \pi \text{ and } \omega'(j) = \begin{cases} -1; & j \in \{i, n + 1 - i\}, \\ \omega(j); & \text{otherwise.} \end{cases}$$

The copula C' is doubly symmetric shuffle. Fig. 2 shows the mass distribution of the difference of copulas C and C', and the graph of the function $C - C'$. The mass of C' is negative in the difference, it is shown dashed.

We have

$$C(u, v) - C'(u, v) = \max\{0, \min\{u - a, v - (1 - a - x), 1 - a - v, a + x - u\}\} + \max\{0, \min\{v - a, u - (1 - a - x), 1 - a - u, a + x - v\}\},$$

so $\delta_C = \delta_{C'}$ and $\phi(C) = \phi(C')$. Furthermore,

$$\rho(C) - \rho(C') = 12 \int_0^1 \int_0^1 (C(u, v) - C'(u, v))dudv = 24\text{Vol},$$

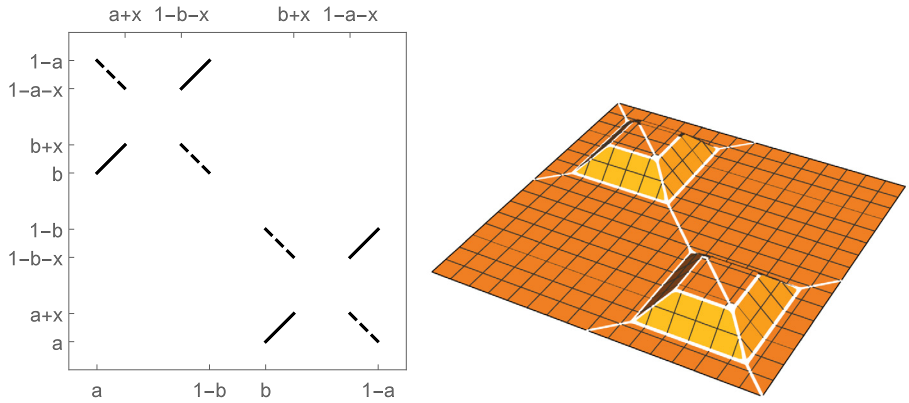

Fig. 3. The mass distribution (left) and the graph (right) of the difference of copulas considered in the proof of Lemma 7 in *Case II*.

where Vol is the volume of a pyramid having the base a square with the side x and height $\frac{x}{2}$, so $\mathrm{Vol} = \frac{x^3}{6}$. We thus have $f(C) = f(C') + 4x^3 > f(C')$, and since the two new segments lie on the opposite diagonal also $q(C') = q(C) + 2x$.

Case II: Suppose that $\frac{n}{2} < \pi(i) < n+1-i$, so $\frac{1}{2} \le b \le 1-a-2x$. Define

$$\pi'(j) = \begin{cases} n+1-j; & j \in \{i, n+1-i, \pi(i), \pi(n+1-i)\}, \\ \pi(j); & \text{otherwise}, \end{cases} \quad \text{and}$$

$$\omega'(j) = \begin{cases} -1; & j \in \{i, n+1-i, \pi(i), \pi(n+1-i)\}, \\ \omega(j); & \text{otherwise}. \end{cases}$$

Fig. 3 shows the mass distribution of the difference of copulas C and C', and the graph of the function $C - C'$.

We have

$$C(u,v) - C'(u,v) = \max\{0, \min\{u-a, v-b, 1-a-v, 1-b-u, x\}\} \\ + \max\{0, \min\{v-a, u-b, 1-a-u, 1-b-v, x\}\},$$

so again $\delta_C = \delta_{C'}$ and $\phi(C) = \phi(C')$. Furthermore, $\rho(C) - \rho(C') = 24\mathrm{Vol}$, where Vol is the volume of a square frustum having the lower base a square with the side $1-a-b$, the upper base a square with the side $1-a-b-2x$ and height x, so

$$\mathrm{Vol} = \tfrac{1}{6}(1-a-b)^3 - \tfrac{1}{6}(1-a-b-2x)^3 = x(1-a-b-x)^2 + \tfrac{1}{3}x^3.$$

We thus have $f(C) = f(C') + 24x(1-a-b-x)^2 + 8x^3 > f(C')$, and since the four new segments lie on the opposite diagonal also $q(C') = q(C) + 4x$.

Case III: Suppose that $\pi(i) \le \frac{n}{2}$, so $b \le \frac{1}{2} - x$ and assume also $b \ge a + \frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}$. Define as in Case II

$$\pi'(j) = \begin{cases} n+1-j; & j \in \{i, n+1-i, \pi(i), \pi(n+1-i)\}, \\ \pi(j); & \text{otherwise}, \end{cases} \quad \text{and}$$

$$\omega'(j) = \begin{cases} -1; & j \in \{i, n+1-i, \pi(i), \pi(n+1-i)\}, \\ \omega(j); & \text{otherwise}. \end{cases}$$

Fig. 4 shows the mass distribution and the graph of the function $C - C'$ in this case.

Again we have

$$C(u,v) - C'(u,v) = \max\{0, \min\{u-a, v-b, 1-a-v, 1-b-u, x\}\} \\ + \max\{0, \min\{v-a, u-b, 1-a-u, 1-b-v, x\}\},$$

so again $\rho(C) = \rho(C') + 24x(1-a-b-x)^2 + 8x^3$. But now

$$\delta_C(u) - \delta_{C'}(u) = 2\max\{0, \min\{u-b, 1-b-u, x\}\},$$

so

$$\phi(C) = \phi(C') + 6\int_0^1 (\delta_C(u) - \delta_{C'}(u))du = 3(1-2b)^2 - 3(1-2b-2x)^2 = 12x(1-2b-x).$$

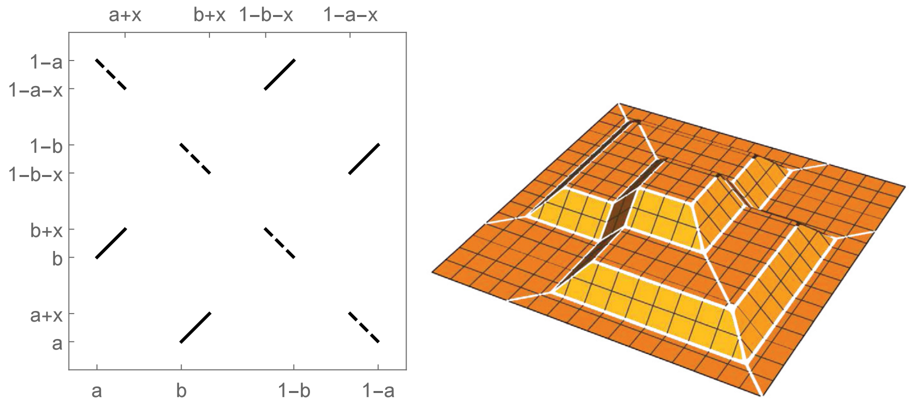

Fig. 4. The mass distribution (left) and the graph (right) of the difference of copulas considered in the proof of Lemma 7 in *Case III*.

Denote by $d = 12x(1-2b-x)$. Now

$$\begin{aligned} f(C)-f(C') &= \rho(C)-\rho(C')-\frac{2}{9}\sqrt{3}(1+2\phi(C))^{3/2}+\frac{2}{9}\sqrt{3}(1+2\phi(C'))^{3/2} \\ &= 24x(1-a-b-x)^2+8x^3+\frac{2}{9}\sqrt{3}\left((1+2\phi(C)-2d)^{3/2}-(1+2\phi(C))^{3/2}\right). \end{aligned}$$

Using Lagrange theorem there exists $t \in [0,d]$ such that

$$(1+2\phi(C)-2d)^{3/2}-(1+2\phi(C))^{3/2} = -3d(1+2\phi(C)-2t)^{1/2} \geq -3d(1+2\phi(C))^{1/2},$$

so

$$\begin{aligned} f(C)-f(C') &\geq 24x(1-a-b-x)^2+8x^3-\frac{2}{9}\sqrt{3}\cdot 3d(1+2\phi(C))^{1/2} \\ &= 24x(1-a-b-x)^2+8x^3-\frac{2d}{\sqrt{3}}\sqrt{1+2\phi(C)} \\ &= 24x(1-a-b-x)^2+8x^3-24x(1-2b-x)\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)} \\ &\geq 24x(1-a-b-x)^2+8x^3-24x(1-a-b-x)\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)} \\ &= 8x^3+24x(1-a-b-x)\left(1-a-b-x-\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}\right) \end{aligned}$$

We now use the assumptions $b \leq \frac{1}{2}-x$, so $1 \geq 2b+2x$, and $b \geq a+\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}$ to estimate further

$$\begin{aligned} f(C)-f(C') &\geq 8x^3+24x(1-a-b-x)\left(2b+2x-a-b-x-\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}\right) \\ &= 8x^3+24x(1-a-b-x)\left(x+b-a-\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}\right) \\ &\geq 8x^3+24x^2(1-a-b-x) > 0. \end{aligned}$$

Finally, since the four new segments lie on the opposite diagonal, we have $q(C') = q(C)+4x$ as in the previous case.

Case IV: Suppose that $i < \pi(i) \leq \frac{n}{2}$, so $a+x \leq b \leq \frac{1}{2}-x$ and assume also $b < a+\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}$. Define

$$\pi'(j) = \begin{cases} j; & j \in \{i, n+1-i, \pi(i), \pi(n+1-i)\}, \\ \pi(j); & \text{otherwise,} \end{cases} \quad \text{and} \quad \omega' = \omega.$$

Fig. 5 shows the mass distribution and the graph of the function $C - C'$.

