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Locally Convex Bialgebroid of an Action Lie Groupoid

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Abstract. Action Lie groupoids are used to model spaces of orbits of actions of Lie groups on manifolds. For each such action groupoid $M \rtimes H$, we construct a locally convex bialgebroid $\mathrm{Dirac}(M \rtimes H)$ with an antipode over $\mathcal{C}_c^\infty(M)$, from which the groupoid $M \rtimes H$ can be reconstructed as its spectral action Lie groupoid $\mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H))$.

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1. Introduction

Our motivation for this paper originates from the Gelfand–Naimark theorem. To any locally compact, Hausdorff topological space X, one assigns the C^*-algebra $C_0(X)$ of all continuous functions on X that vanish at infinity. The set $\mathrm{Spec}(C_0(X))$ of all characters on $C_0(X)$ can be equipped with the weak-* topology, so that it becomes a locally compact Hausdorff space. The Gelfand–Naimark theorem then says that the map $\Phi_X^{\mathrm{lc}} : X \to \mathrm{Spec}(C_0(X))$, which assigns to a point $x \in X$ the evaluation δ_x at x, is a homeomorphism.

We wish to obtain a similar result for the class of geometric spaces which can be represented by Lie groupoids [4, 16, 19, 20]. These spaces include orbifolds, spaces of leaves of foliations, and spaces of orbits of Lie groups actions. In [21], a result in the spirit of the Gelfand–Naimark theorem was established for the class of étale Lie groupoids, which can be used to model orbifolds and spaces of leaves of foliations. To any étale Lie groupoid $\mathscr{G}$, one assigns the Hopf algebroid $\mathcal{C}_c^\infty(\mathscr{G})$ of smooth compactly supported functions on $\mathscr{G}$ (if $\mathscr{G}$ is not Hausdorff, one needs to be careful with the definition). As an algebra $\mathcal{C}_c^\infty(\mathscr{G})$ coincides with the Connes convolution algebra in the

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Hausdorff case [5], while the coalgebra structure is basically induced from the sheaf [22], corresponding to the target map $\mathrm{t} : \mathscr{G} \to M$ of $\mathscr{G}$. Finally, the antipode on $\mathcal{C}_c^\infty(\mathscr{G})$ is induced by the inverse map of $\mathscr{G}$. For each such Hopf algebroid $\mathcal{C}_c^\infty(\mathscr{G})$, one can construct the spectral étale Lie groupoid $\mathcal{G}_{sp}(\mathcal{C}_c^\infty(\mathscr{G}))$, so that there is a natural isomorphism $\Phi_{\mathscr{G}}^{\mathrm{egr}} : \mathscr{G} \to \mathcal{G}_{sp}(\mathcal{C}_c^\infty(\mathscr{G}))$ of Lie groupoids. Similar ideas were used in [10] to extend these results to the semi-direct products of étale Lie groupoids and bundles of Lie groups.

Structure maps of an étale Lie groupoid $\mathscr{G}$ are local diffeomorphisms, so, in particular, the fibers of the target map are discrete. It is, therefore, enough to reconstruct the fibers of such a groupoid just as sets, which can be done by utilizing the coalgebra structure on $\mathcal{C}_c^\infty(\mathscr{G})$. However, in the case of a general groupoid, one needs some additional information, which enables us to recover the topology along the fibers. Let us explain the main idea on a simple example. If Γ is a discrete group, then $\mathcal{C}_c^\infty(\Gamma)$ is just the group Hopf algebra of Γ. Elements of Γ correspond to grouplike elements of $\mathcal{C}_c^\infty(\Gamma)$, while the multiplication and inverse of Γ are encoded in multiplication and antipode of $\mathcal{C}_c^\infty(\Gamma)$. If we now replace Γ with a non-discrete Lie group H, one can still define its group Hopf algebra and reconstruct H, but only as a group. To recover the topology and the smooth structure of H, we need some additional structure. One way to solve this problem is to identify the group Hopf algebra of H with the space $\mathrm{Dirac}(H)$ of distributions on H which is spanned by Dirac distributions. The space $\mathrm{Dirac}(H)$ is a subspace of the space $\mathcal{E}'(H)$ of compactly supported distributions on H. If we equip $\mathrm{Dirac}(H)$ with the induced strong topology from $\mathcal{E}'(H)$, the following two things happen. The group H is naturally homeomorphic to the space of Dirac distributions, which are precisely the grouplike elements of the Hopf algebra $\mathrm{Dirac}(H)$. On the other hand, the space $\mathrm{Dirac}(H)$ is dense in $\mathcal{E}'(H)$. One can show that by combining the well-known fact that $\mathcal{C}_c^\infty(H)$ is dense in $\mathcal{E}'(H)$ with the observation that the Riemann integral can be approximated arbitrarily well by Riemann sums (see Proposition 3.7). As a result, we get that the strong dual $\mathrm{Dirac}(H)'$ is isomorphic to $\mathcal{C}^\infty(H)$. Now, observe that the space $\mathcal{C}^\infty(H)$ is a Fréchet algebra, from which H can be reconstructed as a manifold.

