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On orders of automorphisms of vertex-transitive graphs [☆]

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In this paper we investigate orders, longest cycles and the number of cycles of automorphisms of finite vertex-transitive graphs. In particular, we show that the order of every automorphism of a connected vertex-transitive graph with n vertices and of valence d, $d \leq 4$, is at most $c_d n$ where $c_3 = 1$ and $c_4 = 9$. Whether such a constant c_d exists for valencies larger than 4 remains an unanswered question. Further, we prove that every automorphism g of a finite connected 3-valent vertex-transitive graph Γ, $\Gamma \ncong K_{3,3}$, has a regular orbit, that is, an orbit of $\langle g\rangle$ of length equal to the order of g. Moreover, we prove that in this case either Γ belongs to a well understood family of exceptional graphs or at least $5/12$ of the vertices of Γ belong to a regular orbit of g. Finally, we give an upper bound on the number of orbits of a cyclic group of automorphisms C of a connected 3-valent vertex-transitive

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graph Γ in terms of the number of vertices of Γ and the length of a longest orbit of C.

1. Introduction

For a permutation g of a finite non-empty set Ω, let

$$\begin{aligned} o(g) &= \text{ the order of } g; \\ \ell(g) &= \text{ the length of the longest orbit of } \langle g\rangle; \\ \mu(g) &= \text{ the number of orbits of } \langle g\rangle. \end{aligned}$$

Equivalently, if the permutation g is written as a product of disjoint cycles, then $\mu(g)$ equals the number of cycles (including those of length 1), $\ell(g)$ represents the length of the longest of these cycles, while $o(g)$ equals the least common multiple of the lengths of all the cycles of g.

Further, for a permutation group $G \le \mathrm{Sym}(\Omega)$ we let

$$\mathrm{meo}(G) = \max_{g\in G} o(g), \quad \ell(G) = \max_{g\in G} \ell(g), \quad \mu(G) = \min_{g\in G} \mu(g), \quad \mathrm{meo}_\circ(G) = \max_{\omega\in\Omega} \mathrm{meo}(G_\omega),$$

where G_ω denotes the stabiliser of $\omega \in \Omega$ in G. These parameters are of course not mutually independent (see Lemma 2.1 and Lemma 2.4, for example).

Studying maximal orders of permutations and the longest orbits of cyclic groups have received considerable attention in the theory of finite groups, especially in the context of primitive permutation groups [16–18]. In this paper we propose the investigation of the above parameters in a graph theoretical setting. In particular, for a finite graph Γ with the automorphism group $\mathrm{Aut}(\Gamma)$, we are interested in the following invariants:

$\mathrm{meo}(\Gamma)$	$=$	$\mathrm{meo}(\mathrm{Aut}(\Gamma))$	...	*maximal order of an automorphism*;
$\ell(\Gamma)$	$=$	$\ell(\mathrm{Aut}(\Gamma))$	...	*maximal length of an orbit of an automorphism*;
$\mu(\Gamma)$	$=$	$\mu(\mathrm{Aut}(\Gamma))$	...	*minimal number of orbits of a non-trivial automorphism*;
$\mathrm{meo}_\circ(\Gamma)$	$=$	$\mathrm{meo}_\circ(\mathrm{Aut}(\Gamma))$	...	*maximal order of an automorphism fixing a vertex.*

There is a number of interesting problems related to these parameters, especially in the context of connected vertex-transitive graphs. Let us mention a few.

The first question is related to the existence and possible classification of graphs admitting an automorphism with a large order, that is, graphs Γ with a large value of $\mathrm{meo}(\Gamma)$.

A classical result of Landau [23] states that $\mathrm{meo}(\mathrm{Sym}(n)) = e^{(1+o(1))(n \log n)^{1/2}}$ giving a sub-exponential upper bound on $\mathrm{meo}(\Gamma)$ in terms of the size of the vertex-set $\mathrm{V}(\Gamma)$ of Γ, which is met by the complete graphs and their complements. A more interesting question to ask is whether a better, possibly linear bound holds for connected graphs of fixed valence.

Question 1.1. For which positive integers d does there exist a constant c_d such that every connected d-valent vertex-transitive graph Γ with n vertices satisfies $\mathrm{meo}(\Gamma) \le c_d n$?

In this paper, we answer this question for valencies 3 and 4. In particular, we prove the following:

Theorem 1.2. *Let Γ be a finite connected vertex-transitive graph of valence $d \in \{3, 4\}$ and let n be the number of vertices of Γ. If $d = 3$, then $\mathrm{meo}(\Gamma) \le n$, and if $d = 4$, then $\mathrm{meo}(\Gamma) \le 9n$.*

While the constant $c_3 = 1$ is sharp, as witnessed by cubic circulants, we have no examples of vertex-transitive graphs Γ of valence 4 admitting an automorphism of order $9n$. The smallest possible value of the constant c_4 might thus be as small as 1 (perhaps allowing a finite number of exceptional graphs).

Furthermore, a variant of the Thompson-Wielandt theorem [44, Corollary 3] together with an easy arithmetic argument allowed us to deduce the following fact about arc-transitive locally semiprimitive graphs (recall that Γ is said to be *G-arc-transitive* if $G \le \mathrm{Aut}(\Gamma)$ acts transitively on the set of ordered pairs of adjacent vertices, called *arcs*; for the definition of local semiprimitive, see Section 6).

Theorem 1.3. *For every positive integer d there exists a constant c_d such that every connected d-valent G-arc-transitive G-locally-semiprimitive graph Γ with n vertices satisfies $\mathrm{meo}(G) \le c_d n$.*

While mere existence of the constant c_d as in Question 1.1 often suffices for theoretical applications, finding as small a constant c_d as possible is very desirable for practical applications, such as compiling exhaustive lists of graphs of a given symmetry type (see, for example, [3,30]).

What is more, it is often helpful to investigate structural properties inferred by existence of an automorphism of a large order. The following result, which was recently used together with [31,32] to prove a classification [33] of cubic vertex-transitive graphs Γ with $\mathrm{meo}(\Gamma) \ge |\mathrm{V}(\Gamma)|/3$, gives an explicit upper bound on the number of orbits that an automorphism of relatively large order has. (The term *cubic graph* is used throughout the paper to refer to a finite connected graph every vertex of which has valence 3.)

Theorem 1.4. *Let Γ be a cubic vertex-transitive graph with n vertices and let $g \in \mathrm{Aut}(\Gamma)$. Then $\mu(g) \le \frac{17n}{6\, o(g)}$. In particular, $\mu(\Gamma) \le \frac{17n}{6\, \mathrm{meo}(\Gamma)}$.*

Since clearly $\mathrm{meo}(G) \le \mathrm{meo}_\circ(G)n \le |G_\omega|n$ holds for every transitive permutation group G acting on a set of size n, one way of bounding the parameter $\mathrm{meo}(\Gamma)$ by a linear function of $|\mathrm{V}(\Gamma)|$ is to bound the parameter $\mathrm{meo}_\circ(\Gamma)$ (or even the order of a vertex-stabiliser $\mathrm{Aut}(\Gamma)_v$) by a constant. Bounding the order of $\mathrm{Aut}(\Gamma)_v$ is a classical topic in algebraic graph theory, going back to the work of Tutte [49] where he proved that $|G_v| \le 48$ for every cubic G-arc-transitive graph. This result does not generalise to higher non-prime valencies. However, a long-standing conjecture of Richard Weiss [51] states that for every fixed valence d there exists a constant c_d such that every connected G-arc-transitive G-locally-primitive graph of valence d satisfies $|G_v| \le c_d$, and thus $\mathrm{meo}(G) \le c_d n$. This observation puts Theorem 1.3 into the context of the Weiss Conjecture (and its recent generalisation [29] to locally-semiprimitive graphs).

On the other hand, it is well known that connected vertex-transitive graphs can have an arbitrarily large vertex-stabiliser while still having the order of an element in a vertex-stabiliser bounded by a constant (consider, for example, the family of lexicographic products $C_n[2K_1]$ of a cycle C_n with the edgeless graph on two vertices, where the order of the vertex-stabiliser grows exponentially with n but the automorphisms fixing a vertex have order at most 4). However, there are no known infinite families of connected vertex-transitive graphs $\{\Gamma_n : n \in \mathbb{N}\}$ of fixed valence d such that $\mathrm{meo}_\circ(\Gamma_n) \to \infty$ as $n \to \infty$. This prompted Pablo Spiga to ask whether for every valence d there exists a constant c_d such that every connected d-valent vertex-transitive graph Γ satisfies $\mathrm{meo}_\circ(\Gamma) \le c_d$. Observe that a positive answer to this question would also resolve Question 1.1. Here we consider this question in the case of valence 3 and prove the following result (see Section 3 for the proof).

Theorem 1.5. *If Γ is a connected 3-valent vertex-transitive graph, then* $\mathrm{meo}_\circ(\Gamma) \le 6$.

Let us now discuss the relationship between the parameters $\ell(\Gamma)$ and $\mathrm{meo}(\Gamma)$. Note that $\ell(g)$ divides $o(g)$ for every permutation g. When the equality $\ell(g) = o(g)$ holds, we call an orbit of length $\ell(g)$ a *regular orbit* of $\langle g \rangle$ (or, to simplify terminology, a regular orbit of g). The question which permutation groups have the property that all of their elements possess a regular orbit has received considerable attention (see, for instance, [13,41,42]), especially in the context of primitive permutation groups. This work culminated in [18], where it was proved that a primitive group that does not have this property must preserve a product structure. Since a transitive permutation group $G \le \mathrm{Sym}(\Omega)$ is primitive if and only if every graph Γ with at least one edge such that $\mathrm{V}(\Gamma) = \Omega$ and $G \le \mathrm{Aut}(\Gamma)$ is connected, it is natural to ask to what extent do results about regular orbits in primitive permutation groups extend to automorphism groups of connected vertex-transitive graphs. In this paper, we consider this question in the setting of cubic graphs.

Theorem 1.6. *If Γ is a cubic vertex-transitive graph not isomorphic to the complete bipartite graph $\mathrm{K}_{3,3}$, then every automorphism of Γ has a regular orbit and thus $o(\Gamma) = \ell(\Gamma)$.*

If in addition Γ *is not isomorphic to the complete graph* K_4*, the cube graph* Q_3*, the Petersen graph, the Möbius-Kantor graph, the Pappus graph or the Heawood graph, then for every* $g \in \mathrm{Aut}(\Gamma)$*, either* $\langle g \rangle$ *is transitive on* $\mathrm{V}(\Gamma)$ *or every regular orbit of* g *is adjacent to another regular orbit of* g*.*

Next, we prove that excluding $\mathrm{K}_{3,3}$ and an exceptional family of Split Praeger-Xu graphs, regular orbits of any automorphism cover a large part of every cubic vertex-transitive graph.

Theorem 1.7. *Let* Γ *be a cubic vertex-transitive graph of order* n *and let* $g \in \mathrm{Aut}(\Gamma)$*. If* Γ *is not isomorphic to* $\mathrm{K}_{3,3}$ *or a Split Praeger-Xu graph (defined in Section 5), then at least* $\frac{5}{12}n$ *vertices of* Γ *lie on a regular orbit of* g*.*

The above results all deal with the situation where a graph admits an automorphism of relatively large order. It is, however, a very interesting question what can be said about vertex-transitive graphs the automorphisms of which all have small order. To be more precise, consider the function

$$M_d(n) = \min\{\mathrm{meo}(\Gamma) : \Gamma \text{ is a connected } d\text{-valent vertex-transitive graph on at least } n \text{ vertices}\}.$$

As we show in Section 8, the function $M_d(n)$ is unbounded for every fixed valence $d \geq 3$; or in other words, for every constant c there are only finitely many finite connected d-valent vertex-transitive graphs Γ with $\mathrm{meo}(\Gamma) \leq c$. On the other hand, the function $M_d(n)$ grows extremely slowly; indeed slower than composition of any finite number of logarithms. To quantify this statement, let

$${}^n a = \underbrace{a^{a^{\cdot^{\cdot^{a}}}}}_{n}$$

for every positive integers n and a, and let

$$\mathrm{slog}_a(n) = \max\{i \in \mathbb{N} : {}^i a \leq n\}.$$

The functions $n \mapsto {}^n a$ and $n \mapsto \mathrm{slog}_a(n)$ are sometimes called the *tetration* [15] and the *superlogarithm* [54]. They are clearly inverse to each other in the sense that $\mathrm{slog}_a({}^i a) = i$, or more generally,

$$\mathrm{slog}_a(n) = i \quad \text{if and only if} \quad {}^i a \leq n < {}^{i+1} a.$$

We can now state the following theorem about the asymptotic behaviour of the function $M_d(n)$.

Theorem 1.8. *Let d be an integer greater or equal to 3. Then*

$$\lim_{n\to\infty} M_d(n) = \infty,$$

however, there exists a constant k_d such that

$$M_d(n) \le k_d\, 2^{\mathrm{slog}_2(n)}$$

holds for every positive integer n.

We also like to draw the readers attention to an interesting connection between Theorem 1.8 and the restricted Burnside problem. Observe first that since every finite connected d-valent vertex-transitive graph admits a vertex-transitive subgroup that can be generated by d elements (see Lemma 8.1), the fact that $M_d(n)$ is unbounded follows directly from the solution of the restricted Burnside problem. On the other hand, the crucial step in the proof of the slow growth of the function $M_d(n)$ consists of a construction of an infinite family of d-generated finite groups G_i, $i \in \mathbb{N}$, of exponent 2^i and order larger than ${}^{i}2$, yielding a non-trivial lower bound on the order of the universal finite d-generated group $R(d, 2^i)$ of exponent 2^i. The ideas in our proof of Theorem 1.8 draw heavily from a beautiful paper of Spiga [45], where the solution of the restricted Burnside problem was first applied in the context of vertex-transitive graphs, and the paper of Vaughan-Lee and Zel'manov [50] where bounds on the order of the universal groups $R(d, m)$ were considered.