This time we have $C' \geq C$ so

$$\begin{aligned} C(u,v)-C'(u,v) &= -(C'(u,v)-C(u,v)) \\ &= -\max\{0, \min\{u-a, v-a, b+x-u, b+x-v, x, \\ &\qquad b-a+u-v, b-a+v-u\}\} \end{aligned}$$

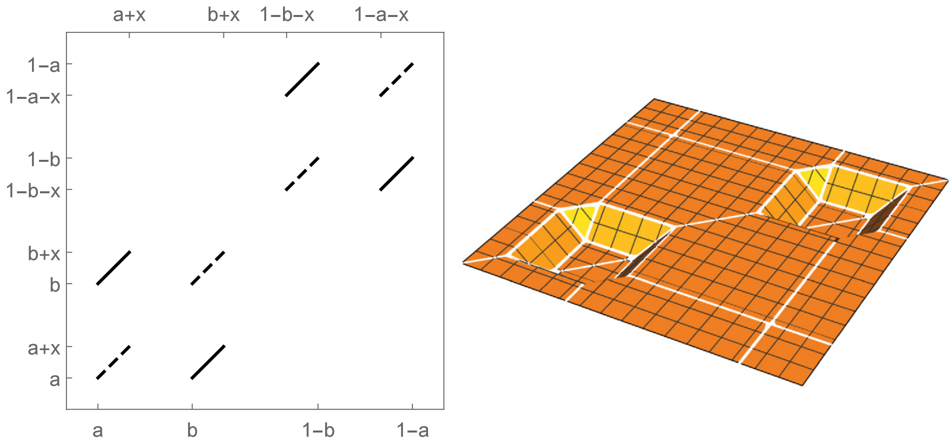

Fig. 5. The mass distribution (left) and the graph (right) of the difference of copulas considered in the proof of Lemma 7 in *Case IV*.

$$- \max\{0, \min\{u-1+b+x, v-1+b+x, 1-a-u, 1-a-v, x, b-a+u-v, b-a+v-u\}\}.$$

It follows that

$$\delta_C(u) - \delta_{C'}(u) = -\max\{0, \min\{u-a, b+x-u, x\}\} - \max\{0, \min\{u-1+b+x, 1-a-u, x\}\},$$

so $\phi(C) = \phi(C') - 12x(b-a)$. Furthermore $\rho(C) = \rho(C') - 24\text{Vol}$, where Vol is the volume of a square frustum having the lower base a square with the side $b-a+x$, the upper base a square with the side $b-a-x$, height x, and two corners cut off, so

$$\text{Vol} = \tfrac{1}{6}(b-a+x)^3 - \tfrac{1}{6}(b-a-x)^3 - 2\cdot\tfrac{1}{6}x^3 = x(b-a)^2$$

and $\rho(C) = \rho(C') - 24x(b-a)^2$. Similarly as in the previous case we estimate

$$\begin{aligned} f(C) - f(C') &= \rho(C) - \rho(C') - \frac{2}{9}\sqrt{3}(1+2\phi(C))^{3/2} + \frac{2}{9}\sqrt{3}(1+2\phi(C'))^{3/2} \\ &= -24x(b-a)^2 + \frac{2}{9}\sqrt{3}\left((1+2\phi(C)+2d)^{3/2} - (1+2\phi(C))^{3/2}\right) \end{aligned}$$

where $d = 12x(b-a)$, so there exists $t \in [0, d]$ such that

$$\begin{aligned} f(C) - f(C') &= -24x(b-a)^2 + \frac{2}{9}\sqrt{3}\cdot 3d(1+2\phi(C)+2t)^{1/2} \\ &\geq -24x(b-a)^2 + \frac{2d}{\sqrt{3}}\sqrt{1+2\phi(C)} \\ &= -24x(b-a)^2 + 24x(b-a)\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)} \\ &= 24x(b-a)\left(a-b+\frac{1}{\sqrt{3}}\sqrt{1+2\phi(C)}\right) > 0. \end{aligned}$$

Finally, since the four new segments lie on the main diagonal, we have $q(C') = q(C) + 4x$ as in the previous case. □

We are now finally ready to prove our the lower bound.

Theorem 8. *For any copula C we have*

$$\frac{2}{9}\sqrt{3}(1+2\phi(C))^{3/2} - 1 \leq \rho(C).$$

For any value $\phi(C) \in [-\frac{1}{2}, 1]$ the bound is attained by some shuffle of M.

Proof. Let C be a copula such that $\phi(C) = p$ for some $p \in [-\frac{1}{2}, 1]$. Define a copula $\widetilde{C} = \frac{1}{4}(C + C^t + \widehat{C} + \widehat{C}^t)$. Since the transformations $C \mapsto C^t$ and $C \mapsto \widehat{C}$ commute it is easy to verify that $\widetilde{C}$ is a doubly symmetric copula. Furthermore, $\phi(\widetilde{C}) = \phi(C)$ and $\rho(\widetilde{C}) = \rho(C)$ because ρ and ϕ preserve convex combinations and $\rho(C^t) = \rho(\widehat{C}) = \rho(C)$

and $\phi(C^t) = \phi(\widehat{C}) = \phi(C)$. Thus, by replacing C with $\widetilde{C}$, we may assume without loss of generality that C is a doubly symmetric copula.

Let $\varepsilon > 0$. By Lemma 4 there exists a straight doubly symmetric shuffle $C' = M(n, J, \pi, \omega)$ such that $\sup_{(u,v)\in\mathbb{I}^2} |C(u, v) - C'(u, v)| < \varepsilon$. Hence,

$$|\rho(C) - \rho(C')| < 12\varepsilon \qquad \text{and} \qquad |\phi(C) - \phi(C')| < 6\varepsilon. \tag{6}$$

By Lagrange theorem we have

$$\begin{aligned} f(C) - f(C') &= \rho(C) - \rho(C') - \Big(\frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} - \frac{2}{9}\sqrt{3}(1 + 2\phi(C'))^{3/2}\Big) \\ &= \rho(C) - \rho(C') - \frac{2}{\sqrt{3}}(1 + 2t)^{1/2}\big(\phi(C) - \phi(C')\big) \end{aligned} \tag{7}$$

for some t between $\phi(C)$ and $\phi(C')$, so that $t \le 1$. Using the estimates (6) in Eq. (7) and the estimate for t we get

$$f(C) - f(C') > -12\varepsilon - \frac{2}{\sqrt{3}}(1 + 2t)^{1/2} \cdot 6\varepsilon \ge -24\varepsilon. \tag{8}$$

By Lemma 3 all the mass of C' is concentrated on squares $J_i \times J_{\pi(i)}$, $i = 1, 2, \ldots, n$, and the squares $J_i \times J_{\pi(i)}$, $J_{\pi(i)} \times J_i$, $J_{n-i+1} \times J_{\pi(n-i+1)}$ and $J_{\pi(n-i+1)} \times J_{n-i+1}$ have the same C'-volume. So, as long as $q(C') < 1$, there exists $i \in \{1, 2, \ldots, n\}$ such that $J_i \times J_{\pi(i)} = [a, a+x] \times [b, b+x]$ with $a < b < 1-a$ and $x > 0$. We now apply Lemma 7 to all squares with this property, one at a time. Each time we apply the lemma, the function $q(C')$ increases and the function $f(C')$ decreases. When this process ends, we are left with a doubly symmetric shuffle C'' such that $q(C'') = 1$ and $f(C'') \le f(C')$. By Proposition 5 we have $f(C'') \ge 0$, and hence $f(C') \ge 0$. Together with the estimate (8), this implies $f(C) > -24\varepsilon$. Sending ε to 0 we obtain $f(C) \ge 0$, which proves the bound.

By Example 6 the bound is attained by a shuffle of M. □

6. Upper bound

We will first prove the upper bound of $\rho(C)$ in terms of $\phi(C)$ in the case when C is the diagonal copula K_δ (see below) and the diagonal is nice enough.

Let C be a doubly symmetric copula. Then its diagonal $\delta(u) = C(u, u)$ satisfies

$$\delta(u) = 2u - 1 + \delta(1 - u). \tag{9}$$

We call such a diagonal *symmetric diagonal*. It is well known that the diagonal is increasing and 2-Lipschitz, so it is differentiable almost everywhere on $\mathbb{I}$ and $\delta'(u) \in [0, 2]$ where it exists. We are going to assume that it is differentiable everywhere, its derivative is continuous, and that

$$0 < \delta'(u) < 2 \quad \text{for all except possibly finitely many points } u \in \mathbb{I}. \tag{10}$$

It follows that δ is strictly increasing, so it is bijective and its inverse $\delta^{-1} : \mathbb{I} \to \mathbb{I}$ exists. Let us introduce three auxiliary functions $\alpha, g, h : \mathbb{I} \to \mathbb{R}$ as follows

$$\begin{aligned} \alpha(u) &= \int_0^u \delta(t)dt, \\ g(u) &= 2u - \delta(u), \\ h(u) &= \delta^{-1}(g(u)). \end{aligned} \tag{11}$$

It holds that $g(0) = h(0) = 0, g(1) = h(1) = 1$. Since $\delta(u) \le u$ for $u \in \mathbb{I}$, we have $g(u) \ge u$ and $h(u) \ge u$ for every $u \in \mathbb{I}$. It follows immediately from (10) that g and h are bijective so that the inverses $g^{-1}, h^{-1} : \mathbb{I} \to \mathbb{I}$ exist. Furthermore, g is differentiable everywhere on $\mathbb{I}$ and its derivative is continuous. Also, by (10), h is differentiable everywhere except possibly in finitely many points $u \in \mathbb{I}$ and its derivative is continuous where it exists.

Fredricks and Nelsen in [44] introduced *diagonal copula* K_δ, given by

$$K_\delta(u, v) = \min\left\{u, v, \frac{\delta(u) + \delta(v)}{2}\right\}.$$

Notice that $u = \frac{1}{2}(\delta(u) + \delta(v))$ holds if and only if $v = h(u)$, so that

$$K_\delta(u, v) = \begin{cases} u; & v \ge h(u), \\ \frac{\delta(u)+\delta(v)}{2}; & v < h(u), u < h(v), \\ v; & u \ge h(v). \end{cases} \tag{12}$$

The following lemma simplifies the double integral of $K_\delta(u, v)$ over the unit square $\mathbb{I}^2$. The proof is given in Appendix.