Judging by the above example, we are led to consider Hopf algebroids not only as purely algebraic objects, but with some additional topological structure. The main idea consists of two parts. First of all, we assign to a Lie groupoid $\mathscr{G}$ a certain Hopf algebroid, from which the algebraic structure of $\mathscr{G}$ can be reconstructed. We then equip this Hopf algebroid with a suitable locally convex structure, which enables us to recover the topology and smooth structure of $\mathscr{G}$.

In this paper, we use this idea on the class of action Lie groupoids, which are used to describe spaces of orbits of Lie groups actions on manifolds. Each such action Lie groupoid $M \rtimes H$ is isomorphic as a groupoid to an étale Lie groupoid $M \rtimes H^\#$, where $H^\#$ is the group H with discrete topology. We use this identification to define the Dirac bialgebroid $\mathrm{Dirac}(M \rtimes H)$ of $M \rtimes H$ as a certain subspace of the space $\mathcal{E}'_{\mathrm{t}}(M \rtimes H)$ of t-transversal distributions on $M \rtimes H$. Transversal distributions on Lie groupoids were studied in [1,2,11,14,26] and, crucially for our problem, it was shown in [14] that the space $\mathcal{E}'_{\mathrm{t}}(M \rtimes H)$

is a locally convex algebra, if we equip it with the strong topology of uniform convergence on bounded subsets. With the induced topology, $\mathrm{Dirac}(M \rtimes H)$ becomes a locally convex bialgebroid with an antipode.

The paper is organized as follows. In Sect. 2, we recall the basic definitions and known results that are used in the rest of the paper. In Sect. 3, we construct for every trivial bundle $\pi : M \times N \to M$ the space $\mathrm{Dirac}_\pi(M \times N)$ of transversal distributions of constant Dirac type. These are families of Dirac distributions, supported on constant sections of π. If the fiber N is discrete, $\mathrm{Dirac}_\pi(M \times N)$ coincides with the LF-space $\mathcal{C}_c^\infty(M \times N)$, while, in general, we show that it is a dense subspace of the space of π-transversal distributions $\mathcal{E}'_\pi(M \times N)$. In Sect. 4, we define on $\mathrm{Dirac}_\pi(M \times N)$ a structure of a locally convex coalgebra over $\mathcal{C}_c^\infty(M)$ and show that its strong $\mathcal{C}_c^\infty(M)$-dual is naturally isomorphic to the Fréchet algebra $\mathcal{C}^\infty(M \times N)$. The combination of the coalgebra structure and locally convex topology enables us to reconstruct from $\mathrm{Dirac}_\pi(M \times N)$ the bundle $\pi : M \times N \to M$ as the spectral bundle $\mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N))$. Finally, in Sect. 5, we use these results to assign to each action Lie groupoid $M \rtimes H$ and its Dirac bialgebroid $\mathrm{Dirac}(M \rtimes H)$. The space $\mathrm{Dirac}(M \rtimes H)$ is a locally convex bialgebroid with an antipode, which coincides with the locally convex Hopf algebra $\mathrm{Dirac}(H)$ in the case when M is a point. Main result of the paper is Theorem 5.7 in which we show that the groupoid $M \rtimes H$ can be reconstructed from the bialgebroid $\mathrm{Dirac}(M \rtimes H)$ as its spectral action Lie groupoid $\mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H))$ (see Definition 5.6).

2. Preliminaries

In this subsection, we will review basic definitions and results that will be needed in the rest of the paper. More details concerning locally convex vector spaces and Lie groupoids can be found for example in [7,13,25], respectively [16,19,20].

We will assume all our manifolds to be smooth, Hausdorff, and paracompact, but not necessarily second-countable. For any such manifold M, we will denote by $\mathcal{C}^\infty(M)$ the vector space of smooth $\mathbb{C}$-valued functions on M. The subspaces of compactly supported and $\mathbb{R}$-valued functions on M will be denoted by $\mathcal{C}_c^\infty(M)$, respectively, $\mathcal{C}^\infty(M, \mathbb{R})$.

2.1. Locally Convex Spaces

All our locally convex vector spaces will be complex and Hausdorff. A subset B of a locally convex space E is bounded if and only if the set $p(B)$ is a bounded subset of $\mathbb{R}$ for any continuous seminorm p on E. For locally convex vector spaces E and F, we will denote by $\mathrm{Hom}(E, F)$ the space of all continuous linear maps from E to F, equipped with the strong topology of uniform convergence on bounded subsets. The basis of neighbourhoods of zero in $\mathrm{Hom}(E, F)$ consists of sets of the form

$$K(B, V) = \{T \in \mathrm{Hom}(E, F) \,|\, T(B) \subset V\},$$

where B is a bounded subset of E and V is a neighbourhood of zero in F. If E and F are modules over an $\mathbb{C}$-algebra A, we will denote by $\mathrm{Hom}_A(E, F)$

the corresponding space of continuous A-module homomorphisms and equip it with the induced topology from $\mathrm{Hom}(E, F)$.