We would also like to draw the reader's attention to a very recent result [1] related to Theorem 1.8, which roughly speaking states that, if we restrict the problem to semiregular automorphisms, then the minimum among all cubic vertex-transitive graphs of the maximum order of a semiregular automorphism is 6.

After proving some auxiliary results in Section 2, we then prove Theorem 1.5 and Theorem 1.8 in Section 3, Theorem 1.6 and Theorem 1.2 for the case of cubic graphs in Section 4, Theorem 1.4 and Theorem 1.7 in Section 5, and Theorem 1.2 for the case of quartic graphs in Section 7. The asymptotic behaviour of the function $M_d(n)$ is discussed in Section 8, where the proof of Theorem 1.8 is given.

2. Auxiliary results

In this section we prove a few easy auxiliary results about the parameters introduced in Section 1. We begin with the following observation.

Lemma 2.1. *Let G be a transitive permutation group on a set Ω, let $g \in G$, let $s(g)$ be the length of a shortest orbit of the cyclic group $\langle g\rangle$ and let $s(G) = \max\{s(g) : g \in G\}$. Then:*

$$\ell(g) \le o(g) \le s(g)\mathrm{meo}_\circ(G) \le \ell(g)\mathrm{meo}_\circ(G); \tag{2.1}$$

$$\ell(G) \le \mathrm{meo}(G) \le s(G)\mathrm{meo}_\circ(G) \le \ell(G)\mathrm{meo}_\circ(G), \tag{2.2}$$

where G_ω is the stabiliser of an arbitrary element $\omega \in \Omega$.

Proof. Let $\omega \in \Omega$, let $C = \langle g\rangle$, let $t = |\omega^C|$ and let $m = \mathrm{meo}_\circ(G)$; since G is transitive on Ω, the parameter m equals $\mathrm{meo}(G_\omega)$ for every $\omega \in \Omega$. Since $s(g) \le \ell(g) \le o(g)$ holds, in order to prove (2.1), it suffices to show that $o(g) \le s(g)m$. To do this, observe that $C_\omega = \langle g^t\rangle$, implying that $|C_\omega| = o(g^t) \le \mathrm{meo}(G_\omega) = m$. By the orbit-stabiliser lemma applied to the action of C on Ω we may thus conclude that

$$o(g) = |C| = |C_\omega|\,|\omega^C| \le m|\omega^C|.$$

If ω was chosen to be in a shortest orbit of C, then $|\omega^C| = s(g)$ and thus $o(g) \le ms(g)$, as required. Part (2.2) now follows easily from (2.1) if we maximise the expressions over all $g \in G$, starting with the rightmost expression in the chain of inequalities of (2.1) and then proceeding towards the left-hand side. □

The following lemma and its corollary give us an elementary but useful tool that can be used to bound the order of a permutation by a linear function of the degree of the permutation group. Corollary 2.3 is then used in Section 6 to prove Theorem 1.3

Lemma 2.2. *If g is a permutation on a set Ω and $C = \langle g\rangle$, then*

$$\frac{o(g)}{\ell(g)} = \frac{\min\{|C_\omega| : \omega \in \Omega\}}{\gcd\{|C_\omega| : \omega \in \Omega\}}.$$

Proof. Recall that $o(g) = \mathrm{lcm}\{|\omega^C| : \omega \in \Omega\}$ and that $|\omega^C| = |C|/|C_\omega|$, implying that

$$o(g) = \mathrm{lcm}\{\frac{|C|}{|C_\omega|} : \omega \in \Omega\} = \frac{|C|}{\gcd\{|C_\omega| : \omega \in \Omega\}}.$$

Similarly, $\ell(g) = \max\{|\omega^C| : \omega \in \Omega\}$ and thus

$$\ell(g) = \frac{|C|}{\min\{|C_\omega| : \omega \in \Omega\}},$$

as claimed. □

Recall that the exponent $\exp(G)$ of a group G is the minimum positive integer e such that $g^e = 1$ for every $g \in G$. Clearly $\exp(G)$ equals the least common multiple of the orders of cyclic subgroup of G.

Corollary 2.3. *Let G be a transitive permutation group on a set Ω, let $\omega \in \Omega$, let p be a prime and let k be an integer coprime to p such that $\exp(G_\omega) = kp^\alpha$ for some $\alpha \geq 1$. Then*

$$o(g) \leq k\ell(g)$$

for every $g \in G$. In particular, if G_ω is a p-group, then every element of G has a regular orbit.

Proof. Let $\omega \in \Omega$. Since C_ω is a cyclic subgroup of G_ω, we see that $|C_\omega|$ divides $\exp(G_\omega)$ and thus $|C_\omega| = tp^\beta$ for some $t \leq k$, $\gcd(t,p) = 1$, and $\beta \leq \alpha$. Now let $\{\omega_1, \ldots, \omega_m\}$ be a complete set of representatives of the orbits of C on Ω indexed in such a way that

$$|C_{\omega_1}| \leq |C_{\omega_2}| \leq \ldots \leq |C_{\omega_m}|.$$

Write $|C_{\omega_i}| = t_i p^{\alpha_i}$ with $\gcd(t_i, p) = 1$. As observed above, it follows that $t_i \leq k$. Let $j \in \{1, \ldots, m\}$ be such that $\alpha_j = \min\{\alpha_i : i \in \{1, \ldots, m\}\}$. Then

$$\gcd\{|C_\omega| : \omega \in \Omega\} = \gcd\{t_i p^{\alpha_i} : i \in \{1, \ldots, m\}\} \geq p^{\alpha_j}.$$

On the other hand, $\min\{|C_\omega| : \omega \in \Omega\} = t_1 p^{\alpha_1} \leq t_j p^{\alpha_j} \leq kp^{\alpha_j}$. The result then follows by Lemma 2.2. □

The final lemma of the section provides an upper bound on the number of orbits of a cyclic subgroup of a transitive permutation group G, depending on the length of its longest orbit and the parameter $\mathrm{meo}_\circ(G)$.

Lemma 2.4. *If G is a transitive permutation group on a set Ω of cardinality n and $g \in G$, then*

$$\mu(g) \leq \left(\frac{n}{\ell(g)} - 1\right)\mathrm{meo}_\circ(G) + 1 \quad \textit{and} \quad \mu(G) \leq \left(\frac{n}{\ell(G)} - 1\right)\mathrm{meo}_\circ(G) + 1. \tag{2.3}$$

Proof. As in Lemma 2.1, let $s(g)$ denote the shortest orbits of the group $\langle g \rangle$ and let $s(g) = n_1 \leq n_2 \leq \ldots \leq n_m = \ell(g)$ be the lengths of the orbits of $\langle g \rangle$, counted with multiplicity. Then $\mu(g) = m$ and

$$s(g) + n_2 + \ldots + n_{m-1} + \ell(g) = n,$$

implying that $(m-1)s(g) \leq s(g) + n_2 + \ldots + n_{m-1} = n - \ell(g)$. By Lemma 2.1, $s(g) \geq \ell(g)/\mathrm{meo}_\circ(G)$ and thus

$$m \leq \frac{n - \ell(g)}{s(g)} + 1 \leq \frac{n - \ell(g)}{\ell(g)}\mathrm{meo}_\circ(G) + 1,$$

proving the first inequality in (2.3). The second inequality now follows if we choose g to be an element of G with the smallest number of orbits. $\square$

3. Vertex-stabilisers of cubic vertex-transitive graphs and proof of Theorem 1.5

In this section we prove Theorem 1.5, which states that $\mathrm{meo}_\circ(\Gamma) \leq 6$ for every cubic vertex-transitive graph. As a side result, we also prove the following lemma of independent interest.

Lemma 3.1. *Let Γ be a cubic graph and let G be a vertex-transitive group of automorphisms of Γ. Then G can be generated by 3 elements.*

Our proof of Theorem 1.5 and Lemma 3.1 is based on the work of Djoković and Miller [7] on cubic arc-transitive graphs, a result of Djoković on 4-valent arc-transitive graphs [6], and the splitting and merging operation, introduced in [30], which links the cubic vertex- but not arc-transitive graphs with a class of 4-valent arc-transitive graphs.

For the rest of this section, let Γ be a cubic vertex-transitive graph, let G be a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$ and let v be a vertex of Γ. To prove Theorem 1.5, we need to prove that $o(g) \leq 6$ for every $g \in G_v$, and to prove Lemma 3.1, we need to show that G can be generated by three elements.

Suppose first that Γ is isomorphic to a prism (that is, the Cartesian product $\mathrm{C}_m \square \mathrm{K}_2$ of a cycle C_m, $m \geq 3$, with the complete graph K_2) or to a Möbius ladder (that is, the circulant graph $\mathrm{Cay}(\mathbb{Z}_{2m}; \{\pm 1, m\})$ with $m \geq 2$). If $\Gamma \cong \mathrm{Cay}(\mathbb{Z}_{2m}; \{\pm 1, m\})$, then either $m = 2$ and $\Gamma \cong K_4$, or $m = 3$ and $\Gamma \cong K_{3,3}$, or $m \geq 4$ and $\mathrm{Aut}(\Gamma)$ is the dihedral group D_{2m} of order $4m$ with the vertex-stabiliser of order 2. Clearly in all of these cases, $o(g) \leq 6$ and G can be generated by at most three elements. On the other hand, if $\Gamma \cong \mathrm{C}_m \square \mathrm{K}_2$, then either $m = 4$ and Γ is the cube Q_3 or $m \neq 3$ and $\mathrm{Aut}(\Gamma) \cong \mathrm{D}_m \times \mathrm{C}_2$ with the vertex-stabiliser having order 2. Again, in all cases, $o(g) \leq 6$ and G can be generated by at most three elements. We will thus assume for the rest of this section that Γ is neither a prism nor a Möbius ladder.

Suppose that G has m orbits in its action on the arcs of Γ. Then, since Γ is vertex-transitive, we have $m \in \{1, 2, 3\}$.

If $m = 3$, then the connectivity of Γ implies that the stabilizer G_v in G of any vertex $v \in \mathrm{V}(\Gamma)$ is trivial. Moreover, Γ is a Cayley graph on the group G and thus G is generated by the three elements mapping a fixed vertex v to the three neighbours of v. Hence $o(g) = 1$ and G can be generated by three elements, as claimed.

If $m = 1$, then Γ is arc-transitive. By a result of Djoković and Miller [7] (which is based on the celebrated work of Tutte [49] on cubic arc-transitive graphs), G_v is isomorphic to either $\mathbb{Z}_3$, S_3, $S_3 \times S_2$, S_4 or $S_4 \times S_2$. In none of the five possible cases, an element of G_v has order greater than 6. Moreover, as was showed [7], the arc-transitive group G can be generated by three carefully selected elements (see also [4]).

We are thus left with the case where $m = 2$. Then G_v must fix an edge x incident to v. Let $\mathcal{T}$ be the orbit of x under the action of G and observe that $\mathcal{T}$ is a perfect matching

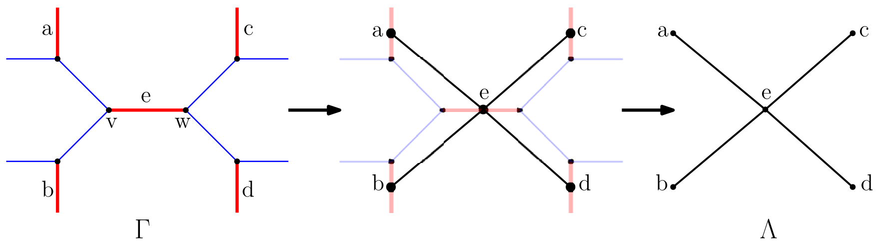

Fig. 3.1. The neighbourhood of an edge with endpoints v and w in Γ, and the neighbourhood of the corresponding vertex in Λ.

in Γ. Moreover, G is transitive on both $\mathrm{A}(\Gamma) \setminus \mathcal{T}^*$ and $\mathcal{T}^*$, where $\mathcal{T}^*$ denotes the arcs of Γ underlying edges in $\mathcal{T}$.

We can thus construct a connected graph Λ as follows. Let $\mathcal{T}$ be the vertex-set of Λ and let $uv \in \mathcal{T}$ be adjacent to $ab \in \mathcal{T} \setminus \{uv\}$ in Λ if and only if there exist an edge $xy \in \mathrm{E}(\Gamma)$ such that $x \in \{u, v\}$ and $y \in \{a, b\}$. Informally, Λ is constructed from Γ by contracting every edge in $\mathcal{T}$ (see [30, Construction 7 in Section 4] for details about this construction and the graph Λ). Since Γ is neither a prism nor a Möbius ladder, then by [30, Lemma 9], Λ is a simple tetravalent graph. Clearly every automorphism in G induces an automorphism of Λ. Since G is transitive on $\mathrm{A}(\Gamma) \setminus \mathcal{T}^*$, we see that G acts arc-transitively on Λ.

Let $e \in \mathcal{T}$ have endpoints v and w and assume the notation in Fig. 3.1. Then $N := \{a, b, c, d\}$ is the set of neighbours of e in Λ. Since G acts transitively on the arcs of Λ, it follows that the permutation group G_e^N induced by the action of G_e on the set N is transitive. Observe also that $\{\{a, b\}, \{c, d\}\}$ is a G_e-invariant partition of N and that G_v is isomorphic to the subgroup of G_e that fixes $\{a, b\}$ and $\{c, d\}$ set-wise. Hence, G_e^N is permutation isomorphic to one of the three imprimitive transitive groups of degree four: D_4, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$.

If $G_e^N \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$, then it follows from the connectivity of Λ that $G_e \cong \mathbb{Z}_4$ or $G_e \cong \mathbb{Z}_2^2$, respectively, and thus $o(g) \leq 4$. Moreover, G is generated by the vertex-stabiliser G_e and any automorphism mapping e to a neighbour of e in Λ, and therefore G is generated by at most 3 elements, as claimed.

