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Oka Domains in Euclidean Spaces

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In this paper, we find surprisingly small Oka domains in Euclidean spaces $\mathbb{C}^n$ of dimension $n > 1$ at the very limit of what is possible. Under a mild geometric assumption on a closed unbounded convex set E in $\mathbb{C}^n$, we show that $\mathbb{C}^n \setminus E$ is an Oka domain. In particular, there are Oka domains only slightly bigger than a halfspace, the latter being neither Oka nor hyperbolic. This gives smooth families of real hypersurfaces $\Sigma_t \subset \mathbb{C}^n$ for $t \in \mathbb{R}$ dividing $\mathbb{C}^n$ in an unbounded hyperbolic domain and an Oka domain such that at $t = 0$, Σ_0 is a hyperplane and the character of the two sides gets reversed. More generally, we show that if E is a closed set in $\mathbb{C}^n$ for $n > 1$ whose projective closure $\overline{E} \subset \mathbb{CP}^n$ avoids a hyperplane $\Lambda \subset \mathbb{CP}^n$ and is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, then $\mathbb{C}^n \setminus E$ is an Oka domain.

1 Introduction

A complex manifold Y is said to be an Oka manifold if every continuous map $X \to Y$ from a Stein manifold X is homotopic to a holomorphic map, with Runge approximation on a compact holomorphically convex subset and interpolation on a closed complex subvariety of X where the given map happens to be holomorphic (see [9, Definition 5.4.1 and Theorem 5.4.4] and [22]). Thus, in the absence of topological obstructions, holomorphic

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maps from Stein manifolds to an Oka manifold satisfy the same approximation and interpolation results as holomorphic functions, that is, maps $X \to \mathbb{C}$. Oka manifolds are at the heart of many existence theorems, with diverse applications. They are at the opposite end of spectrum from Kobayashi hyperbolic manifolds [17], which do not admit any nonconstant holomorphic images of $\mathbb{C}$. Discovering Oka manifolds is a difficult task and progress has been sporadic. The best known examples are complex homogeneous manifolds (Grauert [14]) and Gromov-elliptic manifolds [15].

Most complex manifolds are neither hyperbolic nor Oka, but have a mixture of both properties. For example, a halfspace in $\mathbb{C}^n$ is the product of a halfplane, which is hyperbolic, and the affine space $\mathbb{C}^{n-1}$, which is Oka. Hence, a halfspace does not contain any Oka domains. We show in this paper that most closed convex sets in $\mathbb{C}^n$ for $n > 1$ have Oka complement. In particular, it was a surprise to discover Oka domains only slightly bigger than a halfspace; see Theorems 1.4 and 1.8.

We begin by presenting our main result. Let us consider $\mathbb{C}^n$ as an affine domain in the projective space $\mathbb{CP}^n = \mathbb{C}^n \cup H$, where $H = \mathbb{CP}^n \setminus \mathbb{C}^n \cong \mathbb{CP}^{n-1}$ is the hyperplane at infinity. Given a closed subset $E \subset \mathbb{C}^n$, we denote by $\overline{E}$ its topological closure in $\mathbb{CP}^n$.

Theorem 1.1. If E is a closed subset of $\mathbb{C}^n$ for $n > 1$ and $\Lambda \subset \mathbb{CP}^n$ is a complex hyperplane such that $\overline{E} \cap \Lambda = \varnothing$ and $\overline{E}$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, then $\mathbb{C}^n \setminus E$ is Oka.

Choosing complex coordinates $z = (z', z_n)$ on $\mathbb{C}^n$ in which $\Lambda = \{z_n = 0\}$, it is easily seen that $\overline{E} \cap \Lambda = \varnothing$ if and only if the set $E \cap \{(z', z_n) : |z_n| \le c|z'|\}$ is compact for some $c > 0$.

If E is as in Theorem 1.1 then the domain $\mathbb{CP}^n \setminus \overline{E}$ is also Oka (see [10, Theorem 5.1]).

The proof of Theorem 1.1 (see Section 3) combines the characterization of Oka manifolds by Condition Ell_1, due to Kusakabe [19] (see Theorem 3.1), with a new result proved in this paper concerning the existence of holomorphically varying families of Fatou–Bieberbach domains in $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$, where K is a polynomially convex set in $\mathbb{C}^n$ for $n > 1$; see Theorem 2.3.

Example 1.2. Conditions in Theorem 1.1 are easily verified for the Siegel upper halfspace

$$E = \{z = (z', z_n) \in \mathbb{C}^n : \Im z_n \ge |z'|^2\}. \tag{1.1}$$

(Here, $\Re$ and $\Im$ denote, respectively, the real and the imaginary part of a complex number.) The interior of E is biholomorphic to the ball $\{w \in \mathbb{C}^n : |w| < 1\}$ via the Cayley map

$$z = \Phi(w', w_n) = \mathrm{i}\left(\frac{w'}{1-w_n}, \frac{1+w_n}{1-w_n}\right).$$

Indeed, we have that

$$\Im z_n - |z'|^2 = \frac{1-|w|^2}{|1-w_n|^2} \quad \text{and} \quad w = \Phi^{-1}(z) = \left(\frac{2z'}{z_n+\mathrm{i}}, \frac{z_n-\mathrm{i}}{z_n+\mathrm{i}}\right).$$

(See Rudin [25, Sec. 2.3].) Hence, Φ extends to an automorphism of $\mathbb{CP}^n$ mapping the closed ball $\overline{\mathbb{B}} = \{w \in \mathbb{C}^n : |w| \le 1\}$ onto the projective closure $\overline{E}$ of E so that the hyperplane $\{w_n = 1\}$ gets mapped to the hyperplane at infinity $H = \mathbb{CP}^n \setminus \mathbb{C}^n$ in the z coordinates, while the hyperplane $\Lambda = \{z_n = -\mathrm{i}\}$ is at infinity in the w coordinates. Hence, in the affine w coordinates the set $\overline{E}$ is the closed ball $\overline{\mathbb{B}}$, which is polynomially convex, and $\overline{E} \cap \overline{\Lambda} = \varnothing$.

Theorem 1.1 can be equivalently expressed as follows, considering H as the hyperplane at infinity and letting E be a closed set in $\mathbb{C}^n = \mathbb{CP}^n \setminus H$ and $K = \overline{E}$ its closure in $\mathbb{CP}^n$.

Theorem 1.3. Assume that K is a compact subset of $\mathbb{CP}^n$ for $n > 1$ and $\Lambda \subset \mathbb{CP}^n$ is a complex hyperplane such that $K \cap \Lambda = \varnothing$ and K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$. Then, for every complex hyperplane $H \subset \mathbb{CP}^n$ the manifold $\mathbb{CP}^n \setminus (H \cup K)$ is Oka.

It is natural to look for geometric sufficient conditions on a closed set E in $\mathbb{C}^n$ to satisfy Theorem 1.1. In Section 4, we show that this holds if the topological closure of E in $\mathbb{CP}^n$ is a projectively convex set, meaning that the set of complex hyperplanes contained in $\mathbb{CP}^n \setminus \overline{E}$ is connected and their union equals $\mathbb{CP}^n \setminus \overline{E}$ (see Definition 4.1 and Theorem 4.2).

An important class of sets $E \subset \mathbb{C}^n$ to which Theorem 1.1 applies are convex sets satisfying weak additional conditions. We now describe several results of this type obtained in the paper.

Let E be a closed domain in $\mathbb{C}^n$ with $\mathscr{C}^1$ boundary. We denote by T_pbE the affine tangent hyperplane to bE at $p \in bE$ and by $T_p^{\mathbb{C}}bE$ the unique affine complex hyperplane in T_pbE passing though p. If E is convex, then $E \cap T_pbE \subset bE$. We have the following result.

Theorem 1.4. If E is a closed convex set with $\mathscr{C}^1$ boundary in $\mathbb{C}^n$ for $n > 1$ such that $E \cap T_p^{\mathbb{C}} bE$ does not contain an affine real halfline for any $p \in bE$, then $\mathbb{C}^n \setminus E$ is an Oka domain.

A closed convex set E in $\mathbb{R}^n$ is said to be *strictly convex* if the interior of the line segment connecting any pair of points in E is contained in the interior of E; equivalently, if the boundary of E does not contain any line segment. The following is a corollary to Theorem 1.4.

Corollary 1.5. If E is a closed strictly convex domain with $\mathscr{C}^1$ boundary in $\mathbb{C}^n$ for $n > 1$, then its complement $\mathbb{C}^n \setminus E$ is an Oka domain.