Lemma 9. *Let δ be a symmetric differentiable diagonal with continuous derivative that satisfies (10), $\alpha, g, h : \mathbb{I} \to \mathbb{I}$ auxiliary functions defined by (11), and K_δ diagonal copula. Then*

$$\int_0^1 \delta^{-1}(u)du = 1 - \alpha(1)$$

and

$$\int_0^1 \int_0^1 K_\delta(u,v)dudv = \frac{11}{6} - 2\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))^2 du.$$

Next proposition establishes the upper bound for the diagonal copula.

Proposition 10. *Let δ be a symmetric differentiable diagonal with continuous derivative that satisfies (10) and K_δ diagonal copula. Then*

$$\rho(K_\delta) \le 1 - \frac{2}{3}(1 - \phi(K_\delta))^2.$$

Proof. We will first prove that

$$\int_0^1 \int_0^1 K_\delta(u,v)dudv \le 2\alpha(1) - 2\alpha(1)^2 - \frac{1}{6}.$$

By Lemma 9 we have

$$\int_0^1 \int_0^1 K_\delta(u,v)dudv = \frac{11}{6} - 2\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))^2 du.$$

For the integral on the right we use Jensen's inequality, claiming that

$$\int_0^1 r(s(x))dx \ge r\left(\int_0^1 s(x)dx\right),$$

where $s : \mathbb{I} \to A$ is nonnegative measurable function and $r : A \to \mathbb{R}$ is convex function. So

$$\begin{aligned}\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))^2 du &\ge \left(\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))du\right)^2 \\ &= (1 - \alpha(1) + 1 - \alpha(1))^2 = 4(1-\alpha(1))^2\end{aligned}$$

by Lemma 9, thus

$$\int_0^1 \int_0^1 K_\delta(u,v)dudv \le \frac{11}{6} - 2\alpha(1) - 2(1-\alpha(1))^2 = 2\alpha(1) - 2\alpha(1)^2 - \frac{1}{6}.$$

Finally, to prove the bound for $\rho(K_\delta)$, we use (1) and (5)

$$\begin{aligned}\rho(K_\delta) &= 12\int_0^1 \int_0^1 K_\delta(u,v)dudv - 3 \\ &\le 24\alpha(1) - 24\alpha(1)^2 - 5 \\ &= 24 \cdot \frac{\phi(K_\delta)+2}{6} - 24\left(\frac{\phi(K_\delta)+2}{6}\right)^2 - 5 \\ &= 1 - \frac{2}{3}(1-\phi(K_\delta))^2. \quad \square\end{aligned}$$

We can now prove the same bound for a general copula with arbitrary diagonal.

Theorem 11. *For any copula C we have*

$$\rho(C) \le 1 - \frac{2}{3}(1 - \phi(C))^2.$$

Proof. If $C = M$ then $\rho(C) = \phi(C) = 1$ and the inequality holds. So assume $C \ne M$ so that $\phi(C) < 1$. Let $\varepsilon > 0$ and $\varepsilon < \frac{1-\phi(C)}{6}$. By [19, Theorem 4.1.11] there exists an integer n such that the Bernstein copula

$$B_n^C(u,v) = \sum_{i,j=0}^{n} C\left(\frac{i}{n}, \frac{j}{n}\right)\binom{n}{i}\binom{n}{j}u^i(1-u)^{n-i}v^j(1-v)^{n-j}$$

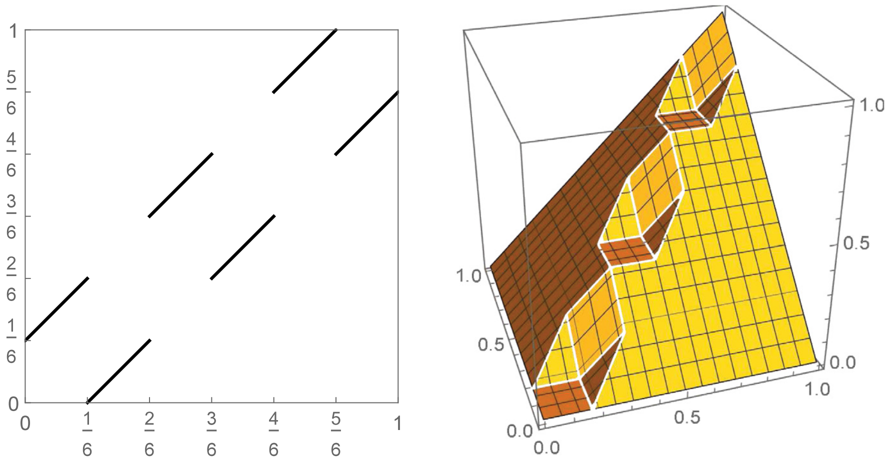

Fig. 6. The mass distribution (left) and the 3D graph (right) of the copula C_3 from Example 12.

differs from $C(u, v)$ by less than ε uniformly. Similarly as in the proof of Theorem 8 we can take $A_n^C = \frac{1}{4}\big(B_n^C + (B_n^C)^t + \widehat{(B_n^C)} + \widehat{(B_n^C)}^t\big)$. Let $\delta_n(u)$ be the diagonal of copula A_n^C. Since the diagonal of B_n^C is a polynomial in u, also $\delta_n(u)$ is a polynomial in u. This implies that δ_n is differentiable with continuous derivative and $\delta_n'(u) = 0$ or $\delta_n'(u) = 2$ for at most finitely many points $u \in \mathbb{I}$. Since A_n^C is a doubly symmetric copula, δ_n is a symmetric diagonal. Furthermore, it follows from [43, Theorem 2 (iv)] that A_n^C is bounded from above by the diagonal copula K_{δ_n}. This together with Proposition 10 implies that

$$\rho(A_n^C) \le \rho(K_{\delta_n}) \le 1 - \frac{2}{3}(1 - \phi(K_{\delta_n}))^2 = 1 - \frac{2}{3}(1 - \phi(A_n^C))^2.$$

Similarly as in the proof of Theorem 8 we estimate

$$\rho(C) \le \rho(B_n^C) + 12\varepsilon = \rho(A_n^C) + 12\varepsilon \le 1 - \frac{2}{3}(1 - \phi(A_n^C))^2 + 12\varepsilon = 1 - \frac{2}{3}(1 - \phi(B_n^C))^2 + 12\varepsilon.$$

Furthermore, $\phi(B_n^C) \le \phi(C) + 6\varepsilon < 1$ by our assumption for ε, hence

$$\rho(C) \le 1 - \frac{2}{3}(1 - \phi(C) - 6\varepsilon)^2 + 12\varepsilon.$$

By sending ε to 0 we obtain the desired bound. □

Next example shows that for certain values of $\phi(C)$ the upper bound given in Theorem 11 is attained.

Let $\{(a_k, b_k), k = 1, 2, \dots, n\}$ be a finite family of disjoint open subintervals of $\mathbb{I}$ and $\{B_k, k = 1, 2, \dots, n\}$ a family of copulas. Then the *ordinal sum* B of $\{B_k, k = 1, 2, \dots, n\}$ with respect to $\{(a_k, b_k), k = 1, 2, \dots, n\}$ is a copula defined by

$$B(u, v) = \begin{cases} a_k + (b_k - a_k)B_k(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}); & (u, v) \in [a_k, b_k]^2, k = 1, 2, \dots, n, \\ \min\{u, v\}; & \text{otherwise}, \end{cases}$$

(see [17, Section 3.2.2]). The Spearman's rho of the ordinal sum B equals

$$\rho(B) = 1 - \sum_{k=1}^{n} (b_k - a_k)^3 (1 - \rho(B_k)).$$

Example 12. Let n be a positive integer and let C_n be a shuffle of M

$$C_n = M(2n, (\tfrac{1}{2n}, \tfrac{2}{2n}, \dots, \tfrac{2n-1}{2n}), (2, 1, 4, 3, \dots, 2n, 2n-1), (1, 1, \dots, 1)).$$

The scatterplot and 3D graph of copula C_3 is shown in Fig. 6. We have

$$\delta_{C_n}(u) = \begin{cases} \frac{2k-2}{2n}; & u \in [\frac{2k-2}{2n}, \frac{2k-1}{2n}], k = 1, 2, \dots, n, \\ 2u - \frac{2k}{2n}; & u \in [\frac{2k-1}{2n}, \frac{2k}{2n}], k = 1, 2, \dots, n, \end{cases}$$

so that

$$\phi(C_n) = 1 - \frac{3}{2n}.$$

The copula C_n is an ordinal sum of n copies of the copula C_1, each of them is of the size $\frac{1}{n}$. Since

$$C_1(u,v)=\begin{cases}0; & 0\le u\le \frac{1}{2}, 0\le v\le \frac{1}{2},\\ u+v-1; & \frac{1}{2}\le u\le 1, \frac{1}{2}\le v\le 1,\\ v-\frac{1}{2}; & 0\le u\le \frac{1}{2}, \frac{1}{2}\le v\le u+\frac{1}{2},\\ u; & \frac{1}{2}\le u\le 1, u+\frac{1}{2}\le v\le 1,\\ u-\frac{1}{2}; & 0\le v\le \frac{1}{2}, \frac{1}{2}\le u\le v+\frac{1}{2},\\ v; & 0\le v\le \frac{1}{2}, v+\frac{1}{2}\le u\le 1,\end{cases}$$

we have $\rho(C_1)=-\frac{1}{2}$ and

$$\rho(C_n)=1-n\cdot\frac{1-\rho(C_1)}{n^3}=1-\frac{3}{2n^2}.$$

Notice that $\rho(C_n)=1-\frac{2}{3}(1-\phi(C_n))^2$, so the point $(\phi(C_n),\rho(C_n))$ lies on the curve $r=1-\frac{2}{3}(1-f)^2$.

In next proposition we show that there is no copula C, such that $\phi(C)=0$ and the point $(\phi(C),\rho(C))$ lies on the curve $r=1-\frac{2}{3}(1-f)^2$. A similar result could be obtained also for some other values of $\phi(C)$.