The space $\mathcal{C}^\infty(\mathbb{R}^l)$ has a structure of a Fréchet algebra for any $l \in \mathbb{N}$. Topology on $\mathcal{C}^\infty(\mathbb{R}^l)$ is generated by a family of seminorms $\{p_{L,m}\}$, indexed by compact subsets L of $\mathbb{R}^l$ and $m \in \mathbb{N}_0$, given by

$$p_{L,m}(F) = \sup_{x \in L, |\alpha| \leqslant m} |D^\alpha(F)(x)|$$

for $F \in \mathcal{C}^\infty(\mathbb{R}^l)$. Here, we denoted $D^\alpha(F) = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \cdots \partial x_l^{\alpha_l}}$, where $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}_0^l$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. If M is a second-countable manifold, one can choose similar seminorms with respect to some open cover of M with local coordinate charts to define the Fréchet topology on $\mathcal{C}^\infty(M)$. This topology coincides with the topology of uniform convergence of all derivatives on compact subsets of M. The strong dual of the space $\mathcal{C}^\infty(M)$ is the space $\mathcal{E}'(M) = \mathrm{Hom}(\mathcal{C}^\infty(M), \mathbb{C})$ of compactly supported distributions on M.

If M is not compact, the subspace $\mathcal{C}_c^\infty(M)$ of $\mathcal{C}^\infty(M)$ is not complete in the Fréchet topology, so we consider a finer LF-topology on $\mathcal{C}_c^\infty(M)$. For any compact subset L of M, we denote by $\mathcal{C}_c^\infty(L)$ the subspace of functions with support contained in L. The space $\mathcal{C}_c^\infty(L)$ is a closed subspace of $\mathcal{C}^\infty(M)$ and hence a Fréchet space itself. The LF-topology on $\mathcal{C}_c^\infty(M)$ is now defined as the inductive limit topology with respect to the family of all subspaces of the form $\mathcal{C}_c^\infty(L)$ for $L \subset M$ compact. The space $\mathcal{C}_c^\infty(M)$ with LF-topology is a complete locally convex space, which is not metrizable, if M is not compact.

If M is a smooth manifold and E is a locally convex vector space, a vector-valued function $u : M \to E$ is smooth if, in local coordinates, all partial derivatives exist and are continuous. We will denote by $\mathcal{C}^\infty(M, E)$ the space of smooth functions on M with values in E and by $\mathcal{C}_c^\infty(M, E)$ its subspace, consisting of compactly supported functions. To make a distinction between scalar functions and vector-valued functions, we will denote by $f(x) \in \mathbb{C}$ the value of a function $f \in \mathcal{C}^\infty(M)$ at x and by $u_x \in E$ the value of a function $u \in \mathcal{C}^\infty(M, E)$ at x.

2.2. Lie Groupoids

A *Lie groupoid* is given by a manifold M of objects and a manifold $\mathscr{G}$ of arrows together with smooth structure maps: target $\mathrm{t} : \mathscr{G} \to M$, source $\mathrm{s} : \mathscr{G} \to M$, multiplication $\mathrm{mlt} : \mathscr{G} \times_M^{\mathrm{s,t}} \mathscr{G} \to \mathscr{G}$, inverse $\mathrm{inv} : \mathscr{G} \to \mathscr{G}$, and unit $\mathrm{uni} : M \to \mathscr{G}$. We assume that the source and the target maps are submersions to ensure that $\mathscr{G} \times_M^{\mathrm{s,t}} \mathscr{G}$ is a smooth manifold. A Lie groupoid is *étale* if all its structure maps are local diffeomorphisms. Note that there exist more general definitions of Lie groupoids, which we will not need.

Example 2.1. We will be mostly interested in action Lie groupoids. Suppose H is a Lie group which acts from the right on the manifold M. The associated action Lie groupoid $\mathscr{G} = M \rtimes H$ is then a Lie groupoid over M with the manifold of arrows $M \times H$ and with the following structure maps:

$$\begin{aligned}
\mathrm{t}(x, h) &= x, \\
\mathrm{s}(x, h) &= xh, \\
\mathrm{mlt}((x, h), (xh, h')) &= (x, h)(xh, h') = (x, hh'), \\
\mathrm{inv}(x, h) &= (x, h)^{-1} = (xh, h^{-1}), \\
\mathrm{uni}(x) &= (x, e).
\end{aligned}$$

Here, $x \in M$ and $h, h' \in H$ are arbitrary, while e is the unit of the Lie group H. The action groupoid $\mathscr{G}$ is étale if and only if the group H is discrete.

2.3. Real Commutative Algebras

Let A be an $\mathbb{R}$-algebra. A real character on A is a nontrivial multiplicative homomorphism from A to $\mathbb{R}$. We will denote by

$$\mathrm{Spec}(A)$$

the space of all real characters on A, equipped with the Gelfand topology (i.e., the relative weak-* topology). If the algebra A satisfies the conditions of the Theorem in [18], the space $\mathrm{Spec}(A)$ also has a natural smooth structure.