Theorem 1.4 follows from Theorem 1.1 by showing that the projective closure $\overline{E} \subset \mathbb{CP}^n$ of any closed convex set $E \subset \mathbb{C}^n$ as in Theorem 1.4 is a compact polynomially convex set in another affine chart on $\mathbb{CP}^n$. This is proved in Section 5 by a combination of complex, convex, and projective geometry.

Theorem 1.4 is new for unbounded convex sets. For compact sets E in $\mathbb{C}^n$ with $n > 1$ it is known that $\mathbb{C}^n \setminus E$ is Oka provided that E is polynomially convex, which includes all compact convex sets (see Kusakabe [18, Theorem 1.2 and Corollary 1.3] and [13]). There are also examples of compact non-polynomially convex sets in $\mathbb{C}^n$ for $n > 1$ with Oka complements; see [10, Theorem 4.10].

In dimension $n = 2$, our results give the first known examples of Oka domains with unbounded complements. For $n \geq 3$, such examples were found by Kusakabe [18, Theorem 1.6], who showed that for any closed polynomially convex set E contained in a set $\{(z', z'') \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : |z''| \leq c(1 + |z'|)\}$ for some $c > 0$, the complement $\mathbb{C}^n \setminus E$ is Oka. However, domains of this type are much bigger than some of those given by Theorem 1.4.

There are many examples satisfying Theorem 1.4, which are of the form

$$E = \{(z', z_n) \in \mathbb{C}^n : \Im z_n \geq \phi(z', \Re z_n)\}, \tag{1.2}$$

where ϕ is a convex function. An example is the Siegel upper halfspace (1.1). Its boundary $\{\Im z_n = |z'|^2\}$ is strongly convex in the z' direction and is foliated by translates of the $\Re z_n$ axis. Hence, Theorem 1.4 shows that $\mathbb{C}^n \setminus E = \{(z', z_n) \in \mathbb{C}^n : \Im z_n < |z'|^2\}$ is an Oka domain.

Theorem 1.4 and Corollary 1.5 imply the following interesting phenomenon. Assume that E is a closed strictly convex set of the form (1.2). Equivalently, ϕ is a strictly convex function, meaning that for every pair of distinct points $a, b \in \mathbb{C}^{n-1} \times \mathbb{R}$ we have that

$$\phi(ta + (1-t)b) < t\phi(a) + (1-t)\phi(b) \text{ for all } 0 < t < 1.$$

Consider the 1-parameter family of real hypersurfaces

$$\Sigma_t = \{(z', z_n) \in \mathbb{C}^n : \Im z_n = t\phi(z', \Re z_n)\} \quad \text{for } t \in \mathbb{R}.$$

If $t > 0$, the convex domain $\Omega_t^+ = \{\Im z_n > t\phi(z', \Re z_n)\}$ above Σ_t does not contain any affine complex line, so it is hyperbolic (see [4, 5]), while the domain $\Omega_t^- = \{\Im z_n < t\phi(z', \Re z_n)\}$ below Σ_t is Oka by Theorem 1.4. For $t < 0$, the picture is reversed, while at $t = 0$ the hyperplane $\Sigma_0 = \{\Im z_n = 0\}$ splits $\mathbb{C}^n$ into a pair of halfspaces. The same conclusion holds if we rescale the Siegel domain (1.1), or a domain of the form $\{\Im z_n > \phi(z')\}$ where ϕ is strictly convex. These are the first known examples of splitting $\mathbb{C}^n$ by a smooth family of hypersurfaces into pairs of an unbounded hyperbolic domain and a (necessarily unbounded) Oka domain such that the nature of the two domains gets reversed at some value of the parameter.

There are examples in the literature of holomorphic families of compact Oka manifolds degenerating to a non-Oka manifold; see [11, Corollary 5]. A recent example with open manifolds (see [10, Theorem 10.1]) is a holomorphic fibration $X \to \mathbb{C}$, with X a Stein domain in $\mathbb{C}^3$, that is trivial over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, with fibres being Fatou–Bieberbach domains in $\mathbb{C}^2$, which degenerate over 0 to the product of a disc with $\mathbb{C}$. However, the reversal of nature of the two sides, observed above, does not occur in this example.

We describe another class of closed unbounded convex sets in $\mathbb{C}^n$ with Oka complements. We shall say that a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *irreducible* if it is not of the form $\phi = \psi \circ P + l$ where $P : \mathbb{R}^n \to \mathbb{R}^m$ is a linear projection with $m < n$, ψ is a convex function on $\mathbb{R}^m$, and l is a linear function on $\mathbb{R}^n$. This means that ϕ is not a convex function of a smaller number of variables, which is linear in the remaining variables.

Corollary 1.6. If ϕ is an irreducible convex function on $\mathbb{C}^{n-1} \times \mathbb{R}$, then the domain

$$\Omega_\phi = \{(z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z', \Re z_n)\}$$

is Oka. The same is true for domains of the form

$$\Omega_\phi = \{(z', z_n) \in \mathbb{C}^n : \Im z_n < \phi(z')\},$$

where $\phi : \mathbb{C}^{n-1} \to \mathbb{R}$ is an irreducible convex function.

Proof. By Azagra [3, Theorem 1 and Proposition 1], the condition that ϕ is irreducible implies that for every $\epsilon > 0$ there is a smooth strictly convex function $\psi : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{R}$ such that $\phi - \epsilon < \psi < \phi$. Hence, the domain $\Omega_\psi = \{\Im z_n < \psi(z', \Re z_n)\}$ is Oka by Corollary 1.5. This gives an increasing sequence $\phi_1 < \phi_2 < \phi_3 < \cdots$ of smooth strictly convex functions on $\mathbb{C}^{n-1} \times \mathbb{R}$ converging uniformly to ϕ such that the sequence of Oka domains Ω_{ϕ_j} increases to the domain Ω_ϕ as $j \to \infty$. By [9, Proposition 5.6.7], it follows that Ω_ϕ is Oka. A similar argument holds in the second case, where the new domain $\Omega_\psi = \{\Im z_n < \psi(z')\}$ is Oka by Theorem 1.4 since the real lines contained in the boundary $b\Omega_\psi = \{\Im z_n = \psi(z')\}$ (in the $\Re z_n$ direction) are not complex tangent to $b\Omega_\psi$. ■

Let us illustrate Corollary 1.6 by an example.

Example 1.7. Every concave wedge in $\mathbb{C}^n$ of the form

$$\Im z_n < c|\Re z_n| + \sum_{j=1}^{n-1} \big(a_j|\Re z_j| + b_j|\Im z_j|\big)$$

for $c \geq 0$ and strictly positive numbers $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$ is an Oka domain.

We also have the following result, which improves Corollary 1.5.

Theorem 1.8. If E is a closed convex set in $\mathbb{C}^n$ for $n > 1$ that does not contain any affine real line, then $\mathbb{C}^n \setminus E$ is an Oka domain.

Proof. By Theorem 6.1, there is a nested sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of smoothly bounded strictly convex sets in $\mathbb{C}^n$ such that $E = \bigcap_{j=1}^\infty E_j$. By Corollary 1.5, the domain $\Omega_j = \mathbb{C}^n \setminus E_j$ is Oka for every $j \in \mathbb{N}$. Hence, $\mathbb{C}^n \setminus E = \bigcup_{j=1}^\infty \Omega_j$ is the increasing union of Oka domains Ω_j, so it is Oka by [9, Proposition 5.6.7]. ■

The above results show that complements of most closed convex sets in $\mathbb{C}^n$ for $n > 1$ are Oka. They provide a partial answer to [10, Problem 4.13], asking whether it is possible to characterise Oka domains in $\mathbb{C}^n$ in terms of geometric properties of their boundaries, in analogy to the classical Levi problem characterising smoothly bounded domains of holomorphy as the Levi pseudoconvex ones. Since the biholomorphically invariant version of strong convexity is strong pseudoconvexity, it is natural to ask the following questions.

Problem 1.9.

(a) Is the complement $\mathbb{C}^n \setminus E$ of every compact strongly pseudoconvex domain $E \subset \mathbb{C}^n$ $(n > 1)$ an Oka domain?

(b) Is every smoothly bounded Oka domain in $\mathbb{C}^n$ Levi pseudoconcave?