Proposition 13. *Suppose that $\phi(C)=0$ for some copula $C\in\mathcal{C}$. Then $\rho(C)<\frac{1}{3}$.*

Proof. We will actually prove that $\rho(C)\le\frac{1}{3}-\frac{12}{1{,}000{,}000}$. Suppose that δ is a symmetric differentiable diagonal with continuous derivative that satisfies (10), $\alpha,g,h:\mathbb{I}\to\mathbb{I}$ auxiliary functions defined by (11), and K_δ diagonal copula. Furthermore suppose that $\phi(K_\delta)=0$, so that $\alpha(1)=\frac{1}{3}$. Let $\varepsilon=\frac{1}{100}$ and denote by $b=\delta(\frac{1}{3}-\varepsilon)$. We will first show that $b\ge\frac{1}{50}$. To this end we may assume that $b\le\frac{1}{6}-2\varepsilon$.

Since δ is increasing, we have $\delta(u)\le b$ for any $u\le\frac{1}{3}-\varepsilon$, and since it is 2-Lipschitz we have $\delta(u)\le 2u-\frac{2}{3}+2\varepsilon+b$ for any $u\ge\frac{1}{3}-\varepsilon$. Since δ is a symmetric diagonal we have $\delta(\frac{2}{3}+\varepsilon)=b+\frac{1}{3}+2\varepsilon$. We can derive similar estimates as above using the point $\frac{2}{3}+\varepsilon$. It follows that for any $u\in\mathbb{I}$

$$\delta(u)\le\delta_1(u)=\min\Big\{u,\max\{b,2u-\frac{2}{3}+2\varepsilon+b\},\max\{b+\frac{1}{3}+2\varepsilon,2u+b-1\}\Big\}.$$

Since $b\le\frac{1}{6}-2\varepsilon$, we have

$$\alpha(1)=\frac{1}{3}\le\int_0^1\delta_1(u)du=\frac{1}{36}(11+36b-36b^2+24\varepsilon+72\varepsilon^2),$$

hence

$$b\ge\frac{1}{2}-\frac{1}{3}\sqrt{2+6\varepsilon+18\varepsilon^2}\approx 0.0214\ge\frac{1}{50}.$$

For any $u\le\frac{1}{50}$ we now have

$$\delta^{-1}(u)\le\delta^{-1}(\tfrac{1}{50})\le\delta^{-1}(b)=\tfrac{1}{3}-\varepsilon,$$

and since $\delta^{-1}(1-u)\le 1$ it follows that

$$\tfrac{4}{3}-\delta^{-1}(u)-\delta^{-1}(1-u)\ge\varepsilon.$$

Thus

$$\int_0^1\left(\frac{4}{3}-\delta^{-1}(u)-\delta^{-1}(1-u)\right)^2du\ge\int_0^{1/50}\left(\frac{4}{3}-\delta^{-1}(u)-\delta^{-1}(1-u)\right)^2du\ge\frac{1}{50}\varepsilon^2.$$

It follows that

$$\begin{aligned}\int_0^1\big(\delta^{-1}(u)+\delta^{-1}(1-u)\big)^2du&=\int_0^1\left(\frac{4}{3}-\delta^{-1}(u)-\delta^{-1}(1-u)\right)^2du\\&\quad+\frac{8}{3}\int_0^1\big(\delta^{-1}(u)+\delta^{-1}(1-u)\big)du-\frac{16}{9}\\&\ge\frac{1}{50}\varepsilon^2+\frac{16}{3}(1-\alpha(1))-\frac{16}{9}\\&=\frac{1}{50}\varepsilon^2+\frac{16}{9}\end{aligned}$$

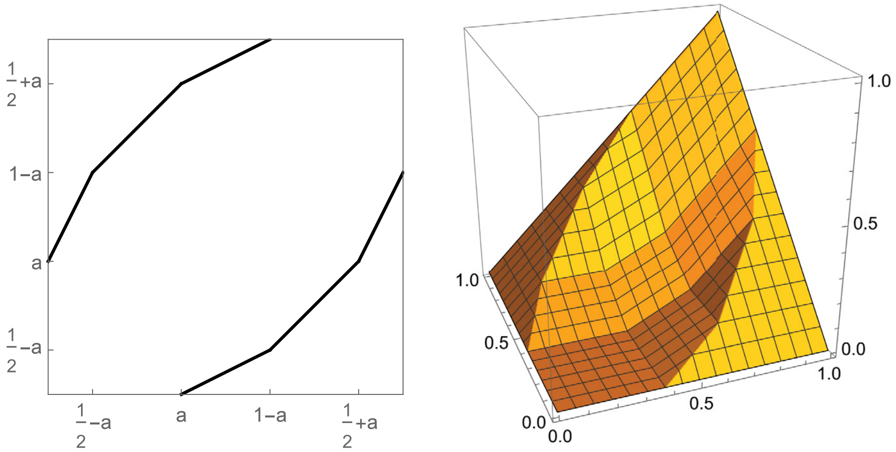

Fig. 7. The mass distribution (left) and the 3D graph (right) of the copula K_{δ_a} from Example 14.

by Lemma 9. Furthermore, by the same lemma we have

$$I = \int_0^1 \int_0^1 K_\delta(u, v)dudv = \frac{11}{6} - 2\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))^2 du$$
$$\leq \frac{11}{6} - \frac{2}{3} - \frac{1}{2}\left(\frac{1}{50}\varepsilon^2 + \frac{16}{9}\right) = \frac{5}{18} - \frac{1}{100}\varepsilon^2.$$

It follows that

$$\rho(K_\delta) = 12I - 3 \leq \frac{1}{3} - \frac{12}{100}\varepsilon^2 = \frac{1}{3} - \frac{12}{1{,}000{,}000}.$$

Finally, a similar argument as in the proof of Theorem 11 shows that for any copula with $\phi(C) = 0$ we have $\rho(C) \leq \frac{1}{3} - \frac{12}{1{,}000{,}000}$. □

Nevertheless, for any value of $\phi(C)$ we can come close to the upper bound proved in Theorem 11 as the next example and proposition demonstrate.

Example 14. Let $a \in [\frac{1}{4}, \frac{1}{2}]$ and let δ_a be a diagonal

$$\delta_a(u) = \begin{cases} 0; & u \leq a, \\ u - a; & a \leq u \leq 1 - a, \\ 2u - 1; & 1 - a \leq u \leq 1. \end{cases}$$

The diagonal copula belonging to δ_a is

$$K_{\delta_a}(u, v) = \begin{cases} 0; & 0 \leq u \leq a, 0 \leq v \leq a, \\ \frac{v-a}{2}; & a \leq v \leq 1-a, \frac{v-a}{2} \leq u \leq a, \\ \frac{u-a}{2}; & a \leq u \leq 1-a, \frac{u-a}{2} \leq v \leq a, \\ v - \frac{1}{2}; & \frac{1}{2} - a \leq u \leq a, 1-a \leq v \leq u + \frac{1}{2}, \\ \frac{u+v}{2} - a; & a \leq u \leq 1-a, a \leq v \leq 1-a, \\ u - \frac{1}{2}; & \frac{1}{2} - a \leq v \leq a, 1-a \leq u \leq v + \frac{1}{2}, \\ \frac{u+2v-a-1}{2}; & a \leq u \leq 1-a, 1-a \leq v \leq \frac{u+a+1}{2}, \\ \frac{2u+v-a-1}{2}; & a \leq v \leq 1-a, 1-a \leq u \leq \frac{v+a+1}{2}, \\ u + v - 1; & 1-a \leq u \leq 1, 1-a \leq v \leq 1, \\ u; & \min\{2u + a, u + \frac{1}{2}, \frac{u+a+1}{2}\} \leq v \leq 1 \\ v; & \min\{2v + a, v + \frac{1}{2}, \frac{v+a+1}{2}\} \leq u \leq 1. \end{cases}$$

The scatterplot and the 3D graph of copula K_{δ_a} is shown in Fig. 7.

It follows that

$$\phi(K_{\delta_a}) = 6a^2 - 6a + 1 \text{ and } \rho(K_{\delta_a}) = 8a^3 - 6a + \tfrac{3}{2},$$

so that

$$\rho(K_{\delta_a}) = -\frac{1}{2} + (1 + 2\phi(K_{\delta_a})) - \frac{\sqrt{3}}{9}(1 + 2\phi(K_{\delta_a}))^{3/2}.$$

Note that $\phi(K_{\delta_a}) \in [-\frac{1}{2}, -\frac{1}{8}]$ for $a \in [\frac{1}{4}, \frac{1}{2}]$. The point $(\phi(K_{\delta_a}), \rho(K_{\delta_a}))$ lies strictly below the curve $r = 1 - \frac{2}{3}(1-f)^2$ for any $a \in [\frac{1}{4}, \frac{1}{2})$.

Let $r : [-\frac{1}{2}, 1] \to [-1, 1]$ be a function defined by

$$r(x) = \begin{cases} 2x + \frac{1}{2} - \frac{\sqrt{3}}{9}(1+2x)^{3/2}; & x \in [-\frac{1}{2}, -\frac{1}{8}], \\ \frac{4}{3}x + \frac{7}{24}; & x \in [-\frac{1}{8}, \frac{1}{4}], \\ \dfrac{2n+1}{n^2+n}x + \dfrac{2n^2-2n+1}{2(n^2+n)}; & x \in [1 - \frac{3}{2n}, 1 - \frac{3}{2(n+1)}] \text{ for } n = 2, 3, \dots, \\ 1; & x = 1. \end{cases} \tag{13}$$

Proposition 15. *For any point $(x, r(x))$ on the graph of function r there exist a copula C, such that $\phi(C) = x$ and $\rho(C) = r(x)$.*

Proof. Example 14 shows that for $x \in [-\frac{1}{2}, -\frac{1}{8}]$ any point on the graph of function r is attained by copula K_{δ_a}. Note that $r(1 - \frac{3}{2n}) = 1 - \frac{3}{2n^2}$, so for $x = 1 - \frac{3}{2n}$ the point on the graph of function r is attained by copula C_n from Example 12. In the interval $[-\frac{1}{8}, \frac{1}{4}]$ the points on the graph of function r are attained by convex combinations of copulas $K_{\delta_{1/4}}$ and C_2 and in the intervals $[1 - \frac{3}{2n}, 1 - \frac{3}{2(n+1)}]$ by convex combinations of copulas C_n and C_{n+1}. □

7. The exact region determined by ϕ and ρ

We can now collect our findings in the following theorem.