If Q is a smooth manifold, we have

$$\mathrm{Spec}(\mathcal{C}^\infty(Q)) = \{\delta_q \,|\, q \in Q\},$$

where δ_q is the Dirac functional, concentrated at the point q. In this case, we can equip the set $\mathrm{Spec}(\mathcal{C}^\infty(Q))$ with a topology and a smooth structure, such that the map $\Phi_Q^{\mathrm{man}} : Q \to \mathrm{Spec}(\mathcal{C}^\infty(Q))$, defined by $\Phi_Q^{\mathrm{man}}(q) = \delta_q$, is a diffeomorphism.

2.4. Coalgebras

Let R be a commutative ring. We say that R has local identities if, for any $r_1, \ldots, r_n \in R$, there exists $r \in R$, such that $rr_i = r_i$ for $i = 1, \ldots, n$. Similarly, a left R-module C is locally unitary if, for any $c_1, \ldots, c_n \in C$, there exists $r \in R$, such that $rc_i = c_i$ for $i = 1, \ldots, n$.

Suppose now that R is an associative, commutative algebra with local identities over the field of complex numbers $\mathbb{C}$. A *coalgebra* over R is a locally unitary left R-module C, equipped with two R-linear maps

$$\begin{aligned}
\Delta &: C \to C \otimes_R C, \\
\epsilon &: C \to R,
\end{aligned}$$

called comultiplication and counit, which satisfy the conditions of being counital and coassociative. This means that equalities $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$, $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$, respectively, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ hold. Note that these conditions make sense, because we can identify C with $R \otimes_R C$ and $C \otimes_R R$, since C is locally unitary. A coalgebra C over R is cocommutative if $\sigma \circ \Delta = \Delta$, where the flip isomorphism $\sigma : C \otimes_R C \to C \otimes_R C$ is given by $\sigma(c \otimes c') = c' \otimes c$.

Our main examples of coalgebras will be coalgebras associated with sheaves, which were introduced in [22].

Example 2.2. Let P and M be manifolds and let $\pi : P \to M$ be a local diffeomorphism (i.e., P is a sheaf over M). The ring $\mathcal{C}_c^\infty(M)$ always has local units, but it is unital if and only if M is compact. It will be convenient to denote for any $f \in \mathcal{C}_c^\infty(M)$ by $1_f \in \mathcal{C}_c^\infty(M)$ a function which satisfies $1_f f = f$. If $\mathrm{supp}(f) \neq M$, the function 1_f is not uniquely defined.

The space $\mathcal{C}_c^\infty(P)$ has a natural structure of a locally unitary left $\mathcal{C}_c^\infty(M)$-module with the multiplication given by

$$(f \cdot F)(p) = f(\pi(p))F(p)$$

for $f \in \mathcal{C}_c^\infty(M)$, $F \in \mathcal{C}_c^\infty(P)$ and $p \in P$. The counit $\epsilon : \mathcal{C}_c^\infty(P) \to \mathcal{C}_c^\infty(M)$ is given by

$$\epsilon(F)(x) = \sum_{p \in \pi^{-1}(x)} F(p)$$

for any $F \in \mathcal{C}_c^\infty(P)$ and any $x \in M$. Note that the above sum is finite, since F has compact support and the fibers of π are discrete.

To describe the comultiplication, let us recall that an open subset $W \subset P$ is π-elementary if $\pi|_W : W \to \pi(W)$ is a diffeomorphism. For any $f \in \mathcal{C}_c^\infty(\pi(W))$, we can consider the function $f \circ \pi|_W \in \mathcal{C}_c^\infty(W)$ as an element of $\mathcal{C}_c^\infty(P)$, if we extend it by zero outside of W. Since any $a \in \mathcal{C}_c^\infty(P)$ has compact support, it can be written (nonuniquely) as a finite sum $a = \sum_{i=1}^n f_i \circ \pi|_{W_i}$, where $W_1, \ldots, W_n$ are π-elementary subsets of P and $f_i \in \mathcal{C}_c^\infty(\pi(W_i))$ for $i = 1, \ldots, n$. The comultiplication $\Delta : \mathcal{C}_c^\infty(P) \to \mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{C}_c^\infty(P)$ can be now defined by

$$\Delta\left(\sum_{i=1}^n f_i \circ \pi|_{W_i}\right) = \sum_{i=1}^n (f_i \circ \pi|_{W_i}) \otimes (1_{f_i} \circ \pi|_{W_i}).$$

One can show that the definition of Δ is independent of the various choices that we have made and that we obtain in this way a cocommutative coalgebra $\mathcal{C}_c^\infty(P)$ over $\mathcal{C}_c^\infty(M)$.