Suppose now that $G_e^N \cong \mathrm{D}_4$. Then, $(G_e, G_{(e,a)}, G_{\{e,a\}})$ (where (e, a) denotes the arc in Λ and $\{e, a\}$ denotes the underlying undirected edge) is a dihedral amalgam of type $(4, 2)$ (see [6]). It follows from the main theorem of [6] that the subgroup H of G_e that fixes both $\{a, b\}$ and $\{c, d\}$ set-wise satisfies the following:

$$
\begin{aligned}
H &= \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle, \\
a_i^2 &= 1, \qquad 0 \leq i \leq n-1, \\
[a_i, a_j] &= r_{i,j} \qquad 0 \leq i, j \leq n-1,
\end{aligned}
$$

where each $r_{i,j}$ has order 2 and lies in the center $\mathrm{Z}(H)$ of H. In particular $N/\mathrm{Z}(H)$ is an elementary abelian 2-group, implying that $g^2 \in \mathrm{Z}(H)$ for every $g \in H$. However since

N is generated by involutions, the exponent of $\mathrm{Z}(H)$ is at most 2, implying that the exponent of N is at most 4. Then, since $G_v \cong H$, we see that G_v has exponent at most 4. Moreover, it follows from the main theorem of [6] that $G = \langle x, y, H\rangle$ where x, y are two elements of G such that the set of conjugates a_0^z with $z \in \langle x, y\rangle$ contains all the generators a_i, $i \in \{0, \ldots, n-1\}$, of H. In particular, $G = \langle x, y, a_0\rangle$.

We have shown that in all possible cases, the order of an element $g \in G_v$ is at most 6 and that G is generated by at most 3 elements, thus concluding the proofs of Theorem 1.5 and of Lemma 3.1.

4. Regular orbits of cubic vertex-transitive graphs and proof of Theorem 1.6

For a graph Γ and a group $G \le \mathrm{Aut}(\Gamma)$, we let Γ/G be the graph whose vertices are the G-orbits of vertices in Γ and two such G-orbits X and Y are adjacent if there is a vertex in X adjacent to a vertex in Y. Recall that a vertex-orbit of the cyclic group $\langle g\rangle$, $g \in \mathrm{Aut}(\Gamma)$, is called *regular* provided that its length equals the order of g; in other words, if its length is divisible by the length of every other orbit of $\langle g\rangle$. We will show that with a few exceptions, every automorphism of a cubic vertex-transitive graph admits a regular orbit (Theorem 4.7), and if the cyclic group generated by such an automorphism is non-transitive, then every regular orbit is adjacent to another regular orbit. These two results combined give us Theorem 1.6. We would like to point out that Theorem 4.7 follows at once from Corollary 2.3 if we restrict it to cubic graphs that are vertex- but not arc-transitive (as in this case the stabiliser of any vertex is a 2-group). However, the arc-transitive case requires a little bit more work. We will need to prove a series of rather simple, but useful lemmas, about the relative sizes of adjacent orbits.

In what follows, we let K_4 denote the complete graph on 4 vertices, $K_{3,3}$ the complete bipartite graph with 3 vertices in each part, and let Q_3 denote the tridimensional cube graph. For integers n and k, the generalised Petersen graph $\mathrm{GP}(n, k)$ is the graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and with edges of the form $x_i x_{i+1}$, $x_i y_i$ and $y_i y_{i+k}$. The well-known Pappus graph and Heawood graph are depicted in Fig. 4.1. The following three lemmas are useful special cases of [35, Theorem 5].

Lemma 4.1. *If Γ is a cubic arc-transitive graph of girth smaller than 6, then Γ is isomorphic to one of the following: K_4, $K_{3,3}$, Q_3, the Petersen graph $\mathrm{GP}(5, 2)$ or the dodecahedron $\mathrm{GP}(10, 2)$.*

Lemma 4.2. *Let Γ be a cubic vertex-transitive graph of girth 4. If there is a vertex u such that all three edges incident to u lie on two distinct 4-cycles, then Γ is isomorphic to $K_{3,3}$ or Q_3.*

Lemma 4.3. *Let Γ be a cubic vertex-transitive graph of girth 3. If any of the following conditions hold, then $\Gamma \cong K_4$:*

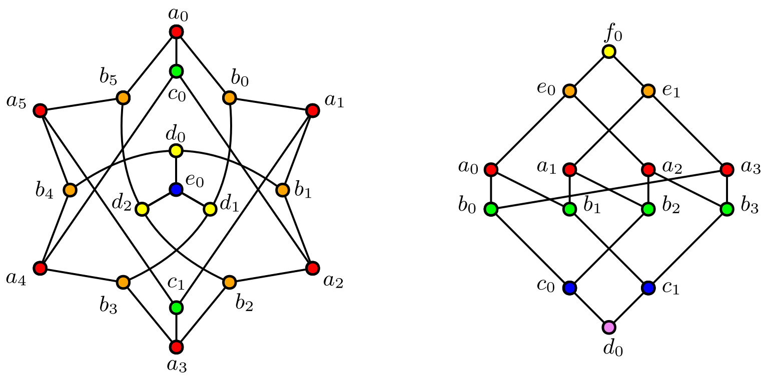

Fig. 4.1. The Pappus graph (left) and the Heawood graph (right). For each graph, vertices of the same colour belong to the same orbit under the action of the automorphism φ that acts by adding 1 to the sub-index of each vertex.

(1) *there is a vertex u such that all three edges incident to u lie on a 3-cycle;*
(2) *there is an edge e that lies on two distinct 3-cycles.*

Lemma 4.4. *Let Γ be a cubic vertex-transitive graph, let $g \in \mathrm{Aut}(\Gamma)$ and let $G = \langle g \rangle$. Let u and v be two adjacent vertices belonging to distinct G-orbits. If $i|u^G| = 3|v^G|$ for some $i \in \{1, 2\}$, then Γ is arc-transitive. In particular, if $i = 2$ then $\Gamma \cong K_{3,3}$.*

Proof. Let $n \in \mathbb{Z}$ be such that $3n = |u^G|$ and $i \cdot n = |v^G|$. Observe that $v = v^{g^{2n}} = v^{g^{4n}}$ but that u, $u^{g^{2n}}$ and $u^{g^{4n}}$ are all distinct. Moreover, since $u \sim v$, we have $u^{g^{2n}} \sim v^{g^{2n}}$ and $u^{g^{4n}} \sim v^{g^{4n}}$. Then v is adjacent to u, $u^{g^{2n}}$ and $u^{g^{4n}}$ and since $u^{g^{4n}} = u^{g^{3n}g^n} = u^{g^n}$, we see that $\Gamma(v) = \{u, u^{g^n}, u^{g^{2n}}\}$. The group $\langle g^{2n} \rangle$ then fixes v while permuting its neighbours. It follows that all three edges incident to v belong to the same $\langle g^{2n} \rangle$-orbit. Then, since Γ is vertex-transitive, it must also be edge-transitive. Moreover, a cubic graph that is both vertex- and edge-transitive must necessarily be arc-transitive.

Now, suppose $i = 2$. Then $|v^G| = 2n$ and thus $v \neq v^{g^n}$. We have shown that $\Gamma(v) = \{u, u^{g^n}, u^{g^{2n}}\}$, and thus $\Gamma(v^{g^n}) = \{u^{g^n}, u^{g^{2n}}, u^{g^{3n}}\} = \{u, u^{g^n}, u^{g^{2n}}\}$. Then every vertex in $\{v, v^{g^n}\}$ is adjacent to every vertex in $\{u, u^{g^n}, u^{g^{2n}}\}$. That is, Γ contains a copy of $K_{2,3}$ and by Lemma 4.1, Γ is isomorphic to K_4, $K_{3,3}$ or Q. However, neither K_4 nor Q contain a subgraph isomorphic to $K_{2,3}$. We conclude that $\Gamma \cong K_{3,3}$. $\square$

Lemma 4.5. *Let Γ be a cubic vertex-transitive graph other than $K_{3,3}$. Let $G \leq \mathrm{Aut}(\Gamma)$ be cyclic, and let u and v be two adjacent vertices belonging to distinct G-orbits. If $|u^G| \geq |v^G|$, then $|u^G| = i|v^G|$ for some $i \in \{1, 2, 3\}$. Moreover, if $i \neq 1$, then every vertex in u^G has exactly one neighbour in v^G, while a vertex in v^G has i neighbours in u^G.*

Proof. Let $n = |u^G|$ and $m = |v^G|$ and note that $u^{g^i} = u$ if and only if $n|i$. Similarly, $v^{g^i} = v$ if and only if $m|i$. It follows that v is adjacent to $u^{g^{im}}$ for all $i \in \mathbb{Z}$. Let λ be the number of neighbours of v in u^G. Clearly, $\lambda \in \{1,2,3\}$. If $\lambda = 1$, then $u^{g^m} = u$ and thus $n|m$. Since by hypothesis $n \geq m$, we see that $n = m$. If $\lambda = 2$, then $u^{g^{2m}} = u$. It follows that $n|2m$ and thus $n = 2m$ or $n = m$. Finally, if $\lambda = 3$, then $u^{g^{3m}} = u$ and $n|3m$. Then $n = 3m$, $n = \frac{3}{2}m$ or $n = m$. However, if $n = \frac{3}{2}m$, then by Lemma 4.4, $\Gamma \cong K_{3,3}$, a contradiction. It follows that $n = 3m$ or $n = m$. Therefore $|u^G| = i|v^G|$ for some $i \in \{1,2,3\}$.

To prove the second part of the statement, suppose $i \neq 1$ and that u is adjacent to a vertex v^{g^j} for some $0 < j < m$. Note that v^{g^j} is also adjacent to u^{g^m}, u^{g^j} and u^{g^j+m}. Since $0 < j < m$ and $|u^G| \geq 2m$, we see that the vertices in $\{u, u^{g^m}, u^{g^j}, u^{g^j+m}\}$ are all distinct. Thus, v has four neighbours, contradicting Γ being a cubic graph. It follows that a vertex in u^G has exactly one neighbour in v^G. In particular, this means there are exactly $|u^G| = im$ edges between u^G and v^G, and thus a vertex in v^G has exactly i neighbours in u^G. $\square$

Corollary 4.6. *Let Γ be a cubic vertex-transitive graph, let $G = \langle g \rangle \in \mathrm{Aut}(\Gamma)$ and let u and v be two adjacent vertices. If $|u^G| = 3|v^G|$ then u^G is the only neighbour of v^G in Γ/G.*

Proof. By Lemma 4.5, every vertex in v^G has three neighbours in u^G, and thus no neighbours in orbits other than u^G. Hence u^G is the only neighbour of v^G in Γ/G. $\square$

In the next theorem, we prove the first part of Theorem 1.6.

Theorem 4.7. *If Γ is a cubic vertex-transitive graph other than $K_{3,3}$, then every $g \in \mathrm{Aut}(\Gamma)$ admits a regular orbit and every orbit of $\langle g \rangle$ has size $\frac{\ell(g)}{k}$ for some $k \in \{1,2,3,4,6\}$.*

Proof. Let $g \in \mathrm{Aut}(\Gamma)$, $G = \langle g \rangle$ and let $u \in \mathrm{V}(\Gamma)$ be such that $\ell(g) = |u^G| \geq |v^G|$ for all $v \in \mathrm{V}(\Gamma)$. We will show that the size of any G-orbit divides $|u^G|$. Let $v \in \mathrm{V}(\Gamma) \setminus \{u^G\}$.

Consider the quotient graph Γ/G. Since Γ is connected so is Γ/G and thus there exists a $u^G v^G$-path $W = u_0^G\, u_1^G\, \ldots\, u_n^G$ where $u_0 = u$, $u_n = v$ and each u_i is adjacent to u_{i+1} in Γ.

Suppose that for some $i, j \in \{0,\ldots,n\}$ and $m \in \mathbb{Z}$, we have $|i-j| = 1$ and $|u_i^G| = 3|u_j^G| = 3m$. Then by Corollary 4.6, u_i^G is the only neighbour of u_j^G in Γ/G. It follows that $j = 0$ (and thus $i = 1$) or $j = n$ (and thus $i = n-1$). However, since $|u_0^G| \geq |u_1^G|$ (by our assumption on u), we see that $j \neq 0$ and thus $j = n$. This together with Lemma 4.5 implies that for all $i \in \{0,\ldots,n-2\}$ we have $|u_i| = k|u_{i+1}|$ for some $k \in \{\frac{1}{2},1,2\}$. Then $|u_{n-1}^G| = \frac{1}{2^r}|u_0^G|$ for some integer r, $0 \leq r \leq n-1$, and thus $|v^G| = |u_n^G| = \frac{1}{3\cdot 2^r}|u_0^G| = \frac{1}{3\cdot 2^r}|u^G|$.

Now, if no two orbits u_i^G and u_j^G satisfy $|u_i^G| = 3|u_j^G|$, then by Lemma 4.5, we have $|u_i| = k_i|u_{i+1}|$ with $k_i \in \{\frac{1}{2},1,2\}$ for all $i \in \{0,\ldots,n-1\}$, and thus $|u^G| = \frac{1}{2^r}|v^G|$ for some $r \in \{0,\ldots,n\}$.