Theorem 16. *The exact region determined by Spearman's rho and Spearman's footrule of all points $\{(\phi(C), \rho(C)) \in [-\frac{1}{2}, 1] \times [-1, 1]; C \in \mathcal{C}\}$ is given by*

$$\frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} - 1 \le \rho(C) \le s(\phi(C))$$

where $s : [-\frac{1}{2}, 1] \to [-1, 1]$ is a concave function satisfying

$$r(x) \le s(x) \le 1 - \frac{2}{3}(1-x)^2$$

and r is the function defined by (13).

Proof. The assertion follows directly from Theorems 8, 11, and Proposition 15. The function s is concave since the exact region is convex. □

Note that the role of ϕ and ρ can be exchanged, so from the theorem one can derive the exact upper bound for $\phi(C)$ in terms of $\rho(C)$ and a tight estimate for the lower bound.

Fig. 8 shows the exact region determined by Spearman's rho and Spearman's footrule. The graph of function $r(x)$ is shown full, and the graph of function $1-\frac{2}{3}(1-x)^2$ is dashed. Note that the exact region determined by Spearman's footrule and Spearman's rho is similar in shape to the exact region determined by Sperman's rho and Kendall's tau, i.e., the upper bound seems to be a piecewise function with finer and finer pieces. However, the exact region determined by Sperman's rho and Kendall's tau is not convex while in our case the region is convex.

Our last proposition estimates κ-similarity measure $\kappa sm(\phi, \rho)$.

Proposition 17. *The (ϕ, ρ)-similarity measure between Spearman's footrule and Spearman's rho satisfies*

$$0.65 = \frac{13}{20} \le \kappa sm(\phi, \rho) \le \frac{121}{64} - \frac{\pi^2}{8} \approx 0.6569.$$

Proof. We have

$$\kappa sm(\phi, \rho) = 1 - \frac{A(\phi, \rho)}{3}$$

from the definition and

$$A(\phi, \rho) \le \int_{-1/2}^{1} \left(1 - \frac{2}{3}(1-x)^2\right) dx - \int_{-1/2}^{1} \left(\frac{2}{9}\sqrt{3}(1+2x)^{3/2} - 1\right) dx = \frac{3}{4} + \frac{3}{10} = \frac{21}{20},$$

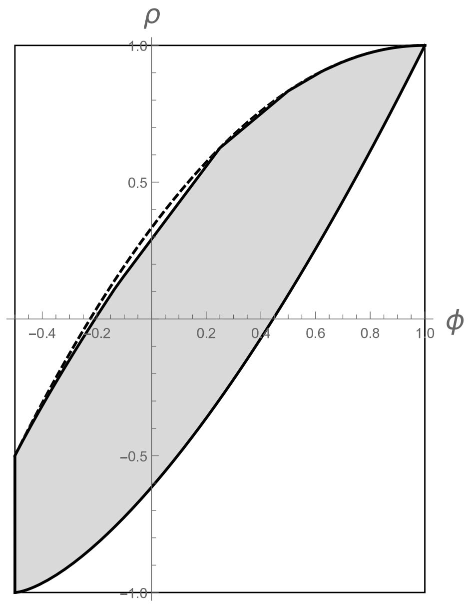

Fig. 8. The exact region determined by Spearman's rho and Spearman's footrule.

so the lower bound follows. On the other hand,

$$
\begin{aligned}
A(\phi, \rho) &\geq \int_{-1/2}^{1} r(x)dx - \int_{-1/2}^{1} \left(\frac{2}{9}\sqrt{3}(1+2x)^{3/2} - 1\right) dx \\
&= \int_{-1/2}^{-1/8} \left(2x + \frac{1}{2} - \frac{\sqrt{3}}{9}(1+2x)^{3/2}\right) dx + \frac{9}{64} \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{2}\left(1 - \frac{3}{2n^2} + 1 - \frac{3}{2(n+1)^2}\right)\left(\frac{3}{2n} - \frac{3}{2(n+1)}\right) + \frac{3}{10} \\
&= -\frac{21}{320} + \frac{9}{64} + \frac{3\pi^2}{8} - \frac{195}{64} + \frac{3}{10} = \frac{3\pi^2}{8} - \frac{171}{64}
\end{aligned}
$$

and the upper bound follows. □

Note that the value $\kappa sm(\rho, \phi) \approx 0.65 < \kappa sm(\rho, \tau) = 0.7114$ suggests that on average the value of Spearman's footrule gives less information about the value of Spearman's rho than the value of Kendall's tau does.

Data availability

No data was used for the research described in the article

Appendix. Some proofs

A.1. Proof of Lemma 3

Let us first prove (b). Note that

$$u_{n-i} = \sum_{k=1}^{n-i}(u_k - u_{k-1}) = \sum_{k=1}^{n-i}(u_{n-k+1} - u_{n-k})$$

by (S4) in Definition 2. By introducing a new index $j = n - k + 1$ we get

$$u_{n-i} = \sum_{j=i+1}^{n}(u_j - u_{j-1}) = u_n - u_i = 1 - u_i.$$

The equality $u_{\frac{n}{2}} = \frac{1}{2}$ follows easily by taking $i = \frac{n}{2}$.

To prove (c) let $v_0 = 0$ and $v_n = 1$ and let $v_1, v_2, \dots, v_{n-1}$ be splitting points on the v-axis such that the mass of C is concentrated on squares $J_i \times J^{\pi^{-1}}_{\pi(i)} = [u_{i-1}, u_i] \times [v_{\pi(i)-1} \times v_{\pi(i)}]$. This means that $v_{\pi(i)} - v_{\pi(i)-1} = u_i - u_{i-1}$ for all i, so that $v_j = v_{j-1} + u_{\pi^{-1}(j)} - u_{\pi^{-1}(j)-1}$. By induction it follows that

$$v_j = \sum_{k=1}^{j} (u_{\pi^{-1}(k)} - u_{\pi^{-1}(k)-1}).$$

Since C is doubly symmetric shuffle of M we have $\pi^{-1} = \pi$, hence

$$v_j = \sum_{k=1}^{j} (u_{\pi(k)} - u_{\pi(k)-1}) = \sum_{k=1}^{j} (u_k - u_{k-1}) = u_j,$$

for all $j = 1, 2, \dots, n-1$, where the second equation follows from (S4). Clearly also $v_0 = u_0$ and $v_n = u_n$. This implies that $J^{\pi^{-1}}_{\pi(i)} = [v_{\pi(i)-1} \times v_{\pi(i)}] = [u_{\pi(i)-1} \times u_{\pi(i)}] = J_{\pi(i)}$ for all $i = 1, 2, \dots, n$ and $V_C(J_i \times J_{\pi(i)}) = u_i - u_{i-1}$. Now $V_C(J_{\pi(i)} \times J_i) = u_{\pi(i)} - u_{\pi(i)-1} = u_i - u_{i-1}$ follows from the above and (S4). Similarly, we prove for the other two volumes which proves (c). Thus, to prove (a) it suffices to show that the set of squares $J_i \times J_{\pi(i)}$ with $\omega(i) = 1$ is invariant under reflection with respect to both diagonals and the same holds for the set of squares $J_i \times J_{\pi(i)}$ with $\omega(i) = -1$. The reflection with respect to the main diagonal reflects the square $J_i \times J_{\pi(i)} = [u_{i-1}, u_i] \times [u_{\pi(i)-1}, u_{\pi(i)}]$ onto $[u_{\pi(i)-1}, u_{\pi(i)}] \times [u_{i-1}, u_i] = J_{\pi(i)} \times J_i = J_{i'} \times J_{\pi(i')}$, where $i' = \pi(i)$ by (S2). Since $\omega(i) = \omega(\pi(i))$ by (S3), the original and the reflected square have the same value of ω. The reflection with respect to the opposite diagonal reflects the square $J_i \times J_{\pi(i)} = [u_{i-1}, u_i] \times [u_{\pi(i)-1}, u_{\pi(i)}]$ onto $[1 - u_{\pi(i)}, 1 - u_{\pi(i)-1}] \times [1 - u_i, 1 - u_{i-1}] = [u_{n-\pi(i)}, u_{n-\pi(i)+1}] \times [u_{n-i}, u_{n-i+1}] = J_{n-\pi(i)+1} \times J_{n-i+1}$ by (b), and $J_{n-\pi(i)+1} \times J_{n-i+1} = J_{\pi(n-i+1)} \times J_{n-i+1} = J_{i''} \times J_{\pi(i'')}$ by (S2), where $i'' = \pi(n-i+1)$. Since $\omega(i) = \omega(n-i+1) = \omega(\pi(n-i+1)) = \omega(i'')$ by (S3), the original and the reflected square have the same value of ω. This finishes the proof of (a). □