The answer to question (a) is negative if E is unbounded; see [10, Example 4.19]. Note that an Oka domain cannot have any local peak points for plurisubharmonic functions, as this would yield a nonconstant bounded plurisubharmonic function on the domain. In particular, an Oka domain has no strongly pseudoconvex boundary points. Hence, the answer to (b) is affirmative in dimension $n = 2$.

Part (b) of the Problem 1.9 may be considered as the *dual Levi problem*. It has been known since Oka's work in 1940s (see [24, Chaps. VI and IX]) that a smoothly bounded domain in $\mathbb{C}^n$ is a domain of holomorphy (equivalently, a Stein domain) if and only if its boundary is Levi pseudoconvex. Oka manifolds are in many ways dual to Stein manifolds, a fact made precise by Lárusson's model category (see [21] and [9, Sect. 7.5]) in which Oka manifolds are fibrant and Stein manifolds are cofibrant. It is therefore natural to expect that these two classes of domains in $\mathbb{C}^n$ are also dual to each other in the geometric sense. If true, this would be an interesting new paradigm in complex analysis.

Here is another open problem.

Problem 1.10. Is there a smooth real hypersurface Σ in $\mathbb{C}^n$ for $n > 1$ such that the connected components of $\mathbb{C}^n \setminus \Sigma$ are Oka domains? The same question for $\mathbb{CP}^n$.

A smooth hypersurface splitting $\mathbb{C}^2$ or $\mathbb{CP}^2$ into pairwise disjoint Oka domains is necessarily Levi-flat, for otherwise one of these Oka domains would admit a bounded nonconstant plurisubharmonic function, which is impossible. For the same reason, a smoothly bounded domain in a complex surface which is both Oka and Stein has Levi-flat boundary. A well-known and long-standing open problem is whether there exists a smooth Levi-flat hypersurface in $\mathbb{CP}^2$; the answer is negative in $\mathbb{CP}^n$ for $n > 2$ (see Siu [26]). On the other hand, Stensønes [27] constructed Fatou–Bieberbach domains in $\mathbb{C}^n$ for any $n > 1$ having smooth boundaries, but it is not known whether the closure of such a domain can have Oka complement.

2 Holomorphic Families of Fatou–Bieberbach Domains Avoiding the Union of a Complex Hyperplane and a Polynomially Convex Set

In this section, we develop the relevant tools that are used in the proof of Theorem 1.1. The main result of the section is Theorem 2.3; see also Corollary 3.2. It gives holomorphic families of Fatou–Bieberbach domains in $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$, where K is a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$. These domains avoid both a complex hyperplane and a polynomially convex set, so they are fairly small. Finding small Fatou–Bieberbach domains is of interest also in connection to Michael's problem; see Dixon and Esterle [6].

Recall that a Lie algebra $\mathfrak{g}$ of holomorphic vector fields on a complex manifold X is said to have the *density property* if the Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$, generated by all $\mathbb{C}$-complete vector fields in $\mathfrak{g}$ (using sums and Lie brackets), is dense in $\mathfrak{g}$ in the compact-open topology (see Varolin [29] or [9, Sect. 4.10]). If X is an algebraic manifold and $\mathfrak{g}$ consists of algebraic vector fields, then $\mathfrak{g}$ is said to have the *algebraic density property* if $\mathfrak{g}_0 = \mathfrak{g}$.

We recall the following result due to Varolin [29, Theorem 5.1 (1)].

Theorem 2.1. If $1 \le k < n$, then the Lie algebra $\mathfrak{g}^{n,k}$ of holomorphic vector fields on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ that vanish on $\mathbb{C}^k \times \{0\}^{n-k}$ has the density property, and the Lie algebra of polynomial vector fields with the same property has the algebraic density property. This holds in particular for the Lie algebra of holomorphic vector fields vanishing on the hyperplane $\{z_n = 0\} = \mathbb{C}^{n-1} \times \{0\}$.

Although the algebraic case of the above result is not explicitly stated in [29], it is evident from [29, proof of Theorem 5.1]. The holomorphic case will suffice for our needs.

The following important application of Theorem 2.1 is seen by [9, proof of Theorem 4.9.2], which was originally proved in [12]. This is the key argument of the Andersén–Lempert approximation theory for isotopies of injective holomorphic maps by holomorphic automorphisms; cf. [1]. (See also Varolin [29, Theorem 2.5] and [7, Theorem 2.12].) To prove the parametric version (the second part of the theorem) in the case at hand, one uses holomorphic vector fields on $\mathbb{C}^n$ depending holomorphically on the parameter ζ and vanishing on the hyperplane $\{z_n = 0\}$ (see Kutzschebauch [20] and [9, Theorem 4.9.10]).

Theorem 2.2. Assume that Ω is a Stein Runge domain in $\mathbb{C}^n$ for $n > 1$ and $\Phi_t : \Omega \to \mathbb{C}^n$ ($t \in [0, 1]$) is an isotopy of injective holomorphic maps such that Φ_0 is the identity map on Ω, and for every $t \in [0, 1]$ the domain $\Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$ and Φ_t agrees with the identity map on $\{z_n = 0\} \cap \Omega$. Then Φ_1 can be approximated uniformly on compacts in Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing $\{z_n = 0\}$ pointwise.

More generally, let Ω be a Stein Runge domain in $\mathbb{C}^N \times \mathbb{C}^n$ with coordinates $\zeta \in \mathbb{C}^N$ and $z \in \mathbb{C}^n$ whose projection to $\mathbb{C}^N$ is a domain $U \subset \mathbb{C}^N$. Assume that

$$\Phi_t(\zeta, z) = (\zeta, \phi_t(\zeta, z)) \ \text{ for } \ (\zeta, z) \in \Omega \ \text{ and } \ t \in [0,1] \tag{2.1}$$

is an isotopy of injective holomorphic maps such that $\phi_0(\zeta, z) = z$, and for all $t \in [0,1]$ the domain $\Phi_t(\Omega)$ is Runge in $\mathbb{C}^N \times \mathbb{C}^n$ and $\phi_t(\zeta, (z', 0)) = (z', 0)$ holds for every $\zeta \in U$ and $z' \in \mathbb{C}^{n-1}$. Then Φ_1 can be approximated uniformly on compacts in Ω by holomorphic maps $F : U \times \mathbb{C}^n \to \mathbb{C}^n$ fixing $U \times \{z_n = 0\}$ pointwise such that $F(\zeta, \cdot) \in \mathrm{Aut}(\mathbb{C}^n)$ for all $\zeta \in U$.

By using Theorem 2.2, we now prove the following result, which is the main analytic ingredient in the proof of Theorem 1.1.

Theorem 2.3. Assume that K is a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$, L is a compact polynomially convex set in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and $f : U \to \mathbb{C}^n$ is a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^N$ of L such that

$$f(\zeta) \in (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \ \text{ holds for all } \zeta \in L.$$

Then there are a neighbourhood $V \subset U$ of L and a holomorphic map $F : V \times \mathbb{C}^n \to \mathbb{C}^n$ such that for every $\zeta \in V$ we have that

$$F(\zeta, 0) = f(\zeta) \text{ and the map } F(\zeta, \cdot) : \mathbb{C}^n \to (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K \text{ is injective.}$$

It follows that

$$B_\zeta = \{F(\zeta, z) : z \in \mathbb{C}^n\} \subset (\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$$

is a family of Fatou–Bieberbach domains depending holomorphically on the parameter $\zeta \in V$. This result is similar in spirit to [13, Theorem 1.1], but the Fatou–Bieberbach domains that we construct here also avoid the hyperplane $\{z_n = 0\}$, which is crucial for our applications.

Proof. Since the set L is polynomially convex, we may assume that U is Stein and Runge in $\mathbb{C}^N$. Then, $X = U \times \mathbb{C}^n$ is a Runge Stein domain in $\mathbb{C}^{N+n}$ and the graph $\Gamma = \{(\zeta, f(\zeta)) \in X : \zeta \in U\}$ of f is a closed Stein submanifold of X. The restricted graph

$$\Gamma_L = \{(\zeta, f(\zeta)) \in X : \zeta \in L\} \subset L \times \Big((\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K\Big) \tag{2.2}$$

is clearly $\mathscr{O}(\Gamma)$-convex (i.e., holomorphically convex in Γ), hence also $\mathscr{O}(X)$-convex as well as polynomially convex in $\mathbb{C}^N \times \mathbb{C}^n$ since X is Runge in $\mathbb{C}^N \times \mathbb{C}^n$. (See Hörmander [16] and Stout [28] for results on holomorphic convexity.) By [8, Lemma 6.5, page 111] the compact set $(L \times K) \cup \Gamma_L$ is $\mathscr{O}(X)$-convex and hence polynomially convex, so it has a basis of Runge Stein neighbourhoods.