Let $m = |v^G|$. We have shown above that $o(g) = \ell(g) = |u^G| = km$ where k is an integer not divisible by any prime larger than 3. In particular, u^G is a regular orbit of g. Observe that $g^m \in G_v$ and that $|u^{g^m}| = k$. In particular, g^m is an element fixing a vertex of order at least k. But then Theorem 1.5, which was proved in Section 3, implies that $k \leq 6$ and thus $k \in \{1, 2, 3, 4, 6\}$. □

Corollary 4.8. *Theorem 1.2 holds in the case $d = 3$. That is,* $\mathrm{meo}(\Gamma) \leq n$ *for every cubic vertex-transitive graph Γ with n vertices.*

Proof. Observe that in the case $\Gamma \ncong K_{3,3}$ this follows directly from the existence of a regular orbit guaranteed by Theorem 4.7. On the other hand, if $\Gamma \cong K_{3,3}$, then $\mathrm{meo}(K_{3,3}) = 6 = n$. □

Remark 4.9. Observe that for every $k \in \{1, 2, 3, 4, 6\}$, there exists a graph Γ and an automorphism $g \in \mathrm{Aut}(\Gamma)$ such that g has an orbit of size $\ell(g)/k$. The Pappus graph admits an automorphism g with orbits of sizes 6, 3, 2 and 1 while the Heawood graph admits an automorphism with orbits of size 4, 2 and 1 (see Fig. 4.1).

Lemma 4.10. *Let Γ be a cubic vertex-transitive graph, let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group, and let $u, v, w \in \mathrm{V}(\Gamma)$ such that $u \sim v$ and $u \sim w$. If $|u^G| = i|v^G| = i|w^G|$ for some $i \in \{2, 3\}$ then the girth of Γ is at most 4 and if $i = 3$, then Γ is isomorphic to K_4, $K_{3,3}$ or the cube graph Q_3.*

Proof. Let g be a generator of G and set $m = |v^G| = |w^G|$. Then $u^{g^m} \neq u$, but $v^{g^m} = v$ and $w^{g^m} = w$. It follows that $u^{g^m} \sim v$ and $u^{g^m} \sim w$ and thus (u, v, u^{g^m}, w) is a 4-cycle. Moreover, if $i = 3$, then by Lemma 4.4, Γ is arc-transitive and by Lemma 4.1, Γ is isomorphic to K_4, $K_{3,3}$ or Q_3. □

The following remark, which will be used in the proof of Theorem 1.4, is a consequence of Corollary 4.6, Lemma 4.5, and Theorem 4.7.

Lemma 4.11. *Suppose Γ is a cubic vertex-transitive graph other than K_4, $K_{3,3}$ or the cube Q_3, let $g \in \mathrm{Aut}(\Gamma)$. Then the following hold.*

(1) *for $i \in \{1, 2\}$, an orbit of size $\ell(g)/3i$ is adjacent to only one orbit, which has size $\ell(g)/i$;*
(2) *for $i \in \{1, 2\}$ an orbit of size $\ell(g)/i$ is adjacent to at most one orbit of size $\ell(g)/3i$;*
(3) *if v^G has size $\ell(g)/4$ then there exists an orbit w^G of size $\ell(g)/2$ (not necessarily adjacent to v^G).*

Proof. Let $u \in \mathrm{V}(\Gamma)$ be such that $|u^G| = \ell(g)$ and let v^G be any other orbit. Since Γ is connected there exists a $v^G u^G$-path $W = v_0^G\, v_1^G\, \dots\, v_n^G$ in Γ/G where $v_0 = v$, $v_n = u$ and each v_i^G is adjacent to v_{i+1}^G.

To show that (1) holds, suppose $|v_0^G| = \ell(g)/3i$. By Lemma 4.5, if $j \in \{0, \ldots, n\}$, then $|v_{j+1}^G| \geq |v_j^G|$ implies $|v_j^G| = k|v_j^G|$ for some $k \in \{1, 2, 3\}$. Then, since $|v_n^G| = 3i|v_0^G|$, there must exist $j \in \{0, \ldots, n\}$ such that $|v_{j+1}^G| = 3|v_j^G|$. By Corollary 4.6, v_{j+1}^G is the only orbit adjacent to v_j^G, which implies that $j = 0$. That is, $v_j^G = v_0^G = v^G$. Thus, v^G is adjacent to only one orbit of size $3|v^G| = 3 \cdot \ell(g)/3i = \ell(g)/i$. Therefore (1) holds.

Now, to show that (3) holds, suppose $|v_0^G| = \ell(g)/4$. Let $j \in \{1, \ldots, n\}$ be the smallest integer for which $|v_j^G| \neq \ell(g)/4$. If $|v_j^G| < |v_{j-1}^G| = \ell(g)/4$, then by Lemma 4.5, $|v_j^G| = |v_{j-1}^G|/k = \ell(g)/4k$ holds for some $k \in \{2, 3\}$, but this contradicts Theorem 4.7. Then $|v_j^G| > |v_{j-1}^G| = \ell(g)/4$, and by Lemma 4.5 we have $|v_j^G| = k \cdot \ell(g)/4$ for some $k \in \{2, 3\}$. However, by Theorem 4.7, $k \neq 3$. It follows that $|v_j^G| = 2\ell(g)/4 = \ell(g)/2$ and thus (3) holds.

Finally, let us show that (2) holds. Suppose $v^G = \ell(g)/i$ for some $i \in \{1, 2\}$. If v has two neighbours w_1 and w_2 such that $|v^G| = 3|w_1^G| = 3|w_2^G|$, then by Lemma 4.10, Γ is isomorphic to K_4, $K_{3,3}$ or Q_3. $\square$

4.1. Proof of Theorem 1.6

We are now ready to finish the proof of Theorem 1.6. For the rest of the section, let Γ be a cubic vertex-transitive graph not isomorphic to $K_{3,3}$, let $g \in \mathrm{Aut}(\Gamma)$ and let $G = \langle g \rangle$. We may assume that G is not transitive on the vertices of Γ. In Theorem 4.7 we have already proved that g has at least one regular orbit, say u^G. Let us now assume in addition that Γ is not isomorphic to K_4, the cube graph Q_3, the Petersen graph, the Möbius-Kantor graph, the Pappus graph or the Heawood graph. We then need to show that every regular orbit of g other than u^G is adjacent to u^G in Γ/G.

Since the order of G is smaller than n, u^G is not the only orbit of G and since Γ is connected, u^G must be adjacent to another orbit in Γ/G. If any of the neighbouring orbits of u^G has size $|u^G|$, then the claim is proved. We shall thus assume that this is not the case. Thus, if v^G is an orbit adjacent to u^G, Lemma 4.5 implies that $|u^G| = k|v^G|$ for some $k \in \{2, 3\}$ and every vertex of u^G is adjacent to precisely k vertices in v^G. There are three cases (which are divided in a total of 7 subcases) to be considered, depending on the numbers of neighbours of u^G in Γ/G.

Case 1: u^G is adjacent to exactly one other orbit v^G, where $u \sim v$. Then $|u^G| = k|v^G|$ for some $k \in \{2, 3\}$.

Case 1.1: *Suppose* $|u^G| = 2|v^G|$. Since u has only one neighbour in v^G and no neighbours in the G-orbits apart from v^G and u^G, it follows that it has two neighbours in u^G, which must be of the form u^{g^i} and $u^{g^{-i}}$ for some $i \in \{1, \ldots, |u^G| - 1|\}$. Let $m = |v^G|$, and observe that $v \sim u$ and $v \sim u^{g^m}$, and the edges vu and vu^{g^m} belong to the same G-orbit. Since Γ is vertex-transitive, there exists $h \in \mathrm{Aut}(\Gamma)$ mapping v to u. Clearly, one edge in $\{vu, vu^{g^m}\}$ is mapped by h to an edge in $\{u^{g^i}, u^{g^{-i}}\}$. It follows that the edges uu^{g^i}, $uu^{g^{-i}}$, vu and v^{g^m} belong to the same $\mathrm{Aut}(\Gamma)$-orbit. That is, Γ is edge-transitive, and since it is cubic and vertex-transitive, it must be arc-transitive. If the girth of Γ is smaller than 6, then 4.1 implies that Γ is isomorphic to K_4, $K_{3,3}$, Q_3, the Petersen

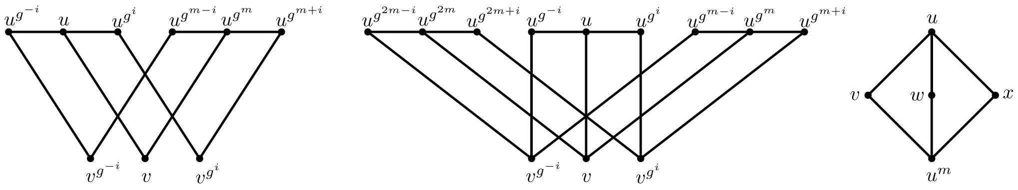

Fig. 4.2. Cycles of length 6 and 4 through u according to Cases 1.1, 1.2 and 3.2.

graph or the dodecahedron graph $\mathrm{GP}(10,2)$. However, among these five graphs only the dodecahedron graph has the property that every automorphism admits two adjacent regular orbits. Thus Γ is isomorphic to K_4, $K_{3,3}$, Q_3, the Petersen graph. Otherwise, observe that for $\alpha \in \{-1,1\}$, $C_\alpha = (v, u, u^{g^{\alpha i}}, v^{g^{\alpha i}}, u^{g^{\alpha i+m}}, u^{g^m})$ is a cycle and thus Γ has girth 6. Moreover, vu and vu^m each lie on both C_0 and C_1 (see Fig. 4.2). Therefore, if we let w be the third neighbour of v (that does not belong to u^G), then the edge vw must lie on one 6-cycle $C \neq C_\alpha$. Clearly C must visit either vu or vu^m, which implies that one edge incident to u (and therefore every edge of Γ by arc-transitivity) lies on at least three 6-cycles. It then follows from [8, Lemma 4.2] that Γ is isomorphic to Pappus Graph, the Heawood graph, the Möbius-Kantor graph $\mathrm{GP}(8,3)$ or the Desargues graph $\mathrm{GP}(10,3)$. I can be verified that every automorphism of $\mathrm{GP}(10,3)$ admits two adjacent regular orbits.

Case 1.2: Suppose $|u^G| = 3|v^G|$. Observe that v has three distinct neighbours in u^G, which implies that u^G and v^G are the only G-orbits of Γ. As in the previous case, u is adjacent to u^{g^i} and $u^{g^{-i}}$ for some $i \in \{1, \dots, |u^G| - 1|\}$. Furthermore v is adjacent to u, u^{g^m} and $u^{g^{2m}}$ where $m = |v^G|$. If $m = 1$ or $m = 2$, then Γ is isomorphic to K_4 or Q_3 respectively. If $m \geq 3$, then the edge uv lies on 4 distinct 6-cycles (see Fig. 4.2), and by [8, Lemma 4.2], Γ is isomorphic to Heawood graph or the Pappus graph.

Case 2: *u^G has exactly two neighbouring orbits v^G and w^G, with $u \sim v$ and $u \sim w$.* We have three subcases.

Case 2.1: Suppose $|u^G| = 3|v^G| = 3|w^G|$. By Lemma 4.10, Γ is isomorphic to either K_4, $K_{3,3}$, or Q.

Case 2.2: Suppose $|u^G| = 2|v^G| = 2|w^G|$. By Lemma 4.5, u has exactly one neighbour in each v^G and w^G. Since the third neighbour of u does not belong to either v^G or w^G, then it has no choice but to belong to u^G. To be more precise, $u \sim u^{g^m}$ where $m = |v^G| = \frac{1}{2}|u^G|$. Then (u, u^{g^m}, v) and (u, u^{g^m}, w) are 3-cycles in Γ, and the edge uu^{g^m} lies on both of them. Then, by Lemma 4.3, $\Gamma \cong K_{3,3}$.

Case 2.3: Suppose $|u^G| = 2|v^G| = 3|w^G|$. Observe that $u \sim u^{g^m}$ where $m = |v^G| = \frac{1}{2}|u^G|$. Then (u, u^{g^m}, v) is a 3-cycle, and since $|u^G| = 3|w^G|$, Γ is arc-transitive by Lemma 4.4. Then by Lemma 4.1, $\Gamma \cong K_4$.

Case 3: *u^G has three distinct neighbouring orbits, v^G, w^G, x^G where u is adjacent to v, w and x.*

Case 3.1: Suppose one of these orbits has size $1/3|u^G|$. Then by Lemma 4.4, Γ is arc-transitive. Furthermore, by the pigeon hole principle, two of these orbits must have size $1/k|u^G|$ for some $k \in \{2,3\}$. It follows that Γ has a 4-cycle and by Lemma 4.1, Γ is isomorphic to K_4, $K_{3,3}$ or Q.

Case 3.2: Suppose $|u^G| = 2|v^G| = 2|w^G| = 2|x^G| = 2m$ for some $m \in \mathbb{Z}$. Observe that each of v, w and x is adjacent to both u and u^{g^m} and thus, every edge incident to u lies on two distinct 4-cycles (see Fig. 4.2). If the girth of Γ is 4, then by Lemma 4.2, Γ is isomorphic to $K_{3,3}$ or the cube graph Q. If the girth of Γ is 3, then u must lie on a 3-cycle. That is, two of the neighbours of u must be adjacent. Without loss of generality, assume $v \sim w$. Then both (u, v, w) and (u^{g^m}, v, w) are 3-cycles. In particular, all three edges incident to v lie on a 3-cycle. Since Γ is vertex-transitive, then every edge incident to u must lie on a 3-cycle. In particular, this implies that u and x have a common neighbour. This leads us to a contradiction, since neither v nor w is adjacent to x (the neighbourhoods of v and w are $\{u, u^{g^m}, w\}$ and $\{u, u^{g^m}, v\}$, respectively).

This finishes the proof of Theorem 1.6.

5. Bounding the number of orbits and proofs of Theorem 1.4 and Theorem 1.7

The aim of this section is to prove Theorems 1.4 and 1.7. Throughout the section, let Γ denote a connected cubic graph of order n, let $g \in \mathrm{Aut}(\Gamma)$ and let $G = \langle g \rangle$. Recall that Theorem 1.4 asserts that

$$\mu(g) \leq \frac{17n}{6\,o(g)} \tag{5.1}$$

(the second claim then follows by applying this inequality to an automorphism g of largest order), and in view of Theorem 4.7, it suffices to prove this inequality with the parameter $o(g)$ substituted by $\ell(g)$ (note that the inequality (5.1) clearly holds when $\Gamma \equiv K_{3,3}$). Furthermore, recall that Theorem 1.7 asserts that unless Γ is isomorphic to $K_{3,3}$, or to a Split Praeger-Xu graph, at least $\frac{5n}{12}$ vertices belong to a regular orbit of g.