For our purposes, we will be mostly interested in trivial sheaves. If Γ is a discrete topological space, then the projection $\pi : M \times \Gamma \to M$ is a local diffeomorphism. We can decompose the vector space $\mathcal{C}_c^\infty(M \times \Gamma)$ as a direct sum

$$\mathcal{C}_c^\infty(M \times \Gamma) = \bigoplus_{y \in \Gamma} \mathcal{C}_c^\infty(M \times \{y\}).$$

Using this decomposition, we can write every element $a \in \mathcal{C}_c^\infty(M \times \Gamma)$ uniquely in the form

$$a = \sum_{i=1}^n f_i \cdot \delta_{y_i}$$

for some $f_1, \ldots, f_n \in \mathcal{C}_c^\infty(M)$ and some $y_1, \ldots, y_n \in \Gamma$. Here, we have denoted for any $f \in \mathcal{C}_c^\infty(M)$ and any $y \in \Gamma$ by $f \cdot \delta_y \in \mathcal{C}_c^\infty(M \times \Gamma)$ the function, given by

$$(f \cdot \delta_y)(x, y') = \begin{cases} f(x); & y' = y, \\ 0; & y' \neq y \end{cases}$$

for $(x, y') \in M \times \Gamma$. The comultiplication and counit are then given on the generators of $\mathcal{C}_c^\infty(M \times \Gamma)$ by the formulas

$$\Delta(f \cdot \delta_y) = (f \cdot \delta_y) \otimes (1_f \cdot \delta_y),$$
$$\epsilon(f \cdot \delta_y) = f.$$

In the rest of this subsection, we will focus on coalgebras over the algebra $\mathcal{C}_c^\infty(M)$ for some manifold M and recall the main results from [22]. For any $x \in M$, we denote by

$$I_x = \{f \in \mathcal{C}_c^\infty(M) \mid f|_U = 0 \text{ for some neighbourhood } U \text{ of } x\}$$

the ideal of $\mathcal{C}_c^\infty(M)$, consisting of all functions with trivial germ at x. The quotient algebra of $\mathcal{C}_c^\infty(M)$ with respect to this ideal will be denoted by

$$\mathcal{C}_c^\infty(M)_x = \mathcal{C}_c^\infty(M)/I_x.$$

Now, let C be a coalgebra over $\mathcal{C}_c^\infty(M)$. The quotient

$$C_x = C/I_xC$$

then inherits a structure (Δ_x, ϵ_x) of a coalgebra over $\mathcal{C}_c^\infty(M)_x$ which is called the local coalgebra of C at x. The image of $c \in C$ in the quotient C_x will be denoted by $c|_x \in C_x$.

An element $c \in C$ is weakly grouplike if $\Delta(c) = c \otimes c'$ for some $c' \in C$. A weakly grouplike element $c \in C$ is normalized on an open subset $U \subset M$ if $\epsilon_x(c|_x) = 1$ for all $x \in U$. Weakly grouplike elements of the sheaf coalgebra $\mathcal{C}_c^\infty(P)$ are precisely elements of the form $f \circ \pi|_W$ for some π-elementary open subset W of P and some $f \in \mathcal{C}_c^\infty(\pi(W))$. Such an element $f \circ \pi|_W$ is normalized on $U \subset \pi(W)$ if $f|_U \equiv 1$.

Since the local ring $\mathcal{C}_c^\infty(M)_x$ is unital for any $x \in M$, we can also define the set of grouplike elements of C_x by

$$G(C_x) = \{\xi \in C_x \mid \Delta_x(\xi) = \xi \otimes \xi,\, \epsilon_x(\xi) = 1\}.$$

An element $\xi \in C_x$ is grouplike if and only if $\xi = c|_x$ for some weakly grouplike element $c \in C$, which is normalized on some open neighbourhood of x.

The spectral sheaf $\mathcal{E}_{sp}(C)$ of a $\mathcal{C}_c^\infty(M)$-coalgebra C is the sheaf

$$\pi_{sp}(C) : \mathcal{E}_{sp}(C) \to M$$

with the stalk

$$\mathcal{E}_{sp}(C)_x = G(C_x).$$

The topology on $\mathcal{E}_{sp}(C)$ is defined by the basis, consisting of $\pi_{sp}(C)$-elementary subsets of $\mathcal{E}_{sp}(C)$ of the form

$$c|_U = \{c|_x \in \mathcal{E}_{sp}(C) \mid x \in U\} \subset \mathcal{E}_{sp}(C),$$

where $c \in C$ is a weakly grouplike element, normalized on an open subset U of M.

Now, let $\pi : P \to M$ be a sheaf over M. By Theorem 2.4 in [22], we have a natural isomorphism of sheaves

$$\Phi_P^{\text{shv}} : P \to \mathcal{E}_{sp}(\mathcal{C}_c^\infty(P))$$

defined by

$$\Phi_P^{\mathrm{shv}}(p) = (f \circ \pi|_W)|_{\pi(p)},$$

where $p \in P$, W is a π-elementary neighbourhood of p in P and $f \in \mathcal{C}_c^\infty(\pi(W))$ is such that $f|_{\pi(p)} = 1 \in \mathcal{C}_c^\infty(M)_{\pi(p)}$. Moreover, by Theorem 2.10 in [22], a coalgebra C is isomorphic to some sheaf coalgebra $\mathcal{C}_c^\infty(P)$ if and only if C is locally grouplike, which by definition means that for every $x \in M$, the $\mathcal{C}_c^\infty(M)_x$-module C_x is free with the basis $G(C_x)$.

2.5. Bialgebroids and Hopf Algebroids

Bialgebroids and Hopf algebroids are generalizations of bialgebras and Hopf algebras over arbitrary rings. In the literature [3,6,12,15,17,23,24,27], one can find several similar definitions, which are in general inequivalent. Our definition follows the one in [21].