Let $\pi : \mathbb{C}^N \times \mathbb{C}^n \to \mathbb{C}^N$ denote the projection on the first factor. Consider the injective π-fibre preserving holomorphic map $\Phi = (\mathrm{Id}, \phi)$ of the form (2.1) on a small Runge Stein neighbourhood $\Omega = \Omega' \cup \Omega''$ of $(L \times K) \cup \Gamma_L$ in $\mathbb{C}^N \times \mathbb{C}^n$, which equals the identity map on a neighbourhood Ω' of $L \times K$ and whose second component equals

$$\phi(\zeta, z) = f(\zeta) + \frac{1}{2}(z - f(\zeta)) = \frac{1}{2} f(\zeta) + \frac{1}{2} z$$

for (ζ, z) in a neighbourhood Ω'' of the graph Γ_L in (2.2). Thus, $\phi(\zeta, \cdot)$ is a contraction by the factor $1/2$ around the point $f(\zeta) \in \mathbb{C}^n$ for every ζ. For a suitable choice of the neighbourhood Ω'' of Γ_L, the map $\phi = \phi_{1/2}$ is connected to $\phi_0(\zeta, z) = z$ by the isotopy

$$\phi_t(\zeta, z) = t f(\zeta) + (1 - t) z \quad \text{for } 0 \le t \le \frac{1}{2}$$

such that $\phi_t(\zeta, z) \in \Omega''$ for every $(\zeta, z) \in \Omega''$ and $t \in [0, 1/2]$. On Ω', we take the constant isotopy $\phi_t(\zeta, z) = \phi_0(\zeta, z) = z$. Clearly, the trace of the isotopy $\Phi_t = (\mathrm{Id}, \phi_t)$ for $t \in [0, 1/2]$ then consists of Runge domains $\Phi_t(\Omega) \subset \Omega$.

By Theorem 2.2, we can approximate Φ as closely as desired on a smaller neighbourhood of $(L \times K) \cup \Gamma_L$ by a holomorphic map

$$\Psi : V \times \mathbb{C}^n \to V \times \mathbb{C}^n, \quad \Psi(\zeta, z) = (\zeta, \psi(\zeta, z)),$$

where $V \subset U$ is a neighbourhood of L, such that for every $\zeta \in V$ we have that

- $\psi(\zeta, \cdot) \in \mathrm{Aut}(\mathbb{C}^n)$,
- $\psi(\zeta, z) = z$ for every $z = (z', 0) \in \mathbb{C}^{n-1} \times \{0\}$, and
- $\psi(\zeta, f(\zeta)) = f(\zeta)$.

Choose a pair of constants $a, b \in \mathbb{R}$ such that

$$0 < a < 1/2 < b < 1 \quad \text{and} \quad b^2 < a.$$

If the approximation of ϕ by ψ is close enough then the estimate

$$a|z - f(\zeta)| \le |\psi(\zeta, z) - f(\zeta)| \le b|z - f(\zeta)| \tag{2.3}$$

holds in a neighbourhood of the graph Γ_L in (2.2). At the same time, we can ensure that ψ is arbitrarily close to the map $(\zeta, z) \mapsto z$ on a neighbourhood of $L \times K$.

It is obvious that this gives a sequence of holomorphic maps ψ_k of the same kind as ψ for $k = 1, 2, \ldots$ such that the estimate (2.3) holds for all of them on the same neighbourhood of Γ_L, and the sequence ψ_k converges to the map $(\zeta, z) \mapsto z$ on a neighbourhood of $L \times K$ as $k \to \infty$.

Consider the sequence of automorphisms

$$\theta_k(\zeta, \cdot) = \psi_k(\zeta, \cdot) \circ \psi_{k-1}(\zeta, \cdot) \circ \cdots \circ \psi_1(\zeta, \cdot) \in \mathrm{Aut}(\mathbb{C}^n)$$

for $k \in \mathbb{N}$ and ζ in a neighbourhood of L. Due to the condition $b^2 < a$ in the estimate (2.3), which holds for all $k \in \mathbb{N}$, the attracting basin $B_\zeta \subset \mathbb{C}^n$ of the sequence θ_k at the fixed point $f(\zeta)$ is biholomorphic to $\mathbb{C}^n$ for every ζ in a neighbourhood $V \subset \mathbb{C}^N$ of L, and there is a holomorphic map $F : V \times \mathbb{C}^n \to \mathbb{C}^n$ such that $F(\zeta, \cdot) : \mathbb{C}^n \xrightarrow{\cong} B_\zeta$ is a biholomorphic map for every $\zeta \in V$ (see Wold [30, Theorem 4]). If the convergence of the sequence ψ_k to the map $(\zeta, z) \mapsto z$ is fast enough on a neighbourhood of $L \times K$, which can be arranged by our construction, then no point of K escapes a given neighbourhood of K, and hence none of the basins B_ζ intersect K. Furthermore, the condition $\psi_k(\zeta, (z', 0)) = (z', 0)$ for all $\zeta \in V$, $z' \in \mathbb{C}^{n-1}$, and $k \in \mathbb{N}$ ensures that the basin B_ζ does not intersect the hyperplane $\mathbb{C}^{n-1} \times \{0\}$. Hence, $B_\zeta = F(\zeta, \mathbb{C}^n)$ is a Fatou–Bieberbach domain in $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$ centred at $f(\zeta) = F(\zeta, 0)$ for every $\zeta \in V$. ∎

3 Proof of Theorem 1.1

We now show how Theorem 2.3 implies Theorem 1.1, and hence also Theorem 1.3. We shall use the following characterization of Oka manifolds due to Kusakabe [19, Theorem 1.3].

Theorem 3.1. A complex manifold Y is an Oka manifold if and only if for every compact convex set $L \subset \mathbb{C}^N$ ($N \in \mathbb{N}$), open set $U \subset \mathbb{C}^N$ containing L, and holomorphic map $f : U \to Y$ there are an open set V in $\mathbb{C}^N$ with $L \subset V \subset U$ and a holomorphic map $F : V \times \mathbb{C}^n \to Y$ for some $n \geq \dim Y$ such that $F(\cdot, 0) = f$ and

$$\left.\frac{\partial}{\partial z}\right|_{z=0} F(\zeta, z) : \mathbb{C}^n \to T_{f(\zeta)} Y \ \text{ is surjective for every } \ \zeta \in V.$$

Such a map F is called a *dominating holomorphic spray* with the core $f = F(\cdot, 0)$.

Note that the map F in Theorem 2.3 is a dominating spray with the given core f. Hence, the following is an immediate corollary to Theorems 2.3 and 3.1.

Corollary 3.2. If K is a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$ then $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus K$ is an Oka manifold.

By using this corollary we infer the following.

Proposition 3.3. Under the assumptions of Theorem 1.1, the domain $\mathbb{C}^n \setminus (E \cup \Lambda)$ is Oka.

Proof. If $\Lambda = H$ then, since $\overline{E} \cap \Lambda = \varnothing$, it follows that E is compact and hence $\mathbb{C}^n \setminus E$ is Oka by [18, Theorem 1.2 and Corollary 1.3] (see also [13]).

Assume now that $\Lambda \neq H$ and $K = \overline{E}$ is a compact polynomially convex set in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$. Choose complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{CP}^n \setminus \Lambda$ such that $H \setminus \Lambda = \{z_n = 0\}$. Then,

$$\mathbb{C}^n \setminus (E \cup \Lambda) = \mathbb{CP}^n \setminus (E \cup H \cup \Lambda) = (\mathbb{CP}^n \setminus \Lambda) \setminus (H \cup K) = \{(z', z_n) : z_n \neq 0\} \setminus K.$$

We are now in the situation of Corollary 3.2, which gives the desired conclusion. ■

We also recall the following result; see [10, Theorem 5.1 and Corollary A.5].

Theorem 3.4. Let K be a compact subset of $\mathbb{CP}^n$ for $n > 1$. If there is a complex hyperplane $\Lambda \subset \mathbb{CP}^n \setminus K$ such that K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$, then K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda'$ for every complex hyperplane $\Lambda' \subset \mathbb{CP}^n \setminus K$ which is connected to Λ by a path of complex hyperplanes in $\mathbb{CP}^n \setminus K$, and $\mathbb{CP}^n \setminus K$ is an Oka domain.