A.2. Proof of Lemma 4

Suppose C is a doubly symmetric copula. Let m be an even positive integer and $n = m^2$, so that n is even as well. Let $J = \{J_1, J_2, \dots, J_n\}$ be a partition of $\mathbb{I}$ such that the intervals $J_1, J_2, \dots, J_n$ are ordered from left to right and the length of J_i is equal to $w_i = V_C([\frac{k-1}{m}, \frac{k}{m}] \times [\frac{j-1}{m}, \frac{j}{m}])$, where $i = m(j-1) + k$, $j = 1, 2, \dots, m$, $k = 1, 2, \dots, m$, and let $\pi \in S_n$ be a permutation given by $\pi(m(j-1) + k) = m(k-1) + j$ for all $j, k \in \{1, 2, \dots, m\}$. Finally, let ω be constantly equal to 1. We claim that $C_m = M(n, J, \pi, \omega)$ is a doubly symmetric shuffle. For the rest of the proof we let $i = m(j-1) + k$, where $j, k \in \{1, 2, \dots, m\}$. Clearly, $\pi^2 = \text{id}$. Furthermore,

$$\begin{aligned}\pi(n-i+1) &= \pi(m^2 - m(j-1) - k + 1) = \pi(m(m-j) + (m-k+1)) \\ &= m(m-k) + (m-j+1) = m^2 - m(k-1) - j + 1 = n - \pi(i) + 1\end{aligned}$$

for all $i = 1, 2, \dots, n$. Since copula C is doubly symmetric, its mass is symmetric with respect to both diagonals. The symmetry with respect to the main diagonal implies

$$w_i = V_C([\tfrac{k-1}{m}, \tfrac{k}{m}] \times [\tfrac{j-1}{m}, \tfrac{j}{m}]) = V_C([\tfrac{j-1}{m}, \tfrac{j}{m}] \times [\tfrac{k-1}{m}, \tfrac{k}{m}]) = w_{m(k-1)+j} = w_{\pi(i)} \tag{14}$$

for all $i \in \{1, 2, \dots, n\}$, while the symmetry with respect to the opposite diagonal implies

$$\begin{aligned} w_i &= V_C([\tfrac{k-1}{m}, \tfrac{k}{m}] \times [\tfrac{j-1}{m}, \tfrac{j}{m}]) = V_C([1 - \tfrac{j}{m}, 1 - \tfrac{j-1}{m}] \times [1 - \tfrac{k}{m}, 1 - \tfrac{k-1}{m}]) \\ &= V_C([\tfrac{m-j}{m}, \tfrac{m-j+1}{m}] \times [\tfrac{m-k}{m}, \tfrac{m-k+1}{m}]) = w_{m(m-k)+m-j+1} = w_{m^2-m(k-1)-j+1} = w_{n-\pi(i)+1} \end{aligned} \tag{15}$$

for all $i \in \{1, 2, \dots, n\}$. By definition of partition J, its splitting points are $u_0 = 0$ and $u_i = \sum_{k=1}^{i} w_k$ for all $i = 1, 2, \dots, n$, where $u_n = \sum_{k=1}^{n} w_k = V_C(\mathbb{I} \times \mathbb{I}) = 1$. Together with equalities (14) and (15) this implies

$$u_i - u_{i-1} = w_i = w_{\pi(i)} = u_{\pi(i)} - u_{\pi(i)-1}$$

and

$$u_i - u_{i-1} = w_i = w_{\pi(i)} = w_{n-\pi^2(i)+1} = w_{n-i+1} = u_{n-i+1} - u_{n-i}.$$

We have thus shown that C_m is a straight doubly symmetric shuffle of M. To finish the proof we note that as m tends to ∞, the copula C_m converges uniformly to C as proved in [42, Theorem 3.1] (cf. also [17, §3.2.3]). □

A.3. Proof of Proposition 5

Let $\delta_C(u) = C(u, u)$ be the diagonal of C and define

$$\alpha_C(u) = \int_0^u \delta_C(t)dt.$$

Let $\phi(C) = p$ for some $p \in [-\frac{1}{2}, 1]$ and $u_0 = \frac{1}{2}(1 - \frac{1}{\sqrt{3}}\sqrt{2p+1}) \in [0, \frac{1}{2}]$. We will prove that the following holds:

(a)

$$C(u,v)=\begin{cases}\delta_C(u); & 0\le u\le \frac{1}{2}, u\le v\le 1-u,\\ \delta_C(v); & 0\le v\le \frac{1}{2}, v\le u\le 1-v,\\ \delta_C(u)+v-u; & \frac{1}{2}\le u\le 1, 1-u\le v\le u,\\ \delta_C(v)+u-v; & \frac{1}{2}\le v\le 1, 1-v\le u\le v,\end{cases}$$

and in particular $C=C^t$.

(b)

$$\int_0^1\int_0^1 C(u,v)dudv = 4\int_0^{\frac{1}{2}}\alpha_C(u)du - 4\int_{\frac{1}{2}}^1 \alpha_C(u)du + 2\alpha_C(1)-\frac{1}{6}.$$

(c) The function δ_C is increasing and 1-Lipschitz on the interval $[0,\frac{1}{2}]$.

(d) For every $u\in[0,\frac{1}{2}]$ we have

$$\delta_C(1-u)=1-2u+\delta_C(u),$$

and in particular, $C=\widehat{C}$.

(e) For every $u\in[0,\frac{1}{2}]$ we have

$$\alpha_C(1-u)=2\alpha_C(\tfrac{1}{2})-\alpha_C(u)+\frac{(1-2u)^2}{4}.$$

(f) $\alpha_C(1)=\frac{p+2}{6}$ and $\alpha_C(\frac{1}{2})=\frac{2p+1}{24}$.

(g)

$$\int_0^1\int_0^1 C(u,v)dudv = 8\int_0^{\frac{1}{2}}\alpha_C(u)du+\frac{1}{6}.$$

(h) For every $u\in[0,\frac{1}{2}]$ we have

$$\alpha_C(u)\ge\alpha_0(u):=\begin{cases}0; & 0\le u\le u_0,\\ \frac{1}{2}(u-u_0)^2; & u_0<u\le\frac{1}{2}.\end{cases}$$

From the condition $P(U=V)+P(U=1-V)=1$ it follows

$$P\left(U\le\frac{1}{2}, U<V<1-U\right)=0, \qquad P\left(U\ge\frac{1}{2}, 1-U<V<U\right)=0,$$
$$P\left(Y\le\frac{1}{2}, V<U<1-V\right)=0, \qquad P\left(Y\ge\frac{1}{2}, 1-V<U<V\right)=0.$$

If $u\in[0,\frac{1}{2}]$ we have for any $v\in[u,1-u]$ that $C(u,v)=C(u,u)=\delta_C(u)$. If $u\in[\frac{1}{2},1]$ we have for any $v\in[1-u,u]$ that $C(u,v)=C(u,u)+v-u=\delta_C(u)+v-u$. Similar equalities hold if we interchange the roles of u and v, so the copula C is symmetric, which proves (a).

To prove (b) we compute

$$\begin{aligned}\int_0^1\int_0^1 C(u,v)dudv &= 2\int_0^{\frac{1}{2}}\left(\int_u^{1-u}C(u,v)dv\right)du+2\int_{\frac{1}{2}}^1\left(\int_{1-u}^{u}C(u,v)dv\right)du\\ &= 2\int_0^{\frac{1}{2}}\left(\int_u^{1-u}\delta_C(u)dv\right)du+2\int_{\frac{1}{2}}^1\left(\int_{1-u}^{u}(\delta_C(u)+v-u)dv\right)du\\ &= 2\int_0^{\frac{1}{2}}\delta_C(u)(1-2u)du+2\int_{\frac{1}{2}}^1\delta_C(u)(2u-1)du-\frac{1}{6}\\ &= 2\alpha_C(\tfrac{1}{2})-4\int_0^{\frac{1}{2}}u\delta_C(u)du-2\alpha_C(1)+2\alpha_C(\tfrac{1}{2})+4\int_{\frac{1}{2}}^1 u\delta_C(u)du-\frac{1}{6}.\end{aligned}$$

Using integration by parts we obtain

$$\int_0^{\frac{1}{2}}u\delta_C(u)du=\tfrac{1}{2}\alpha_C(\tfrac{1}{2})-\int_0^{\frac{1}{2}}\alpha_C(u)du$$

and

$$\int_{\frac{1}{2}}^{1} u\delta_C(u)du = \alpha_C(1) - \tfrac{1}{2}\alpha_C(\tfrac{1}{2}) - \int_{\frac{1}{2}}^{1} \alpha_C(u)du.$$

Plugging into the previous equation we obtain (b).

The diagonal of a copula is obviously increasing, so to prove (c) suppose $0 \le u \le v \le \frac{1}{2}$. Since $C(u, v) = \delta_C(u)$ in this case, we have $\delta_C(v) - \delta_C(u) = C(v, v) - C(u, v) \le v - u$. If $u \in [0, \frac{1}{2}]$ we obtain $\delta_C(1-u) = 1 - 2u + \delta_C(u)$ from $C(1-u, 1-u) = C(1-u, u) + 1 - 2u = C(u, u) + 1 - 2u$. If $u \in [0, \frac{1}{2}]$ and $v \in [u, 1-u]$, we have

$$\begin{aligned}\widehat{C}(u, v) &= u + v - 1 + C(1-u, 1-v) = u + v - 1 + \delta_C(1-u) + (1-v) - (1-u) \\ &= \delta_C(1-u) - 1 + 2u = \delta_C(u) = C(u, v).\end{aligned}$$

Similarly other cases are treated to prove (d).

To prove (e) assume $u \in [0, \frac{1}{2}]$ and compute

$$\begin{aligned}\alpha_C(1-u) &= \int_0^{1-u} \delta_C(t)dt = \alpha_C(\tfrac{1}{2}) + \int_{\frac{1}{2}}^{1-u} \delta_C(t)dt = \alpha_C(\tfrac{1}{2}) + \int_{\frac{1}{2}}^{1-u} (2t - 1 + \delta_C(1-t))dt \\ &= \alpha_C(\tfrac{1}{2}) + \frac{(1-2u)^2}{4} + \int_{\frac{1}{2}}^{1-u} \delta_C(1-t)dt = \alpha_C(\tfrac{1}{2}) + \frac{(1-2u)^2}{4} - \int_{\frac{1}{2}}^{u} \delta_C(t)dt \\ &= 2\alpha_C(\tfrac{1}{2}) + \frac{(1-2u)^2}{4} - \int_0^u \delta_C(t)dt = 2\alpha_C(\tfrac{1}{2}) + \frac{(1-2u)^2}{4} - \alpha_C(u).\end{aligned}$$

The property $\alpha_C(1) = \frac{p+2}{6}$ is immediate from the definitions of $\phi(C)$ and α_C, and we get $\alpha_C(\frac{1}{2}) = \frac{2p+1}{24}$ by plugging $u = 0$ into (e).