The validity of both theorems for graphs of small order can easily be confirmed by consulting the census of cubic vertex-transitive graphs [30]. We may thus assume henceforth that $n \geq 20$.

We begin by proving the inequality (5.1) for the case where Γ belongs to the family of *Split Praeger-Xu graphs*, which we now introduce (note that Theorem 1.7 holds trivially in this case).

For an integer $r \geq 3$, let $\vec{\mathrm{PX}}(r,1)$ denote the directed graph with the vertex-set $\mathbb{Z}_r \times \mathbb{Z}_2$ and with a directed edge pointing from (x,i) to $(x+1,j)$ for every $x \in \mathbb{Z}_r$ and $i,j \in \mathbb{Z}_2$. For an integer s, $2 \leq s \leq r-1$, we let $\vec{\mathrm{PX}}(r,s)$ be the directed graph whose vertex-set is the set of all directed paths of length s in $\vec{\mathrm{PX}}(r,1)$ and with a directed edge pointing from an s-path $(u_0, u_1, \ldots, u_s)$ to the successor s-paths $(u_1, \ldots, u_s, u_{s+1})$ and $(u_1, \ldots, u_s, v_{s+1})$, where u_{s+1} and v_{s+1} are the two out-neighbours of u_s in $\vec{\mathrm{PX}}(r,1)$. We should point out that the directed graphs $\vec{\mathrm{PX}}(r,s)$ were first introduced in [37] in a slightly different way and were denoted $C_2(r,s)$. Several equivalent descriptions of the directed graphs $\vec{\mathrm{PX}}(r,s)$ and their undirected counterparts were discussed in [22].

The *Split Praeger-Xu graph* $\mathrm{SPX}(r,s)$ is the graph obtained from $\vec{\mathrm{PX}}(r,s)$ by splitting each vertex u of $\vec{\mathrm{PX}}(r,s)$ into two vertices, denoted u_- and u_+, and by connecting each u_- with u_+ for every vertex u of $\vec{\mathrm{PX}}(r,s)$, and every v_+ to u_- for every directed edge (v,u) of $\vec{\mathrm{PX}}(r,s)$. The splitting operation was introduced in [34], where the graphs $\mathrm{SPX}(r,s)$ appeared under the name $\mathrm{Pl}^{s-1}(\vec{\mathrm{W}}(r,2))$.

Observe that the automorphism group of $\vec{\mathrm{PX}}(r,1)$ is isomorphic to the semidirect product $\mathrm{C}_2^r \rtimes \mathrm{C}_r$ with the elementary abelian group C_2^r being generated by automorphisms τ_i, $i \in \mathbb{Z}_r$, interchanging the vertices $(i,0)$ and $(i,1)$ while fixing all other vertices. Moreover, $\mathrm{Aut}(\vec{\mathrm{PX}}(r,1))$ acts in an obvious way as a vertex-transitive group of automorphisms on $\vec{\mathrm{PX}}(r,s)$ for every s, $1 \le s \le r-1$, as well as on $\mathrm{SPX}(r,s)$. In fact, one can easily see that $\mathrm{Aut}(\vec{\mathrm{PX}}(r,s)) \cong \mathrm{Aut}(\vec{\mathrm{PX}}(r,1))$ and $|\mathrm{Aut}(\vec{\mathrm{PX}}(r,s)) : \mathrm{Aut}(\mathrm{SPX}(r,s))| = 2$. In this correspondence, an automorphism fixing a vertex of $\vec{\mathrm{PX}}(r,s)$ corresponds to an automorphism of $\vec{\mathrm{PX}}(r,1)$ fixing a directed s-path of $\vec{\mathrm{PX}}(r,1)$ and thus belongs to the group C_2^r. Similarly, an element $g \in \mathrm{Aut}(\mathrm{SPX}(r,s))$ fixing a vertex v_+ of $\mathrm{SPX}(r,s)$ corresponds to an automorphism of $\vec{\mathrm{PX}}(r,s))$ which fixes v. In particular, the exponent of the vertex-stabiliser in $\mathrm{Aut}(\mathrm{SPX}(r,s))$ is 2, and so $\mathrm{meo}_\circ(\mathrm{SPX}(r,s)) = 2$. We can now apply the inequality (2.3) in Lemma 2.4 to conclude that

$$\mu(g) \le 2\left(\frac{n}{\ell(g)} - 1\right) + 1 < \frac{17n}{6\,o(g)},$$

where the second inequality follows from the fact that $\ell(g) = o(g)$ by Theorem 1.6. We have thus proved the following.

Lemma 5.1. *If $\Gamma \cong \mathrm{SPX}(r,s)$ with $r \ge 3$ and $1 \le s \le r-1$, then the inequality (5.1) holds for every $g \in \mathrm{Aut}(\Gamma)$. In particular, Theorem 1.4 holds for the Split Praeger-Xu graphs.*

This lemma, together with a recent result of Pablo Spiga and the first-named author of this paper [28], which bounds the number of vertices that can be fixed by a non-trivial automorphism in a cubic vertex-transitive graph, yields the following.

Corollary 5.2. *Let Γ be a cubic vertex-transitive graph on n vertices admitting a non-identity automorphism fixing more than $\frac{n}{3}$ vertices of Γ. Then the inequality (5.1) holds for every $g \in \mathrm{Aut}(\Gamma)$. In particular, Theorems 1.4 and 1.7 hold for such a graph Γ.*

Proof. As observed, both theorems hold for graphs on at most 20 vertices, hence we may thus assume that $n > 20$. By [28, Theorem 1.2], Γ is then isomorphic to a Split Praeger-Xu graph $\mathrm{SPX}(r,s)$ with $r \ge 3$ and $s \le 2r/3$. The result then follows by Lemma 5.1. $\square$

We shall thus assume henceforth that $n > 20$ and that Γ is not a Split Praeger-Xu graph. Let $k = \frac{n}{o(g)} = \frac{n}{\ell(g)}$ and let $m = o(g)$. For an integer i dividing m, let Ω_i be the set of vertices of Γ contained in a G-orbit of size $\frac{m}{i}$ and let N_i denote the number of G-orbits of size $\frac{m}{i}$. Note that $|\Omega_i| = \frac{m}{i}N_i$. Moreover, by Lemma 4.5, $\Omega_i = \emptyset$ unless $i \in \{1,2,3,4,6\}$. Recall that our aim is to show that G has at most $17k/6$ orbits on $\mathrm{V}(\Gamma)$ and that $|\Omega_1| \geq \frac{5n}{12}$.

By applying line (1) of Lemma 4.11 with $i \in \{1,2\}$, we see that every G-orbit X of length $m/3i$ is adjacent to a G-orbit Y of size m/i, and by line (2), X is the only orbit of length $m/3i$ which is adjacent to Y. This implies that the number of G-orbits of size $m/3i$ is at most the number of G-orbits of size m/i; that is:

$$N_6 \leq N_2, \tag{5.2}$$

$$N_3 \leq N_1. \tag{5.3}$$

Now observe that $mN_1 + \frac{m}{3}N_3 = |\Omega_1| + |\Omega_3| \leq n = mk$ and thus $N_1 + \frac{1}{3}N_3 \leq k$. From this and inequality (5.3), we obtain

$$N_3 \leq \frac{3}{4}k. \tag{5.4}$$

Moreover, since $N_1 \leq k - \frac{1}{3}N_3$ and thus $N_1 + N_3 \leq k + \frac{2}{3}N_3$. It follows from (5.4) that

$$N_1 + N_3 \leq \frac{3}{2}k. \tag{5.5}$$

Clearly if m is odd, then G can only have orbits of size m and $\frac{m}{3}$. That is, $\mathrm{V}(\Gamma) = \Omega_1 \cup \Omega_3$ and $\mu(g) = N_1 + N_3$. By inequality (5.5) we have $\mu(g) \leq \frac{3}{2}k < \frac{17}{6}k$, and Theorem 1.4 holds in this case. Furthermore,

$$|\Omega_1| = mN_1 = n - \frac{m}{3}N_3 \geq n - \frac{m}{3}N_1 = n - \frac{1}{3}|\Omega_1|,$$

implying that $|\Omega_1| \geq 3n/4 \geq 5n/12$, showing that also Theorem 1.7 holds in this case. Hence, we may assume that m is even.

Now, consider the automorphism $g^{\frac{m}{2}}$ and observe that it fixes every orbit of size $\frac{m}{2}$ as well as every orbit of size $\frac{m}{4}$. That is, the set $\Omega_2 \cup \Omega_4$ is fixed point-wise by $g^{\frac{m}{2}}$. Since the order of g is m, the automorphism $g^{\frac{m}{2}}$ is non-trivial, and since, by our assumption, no non-trivial automorphism fixes more than one third of the vertices, it follows that $|\Omega_2 \cup \Omega_4| \leq \frac{n}{3}$. That is

$$N_2\frac{m}{2} + N_4\frac{m}{4} \leq \frac{n}{3} = \frac{mk}{3}, \tag{5.6}$$

and thus

$$N_4 \leq \frac{4}{3}k - 2N_2. \tag{5.7}$$

Now, $\mu(g) = N_1+N_2+N_3+N_4+N_6$, but $N_4 \le \frac{4}{3}k-2N_2$ and $N_6 \le N_2$ by inequalities (5.7) and (5.2). Then

$$\mu(g) \le N_1 + N_2 + N_3 + \left(\frac{4}{3}k - 2N_2\right) + N_2 = N_1 + N_3 + \frac{4}{3}k = \frac{17}{6}k,$$

where the last equality follows from (5.5). This completes the proof of Theorem 1.4.

To finish the proof of Theorem 1.7, recall that $\Omega_6 = \frac{m}{6}N_6$ and $N_6 \le N_2$. Moreover, by inequality (5.6), we have $N_2 \le \frac{2}{3}k$ and thus

$$|\Omega_6| = \frac{m}{6}N_6 \le \frac{m}{6}N_2 \le \frac{m}{6}\cdot\frac{2}{3}k = \frac{1}{9}mk = \frac{1}{9}n. \tag{5.8}$$

Now, $n = |\Omega_1| + |\Omega_2| + |\Omega_3| + |\Omega_4| + |\Omega_6|$ but $|\Omega_2| + |\Omega_4| \le \frac{1}{3}n$ and $|\Omega_6| \le \frac{1}{9}n$ by inequalities (5.6) and (5.8), respectively. Therefore

$$|\Omega_1| = n - (|\Omega_2| + |\Omega_4|) - |\Omega_6| - |\Omega_3| \ge n - \frac{1}{3}n - \frac{1}{9}n - |\Omega_3| = \frac{5}{9}n - |\Omega_3| \tag{5.9}$$

Since $N_3 \le N_1$ we have $\Omega_3 = \frac{m}{3}N_3 \le \frac{m}{3}N_1 = \frac{1}{3}\Omega_1$. From this and (5.9), we have $|\Omega_1| \ge \frac{5}{9}n - |\Omega_3| \ge \frac{5}{9}n - \frac{1}{3}|\Omega_1|$. We conclude that $|\Omega_1| \ge \frac{5n}{12}$. This concludes the proof of Theorem 1.7.

Remark 5.3. We would like to point out that the constant $\frac{17}{6}$ featuring in Theorem 1.4 could most likely be substituted with a smaller value. However, it is not possible to do better than replacing it by 2, as the following two extreme examples show. First, consider the case where $G := \langle g\rangle$ is transitive (that is, when the graph is a circulant). Then G has a single orbit of length $n = |\mathrm{V}(\Gamma)|$, $n/o(g) = 1$ and thus $\mu(g) = 1 = 2n/o(g) - 1$. On the other side of the spectrum, the Split Praeger-Xu graphs $\mathrm{SPX}(n/2, 1)$ admit a group of automorphisms of order 2 having $n-2$ orbits on $\mathrm{V}(\Gamma)$, two of which have size 2 (while all others consist of a single vertex). Here the number of orbits equals $2n/o(g) - 2$.

Conjecture 5.4. *Let Γ be a cubic vertex-transitive graph of order n, and let $g \in \mathrm{Aut}(\Gamma)$. Then $\langle g\rangle$ has at most $\frac{2n}{o(g)} - 1$ orbits on $\mathrm{V}(\Gamma)$.*

Remark 5.5. As with the bound given in Theorem 1.4, here too we suspect that the constant 12/5 appearing in Theorem 1.7 can be improved. It is not possible to do better than 2/3 as the following family of examples show. For an even integer $r > 0$, let $\Psi(r)$ be the graph with vertex set $\mathbb{Z}_r \times \mathbb{Z}_3$ and with edges of the form $\{(i, j), (i-1, j)\}$, $\{(i, j), (i+1, j+1)\}$ and $\{(i, j), (i+1, j+2)\}$ for all even $i \in \mathbb{Z}_r$. The graph $\Psi(r)$ is a cubic vertex-transitive graph of order $3r$ (for more details see [36, Section 2.4] where the graph $\Psi(r)$ is called Σ_r). Observe that the permutation g that interchanges $(i, 0)$ with $(i, 1)$ while fixing $(i, 2)$ for all $i \in \mathbb{Z}_r$ is an automorphism of $\Psi(r)$. The regular orbits of g are the sets of the form $\{(i, 0), (i, 1)\}$. It follows that two thirds of the vertices of $\Psi(r)$

lie on a regular orbit of g. We believe that, excluding the family of the Split Praeger-Xu graphs, this is an extreme case.