Let R be a commutative $\mathbb{C}$-algebra with local units. We say that a $\mathbb{C}$-algebra A extends R if R is a subalgebra of A and A has local units in R. We do not assume that R is a central subalgebra of A. Any $\mathbb{C}$-algebra A which extends R, is naturally an R-R-bimodule. We will denote by $A \otimes_R A = A \otimes_R^{ll} A$ the tensor product of left R-modules, which has two natural right R-module structures. To be precise, for $a, a' \in A$ and $r \in R$, the two right module structures are given by $(a \otimes a')r = ar \otimes a'$, respectively, $(a \otimes a')r = a \otimes a'r$.

A *bialgebroid* over R is a $\mathbb{C}$-algebra A which extends R, together with structure maps $\Delta : A \to A \otimes_R A$ and $\epsilon : A \to R$ for which (A, Δ, ϵ) is a cocommutative coalgebra, such that:

(i) $\Delta(A) \subset A \overline{\otimes}_R A$, where $A \overline{\otimes}_R A$ is the algebra consisting of those elements of $A \otimes_R A$, on which both right R-actions coincide,
(ii) $\epsilon|_R = \mathrm{id}$ and $\Delta|_R$ is the canonical embedding $R \subset A \otimes_R A$,
(iii) $\epsilon(ab) = \epsilon(a\epsilon(b))$ and $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a, b \in A$.

Antipode on a bialgebroid A is a $\mathbb{C}$-linear involution $S : A \to A$ which satisfies the conditions:

(i) $S|_R = \mathrm{id}$ and $S(ab) = S(b)S(a)$ for any $a, b \in A$,
(ii) $\mu_A \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon \circ S$, where μ_A denotes the multiplication in A.

A *Hopf algebroid* over R is a bialgebroid A over R with an antipode. Note that in the case when R is a central subalgebra of A, the notions of bialgebroid and Hopf algebroid coincide with the more familiar notions of bialgebra and Hopf algebra.

In the next example, we will recall from [21] the Hopf algebroid associated with an étale Lie groupoid.

Example 2.3. Let $\mathscr{G}$ be an étale Lie groupoid over M. Multiplication on $\mathscr{G}$ induces a convolution product [5] on $\mathcal{C}_c^\infty(\mathscr{G})$, given by the formula

$$(a_1 * a_2)(g) = \sum_{g = g_1 g_2} a_1(g_1) a_2(g_2)$$

for $a_1, a_2 \in \mathcal{C}_c^\infty(\mathscr{G})$. Note that this sum is always finite as a_1 and a_2 are compactly supported. Since $\mathrm{t} : \mathscr{G} \to M$ is a sheaf, we can also consider the space $\mathcal{C}_c^\infty(\mathscr{G})$ as a locally grouplike coalgebra over $\mathcal{C}_c^\infty(M)$. Finally, the antipode $S : \mathcal{C}_c^\infty(\mathscr{G}) \to \mathcal{C}_c^\infty(\mathscr{G})$ is defined by the formula

$$S(a) = a \circ \mathrm{inv}$$

for $a \in \mathcal{C}_c^\infty(\mathscr{G})$. In this way, $\mathcal{C}_c^\infty(\mathscr{G})$ becomes a Hopf algebroid over $\mathcal{C}_c^\infty(M)$.

Suppose now that Γ is a discrete group which acts from the right on the manifold M and denote by $M \rtimes \Gamma$ the associated action groupoid. We then have the decomposition

$$\mathcal{C}_c^\infty(M \rtimes \Gamma) = \bigoplus_{g \in \Gamma} \mathcal{C}_c^\infty(M \times \{g\}).$$

For any $g \in \Gamma$ and $f \in \mathcal{C}_c^\infty(M)$, let us denote by $gf \in \mathcal{C}_c^\infty(M)$ the function, given by $(gf)(x) = f(xg)$ for $x \in M$. If we use the notation from Example 2.2, the convolution product and the antipode on $\mathcal{C}_c^\infty(M \rtimes \Gamma)$ can be described on the set of generators by the formulas

$$(f_1 \cdot \delta_{g_1}) * (f_2 \cdot \delta_{g_2}) = (f_1(g_1 f_2)) \cdot \delta_{g_1 g_2},$$
$$S(f \cdot \delta_g) = (g^{-1} f) \cdot \delta_{g^{-1}}.$$

Now, let A be a Hopf algebroid over $\mathcal{C}_c^\infty(M)$. We will next recall from [21] the construction of the spectral étale Lie groupoid $\mathcal{G}_{sp}(A)$, associated with A. Note that A is a coalgebra over $\mathcal{C}_c^\infty(M)$, so we have the notion of weakly grouplike elements. We say that a weakly grouplike element $a \in A$ is S-invariant if there exists $a' \in A$, such that $\Delta(a) = a \otimes a'$ and $\Delta(S(a)) = S(a') \otimes S(a)$. In the case of the Hopf algebroid $\mathcal{C}_c^\infty(\mathscr{G})$ of an étale Lie groupoid $\mathscr{G}$, an element $a \in \mathcal{C}_c^\infty(\mathscr{G})$ is S-invariant weakly grouplike if and only if it is of the form $f \circ \mathrm{t}|_W$ for some bisection W of $\mathscr{G}$ and some $f \in \mathcal{C}_c^\infty(\mathrm{t}(W))$ (a bisection of an étale Lie groupoid $\mathscr{G}$ is an open subset W of $\mathscr{G}$ which is both t-elementary and s-elementary).