Proof of Theorem 1.1. By Theorem 3.4 applied to the compact set $K = \overline{E} \subset \mathbb{CP}^n$ there are hyperplanes $\Lambda_0 = \Lambda, \Lambda_1, \ldots, \Lambda_n$ in $\mathbb{CP}^n \setminus \overline{E}$ close to Λ such that $\bigcap_{i=0}^n \Lambda_i = \varnothing$ and $\overline{E}$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda_i$ for $i = 0, 1, \ldots, n$. Let $\mathbb{C}^n = \mathbb{CP}^n \setminus H$. By Proposition 3.3 the domain $\mathbb{C}^n \setminus (E \cup \Lambda_i)$ is Oka for every $i = 0, \ldots, n$. Note that

$$\mathbb{C}^n \setminus E = \bigcup_{i=0}^n \mathbb{C}^n \setminus (E \cup \Lambda_i).$$

Since every Oka domain $\mathbb{C}^n \setminus (E \cup \Lambda_i) = (\mathbb{C}^n \setminus E) \setminus \Lambda_i$ is Zariski open in $\mathbb{C}^n \setminus E$, the localization theorem for Oka manifolds [19, Theorem 1.4] (see also [10, Theorem 3.6]) implies that $\mathbb{C}^n \setminus E$ is an Oka domain. ■

4 Projectively Convex Sets

This section contains some preparatory results used in the proof of Theorem 1.4.

There are several notions of convexity for subsets of complex projective spaces. We shall be interested in the following ones; see [2, Definition 2.1.2].

Definition 4.1. Let K be a compact set in $\mathbb{CP}^n$.

(i) The set K is *linearly convex* if for every point $p \in \mathbb{CP}^n \setminus K$ there is a complex hyperplane $\Lambda \subset \mathbb{CP}^n$ with $p \in \Lambda$ and $\Lambda \cap K = \varnothing$.

(ii) The set K is *projectively convex* if it is linearly convex and the space of all complex hyperplanes contained in $\mathbb{CP}^n \setminus K$ is connected.

Note that complex hyperplanes in $\mathbb{CP}^n$ are parameterized by the dual projective space $\mathbb{CP}^{n*}$. The set of hyperplanes lying in $\mathbb{CP}^n \setminus K$ is clearly open, but it need not be connected in general. Clearly, both notions are invariant under holomorphic automorphisms of $\mathbb{CP}^n$. Our interest in projective convexity is that it gives a geometric sufficient condition for validity of Theorem 1.1. This gives many new examples of Oka domains in Euclidean and projective spaces.

Theorem 4.2. The following hold for any compact projectively convex set $K \subsetneq \mathbb{CP}^n$ with $n > 1$.

(a) For every complex hyperplane $\Lambda \subset \mathbb{CP}^n$ with $\Lambda \cap K = \varnothing$ the set K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda \cong \mathbb{C}^n$.

(b) $\mathbb{CP}^n \setminus K$ is an Oka manifold.

(c) For every complex hyperplane $H \subset \mathbb{CP}^n$ the set $\mathbb{CP}^n \setminus (H \cup K)$ is an Oka manifold.

Proof. To prove (a) we proceed as follows. (See also [10, Corollary A.5].) By the assumption, the domain $\mathbb{CP}^n \setminus K$ is a union of complex hyperplanes that form an open connected family $\mathcal{H}_K \subset \mathbb{CP}^{n*}$. Fix $\Lambda_0 \in \mathcal{H}_K$. Given a point $p \in \mathbb{CP}^n \setminus (K \cup \Lambda_0)$, there is a path $\Lambda_t \in \mathcal{H}_K$ for $t \in [0, 1]$ connecting Λ_0 to a hyperplane Λ_1 with $p \in \Lambda_1$. We may assume that $\Lambda_t \neq \Lambda_0$ for $t \in (0, 1]$. Note that $K \subset \mathbb{CP}^n \setminus \Lambda_0 = \mathbb{C}^n$, and $\Sigma_t := \Lambda_t \setminus \Lambda_0 \subset \mathbb{C}^n$ for $t \in (0, 1]$ is a path of affine complex hyperplanes in $\mathbb{C}^n \setminus K$ such that $p \in \Sigma_1$ and Σ_t diverges to infinity as $t \to 0$. By Oka's criterion (see Oka [23], Stout [28, Theorem 2.1.3]), and [10, Corollary A.2] this implies that p does not belong to the polynomial hull of K in $\mathbb{CP}^n \setminus \Lambda_0$. Since this holds for every point $p \in \mathbb{CP}^n \setminus (K \cup \Lambda_0)$, we conclude that K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda_0$. This proves (a).

To prove (b), choose hyperplanes $\Lambda_0, \Lambda_1, \ldots, \Lambda_n \in \mathcal{H}_K$ such that $\bigcap_{i=0}^n \Lambda_i = \varnothing$. (It suffices to take small generic perturbations of any given hyperplane $\Lambda_0 \in \mathcal{H}_K$.) Then,

$$\mathbb{CP}^n \setminus K = \bigcup_{i=0}^{n} \mathbb{CP}^n \setminus (K \cup \Lambda_i).$$

By part (a), the set K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda_i \cong \mathbb{C}^n$ for every $i = 0, \ldots, n$, and hence the domain $\mathbb{CP}^n \setminus (K \cup \Lambda_i) = (\mathbb{CP}^n \setminus \Lambda_i) \setminus K$ is Oka by [18, Theorem 1.2 and Corollary 1.3]. This gives a covering of $\mathbb{CP}^n \setminus K$ by Zariski open Oka domains, so $\mathbb{CP}^n \setminus K$ is Oka by Kusakabe's localization theorem [19, Theorem 1.4].

Part (c) follows from (a) and Theorem 1.3, which is equivalent to Theorem 1.1 proved in the previous section. ■

Example 4.3. In $\mathbb{C}^n$ with coordinates (z', z_n), we consider a domain of the form

$$\Omega = \left\{(z', z_n) \in \mathbb{C}^n : |z_n|^2 < c(1 + |z'|^2)\right\} \quad \text{for } c > 0.$$

Let H denote the hyperplane at infinity and $\Lambda = \{z_n = 0\}$. In suitable affine coordinates $w = (w', w_n)$ on $\mathbb{CP}^n \setminus \Lambda = (\mathbb{C}^n \cup H) \setminus \Lambda \cong \mathbb{C}^n$ in which Λ is the hyperplane at infinity and $H = \{w_n = 0\}$, the domain Ω is the complement of $H \cup \overline{\mathbb{B}}$ where $\mathbb{B}$ is a ball centred at the origin. Hence, Ω is an Oka domain by part (c) of Theorem 4.2.

Remark 4.4. The use of hyperplanes in the proof of Theorem 4.2 can be replaced by more general hypersurfaces to obtain the following criterion for validity of Theorem 3.4. Let K be a compact subset of $\mathbb{CP}^n$ and $\Lambda \subset \mathbb{CP}^n \setminus K$ be a hyperplane. Assume that for every $p \in \mathbb{CP}^n \setminus (K \cup \Lambda)$ there is a continuous 1-parameter family of compact complex hypersurfaces $A_t \subset \mathbb{CP}^n \setminus K$ $(t \in [0, 1])$ of the same degree $k \in \mathbb{N}$ such that $p \in A_0$ and A_t converges to Λ as $t \to 1$, that is, for every neighbourhood $U \subset \mathbb{CP}^n$ of Λ there is a $c \in (0, 1)$ such that $A_t \subset U$ for all $t \in (c, 1)$. Then K is polynomially convex in $\mathbb{CP}^n \setminus \Lambda$, so Theorem 3.4 applies. This is seen by the argument in the proof of Theorem 4.2, using Oka's criterion for polynomial convexity.

We now give a geometric characterization of closed sets in $\mathbb{C}^n$ having projectively convex closures. Assume that Λ is a complex affine subspace of dimension $k \in \{1, \ldots, n-1\}$ in $\mathbb{C}^n$ and $p \in \Lambda$. In suitable affine complex coordinates $z = (z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, we have that $p = 0$ and $\Lambda = \{z'' = 0\}$. Given $c > 0$, we define

$$C(\Lambda, p, c) = \{(z', z'') \in \mathbb{C}^n : |z''| \le c|z'|\}. \tag{4.1}$$

This is a closed cone with the axis Λ and vertex p. The analogous definition makes sense for real affine subspaces of $\mathbb{R}^n$.

The proof of the following elementary observation is left to the reader.