Conjecture 5.6. *Let Γ be a cubic vertex-transitive graph of order n not isomorphic to $\mathrm{K}_{3,3}$ or a Split Praeger-Xu graph, and let $g \in \mathrm{Aut}(\Gamma)$. Then at least $\frac{2}{3}n$ vertices lie on a regular orbit of g.*

6. The order of automorphisms of locally-semiprimitive graphs

Let Γ be a graph, let $G \leq \mathrm{Aut}(\Gamma)$ and let $v \in \mathrm{V}(\Gamma)$. The permutation group induced by the action of the stabiliser G_v on the neighbourhood $\Gamma(v)$ of the vertex v will be denoted $G_v^{\Gamma(v)}$. Observe that if G acts transitively on $\mathrm{V}(\Gamma)$, then up to permutation isomorphism the group $G_v^{\Gamma(v)}$ is independent of the choice of v. In this case, if L is an arbitrary permutation group permutation isomorphic to $G_v^{\Gamma(v)}$, we say that Γ is of local G-action L; if $G = \mathrm{Aut}(\Gamma)$, then the prefix G can be omitted.

A transitive permutation group $G \leq \mathrm{Sym}(\Omega)$ is called semiregular provided that $G_\omega = 1$ for every $\omega \in \Omega$. Furthermore, following [2], we call a transitive permutation group $G \leq \mathrm{Sym}(\Omega)$ *semiprimitive* provided that every normal subgroup N of G is either transitive or semiregular; see [12] for more results on semiprimitive groups. Observe that every primitive, as well as every quasiprimitive permutation group is semiprimitive. For a graph Γ and $G \leq \mathrm{Aut}(\Gamma)$ we say that Γ is *locally G-semiprimitive* (*locally G-primitive*) whenever the permutation group $G_v^{\Gamma(v)}$ is semiprimitive (primitive, respectively) for every vertex v.

A still unresolved conjecture of Richard Weiss [51] states that for every valence d there exists a constant c_d such that for every finite connected G-arc-transitive locally G-primitive graph Γ the order of the vertex-stabiliser G_v is bounded by c_d, and in particular, $\mathrm{meo}_\circ(G) \leq c_d$ and $\mathrm{meo}(G) \leq c_d|\mathrm{V}(\Gamma)|$. Even though several partial results were proved (see, for example [25,39,43,46,48]), this conjecture is still wide open. Weiss' conjecture was strengthened first by Cheryl Praeger [38], who relaxed the condition of local primitivity to local quasi-primitivity, and then by Spiga, Verret and the first-named author of this paper [29], who relaxed the condition to local semiprimitivity (see also [10,11]).

A starting point to most attempts to prove Weiss's conjecture is the so-called Thompson-Wielandt Theorem (see [53, Theorem 6.6]). Here is a variant for locally semiprimitive graphs which was proved in [44].

Theorem 6.1. *[44, Corollary 3] Let Γ be a connected G-arc-transitive locally G-semiprimitive graph, let $\{u, v\}$ be an edge of Γ and let $G_{uv}^{[1]}$ be the point-wise stabiliser of all the vertices at distance 1 from u or v. Then $G_{uv}^{[1]}$ is trivial or a p-group.*

Lemma 6.2. *Let Γ be a connected G-arc-transitive d-valent graph and let $\{u, v\}$ be an edge of Γ. Then $|G_v| \leq d!(d-1)!|G_{uv}^{[1]}|$.*

Proof. Let $G_v^{[1]}$ be the point-wise stabiliser of the action of G_v on $\Gamma(v)$. Observe that $G_{uv}^{[1]}$ is the kernel of the action of $G_v^{[1]}$ on $\Gamma(u)\setminus\{v\}$. But then $|G_v| \leq d!|G_v^{[1]}| \leq d!(d-1)!|G_{uv}^{[1]}|$, as claimed. □

Theorem 1.3 now easily follows from the above two results and Corollary 2.3 in the following way. Let Γ be a connected G-arc-transitive locally G-semiprimitive graph of valence d and let $\{u, v\}$ be an edge of Γ. By Lemma 6.2, $|G_v| =\leq d!(d-1)!|G_{uv}^{[1]}|$. Since, by Theorem 6.1, $G_{uv}^{[1]}$ is trivial or a p-group, Corollary 2.3 now implies that $o(g) \leq d!(d-1)!\ell(g)$ for every $g \in G$. We have thus shown that $\mathrm{meo}(G) \leq c_d|\mathrm{V}(\Gamma)|$, where $c_d = d!(d-1)!$. This completes the proof of Theorem 1.3.

7. The order of automorphisms of quartic vertex-transitive graphs

In this section we prove Theorem 1.2 for the case of quartic graphs. That is, we prove that $\mathrm{meo}(\Gamma) \leq 9|\mathrm{V}(\Gamma)|$ holds for every finite connected vertex-transitive graph of valence 4. As we shall see, the proof quickly reduces to proving a bound on the exponent of a Sylow 3-subgroup of a vertex-stabiliser in a finite connected 6-valent arc-transitive graph. We thus begin by proving the following result, which is a generalisation of [47, Theorem 4.9].

Proposition 7.1. *Let L be a permutation group on Ω, let p be a prime and let H a p-subgroup of L. Suppose that there exist $x, y \in \Omega$ such that*

- $H = \langle H_x, H_y\rangle$,
- $x^H \cup y^H = \Omega$, *and*
- $|H : H_x| = |H : H_y| = p$.

Let Γ be a connected G-vertex-transitive and G-edge-transitive graph with local G-action L, let v be a vertex of Γ and identify $G_v^{\Gamma(v)}$ with L. If S is a p-subgroup of G_v that projects to H along the natural projection $G_v \to L$, then S has the following properties:

(1) *S has nilpotency class at most 3;*
(2) *S contains an elementary abelian p-subgroup of order at least $|S|^{2/3}$;*
(3) $|\mathbf{Z}(S)|^3 \geq |S|$;
(4) *S has exponent at most p^2.*

Proof. Let u and w be the neighbours of v corresponding to x and y under the identification of $\Gamma(v)$ with Ω.

We show that the arcs (u, v) and (v, w) are in the same G-orbit. We argue by contradiction and we suppose that this is not the case. Since Γ is G-edge-transitive, it follows that (u, v) is in the same G-orbit as (w, v). This implies that u and w are in the same G_v-orbit and hence x and y are in the same L-orbit. This implies that L is transitive, so Γ is G-arc-transitive and hence (u,v) and (v,w) are in the same G-orbit, which is a contradiction.

Let $\phi \in G$ such that $(u,v)^\phi = (v,w)$. We show that $\langle S, \phi\rangle$ is transitive on $\mathrm{V}(\Gamma)$. For $i \in \mathbb{Z}$, let $v_i = v^{\phi^i}$ and let $S_i = S^{\phi^i}$. Note that $(v_{-1}, v_0, v_1) = (u,v,w)$ and hence $\Gamma(v_0) = (v_{-1})^{S_0} \cup (v_1)^{S_0}$. Conjugating by ϕ^i, we obtain that $\Gamma(v_i) = (v_{i-1})^{S_i} \cup (v_{i+1})^{S_i}$ for every $i \in \mathbb{Z}$. Let $G^* = \langle S_i \mid i \in \mathbb{Z}\rangle$ and let $X = v^{\langle\phi\rangle} = \{v_i \mid i \in \mathbb{Z}\}$. Note that $G^* \leq \langle S, \phi\rangle$, and hence it suffices to show that $X^{G^*} = \mathrm{V}(\Gamma)$. By contradiction, suppose that there exists a vertex not in X^{G^*} and choose one with minimum distance to X. Call this vertex α and let $(p_0, \ldots, p_{n-1}, p_n)$ be a shortest path from α to a vertex of X. In particular, $p_0 = \alpha$ and $p_n = v_i$ for some $i \in \mathbb{Z}$. Since $\Gamma(v_i) = (v_{i-1})^{S_i} \cup (v_{i+1})^{S_i}$, there exists $\sigma \in S_i \leq G^*$ such that $(p_{n-1})^\sigma \in \{v_{i-1}, v_{i+1}\} \subseteq X$. Since α is not in S^{G^*}, neither is α^σ, but α^σ is closer to X than α is, which is a contradiction.

From now on, we follow the notation of [5] and [14] as closely as possible. Let $P = S$, let $R = S_u$ and let $Q = S_w$. Note that $R^\phi = Q$, and R and Q both have index p in P.

Let N be the subgroup of P generated by all the subgroups of R that are normalised by ϕ. By [14, Proposition 2.1], N is normal in P. By definition, N is normalised by ϕ and hence N is normalised by $\langle P, \phi\rangle$. On the other hand, we have shown that $\langle P, \phi\rangle$ is transitive on $\mathrm{V}(\Gamma)$. Since $N \leq P \leq G_v$, it follows that $N = 1$. This shows that condition (1.1) of [5] is satisfied.

Let $|S| = p^t$ and let u, v and $x_1, \ldots, x_t$ be as in [5, Theorem 1] and let $E = \langle x_1, \ldots, x_u\rangle$. By [5, Lemma 2.2 (d,f,g)], P has nilpotency class at most 3, $\mathbf{Z}(P) = \langle x_{v+1}, \ldots, x_u\rangle$ and $[P,P]$ is elementary abelian. It follows from [5, Lemma 2.1] that P is generated by elements of order p and that $|P| = p^t$, $|E| = p^u$ and $|\mathbf{Z}(P)| = p^{u-v}$. It also follows from [5, Theorem 1 (1.2-1.5)] that $v = t - u$, $u \geq \frac{2}{3}t$, $E \leq P$ and E is elementary abelian which concludes the proof of (2). Since $v = t-u$, $|\mathbf{Z}(P)| = p^{u-v} = p^{2u-t} \geq p^{t/3} = |P|^{1/3}$ and (3) follows. Finally, P has exponent at most p^2 by [47, Lemma 3.6]. □

Lemma 7.2. *Let p be an odd prime and let L be a permutation group of degree $2p$, either transitive or having two orbits of size p, and let H be a Sylow p-subgroup of L. One of the following holds:*

(1) *H is semiregular of order p, or*
(2) *H satisfies the first part of the hypothesis of Proposition 7.1.*

Proof. Since p is odd, $|H|$ divides p^2. If $|H| = p^2$, then H must be isomorphic to $\mathrm{C}_p \times \mathrm{C}_p$ acting naturally with two orbits of size p and it is easy to check that the first part of the hypothesis of Proposition 7.1 is satisfied. Since p divides the size of an orbit of L, $H \neq 1$ hence $|H| = p$.

If L is primitive then, a well known consequence of the classification of finite simple groups states that either $\mathrm{Alt}(2p) \leq L$ or $p = 5$ and $\mathrm{Alt}(5) \leq L \leq \mathrm{Sym}(5)$ (see for example [24]). In the former case, $|H| = p^2$ while in the later case, p does not divide $|L_x|$ hence H is semiregular. If L admits a system of p blocks of size 2, then again p does not divide $|L_x|$. If L admits a system of 2 blocks of size p, then H is contained in the setwise stabiliser of the blocks, which has two orbits of size p. So we may assume that L has two orbits of size p.

We may assume that H is not semiregular, and thus $|H_x| = p$ for some $x \in \Omega$. Since $|H| = p$, H fixes x, so H fixes p points and has one orbit of size p, with representative y, say. Now, x and y must be representatives of the two orbits of L and there must be another Sylow subgroup H' that is transitive on x^L and, since it is conjugate to H, it must fix y^L pointwise. Now, $|\langle H, H'\rangle| = p^2$, which is a contradiction. □

Note that the hypothesis that p is odd is necessary as if $p = 2$, then $L = H = \mathrm{D}_4$ in its natural action is a counterexample. (It is neither semiregular nor generated by point-stabilisers.)

Corollary 7.3. *Let p be an odd prime and let Γ be a connected G-vertex-transitive and G-edge-transitive graph of valency $2p$, and let S be a Sylow p-subgroup of a vertex-stabiliser G_v. Then S has the properties at the end of Proposition 7.1.*

Proof. Let L be the local action at v. This is a permutation group of degree $2p$. It is either transitive, or has two orbits of size p. Let H be the projection of S onto L. Note that H is a Sylow p-subgroup of L. We apply Lemma 7.2 to conclude that H is semiregular or H satisfies the first part of Proposition 7.1. If H is semiregular, then S is arc-semiregular and $|S| = p$ and clearly it satisfies all the properties. Otherwise, we apply Proposition 7.1. □

Equipped with Corollary 7.3, we can now finish the proof of Theorem 1.2 for the case of quartic graphs. For the rest of the section, let Γ be a finite connected vertex-transitive graph of valence 4, let $G = \mathrm{Aut}(\Gamma)$ and let $v \in \mathrm{V}(\Gamma)$. Then $G_v^{\Gamma(v)}$ is permutation isomorphic to one of the groups:

(1) the doubly transitive permutation groups $\mathrm{Sym}(4)$, $\mathrm{Alt}(4)$ of degree 4;
(2) the transitive groups D_4, C_4, $\mathrm{C}_2 \times \mathrm{C}_2$ of degree 4;
(3) $\{\mathrm{id}\}$, $\mathrm{C}_2 \times \mathrm{C}_2$ or C_3 in their unique intransitive faithful actions on 4 points, or C_2 either its action with two orbits of length 2 or in its action with three orbits, one of length 2 two of length 1;
(4) $\mathrm{Sym}(3)$ in its unique faithful action on 4 points.

If (1) occurs, then G acts transitively on the 2-arcs of Γ (where a 2-arc is a triple (w, u, v) of distinct vertices such that u is adjacent to both w and v). Based on the results in [9,52] it was proved in [27] that G_v is then isomorphic to one of 7 finite groups. In particular, it follows from [27, Theorem 4] that $\exp(G_v)$ divides $2^3 \cdot 3^2$. It then follows from Corollary 2.3 that $o(g) \le 9\ell(g)$ for every $g \in G$.