To prove (g) we apply (e) and (b):

$$\begin{aligned}\int_0^1\int_0^1 C(u, v)dudv &= 4\int_0^{\frac{1}{2}} \alpha_C(u)du - 4\int_{\frac{1}{2}}^1 \alpha_C(u)du + 2\alpha_C(1) - \frac{1}{6} \\ &= 4\int_0^{\frac{1}{2}} \alpha_C(u)du - 4\int_{\frac{1}{2}}^1 \left(2\alpha_C(\tfrac{1}{2}) - \alpha_C(1-u) + \frac{(1-2u)^2}{4}\right)du + 2\alpha_C(1) - \frac{1}{6} \\ &= 4\int_0^{\frac{1}{2}} \alpha_C(u)du + 4\int_{\frac{1}{2}}^1 \alpha_C(1-u)du - 4\alpha_C(\tfrac{1}{2}) + 2\alpha_C(1) - \frac{1}{3} \\ &= 8\int_0^{\frac{1}{2}} \alpha_C(u)du - 4\alpha_C(\tfrac{1}{2}) + 2\left(2\alpha_C(\tfrac{1}{2}) + \frac{1}{4}\right) - \frac{1}{3} \\ &= 8\int_0^{\frac{1}{2}} \alpha_C(u)du + \frac{1}{6}.\end{aligned}$$

We will prove (h) by contradiction. First notice that $\alpha_C(u) \ge 0$ and $\alpha_C(\frac{1}{2}) = \alpha_0(\frac{1}{2}) = \frac{2p+1}{24}$ by the definition of u_0. So suppose that $\alpha_C(u_1) < \alpha_0(u_1) = \frac{1}{2}(u_1 - u_0)^2$ for some $u_1 \in (u_0, \frac{1}{2})$. We will first prove that $\delta_1 := \delta_C(u_1) < u_1 - u_0$. Since δ_C is 1-Lipschitz, we have for every $t \in [u_0, u_1]$ that $\delta_C(t) \ge \delta_1 + t - u_1$. Thus

$$\alpha_C(u_1) = \int_0^{u_1} \delta_C(t)dt \ge \int_{u_0}^{u_1} \delta_C(t)dt \ge \int_{u_0}^{u_1} (\delta_1 + t - u_1)dt = \delta_1(u_1 - u_0) - \frac{1}{2}(u_1 - u_0)^2$$

and

$$\delta_1 \le \frac{\alpha_C(u_1) + \frac{1}{2}(u_1 - u_0)^2}{u_1 - u_0} < \frac{\frac{1}{2}(u_1 - u_0)^2 + \frac{1}{2}(u_1 - u_0)^2}{u_1 - u_0} = u_1 - u_0.$$

Now we have for every $t \in [u_1, \frac{1}{2}]$ that $\delta_C(t) \le \delta_1 + t - u_1 < t - u_0$. Hence

$$\begin{aligned}\alpha_C(\tfrac{1}{2}) &= \alpha_C(u_1) + \int_{u_1}^{\frac{1}{2}} \delta_C(t)dt \\ &< \alpha_C(u_1) + \int_{u_1}^{\frac{1}{2}} (t - u_0)dt\end{aligned}$$

$$\begin{aligned}&= \alpha_C(u_1) + \tfrac{1}{8} - \tfrac{1}{2}u_1^2 - \tfrac{1}{2}u_0(1-2u_1)\\&< \tfrac{1}{2}(u_1-u_0)^2 + \tfrac{1}{8} - \tfrac{1}{2}u_1^2 - \tfrac{1}{2}u_0(1-2u_1)\\&= \tfrac{1}{2}(\tfrac{1}{2}-u_0)^2 = \alpha_0(\tfrac{1}{2}),\end{aligned}$$

a contradiction.

Finally, using the definition of ρ, (g), and (h), we have

$$\begin{aligned}\rho(C) &= 12\int_0^1\int_0^1 C(u,v)dudv - 3 = 96\int_0^{\frac{1}{2}}\alpha_C(u)du - 1\\&\geq 96\int_0^{\frac{1}{2}}\alpha_0(u)du - 1 = 48\int_{u_0}^{\frac{1}{2}}(u-u_0)^2du - 1\\&= 2(1-2u_0)^3 - 1 = \frac{2}{9}\sqrt{3}(1+2p)^{3/2} - 1 = \frac{2}{9}\sqrt{3}(1+2\phi(C))^{3/2} - 1. \quad \square\end{aligned}$$

A.4. Proof of Lemma 9

We will prove the following:

(a)
$$g^{-1}(u) = 1 - \delta^{-1}(1-u) \quad \text{for all } u \in \mathbb{I}.$$

(b)
$$\int_0^1 u\delta(u)du = \frac{1}{2}\alpha(1) + \frac{1}{12}.$$

(c)
$$\int_0^1 \alpha(u)du = \frac{1}{2}\alpha(1) - \frac{1}{12}.$$

(d)
$$\int_0^1 \delta^{-1}(u)du = 1 - \alpha(1).$$

(e)
$$\int_0^1 (\delta^{-1}(u))^2du = \frac{5}{6} - \alpha(1).$$

(f)
$$\int_0^1 \alpha(h(u))du = \alpha(1) - 1 + \int_0^1 (4u - \delta(u) - u\delta'(u))h(u)du.$$

(g)
$$\int_0^1\int_0^1 K_\delta(u,v)dudv = \int_0^1 g^{-1}(u)\delta^{-1}(u)du.$$

(h)
$$\int_0^1\int_0^1 K_\delta(u,v)dudv = \frac{11}{6} - 2\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u) + \delta^{-1}(u))^2du.$$

We first compute

$$\begin{aligned}g(1-\delta^{-1}(1-u)) &= 2(1-\delta^{-1}(1-u)) - \delta(1-\delta^{-1}(1-u))\\&= 2 - 2\delta^{-1}(1-u) - \big(1 - 2\delta^{-1}(1-u) + \delta(\delta^{-1}(1-u))\big)\end{aligned}$$

by equality (9), so that

$$g(1-\delta^{-1}(1-u)) = 2 - 2\delta^{-1}(1-u) - 1 + 2\delta^{-1}(1-u) - (1-u) = u,$$

which proves (a).

Next we use equality (9) in the integral

$$\int_0^1 u\delta(u)du = \int_0^1 u(2u-1+\delta(1-u))du = \frac{1}{6} + \int_0^1 u\delta(1-u)du.$$

We introduce a new variable $t = 1-u$ to get

$$\begin{aligned}\int_0^1 u\delta(u)du &= \frac{1}{6} + \int_1^0 (1-t)\delta(t)(-dt) = \frac{1}{6} + \int_0^1 \delta(t)dt - \int_0^1 t\delta(t)dt \\ &= \frac{1}{6} + \alpha(1) - \int_0^1 u\delta(u)du.\end{aligned}$$

We express the integral from the obtained equation to get (b).

To prove (c) we integrate by parts and use (b)

$$\int_0^1 \alpha(u)du = u\alpha(u)\Big|_0^1 - \int_0^1 u\delta(u)du = \alpha(1) - \left(\frac{1}{2}\alpha(1) + \frac{1}{12}\right) = \frac{1}{2}\alpha(1) - \frac{1}{12}.$$

To prove (d) we first introduce a new variable by $u = \delta(t)$ in the integral and then integrate by parts

$$\int_0^1 \delta^{-1}(u)du = \int_0^1 t\delta'(t)dt = t\delta(t)\Big|_0^1 - \int_0^1 \delta(t)dt = 1 - \alpha(1).$$

In a similar way we prove (e), in the final step we use (b)

$$\int_0^1 (\delta^{-1}(u))^2 du = \int_0^1 t^2\delta'(t)dt = t^2\delta(t)\Big|_0^1 - \int_0^1 2t\delta(t)dt = 1 - 2\left(\frac{1}{2}\alpha(1) + \frac{1}{12}\right) = \frac{5}{6} - \alpha(1).$$

To prove (f) we integrate by parts twice

$$\begin{aligned}\int_0^1 \alpha(h(u))du &= u\alpha(h(u))\Big|_0^1 - \int_0^1 u\alpha'(h(u))h'(u)du = \alpha(1) - \int_0^1 ug(u)h'(u)du \\ &= \alpha(1) - ug(u)h(u)\Big|_0^1 + \int_0^1 (g(u) + ug'(u))h(u)du \\ &= \alpha(1) - 1 + \int_0^1 (4u - \delta(u) - u\delta'(u))h(u)du.\end{aligned}$$

To prove (g) we first use the symmetry of copula K_δ and then Eq. (12)

$$\begin{aligned}I &= \int_0^1\int_0^1 K_\delta(u,v)dudv = 2\int_0^1 \left(\int_u^1 K_\delta(u,v)dv\right)du \\ &= 2\int_0^1 \left(\int_u^{h(u)} \frac{\delta(u)+\delta(v)}{2}dv + \int_{h(u)}^1 udv\right)du \\ &= \int_0^1 (\delta(u)(h(u)-u) + \alpha(h(u)) - \alpha(u))\,du + 2\int_0^1 u(1-h(u))du \\ &= \int_0^1 \delta(u)h(u)du - \int_0^1 u\delta(u)du + \int_0^1 \alpha(h(u))du - \int_0^1 \alpha(u)du - \int_0^1 2uh(u)du + 1.\end{aligned}$$

Next we use (b), (c), and (f) to get

$$\begin{aligned}I &= \int_0^1 \delta(u)h(u)du - \left(\frac{1}{2}\alpha(1) + \frac{1}{12}\right) + \alpha(1) - 1 + \int_0^1 (4u - \delta(u) - u\delta'(u))h(u)du \\ &\quad - \left(\frac{1}{2}\alpha(1) - \frac{1}{12}\right) - \int_0^1 2uh(u)du + 1 \\ &= \int_0^1 (2u - u\delta'(u))h(u)du = \int_0^1 ug'(u)h(u)du.\end{aligned}$$

Finally, we introduce a new variable $t = g(u)$ to get

$$I = \int_0^1 g^{-1}(t)h(g^{-1}(t))dt = \int_0^1 g^{-1}(t)\delta^{-1}(t)dt.$$

To prove (h) we first use (g), (a), and (d) to get

$$I = \int_0^1 g^{-1}(u)\delta^{-1}(u)du = \int_0^1 (1-\delta^{-1}(1-u))\delta^{-1}(u)du$$
$$= 1-\alpha(1) - \int_0^1 \delta^{-1}(1-u)\delta^{-1}(u)du.$$