Lemma 4.5. Let E be a closed set in $\mathbb{C}^n$ and Λ be a complex affine subspace of $\mathbb{C}^n$. Consider $\mathbb{C}^n$ as a domain in $\mathbb{CP}^n$ and set $H = \mathbb{CP}^n \setminus \mathbb{C}^n$. The following are equivalent.

(i) There is a point $p \in \Lambda$ and a number $c > 0$ such that $C(\Lambda, p, c) \cap E$ is compact.

(ii) The projective closure of Λ does not intersect $\overline{E}$ along H, that is, $\overline{E} \cap \overline{\Lambda} \cap H = \varnothing$.

We shall say that Λ is *E-stable* if these equivalent conditions hold.

Remark 4.6. Let E and Λ be as in Lemma 4.5.

(a) If the condition in Lemma 4.5 (i) holds for a point $p_0 \in \Lambda$, then it holds for every point $p \in \Lambda$, with a constant $c > 0$ depending on p.

(b) If Λ is E-stable then so is any parallel translate Λ' of Λ. In fact, $\overline{\Lambda}' \cap H = \overline{\Lambda} \cap H$.

(c) The space of E-stable k-dimensional affine subspaces $\Lambda \subset \mathbb{C}^n \setminus E$ is open.

Proposition 4.7. Let E be a closed subset of $\mathbb{C}^n$. Then the closure $\overline{E} \subset \mathbb{CP}^n$ is projectively convex if and only if the following three conditions hold.

(i) Every point $p \in \mathbb{C}^n \setminus E$ lies in an E-stable hyperplane $\Lambda \subset \mathbb{C}^n$ such that $E \cap \Lambda = \varnothing$.

(ii) Every E-stable affine complex line $L \subset \mathbb{C}^n$ has a parallel translate contained in an E-stable complex hyperplane $\Lambda \subset \mathbb{C}^n \setminus E$.

(iii) The space of E-stable affine complex hyperplanes in $\mathbb{C}^n \setminus E$ is connected.

Proof. Let $H = \mathbb{CP}^n \setminus \mathbb{C}^n$. By Lemma 4.5, condition (i) means that $\mathbb{C}^n \setminus E$ is a union of affine hyperplanes whose projective closures do not intersect $\overline{E}$. Condition (ii) means that for every point $p \in H \setminus \overline{E}$ there is an affine hyperplane $\Lambda \subset \mathbb{C}^n \setminus E$ such that $p \in \overline{\Lambda} \subset \mathbb{CP}^n \setminus \overline{E}$. Hence, (i) and (ii) together are equivalent to $\mathbb{CP}^n \setminus \overline{E}$ being a union of complex hyperplanes. Finally, (iii) means that the set of complex hyperplanes in $\mathbb{CP}^n \setminus \overline{E}$ is connected. Hence, the three conditions together are equivalent to $\overline{E}$ being projectively convex. ∎

The following is an immediate corollary to Theorem 1.1 and Proposition 4.7.

Corollary 4.8. If a closed set E in $\mathbb{C}^n$ for $n > 1$ satisfies the hypotheses in Proposition 4.7 then $\mathbb{C}^n \setminus E$ is an Oka domain.

As an application of this result, we show the following.

Proposition 4.9. The complement of an affine real line in $\mathbb{C}^2$ is Oka.

Proof. We may assume that the affine real line in question is $\mathbb{R}\times\{0\}\subset\mathbb{C}^2$. Let Ω denote its complement in $\mathbb{C}^2$. Consider the following subsets of $\mathbb{C}^2$:

$$\Delta_+ = \{(z,0) : \Re z \geq 0\}, \quad \Delta_- = \{(z,0) : \Re z \leq 0\}.$$

Then $\Omega = (\Omega \setminus \Delta_+) \cup (\Omega \setminus \Delta_-)$ and both domains $\Omega \setminus \Delta_\pm = \mathbb{C}^2 \setminus \Delta_\pm$ are Zariski open in Ω. In view of the localization theorem [19, Theorem 1.4] it suffices to show that $\mathbb{C}^2 \setminus \Delta_\pm$ are Oka.

Consider $\mathbb{C}^2 \setminus \Delta_+$. It is immediate that a complex line $L \subset \mathbb{C}^2 \setminus \Delta_+$ is Δ_+-stable if and only if it intersects the complex line $\mathbb{C} \times \{0\}$ at a point $(a,0)$ with $\Im a < 0$, that is, in the interior of Δ_-. It is easily seen that the set of such lines is connected and its union equals $\mathbb{C}^2 \setminus \Delta_+$, so this domain is Oka by Corollary 4.8. (An alternative argument is that the closure of Δ_+ in $\mathbb{CP}^2$ is an embedded closed holomorphic disk D_+. Such a disk is polynomially convex in $\mathbb{CP}^2 \setminus \Lambda \cong \mathbb{C}^2$ for any complex line Λ not intersecting D_+, and hence $\mathbb{C}^2 \setminus \Delta_+$ is Oka by Theorem 1.1.) The analogous argument applies to the domain $\mathbb{C}^2 \setminus \Delta_-$. ■

Remark 4.10. It was shown by Kusakabe [18, Corollary 1.7] that if (n,k) is a pair of integers with $1 \leq k \leq n$, $n \geq 3$, and $(n,k) \neq (3,3)$ then for any closed set E in $\mathbb{R}^k \subset \mathbb{C}^n$ the complement $\mathbb{C}^n \setminus E$ is Oka; in particular, $\mathbb{C}^n \setminus \mathbb{R}^k$ is Oka for these pairs of values (n,k). To prove this, Kusakabe used his theorem (see [18, Theorem 1.6]) saying that if E is a closed, possibly unbounded polynomially convex subset of $\mathbb{C}^n = \mathbb{C}^{n-2} \times \mathbb{C}^2$, which is contained in a set of the form $\{(z',z'') : |z''| \leq c(1+|z'|)\}$ with respect to some holomorphic coordinates $z = (z',z'')$ on $\mathbb{C}^{n-2} \times \mathbb{C}^2$ and $c > 0$, then $\mathbb{C}^n \setminus E$ is Oka. This approach does not work for the exceptional cases $(n,k) \in \{(2,1),(2,2),(3,3)\}$. Proposition 4.9 settles the case $n=2$, $k=1$ by a completely different method. The remaining two cases $(2,2)$ and $(3,3)$ are not amenable to this method either, so they remain an open problem.

5 Convex Domains in $\mathbb{C}^n$ With Oka Complements

In this section, we prove Theorem 1.4. We will show that for every closed convex set $E \subset \mathbb{C}^n$ satisfying the conditions of that theorem, its projective closure $\overline{E} \subset \mathbb{CP}^n$ is projectively convex (see Definition 4.1), so the result will follow from Theorem 4.2.

Given a domain $E \subset \mathbb{C}^n$ with $\mathscr{C}^1$ boundary and a point $p \in bE$, we denote by $T_p^{\mathbb{C}} bE$ the maximal complex subspace (a complex hyperplane) in the real tangent space $T_p bE$. (Both tangent spaces are considered as affine spaces passing through the point p.) Recall that a real affine subspace $\Lambda \subset \mathbb{C}^n$ with $p \in \Lambda \cap bE$ is said to be supporting for E at p if $\Lambda \cap E \subset bE$. If E is convex and bE is of class $\mathscr{C}^1$, then this holds if and only if $\Lambda \subset T_p bE$, and if Λ is complex then it holds if and only if $\Lambda \subset T_p^{\mathbb{C}} bE$.

The notion of an E-stable affine subspace was introduced in Lemma 4.5.

Lemma 5.1. Let E be a closed convex set in $\mathbb{C}^n$ and $p \in bE$. Assume that $\Lambda \subset \mathbb{C}^n$ is a supporting affine complex subspace for E at p. Then Λ is E-stable if and only if $E \cap \Lambda$ does not contain a real halfline. The analogous result holds in the real setting.

Proof. Choose affine coordinates $z = (z', z'')$ on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ with $k = \dim \Lambda$ such that $p = 0$ and $\Lambda = \{z'' = 0\}$.

If $E \cap \Lambda$ contains a real halfline L, then the terminal point of L at infinity lies in the projective closure $\overline{E} \subset \mathbb{CP}^n$ of E, so Λ is not E-stable. Note also that, since E is closed and convex, the line segments l_q connecting the given initial point p to points $q \in L$ moving to infinity converge to a halfline in $E \cap \Lambda$ with the finite endpoint p.