If either (2) or (3) occurs, then the fact that $G_v^{\Gamma(v)}$ is a p-group for $p = 2$ or $p = 3$, the connectivity of Γ implies that G_v is a p-group, and Corollary 2.3 then yields that $o(g) = \ell(g)$.

For the rest of the section we will assume that (4) occurs. Then every vertex $v \in \mathrm{V}(\Gamma)$ has a unique neighbour v' which is fixed by every automorphism in G_v. Observe that $v'' = v$ for every $v \in \mathrm{V}(\Gamma)$ and that the set $M = \{\{v, v'\} : v \in \mathrm{V}(\Gamma)\}$ forms a complete matching of Γ invariant under the action of G.

Think of the edges in M red and the edges outside M blue. Similarly, call an arc blue or red if its underlying edge is blue or red. Observe that G has two orbits on the arc-set of Γ, one consisting of all blue arcs and of all red arcs.

Let Λ be the graph with vertex-set M and with two red edges vv', $uu' \in M$ adjacent in Λ whenever one of v, v' is adjacent to one of u, u'. Clearly, every element of G (in its action on M) induces an automorphism of Λ. Let K denote the kernel of the action of G on M. Then G/K is a vertex-transitive group of automorphisms of Λ.

Let $e := vv'$ be a red edge of Γ, let a, b, c be the three neighbours of v distinct from v' and let x, y, z be the three neighbours of v' distinct from v. Then the neighbourhood of e in Λ is $\{aa', bb', cc', dd', xx', yy', zz'\}$. Observe also that the stabiliser G_e acts transitively on the set $\{a, b, c, x, y, z\}$, and thus also on the neighbourhood of e in Λ. In particular, G/K is not only vertex-transitive group of automorphism of Λ but in fact arc-transitive.

Since a, b, c are pairwise distinct, so are aa', bb' and cc'. The valence of Λ is thus at most 6 and at least 3. Observe also that since $G_v^{\Gamma(v)} \cong \mathrm{Sym}(3)$ the vertex-stabiliser G_v contains an element g acting on $\{a, b, c\}$ as the permutation $(a\,b\,c)$ and on $\{x, y, z\}$ as $(x\,y\,z)$.

Suppose first that the valence of Λ is less than 6. Then the red edge aa' equals one of the edges xx', yy' or zz', say xx'. By applying the automorphism g twice, we see that $yy' = bb'$ and $zz' = cc'$, implying that the valence of Λ is 3 in this case. In particular, Λ is a cubic G/K-arc-transitive graph. By [49], the order of the vertex-stabiliser $(G/K)_e$ is then of the form $3 \cdot 2^{s-1}$ for some $s \le 5$. Let us now consider the kernel K. Since G acts transitively on the arcs of Λ, the subgraph B of Γ induced by the blue edges between two adjacent distinct red edges uu' and ww' is independent of the choice of uu' and ww' and admits an automorphism swapping the pair $\{u, u'\}$ with the pair $\{w, w'\}$. Since there are precisely six blue edges adjacent to any given red edge it follows that B consists of two edges and must thus be isomorphic to $2K_2$. By the connectivity of Γ, this implies that every element of K either fixes each pair $\{w, w'\}$, $w \in \mathrm{V}(\Gamma)$ point-wise or swaps the two vertices in each such pair. In particular, the order of K is at most 2. This implies that the order of G_e is at most twice the order of $(G/K)_e$ and thus equal to $3 \cdot 2^s$ for some $s \le 5$. Since G_v is of index 2 in G_e, Corollary 2.3 implies that $o(g) \le 3\ell(g)$ for every $g \in G$.

We are thus left with the case where the valence of Λ is 6. Observe that then there is at most one blue edge between every two red edges, implying that $K = 1$ and $G \le \mathrm{Aut}(\Lambda)$. Moreover, the group $G_e^{\Lambda}(e)$ is permutation isomorphic to the group induced by the action of G_e on the vertices $\{a, b, c, x, y, z\}$. Observe that the latter group is imprimitive with

$\{a, b, c\}$ being a block of imprimitivity, implying that its order is a divisor of $2|\mathrm{Sym}(3)| = 12$. The connectivity of Λ then implies that G_e is a $\{2,3\}$-group. On the other hand, by Corollary 7.3, the Sylow 3-subgroup has exponent at most 9. In particular, $\exp(G_e)$ divides $9 \cdot 2^\alpha$ with α a positive integer. Since G_v has index 2 in G_e, Corollary 2.3 then implies that $o(g) \le 9\ell(g)$ holds for every $g \in G$. This completes the proof of Theorem 1.2.

8. Vertex-transitive graphs with bounded order of automorphisms

This section is devoted to the proof of Theorem 1.8. We begin by proving that $M_d(n) \to \infty$ as $n \to \infty$ for every integer $d \ge 3$. Observe that this claim is equivalent to the statement that for every constant c there exists a constant n_c such that every finite connected d-valent vertex-transitive graphs Γ with $\mathrm{meo}(\Gamma) \le c$ satisfies $|\mathrm{V}(\Gamma)| \le n_c$.

Before proceeding, let us prove the following folklore lemma, which was stated and proved in the cubic case in [45]; the proof provided there easily extends to an arbitrary valence.

Lemma 8.1. *If Γ is a connected vertex-transitive graph of valence d, then* $\mathrm{Aut}(\Gamma)$ *contains a d-generated vertex-transitive subgroup.*

Proof. Let v be a vertex of Γ and, for every $u \in \Gamma(v)$, let $g_u \in \mathrm{Aut}(\Gamma)$ such that $v^{g_u} = u$. Let $X = \{g_u : u \in \Gamma(v)\}$ and let $G = \langle X \rangle$. We claim that G acts transitively on $\mathrm{V}(\Gamma)$. Suppose the contrary and let w be a vertex which is closest to v among all the vertices of Γ not contained in the orbit v^G. Then clearly w has a neighbour z contained in v^G. Let $g \in G$ be such that $z^g = v$ and let $u = w^g$. Then u is a neighbour of v. Moreover, $v^{g_u g^{-1}} = u^{g^{-1}} = w$. Since $g_u \in X$ and $g \in G$, we see that $g_u g^{-1} \in G$ and thus $w \in v^G$, a contradiction. □

Remark 8.2. Note that Lemma 8.1 proves the existence of a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$ with a generating set of bounded cardinality, but says nothing about the minimum generating set for $\mathrm{Aut}(\Gamma)$. The question whether the cardinality of a minimum generating set of the automorphism group of a finite connected d-valent vertex-transitive graph can be bounded above by a function of d is an interesting open problem, which was (to the best of our knowledge) first posed by Pablo Spiga.

Now let c be an arbitrary constant and let Γ be a finite connected d-valent vertex-transitive graph with $\mathrm{meo}(\Gamma) \le c$. By Lemma 8.1, there exists a vertex-transitive subgroup G of $\mathrm{Aut}(\Gamma)$ generated by d elements. Since $\mathrm{meo}(\Gamma) \le c$, we see that $\mathrm{meo}(G) \le c$ and thus $\exp(G) \le c!$. By the solution of the restricted Burnside problem [20,55,56], there exists a constant n_c such that every finite d-generated group of exponent at most $c!$ has order less than n_c. In particular, $|G| \le n_c$. Since G acts transitively on $\mathrm{V}(\Gamma)$, this implies that $|\mathrm{V}(\Gamma)| \le n_c$, as claimed. This finishes the proof that $M_d(n) \to \infty$ as $n \to \infty$.

In order to complete the proof of Theorem 1.8, we shall now construct a sequence of connected d-valent vertex-transitive graphs Γ_i whose order is large with respect to $\exp(\mathrm{Aut}(\Gamma_i))$ (see Theorem 8.6 for details). But we first need to prove a few auxiliary results.

Lemma 8.3. *If G is a group with a subgroup H of finite exponent and finite index $|G : H| = n$, then $\exp(G) \le \exp(H)\exp(\mathrm{Sym}(n)) \le n!\exp(H)$.*

Proof. Let K be the core of H in G. We can view G/K as a subgroup of $\mathrm{Sym}(n)$ and the result follows easily. □

For a group G and a prime p, let $\mathrm{O}_p(G)$ be the largest normal p-subgroup of G. The following lemma is another folklore result.

Lemma 8.4. *Let Γ be a connected G-vertex-transitive and G-edge-transitive graph and let $\{u, v\}$ be an edge of Γ. If $\mathrm{O}_p(G_{uv}^{\Gamma(u)}) = 1$ and $\mathrm{O}_p(G_{uv}^{\Gamma(v)}) = 1$, then $\mathrm{O}_p(G_{uv}^{[1]}) = 1$.*

Proof. We have $1 = \mathrm{O}_p(G_{uv}^{\Gamma(v)}) \cong \mathrm{O}_p(G_{uv}/G_v^{[1]})$ hence $\mathrm{O}_p(G_{uv}) \trianglelefteq G_v^{[1]}$ and thus $\mathrm{O}_p(G_{uv}) \le \mathrm{O}_p(G_v^{[1]})$. Since $\mathrm{O}_p(G_v^{[1]})$ is characteristic in $G_v^{[1]}$ and thus normal in G_{uv}, we have $\mathrm{O}_p(G_v^{[1]}) = \mathrm{O}_p(G_{uv})$. On the other hand, $G_v^{[1]}/G_{uv}^{[1]} \cong (G_v^{[1]})^{\Gamma(u)} \trianglelefteq G_{uv}^{\Gamma(u)}$ hence $\mathrm{O}_p(G_v^{[1]}/G_{uv}^{[1]}) \le \mathrm{O}_p(G_{uv}^{\Gamma(u)}) = 1$. It follows that $\mathrm{O}_p(G_v^{[1]}) \le \mathrm{O}_p(G_{uv}^{[1]})$, but $\mathrm{O}_p(G_{uv}^{[1]}) \le \mathrm{O}_p(G_{uv})$ and thus $\mathrm{O}_p(G_{uv}^{[1]}) = \mathrm{O}_p(G_{uv}) = \mathrm{O}_p(G_v^{[1]})$. Since Γ is connected, G-vertex-transitive and G-edge-transitive, it follows that $\langle G_v, G_{uv}\rangle = G$, implying that $\mathrm{O}_p(G_{uv}^{[1]}) \trianglelefteq G$ and $\mathrm{O}_p(G_{uv}^{[1]}) = 1$. □

Lemma 8.5. *If Γ is a connected finite G-arc-transitive d-valent graph such that $G_v^{\Gamma(v)} \cong \mathrm{Sym}(d)$, then $|G_v| \le c(d)$ where $c(1) = 1$, $c(2) = 2$, $c(3) = 2^4 \cdot 3$, $c(4) = 2^4 \cdot 3^6$, $c(5) = 3^2 \cdot 2^9 \cdot 5$ and $c(d) = d!(d-1)!$ for $d \ge 6$.*

Proof. The claim is trivial for $d \in \{1, 2\}$. For $d = 3$, the result follows by Tutte's classical result on vertex-stabilisers in cubic arc-transitive graphs [49]. For $d = 4$, it follows from work of Gardiner, more specifically, from Theorems 3.9 and 3.15 and the remark following Lemma 3.3 in [9]. For $d = 5$, the result is given explicitly in [19, Theorem 1.1], but most of the work was done by Weiss [52].

Finally, if $\{u, v\}$ is an edge of Γ, then $G_{uv}^{\Gamma(v)} \cong (G_v^{\Gamma(v)})_u \cong \mathrm{Sym}(d)_u \cong \mathrm{Sym}(d-1)$. In particular, if $d \ge 6$, then $\mathrm{O}_p(G_{uv}^{\Gamma(u)}) = 1$ for every prime p. By Lemma 8.4, this implies $\mathrm{O}_p(G_{uv}^{[1]}) = 1$. By the Thompson-Wielandt Theorem (see for example [9, Corollary 2.3]), it follows that $G_{uv}^{[1]} = 1$. Now, $|G_v/G_v^{[1]}| = |\mathrm{Sym}(d)| = d!$, while $G_v^{[1]} \cong G_v^{[1]}/G_{uv}^{[1]} \cong (G_v^{[1]})^{\Gamma(u)} \le (G_{uv})^{\Gamma(u)} \cong \mathrm{Sym}(d-1)$, so $|G_v^{[1]}| \le (d-1)!$ and $|G_v| \le d!(d-1)!$, as claimed. □

Let us now recursively define the values $r_d(i)$ for $i \in \mathbb{N} \cup \{0\}$ by setting:

$$r_d(0) = d - 1 \quad \text{and} \quad r_d(i+1) = 1 + 2^{r_d(i)}(r_d(i) - 1) \;\; \text{for} \;\; i \geq 0.$$

Observe that $r_d(i) \geq {}^{(i+1)}2$ holds for every $i \geq 0$ and $d \geq 3$. We can now prove the following.

Theorem 8.6. *Let $d \geq 3$ and let $c(d)$ be as in Lemma 8.5. For every $i \geq 1$, there exists a connected Cayley graph Γ_i of valence d such that*

$$|\mathrm{V}(\Gamma_i)| = 2^{1+r_d(0)+\cdots+r_d(i-1)} \geq {}^{i+1}2$$

while $\exp(\mathrm{Aut}(\Gamma_i)) \leq 2^i(c(d)!)$.

Proof. Let $G = \langle a_1, \ldots, a_d \mid a_1^2, \ldots, a_d^2 \rangle$ and let $F = \langle a_1a_2, \ldots, a_1a_d \rangle \leq G$. For all $i, j \in \{1, \ldots, d\}$, we have $a_ia_j = (a_1a_i)^{-1}a_1a_j \in F$. It follows that F consists of all the elements of G that can be written as words of even length in $\{a_1, \ldots, a_n\}$. Since all generators of F and all relators of G have even length, F does not contain any element of odd length, implying that $|G : F| = 2$. There are many ways to see that F is a free group. For example, one may use the Kurosh Subgroup Theorem [26, Corollary 2], the Reidemeister-Schreier Rewriting Process (see for example [21, Section III.8]) or note that F acts semiregularly on the vertices and edges of the tree $\mathrm{Cay}(G, \{a_1, \ldots, a_d\})$ and apply Serre's Theorem [40, Theorem 4, Section 3.3]. As $\{a_1a_2, \ldots, a_1a_d\}$ is a minimal generating set for F (a_1a_i is the only generator of F involving a_i and no relator of G involves multiple generators), F has rank $d - 1$.