Now

$$\delta^{-1}(1-u)\delta^{-1}(u) = \frac{1}{2}\left((\delta^{-1}(1-u)+\delta^{-1}(u))^2 - (\delta^{-1}(1-u))^2 - (\delta^{-1}(u))^2\right),$$

thus

$$I = 1-\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u)+\delta^{-1}(u))^2 du + \frac{1}{2}\int_0^1 (\delta^{-1}(1-u))^2 du + \frac{1}{2}\int_0^1 (\delta^{-1}(u))^2 du$$
$$= 1-\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u)+\delta^{-1}(u))^2 du + \frac{1}{2}\left(\frac{5}{6}-\alpha(1)\right) + \frac{1}{2}\left(\frac{5}{6}-\alpha(1)\right)$$
$$= \frac{11}{6} - 2\alpha(1) - \frac{1}{2}\int_0^1 (\delta^{-1}(1-u)+\delta^{-1}(u))^2 du$$

by (e). □

References

[1] Marco Scarsini, On measures of concordance, Stochastica 8 (3) (1984) 201–218, 796650.
[2] H.H. Edwards, M.D. Taylor, Characterizations of degree one bivariate measures of concordance, J. Multivariate Anal. 100 (8) (2009) 1777–1791, 2535386.
[3] Gregory A. Fredricks, Roger B. Nelsen, On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables, J. Statist. Plann. Inference 137 (7) (2007) 2143–2150, 2325421.
[4] Sebastian Fuchs, Klaus D. Schmidt, Bivariate copulas: Transformations, asymmetry and measures of concordance, Kybernetika (Prague) 50 (1) (2014) 109–125, 3195007.
[5] Damjana Kokol Bukovšek, Tomaž Košir, Blaž Mojškerc, Matjaž Omladič, Spearman's footrule and Gini's gamma: Local bounds for bivariate copulas and the exact region with respect to Blomqvist's beta, J. Comput. Appl. Math. 390 (2021) 23, 4205170.
[6] Eckhard Liebscher, Copula-based dependence measures, Depend. Model. 2 (1) (2014) 49–64.
[7] Roger B. Nelsen, José Juan Quesada-Molina, José Antonio Rodríguez-Lallena, Manuel Úbeda-Flores, Bounds on bivariate distribution functions with given margins and measures of association, Comm. Statist. Theory Methods 30 (6) (2001) 1155–1162, 1862170.
[8] Roger B. Nelsen, José Juan Quesada-Molina, José Antonio Rodríguez-Lallena, Manuel Úbeda-Flores, Distribution functions of copulas: A class of bivariate probability integral transforms, Statist. Probab. Lett. 54 (3) (2001) 277–282, 1857942.
[9] Javad Behboodian, Ali Dolati, Manuel Úbeda-Flores, A multivariate version of Gini's rank association coefficient, Statist. Papers 48 (2) (2007) 295–304, 2295817.
[10] Fabrizio Durante, Sebastian Fuchs, Reflection invariant copulas, Fuzzy Sets and Systems 354 (2019) 63–73, 3906654.
[11] M.D. Taylor, Multivariate measures of concordance for copulas and their marginals, Depend. Model. 4 (1) (2016) 224–236, 3555170.
[12] Manuel Úbeda-Flores, Multivariate versions of Blomqvist's beta and Spearman's footrule, Ann. Inst. Statist. Math. 57 (4) (2005) 781–788.
[13] Damjana Kokol Bukovšek, Blaž Mojškerc, On the exact region determined by Spearman's footrule and Gini's gamma, J. Comput. Appl. Math. 410 (2022) 13, 4395320.
[14] H.E. Daniels, Rank correlation and population models, J. R. Stat. Soc. Ser. B Stat. Methodol. 12 (1950) 171–181, 40629.
[15] J. Durbin, A. Stuart, Inversions and rank correlation coefficients, J. R. Stat. Soc. Ser. B Stat. Methodol. 13 (1951) 303–309, 47994.
[16] Manuela Schreyer, Roland Paulin, Wolfgang Trutschnig, On the exact region determined by Kendall's τ and Spearman's ρ, J. R. Stat. Soc. Ser. B Stat. Methodol. 79 (2) (2017) 613–633, 3611762.
[17] Roger B. Nelsen, An introduction to copulas, in: Springer Series in Statistics, second ed., Springer, New York, 2006, p. xiv+269, 2197664.
[18] Damjana Kokol Bukovšek, Nik Stopar, On the exact regions determined by Kendall's tau and other concordance measures, Mediterr. J. Math. 20 (3) (2023) 16, 4554107.
[19] Fabrizio Durante, Carlo Sempi, Principles of Copula Theory, CRC Press, Boca Raton, FL, 2016, p. xvi+315, 3443023.
[20] Roger B. Nelsen, Manuel Úbeda Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, C. R. Math. Acad. Sci. Paris 341 (9) (2005) 583–586, 2182439.
[21] Ali Dolati, Manuel Úbeda-Flores, On measures on multivariate concordance, JPSS J. Probab. Stat. Sci. 4 (2) (2006) 147–163, 2488161.
[22] M.D. Taylor, Multivariate measures of concordance, Ann. Inst. Statist. Math. 59 (4) (2007) 789–806, 2397737.
[23] Damjana Kokol Bukovšek, Tomaž Košir, Blaž Mojškerc, Matjaž Omladič, Relation between non-exchangeability and measures of concordance of copulas, J. Math. Anal. Appl. 487 (1) (2020) 123951, 26, 4066734.
[24] Roger B. Nelsen, Concordance and copulas: A survey, in: Distributions with Given Marginals and Statistical Modelling, Kluwer Acad. Publ., Dordrecht, 2002, pp. 169–177, 2058990.
[25] N. Kamnitui, C. Genest, P. Jaworski, W. Trutschnig, On the size of the class of bivariate extreme-value copulas with a fixed value of Spearman's rho or Kendall's tau, J. Math. Anal. Appl. 472 (1) (2019) 920–936, 3906404.
[26] Sarah Nogueira, Konstantinos Sechidis, Gavin Brown, On the use of Spearman's rho to measure the stability of feature rankings, in: Pattern Recognition and Image Analysis, in: Lecture Notes in Comput. Sci, vol. 10255, Springer, Cham, 2017, pp. 381–391, 3673885.
[27] S. Soualhi, Z. Mokhtari, A. Boussaad, Blind source separation based on wavelet and Spearman's rho, J. Numer. Math. Stoch. 8 (1) (2016) 1–8, 3519675.
[28] Włodzimierz Wysocki, Kendall's tau and Spearman's rho for n-dimensional Archimedean copulas and their asymptotic properties, J. Nonparametr. Stat. 27 (4) (2015) 442–459, 3406321.
[29] Sebastian Fuchs, Klaus D. Schmidt, On order statistics and Kendall's tau, Statist. Probab. Lett. 169 (2021) 7, 4168327.

[30] Sneha Jadhav, Shuangge Ma, An association test for functional data based on Kendall's tau, J. Multivariate Anal. 184 (2021) 9, 4233414.

[31] P.L. Conti, Ya. Yu. Nikitin, Rates of convergence for a class of rank tests for independence, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 260 (Veroyatn. i Stat. 3) (1999) 155–163, 319–320, 1759160.

[32] Christian Genest, Johanna Nešlehová, Noomen Ben Ghorbal, Spearman's footrule and Gini's gamma: A review with complements, J. Nonparametr. Stat. 22 (8) (2010) 937–954, 2738876.

[33] Roger B. Nelsen, Concordance and Gini's measure of association, J. Nonparametr. Stat. 9 (3) (1998) 227–238, 1649514.

[34] Persi Diaconis, R.L. Graham, Spearman's footrule as a measure of disarray, J. R. Stat. Soc. Ser. B Stat. Methodol. 39 (2) (1977) 262–268, 652736.

[35] Pranab K. Sen, Ibrahim A. Salama, Dana Quade, Spearman's footrule: Asymptotics in applications, Chil. J. Stat. 2 (1) (2011) 3–20, 2820015.

[36] Javad Behboodian, Ali Dolati, Manuel Úbeda-Flores, Measures of association based on average quadrant dependence, J. Probab. Stat. Sci. 3 (1) (2005) 161–173.

[37] Donato Michele Cifarelli, Pier Luigi Conti, Eugenio Regazzini, On the asymptotic distribution of a general measure of monotone dependence, Ann. Statist. 24 (3) (1996) 1386–1399, 1401856.

[38] H.H. Edwards, P. Mikusiński, M.D. Taylor, Measures of concordance determined by D_4-invariant copulas, Int. J. Math. Math. Sci. 2004 (69–72) (2004) 3867–3875, 2129414.

[39] H.H. Edwards, P. Mikusiński, M.D. Taylor, Measures of concordance determined by D_4-invariant measures on $(0,1)^2$, Proc. Amer. Math. Soc. 133 (5) (2005) 1505–1513, 2111952.

[40] Sebastian Fuchs, Copula-induced measures of concordance, Depend. Model. 4 (1) (2016) 205–214, 3555168.

[41] Harry Joe, Multivariate concordance, J. Multivariate Anal. 35 (1) (1990) 12–30, 1084939.

[42] P. Mikusiński, H. Sherwood, M.D. Taylor, Shuffles of min, Stochastica 13 (1) (1992) 61–74, 1197328.

[43] Roger B. Nelsen, José Juan Quesada-Molina, José Antonio Rodríguez-Lallena, Manuel Úbeda-Flores, On the construction of copulas and quasi-copulas with given diagonal sections, Insurance Math. Econom. 42 (2) (2008) 473–483, 2404309.

[44] G.A. Fredricks, R.B. Nelsen, Copulas constructed from diagonal sections, in: Distributions with Given Marginals and Moment Problems (Prague, 1996), Kluwer Acad. Publ., Dordrecht, 1997, pp. 129–136, 1614666.