To prove the converse, assume that Λ is not E-stable, so the intersection of E with the closed cone $C(\Lambda, p, c)$ in (4.1) is unbounded for every $c > 0$. Letting $c \to 0$ we obtain a sequence of unit vectors $v_j = (v_j', v_j'') \in \mathbb{C}^n$ and numbers $t_j > 0$ such that $t_j v_j \in E$ for all $j \in \mathbb{N}$, $\lim_{j\to\infty} t_j = +\infty$, and $\lim_{j\to\infty} |v_j''|/|v_j'| = 0$. By passing to a subsequence, we may assume that the sequence $v_j'/|v_j'|$ converges to a unit vector $v' \in \mathbb{C}^k$. The line segments $l_j \subset E$ connecting $p = 0$ to the points $t_j v_j \in E$ then converge to the halfline $\mathbb{R}_+ v' \in E \cap \Lambda$ terminating at p. ■

Recall that affine subspaces Λ and V in $\mathbb{R}^n$ are said to be complementary if and only if $\dim \Lambda + \dim V = n$ and their intersection is a point. For vector subspaces, this holds if and only if $\mathbb{R}^n$ is their direct sum $\Lambda \oplus V$.

Lemma 5.2. Let E be a closed convex set in $\mathbb{R}^n$, and let $\Lambda \subset \mathbb{R}^n$ be an affine subspace such that $E \cap \Lambda$ is bounded. Then the following are equivalent.

(i) Every parallel translate Λ' of Λ intersects E.

(ii) There is a vector subspace $V \subset \mathbb{R}^n$ complementary to Λ such that $E = E \cap \Lambda + V$.

If these equivalent conditions fail, then there is a translate Λ' of Λ, which is a supporting subspace for E at a point $q \in bE \cap \Lambda'$.

Note that the set $E \cap \Lambda + V$ in part (ii) is a tube with basis $E \cap \Lambda$ and fibre V.

Proof of Lemma 5.2. The implication (ii) $\Rightarrow$ (i) is obvious.

Let us now prove that (i) $\Rightarrow$ (ii). We begin with the case when Λ is a hyperplane. Choose coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ such that $\Lambda = \{x_n = 0\}$. By the assumption, the closed convex set $E' = E \cap \Lambda$ is bounded. Let $H^\pm = \{\pm x_n \geq 0\}$. The assumption that every translate of Λ intersects the set $E^+ := E \cap H^+$ gives a sequence of unit vectors $v_j \in H^+$ and numbers $t_j > 0$ with $\lim_{j\to\infty} t_j = +\infty$ such that $t_j v_j \in E^+$. By compactness of the unit sphere we may pass to a subsequence and assume that $\lim_{j\to\infty} v_j = v$. Fix a point $x' \in E'$. The line segments connecting x' to the points $t_j v_j \in E^+$ converge to the halfline $x' + \mathbb{R}_+ v$ as $j \to \infty$. Since E is closed and convex, this halfline belongs to E^+. In particular, $v \notin \Lambda$ since $E \cap \Lambda$ is bounded. This shows that E^+ contains the tube $E' + \mathbb{R}_+ v$. Since $E^- := E \cap H^-$ is unbounded as well, the analogous argument shows that it contains a tube $E' + \mathbb{R}_+ w$ for some unit vector $w \in H^-$. Note that $w = -v$, for otherwise the convex hull of the union of these tubes contains a point in $\Lambda \setminus E'$. Hence, E contains the tube $E' + \mathbb{R}v$, and the same argument as above shows that $E = E' + \mathbb{R}v$.

The above argument also shows that if a set E as in the lemma is not a tube of the form $E' + \mathbb{R}v$ then at least one of the closed convex sets $E^\pm := E \cap H^\pm$ is bounded, and hence there is a parallel translate of Λ satisfying the last statement in the lemma.

Consider now the general case. We may assume that $p = 0$ and

$$\Lambda = \{x \in \mathbb{R}^n : x_{k+1} = 0, \ldots, x_n = 0\}.$$

For every $j \in \{k+1, \ldots, n\}$ let $V_j \subset \mathbb{R}^n$ denote the subspace of dimension $k+1$ spanned by the coordinate directions $1, \ldots, k$ and j. Then Λ is a hyperplane in V_j, so the special case proved above gives $E \cap V_j = E' + \mathbb{R} \cdot v_j$ for some unit vector $v_j \in V_j \setminus \Lambda$. Due to convexity it follows that E contains the tube $E' + V$ with $V = \mathrm{span}\{v_{k+1}, \ldots, v_n\}$. Thus, V is an $(n-k)$-dimensional subspace of $\mathbb{R}^n$ complementary to Λ. If E contains a vector $w \in \mathbb{R}^n$ not in $E' + V$, then convex combinations of w and vectors from $E' + V$ give points in $E \cap \Lambda$ which are not contained in E', a contradiction. ■

Corollary 5.3. If the set E is as in Theorem 1.4 and $\Lambda \subset \mathbb{C}^n$ is an affine complex subspace such that $E \cap \Lambda$ is bounded, then there is a parallel translate Λ' of Λ with $E \cap \Lambda' = \varnothing$, and also one satisfying $E \cap \Lambda' = \{p\}$ with $p \in bE$. Furthermore, Λ is E-stable.

Proof. If no translate of Λ avoids E then by Lemma 5.2 the set E is a tube $E' + V$, where $E' = E \cap \Lambda$ and $V \subset \mathbb{C}^n$ is a real subspace complementary to Λ. Hence, at every point $p \in bE$ the tangent hyperplane $T_p bE$ contains the affine real subspace $p + V$ of dimension at least two. Since $T_p^{\mathbb{C}} bE$ is a real hyperplane in $T_p bE$, its intersection with $p + V$ contains a real line, contradicting the hypothesis in Theorem 1.4. This shows that there is a translate Λ' of Λ avoiding E, and also one which is a supporting subspace for E at a point $p \in bE$ (see Lemma 5.2). In the latter case, Λ' (being a complex affine subspace) is contained in $T_p^{\mathbb{C}} bE$. The assumption in Theorem 1.4 that $E \cap T_p^{\mathbb{C}} bE$ contains no halfline implies by Lemma 5.1 that $T_p^{\mathbb{C}} bE$ is E-stable. Hence, $\Lambda' \subset T_p^{\mathbb{C}} bE$ is also E-stable, and the same holds for Λ since this property is translation invariant. ■

Proof of Theorem 1.4. We claim that $\overline{E} \subset \mathbb{CP}^n$ is projectively convex, so the result will follow from Theorem 4.2. By Proposition 4.7 we must verify the following conditions:

(i) $\mathbb{C}^n \setminus E$ is a union of E-stable affine complex hyperplanes,

(ii) every E-stable complex line $L \subset \mathbb{C}^n$ has a parallel translate contained in an E-stable complex hyperplane $\Lambda \subset \mathbb{C}^n \setminus E$, and

(iii) the set of all E-stable affine complex hyperplanes in $\mathbb{C}^n \setminus E$ is connected.

Proof of (i): Choose a point $q \in \mathbb{C}^n \setminus E$. Let $p \in bE$ be the closest point to q in E. The tangent plane $T_p bE$ is then a supporting hyperplane for E and is orthogonal to the real line through p and q. The affine complex tangent plane $T_p^{\mathbb{C}} bE$ is E-stable by Lemma 5.1. The parallel translate Λ of $T_p^{\mathbb{C}} bE$ to the point q is then contained in $\mathbb{C}^n \setminus E$, and it is E-stable since this property is translation invariant.

Proof of (ii): Let L be an E-stable affine complex line. Then $E \cap L$ is bounded, and Lemma 5.2 shows that a parallel translate L' of L is tangent to bE at some point $p \in bE$. Then, $L' \subset T_p^{\mathbb{C}} bE$, which is an E-stable complex hyperplane by Lemma 5.1. Translating $T_p^{\mathbb{C}} bE$ away from E gives an E-stable hyperplane $\Lambda \subset \mathbb{C}^n \setminus E$ containing a translate of L.

Proof of (iii): We claim that a complex affine hyperplane Λ in $\mathbb{C}^n$ is E-stable if and only if it is parallel to the complex tangent space $T_p^{\mathbb{C}} bE$ for some point $p \in bE$. In one direction, Lemma 5.1 shows that for every $p \in bE$ the complex tangent space $T_p^{\mathbb{C}} bE$ is E-stable. Conversely, if Λ is E-stable then $\Lambda \cap E$ is a bounded set, and Corollary 5.3 shows that a parallel translate of Λ is tangent to bE at some point $p \in bE$, so this translate equals $T_p^{\mathbb{C}} bE$. It is easily seen that if E is as in the theorem then its boundary bE is connected, so the above shows that the set of E-stable hyperplanes is connected.