An arrow of A with target $y \in M$ is an element $g \in G(A_y)$, for which there exists an S-invariant weakly grouplike element $a \in A$, such that $g = a|_y$. The set of all arrows of A with target y will be denoted by $\mathcal{G}_{sp}(A)_y$. All arrows of A form a subsheaf $\mathcal{G}_{sp}(A)$ of $\mathcal{E}_{sp}(A)$, whose projection will be denoted by

$$\mathrm{t} = \pi_{sp}(A)|_{\mathcal{G}_{sp}(A)} : \mathcal{G}_{sp}(A) \to M.$$

To describe the source map of $\mathcal{G}_{sp}(A)$, we first recall that each S-invariant weakly grouplike element $a \in A$ induces a $\mathbb{C}$-linear map $T_a : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)$, given by $T_a(f) = \epsilon(S(fa))$. If a is normalized on an open subset U of M, one can find an open subset U' of M and a unique diffeomorphism $\tau_{U,a} : U' \to U$, such that $T_a(\mathcal{C}_c^\infty(U)) \subset \mathcal{C}_c^\infty(U')$ and $T_a(f) = f \circ \tau_{U,a}$ for any $f \in \mathcal{C}_c^\infty(U)$. Furthermore, the element $S(a)$ is S-invariant weakly grouplike, normalized on U', and we have that $\tau_{U',S(a)} = \tau_{U,a}^{-1}$. The source map $\mathrm{s} : \mathcal{G}_{sp}(A) \to M$ is now defined by

$$\mathrm{s}(a|_y) = \tau_{U,a}^{-1}(y),$$

where $a \in A$ is an S-invariant weakly grouplike element, normalized on U. Now, choose elements $a, b \in A$ which represent an arrow $a|_y \in \mathcal{G}_{sp}(A)_y$ and an arrow $b|_x \in \mathcal{G}_{sp}(A)_x$ such that $\mathrm{s}(a|_y) = x$. The product of $a|_y$ and $b|_x$ is then defined by

$$a|_y b|_x = (ab)|_y.$$

The unit $\mathrm{uni}(x) \in \mathcal{G}_{sp}(A)$ at the point $x \in M$ is given by

$$\mathrm{uni}(x) = f|_x,$$

where $f \in \mathcal{C}_c^\infty(M) \subset A$ is any function with germ $f|_x = 1 \in \mathcal{C}_c^\infty(M)_x$. Finally, the inverse of an arrow $a|_y \in \mathcal{G}_{sp}(A)_y$ with $\mathrm{s}(a|_y) = x$ is defined by

$$a|_y^{-1} = S(a)|_x.$$

The groupoid $\mathcal{G}_{sp}(A)$ is an étale Lie groupoid over M, called the spectral étale Lie groupoid of the Hopf algebroid A. For any étale Lie groupoid $\mathscr{G}$ over M, we have a natural isomorphism of Lie groupoids

$$\Phi_{\mathscr{G}}^{\mathrm{egr}} : \mathscr{G} \to \mathcal{G}_{sp}(\mathcal{C}_c^\infty(\mathscr{G})),$$

defined by

$$\Phi_{\mathscr{G}}^{\mathrm{egr}}(g) = (f \circ \mathrm{t}|_W)|_{t(g)},$$

where W is any bisection of $\mathscr{G}$ which contains g and $f \in \mathcal{C}_c^\infty(\mathrm{t}(W))$ is any function with $f|_{\mathrm{t}(g)} = 1 \in \mathcal{C}_c^\infty(M)_{\mathrm{t}(g)}$.

Hopf algebroid A is isomorphic to a Hopf algebroid of the form $\mathcal{C}_c^\infty(\mathscr{G})$ for an étale Lie groupoid $\mathscr{G}$ if and only if it is locally grouplike, which means that for every $y \in M$, the $\mathcal{C}_c^\infty(M)_y$-coalgebra A_y is a free $\mathcal{C}_c^\infty(M)_y$-module with the basis consisting of arrows of A at the point y.

3. Transversal Distributions of Constant Dirac Type

Let $\pi : M \times N \to M$ be a trivial bundle over M with fiber N. In the spirit of the Gelfand–Naimark theorem, we will assign to it a locally convex $\mathcal{C}_c^\infty(M)$-module $\mathrm{Dirac}_\pi(M \times N)$ of distributions of constant Dirac type on $M \times N$ and show that its strong $\mathcal{C}_c^\infty(M)$-dual is isomorphic to the space $\mathcal{C}^\infty(M \times N)$. In general, the space $\mathrm{Dirac}_\pi(M \times N)$ is a dense subspace of the space $\mathcal{E}'_\pi(M \times N)$ of compactly supported transversal distributions. However, in the case when $N = \Gamma$ is discrete, the space $\mathrm{Dirac}_\pi(M \times \Gamma)$ is complete and isomorphic to the LF-space $\mathcal{C}_c^\infty(M \times \Gamma)$.