Consider the series

$$F = F_0 \geq F_1 \geq F_2 \geq \cdots$$

where $F_{i+1} = (F_i)^2$ for $i \geq 0$. (That is, F_{i+1} is the group generated by squares of elements in F_i.) Note that F_{i+1} is the smallest normal subgroup of F_i such that F_i/F_{i+1} is an elementary abelian 2-group. It follows that F/F_i has exponent at most 2^i. By the Nielsen–Schreier Theorem [40, Theorem 5], F_i is free for every $i \geq 1$.

We claim that, for every $i \geq 0$, F_i is free of rank $r_d(i)$ and $|F_i/F_{i+1}| = 2^{r_d(i)}$. We prove this by induction on i. For $i = 0$, we already saw that $F_0 = F$ is free of rank $d - 1 = r_d(0)$ and thus $|F_0/F_1| = 2^{d-1}$. Now assume the result is true for some $i \geq 0$. Since F_i is free of rank $r_d(i)$ and $F_{i+1} = (F_i)^2$, we have $|F_i/F_{i+1}| = 2^{r_d(i)}$ and then the Schreier Index Formula gives that F_{i+1} is free of rank $1 + 2^{r_d(i)}(r_d(i) - 1) = r_d(i + 1)$, as required (this argument was inspired by an argument of Mike Newman found in [50, Section 2]).

Since $|G : F| = 2$, we have $F^2 \leq G^2 \leq F$. On the other hand, $|G : G^2| = 2^d$ and $|F : F^2| = 2^{d-1}$ so $G^2 = F^2 = F_1$. It follows that F_i is characteristic in G for every $i \geq 1$. Let $i \geq 1$ and let $Q_i = G/F_i$. Observe that $|Q_i| = 2^{1+r_d(0)+\cdots+r_d(i-1)}$ and

$$\exp(Q_i) \le \exp(F_0/F_1)\cdots\exp(F_{i-1}/F_i) \le 2^i.$$

For $x \in G$, let $\overline{x} = F_i x$ denote the image of x under the natural projection $G \to Q_i$. Let $S_i = \{\overline{a_1}, \dots, \overline{a_d}\} \subseteq Q_i$, $\Gamma_i = \mathrm{Cay}(Q_i, S_i)$ and $A_i = \mathrm{Aut}(\Gamma_i)$. If $\overline{a_j} = \overline{a_k}$ with $j \neq k$, then $F_i a_j = F_i a_k$ and hence $F_1 a_j = F_1 a_k$, contradicting the fact that $G/F_1 \cong \mathbb{Z}_2^d$. A similar contradiction is obtained under the assumption that $\overline{a_j}$ is trivial for some $j \in \{1, \dots, d\}$. This implies that Γ_i is a connected Cayley graph of valency d on the group Q_i. In particular, $|\mathrm{V}(\Gamma_i)| = |Q_i| = 2^{1+r_d(0)+\cdots+r_d(i-1)}$. Now recall that $r_d(i-1) \ge^i 2$ and thus

$$|\mathrm{V}(\Gamma)_i| \ge 2^{({}^i 2)} = {}^{i+1}2,$$

as required. Since F_i is characteristic in G, every permutation π in $\mathrm{Sym}(d)$ yields an automorphism of Q_i mapping $\overline{a_i} \in S_i$ to $\overline{a_{i^\pi}} \in S_i$. Since such an automorphism of Q_i preserves the set S_i, it induces an automorphism of Γ_i. In particular, the local action $(A_i)_{\mathbf{1}}^{\Gamma_i(\mathbf{1})}$ is permutation isomorphic to $\mathrm{Sym}(d)$. Since Q_i acts regularly on the vertices of Γ_i, it follows by Lemma 8.5 that $|A_i : Q_i| = |(A_i)_{\mathbf{1}}| \le c(d)$. By Lemma 8.3, this implies $\exp(A_i) \le \exp(Q_i)c(d)! \le 2^i(c(d)!)$. □

It is now easy to finish the proof of Theorem 1.8. Let n be an arbitrary positive integer, let $i = \mathrm{slog}_2(n)$ and let Γ_i be as in Theorem 8.6. Then $|\mathrm{V}(\Gamma_i)| \ge^{i+1} 2$ and $\exp(\mathrm{Aut}(\Gamma_i)) \le 2^i(c(d))!)$. Let $k_d = (c(d))!)$. Since ${}^i 2 \le n <^{i+1} 2$, the above implies that

$$M_d({}^i 2) \le M(n) \le M_d({}^{i+1}2) \le M_d(|\mathrm{V}(\Gamma_i)|) \le \mathrm{meo}(\Gamma_i) \le \exp(\mathrm{Aut}(\Gamma_i)) \le k_d\, 2^{\mathrm{slog}_2(n)},$$

finishing the proof of Theorem 1.8.

Data availability

No data was used for the research described in the article.

References

[1] M. Barbieri, V. Grazian, P. Spiga, On the order of semiregular automorphisms of cubic vertex-transitive graphs, arXiv:2302.00034.
[2] Á. Bereczky, A. Maróti, On groups with every normal subgroup transitive or semi-regular, J. Algebra 319 (2008) 1733–1751.
[3] M. Conder, P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, J. Comb. Math. Comb. Comput. 40 (2002) 41–63.
[4] M. Conder, P. Lorimer, Automorphism groups of symmetric graphs of valency 3, J. Comb. Theory, Ser. B 47 (1989) 60–72.
[5] J. Currano, Finite p-groups with isomorphic subgroups, Can. J. Math. 25 (1973) 1–13.
[6] D. Djoković, A class of finite group-amalgams, Proc. Am. Math. Soc. 80 (1980) 22–26.
[7] D. Djoković, G.L. Miller, Regular groups of automorphisms of cubic graphs, J. Comb. Theory, Ser. B 29 (1980) 195–230.
[8] Y.-Q. Feng, R. Nedela, Symmetric cubic graphs of girth at most 7, Acta Univ. M. Belii Ser. Math. 13 (2006) 33–35.
[9] A. Gardiner, Arc transitivity in graphs, Q. J. Math. Oxf. Ser. 24 (1973) 399–407.

[10] M. Giudici, L. Morgan, A class of semiprimitive groups that are graph-restrictive, Bull. Lond. Math. Soc. 46 (2014) 1226–1236.
[11] M. Giudici, L. Morgan, On locally semiprimitive graphs and a theorem of Weiss, J. Algebra 427 (2015) 104–117.
[12] M. Giudici, L. Morgan, A theory of semiprimitive groups, J. Algebra 503 (2018) 146–185.
[13] M. Giudici, C. Praeger, P. Spiga, Finite primitive permutation groups and regular cycles of their elements, J. Algebra 421 (2015) 27–55.
[14] G. Glauberman, Isomorphic subgroups of finite p-groups. I, Can. J. Math. 23 (1971) 983–1022.
[15] R.L. Goodstein, Transfinite ordinals in recursive number theory, J. Symb. Log. 12 (1947) 123–129.
[16] S. Guest, J. Morris, C.E. Praeger, P. Spiga, On the maximum orders of elements of finite almost simple groups and primitive permutation groups, Trans. Am. Math. Soc. 367 (2015) 7665–7694.
[17] S. Guest, J. Morris, C.E. Praeger, P. Spiga, Finite primitive permutation groups containing a permutation having at most four cycles, J. Algebra 454 (2016) 233–251.
[18] S. Guest, P. Spiga, Finite primitive groups and regular orbits of group elements, Trans. Am. Math. Soc. 369 (2017) 997–1024.
[19] S.T. Guo, Y.Q. Feng, A note on pentavalent s-transitive graphs, Discrete Math. 312 (2012) 2214–2216.
[20] P. Hall, G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem, Proc. Lond. Math. Soc. 6 (1956) 1–42.
[21] A. Hulpke, Notes on computational group theory, https://www.math.colostate.edu/~hulpke/CGT/cgtnotes.pdf. (Accessed September 2021).
[22] R. Jajcay, P. Potočnik, S. Wilson, The Praeger-Xu graphs: cycle structures, maps and semitransitive orientations, Acta Math. Univ. Comen. 88 (2019) 269–291.
[23] E. Landau, Über die Maximalordnung der Permutationen gegebenen Grades, Arch. Math. Phys. 5 (1903) 92–103.
[24] M.W. Liebeck, J. Saxl, Primitive permutation groups containing an element of large prime order, J. Lond. Math. Soc. 31 (1985) 237–249.
[25] L. Morgan, Vertex-transitive graphs with local action the symmetric group on ordered pairs, J. Group Theory 26 (2023) 519–531.
[26] E.T. Ordman, On subgroups of amalgamated free products, Math. Proc. Camb. Philos. Soc. 69 (1971) 13–23.
[27] P. Potočnik, A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index $(4, 2)$, Eur. J. Comb. 30 (2009) 1323–1336.
[28] P. Potočnik, P. Spiga, On the number of fixed points of automorphisms of vertex-transitive graphs, Combinatorica 102 (2021) 703–747.
[29] P. Potočnik, P. Spiga, G. Verret, On graph-restrictive permutation groups, J. Comb. Theory, Ser. B 102 (2012) 820–831.
[30] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, J. Symb. Comput. 50 (2013) 465–477.
[31] P. Potočnik, M. Toledo, Classification of cubic vertex-transitive tricirculants, Ars Math. Contemp. 18 (2020) 1–31.
[32] P. Potočnik, M. Toledo, Finite cubic graphs admitting a cyclic group of automorphisms with at most three orbits on vertices, Discrete Math. 344 (2021) 112195.
[33] P. Potočnik, M. Toledo, Cubic vertex-transitive graphs admitting an automorphism with a long orbit, Bull. Malays. Math. Sci. Soc. 46 (2023) 133.
[34] P. Potočnik, G. Verret, On the vertex-stabiliser in arc-transitive digraphs, J. Comb. Theory, Ser. B 100 (2010) 497–509.
[35] P. Potočnik, J. Vidali, Girth-regular graphs, Ars Math. Contemp. 17 (2019) 249–368.
[36] P. Potočnik, J. Vidali, Cubic vertex-transitive graphs of girth 6, arXiv:2005.01635.
[37] C.E. Praeger, Highly arc transitive digraphs, Eur. J. Comb. 10 (1989) 281–292.
[38] C.E. Praeger, Finite quasiprimitive group actions on graphs and designs, in: Groups–Korea '98, de Gruyter, 2000, pp. 319–331.
[39] C.E. Praeger, L. Pyber, P. Spiga, E. Szabo, Graphs with automorphisms groups admitting composition factors of bounded rank, Proc. Am. Math. Soc. 140 (2012) 2307–2318.
[40] J.-P. Serre, Trees, Springer-Verlag, 1980 (Translated from the French by John Stillwell).
[41] J. Siemons, A. Zalesskii, Intersections of matrix algebras and permutation representations of $\mathrm{PSL}(n, q)$, J. Algebra 226 (2000) 451–478.
[42] J. Siemons, A. Zalesskii, Regular orbits of cyclic subgroups in permutation representations of certain simple groups, J. Algebra 256 (2002) 611–625.

[43] P. Spiga, On G-locally primitive graphs of locally twisted wreath type and a conjecture of Weiss, J. Comb. Theory, Ser. A 118 (2011) 2257–2260.

[44] P. Spiga, Two local conditions on the vertex stabiliser of arc-transitive graphs and their effect on the Sylow subgroups, J. Group Theory 15 (2012) 23–35.

[45] P. Spiga, Semiregular elements in cubic vertex-transitive graphs and the restricted Burnside problem, Math. Proc. Camb. Philos. Soc. 157 (2014) 45–61.

[46] P. Spiga, An application of the local $C(G,T)$ theorem to a conjecture of Weiss, Bull. Lond. Math. Soc. 48 (2016) 12–18.

[47] P. Spiga, G. Verret, On the order of vertex-stabilisers in vertex-transitive graphs with local group $\mathrm{C}_p \times \mathrm{C}_p$ or $\mathrm{C}_p \wr \mathrm{C}_2$, J. Algebra 48 (2016) 174–209.

[48] V.I. Trofimov, Supplement to "The group $E_6(q)$ and graphs with a locally linear group of automorphism" by V.I. Trofimov and R.M. Weiss, Math. Proc. Camb. Philos. Soc. 148 (2010) 1–32.

[49] W.T. Tutte, A family of cubical graphs, Proc. Camb. Philos. Soc. 43 (1947) 459–474.

[50] M. Vaughan-Lee, E.I. Zel'manov, Bounds in the restricted Burnside problem, J. Aust. Math. Soc. 67 (1999) 261–271.

[51] R. Weiss, s-transitive graphs, Colloq. Math. Soc. János Bolyai 25 (1978) 827–847.

[52] R. Weiss, Presentation for (G,s)-transitive graphs of small valency, Math. Proc. Philos. Soc. 101 (1987) 7–20.

[53] H. Wielandt, Subnormal Subgroups and Permutation Groups, Ohio State University Lecture Notes, Columbus, OH, 1971.

[54] Wikipedia contributors, Super-logarithm, Wikipedia, the free encyclopedia, https://en.wikipedia.org/w/index.php?title=Super-logarithm&oldid=977146425. (Accessed 14 September 2021).

[55] E.I. Zelmanov, The solution of the restricted Burnside problem for groups of odd exponent, Izv. Math. USSR 36 (1991) 4–60.

[56] E.I. Zelmanov, The solution of the restricted Burnside problem for 2-groups, Mat. Sb. 182 (1991) 568–592.