It remains to see that the set $\mathcal{E}$ of E-stable complex affine hyperplanes in $\mathbb{C}^n$ which do not intersect E is also connected. If a hyperplane $\Lambda \subset \mathbb{C}^n \setminus E$ is E-stable, then by Corollary 5.3 we can translate Λ within $\mathbb{C}^n \setminus E$ until it hits bE for the first time at some point $p \in bE$, and this new hyperplane Λ' is then equal to $T_p^{\mathbb{C}} bE$ and $\Lambda' \cap E \subset bE$. This shows that the set of E-stable affine complex hyperplanes contained in $\mathbb{C}^n \setminus \mathring{E}$ is connected. Take an interior point $q \in \mathring{E} = E \setminus bE$; we may assume that $q = 0$. Consider the decreasing family of closed convex domains $E_k = (1 + 1/k)E$, that is, we dilate E by the factor $1 + 1/k$. Note that $E \subset \mathring{E}_k$ for all k and $E = \bigcap_{k=1}^{\infty} E_k$. Clearly, every E_k has the same properties as E and the same set of stable hyperplanes. From what has been shown above, the set $\mathcal{E}_k$ of E-stable hyperplanes avoiding $\mathbb{C}^n \setminus \mathring{E}_k$ is connected. As $k \to \infty$, the domains E_k shrink down to E, and hence $\mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k$ is an increasing union of connected sets, so it is connected. ■

6 Intersections of Strictly Convex Sets

It is well known that every closed convex set in a Euclidean space $\mathbb{R}^m$ is an intersection of halfspaces. In this section, we characterise those closed sets that are intersections of strongly convex sets. A closed set in $\mathbb{R}^m$ is called *strongly convex* if it has $\mathscr{C}^2$ boundary and the principal normal curvatures of the boundary are strictly positive at every point. Clearly, a strongly convex domain is also strictly convex, but the converse fails in general.

The following result is used in the proof of Theorem 1.8.

Theorem 6.1. If E is a closed convex set in $\mathbb{R}^m$, which does not contain an affine line, then $E = \bigcap_j E_j$ where every E_j is a closed strongly convex set and $E_{j+1} \subset E_j$ for $j = 1, 2, \ldots$.

Conversely, a convex set containing a line is not contained in any strictly convex set.

Proof. In [2, Section 1.3], a set $K \subset \mathbb{RP}^m$ is called convex if (*i*) K does not contain a projective line, and (*ii*) the intersection of any projective line with K is connected. (There are other notions of convexity in projective spaces, but this is the one relevant here.) If E is a closed convex set in $\mathbb{R}^m$ that does not contain any affine real line then its closure $K = \overline{E}$ in $\mathbb{RP}^m$ is convex.

Fix a point $p \in \mathbb{RP}^m \setminus K$. By [2, Theorem 1.3.11], there is a hyperplane $H \subset \mathbb{RP}^m \setminus K$ with $p \in H$. If $p \in \mathbb{R}^m$, then the real hyperplane $H' = H \cap \mathbb{R}^m$ contains p and a proper cone Q around H' avoids E. (In the terminology introduced in Lemma 4.5, H' is E-stable.) We may assume that p is the origin, $H' = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 0\}$, and E is contained in the halfspace $\{x_m > 0\}$. Then there exists a smooth strongly convex function $f(x_1,\ldots,x_{m-1})$ (i.e., with positive definite Hessian at every point) such that $f(0)=0$ and the graph of f is contained in a strictly smaller cone $\Omega'\subset\Omega$. For such f and $\epsilon>0$ small enough, the set

$$E'=\{x\in\mathbb{R}^m : x_m\geq f(x_1,\ldots,x_{m-1})+\epsilon\}$$

is strongly convex with $E\subset E'$ and $0\in\mathbb{R}^m\setminus E'$. Hence, the function

$$\rho(x_1,\ldots,x_m)=\mathrm{e}^{f(x_1,\ldots,x_{m-1})-x_m}-1$$

is strongly convex on $\mathbb{R}^m$, $\rho(0)=0$, and $\rho\leq \mathrm{e}^{-\epsilon}-1<0$ on E.

By scaling, it follows that for every compact set $K\subset\mathbb{R}^m\setminus E$ there exist strongly convex functions $\rho_1,\ldots,\rho_k$ on $\mathbb{R}^m$ such that $E\subset\{x\in\mathbb{R}^m : \rho_j(x)\leq -1, j=1,..,k\}$ such that for every $x\in K$ we have that $\rho_j(x)>0$ for some $j\in\{1,\ldots,k\}$. Exhausting $\mathbb{R}^m\setminus E$ by compact sets, it follows that there exist strongly convex functions $\{\rho_j\}_{j=1}^{\infty}$ such that $E=\bigcap_{j=1}^{\infty}\{x\in\mathbb{R}^m : \rho_j(x)\leq 0\}$ and $\rho_j(x)\leq -1$ for $x\in E$ for all j.

It remains to find a decreasing sequence of smoothly bounded strongly convex sets $E_1\supset E_2\supset\cdots\supset\bigcap_{k=1}^{\infty}E_k=E$. We begin by taking $\tau_1=\rho_1$ and $E_1=E_1'=\{\tau_1\leq 0\}$. To get E_2, we take the convex set $E_1\cap E_2'=\{\max\{\tau_1,\rho_2\}\leq 0\}$ and smoothen the corners. This is done by a *regularized maximum function* defined as follows (see [9, page 69]). Given a number $\delta>0$, we select a nonnegative smooth even function $\xi\geq 0$ on $\mathbb{R}$ with support in $[-\frac{\delta}{2},\frac{\delta}{2}]$ such that $\int\xi(t)\mathrm{d}t=1$, and we set for $(u_1,u_2)\in\mathbb{R}^2$:

$$\mathrm{rmax}\{u_1,u_2\}=\int_{\mathbb{R}^2}\max\{t_1+u_1,t_2+u_2\}\,\xi(t_1)\xi(t_2)\mathrm{d}t_1\,\mathrm{d}t_2. \tag{6.1}$$

It is easily verified that the function rmax is smooth, increasing in every variable, and convex jointly in both variables. Hence, if $u_1(x)$ and $u_2(x)$ are (strongly) convex functions then $\mathrm{rmax}\{u_1(x),u_2(x)\}$ is also (strongly) convex. Moreover, we have that

$$\max\{u_1,u_2\}\leq\mathrm{rmax}\{u_1,u_2\}\leq\max\{u_1,u_2\}+\delta\ \text{ for all } (u_1,u_2)\in\mathbb{R}^2$$

and

$$\mathrm{rmax}\{u_1,u_2\}=\begin{cases}u_1, & \text{if } u_2\leq u_1-\delta,\\ u_2, & \text{if } u_1\leq u_2-\delta.\end{cases}$$

In a similar one defines rmax for any finite number of variables.

This shows that for a suitable choice of ξ in the definition of rmax the function

$$\tau_2 := \mathrm{rmax}\{\tau_1, \rho_2\} : \mathbb{R}^m \to \mathbb{R}$$

is smooth strongly convex and satisfies $\tau_2(a_1) > 0$, $\tau_2(a_2) > 0$, and $\tau_2 \le -c_2 < 0$ on E for some $c_2 > 0$. The set $E_2 = \{\tau_2 \le 0\}$ is strongly convex with smooth boundary, and it satisfies $E \subset \mathring{E}_2 \subset E_2 \subset E_1 \cap E_2'$ and $a_2 \notin E_2$.

Assume inductively that we have constructed strongly convex functions $\tau_j : \mathbb{R}^m \to \mathbb{R}$ for $j = 1, \ldots, k$ such that the sets $E_j = \{\tau_j \le 0\}$ satisfy $E \subset E_k \subset E_{k-1} \cdots \subset E_1$, $\tau_j \le c_j < 0$ on E for all $j = 1, \ldots, k$, and $\tau_j(a_i) > 0$ for $1 \le i \le j$). We then take

$$\tau_{k+1} = \mathrm{rmax}\{\tau_k, \rho_{k+1}\}$$

for a suitable choice of the weight function ξ in rmax to obtain the next strongly convex function τ_{k+1} and the next set $E_{k+1} = \{\tau_{k+1} \le 0\}$ in the sequence. ■

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