We start with the definition of transversal distributions on a trivial bundle.

Definition 3.1. Let M and N be second-countable manifolds and let $M \times N$ be the trivial bundle over M with fiber N and projection $\pi : M \times N \to M$. The space of **π-transversal distributions with compact support** is the space

$$\mathcal{E}'_\pi(M \times N) = \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{C}^\infty(M \times N), \mathcal{C}_c^\infty(M)).$$

In other words, $\mathcal{E}'_\pi(M \times N)$ is the space of continuous $\mathcal{C}_c^\infty(M)$-linear maps from $\mathcal{C}^\infty(M \times N)$ to $\mathcal{C}_c^\infty(M)$, where the $\mathcal{C}_c^\infty(M)$-module structure on $\mathcal{C}^\infty(M \times N)$ is given by

$$(f \cdot F)(x, y) = f(x)F(x, y)$$

for $f \in \mathcal{C}_c^\infty(M)$, $F \in \mathcal{C}^\infty(M \times N)$ and $(x, y) \in M \times N$. The space $\mathcal{E}'_\pi(M \times N)$ is a $\mathcal{C}_c^\infty(M)$-module as well, with module structure given by

$$(f \cdot T)(F) = T(f \cdot F)$$

for $f \in \mathcal{C}_c^\infty(M)$, $F \in \mathcal{C}^\infty(M \times N)$ and $T \in \mathcal{E}'_\pi(M \times N)$. If we equip the space $\mathcal{E}'_\pi(M \times N)$ with the strong topology of uniform convergence on bounded subsets, it becomes a complete locally convex space.

Remark 3.2. (1) It was shown in [14] that there is an isomorphism

$$\mathcal{E}'_\pi(M \times N) \cong \mathcal{C}_c^\infty(M, \mathcal{E}'(N)),$$

which enables us to identify a π-transversal distribution $T \in \mathcal{E}'_\pi(M \times N)$ with a smooth, compactly supported family $u = u(T) \in \mathcal{C}_c^\infty(M, \mathcal{E}'(N))$. We will denote the value of u at $x \in M$ by $u_x \in \mathcal{E}'(N)$. If we denote by $\pi_N : M \times N \to N$ the projection to N, the distribution u_x is given by the formula

$$u_x(F) = T(F \circ \pi_N)(x)$$

for any $F \in \mathcal{C}^\infty(N)$. We can also view any $u \in \mathcal{C}_c^\infty(M, \mathcal{E}'(N))$ as a π-transversal distribution $T = T(u)$, if we define

$$T(F)(x) = u_x(F \circ \iota_x)$$

for $F \in \mathcal{C}^\infty(M \times N)$. Here, $\iota_x : N \to \{x\} \times N$ is given by $\iota_x(y) = (x, y)$ for $x \in M$. Different letters T and u are used intentionally to make a slight distinction between transversal distributions and smooth families of distributions along the fibers.

(2) We can define a support of a family $u \in \mathcal{C}_c^\infty(M, \mathcal{E}'(N))$ either as a subspace of M or a subspace of $M \times N$. The support of u is the subset $\operatorname{supp}(u)$ of M, defined by $\operatorname{supp}(u) = \overline{\{x \in M \mid u_x \neq 0\}}$. On the other hand, the total support of u is the subset $\operatorname{supp}_{M\times N}(u)$ of $M \times N$, consisting of all points $(x, y) \in M \times N$, which satisfy the condition that for every open neighbourhood U of (x, y), there exists $F \in \mathcal{C}_c^\infty(U)$, such that $u(F) \neq 0$. For $u \in \mathcal{C}_c^\infty(M, \mathcal{E}'(N))$, both supports are compact and we have $\pi(\operatorname{supp}_{M\times N}(u)) = \operatorname{supp}(u)$. We can also define the support of a π-transversal distribution $T \in \mathcal{E}'_\pi(M \times N)$ as the subset $\operatorname{supp}(T) = \operatorname{supp}_{M\times N}(u(T))$ of $M \times N$. For more details about supports, we refer the reader to [9].

(3) If $\dim(M) > 0$, the space $\operatorname{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{C}^\infty(M \times N), \mathcal{C}_c^\infty(M))$ in fact coincides with the space $\operatorname{Lin}_{\mathcal{C}_c^\infty(M)}(\mathcal{C}^\infty(M \times N), \mathcal{C}_c^\infty(M))$ of $\mathcal{C}_c^\infty(M)$-linear maps, without any assumption on continuity (see [8]).

Let us take a look at some important examples of transversal distributions that will be used throughout the paper.

Example 3.3. (1) Let $\pi : M \times N \to M$ be a trivial bundle and denote for any $y \in N$ by $E_y = M \times \{y\}$ the horizontal subspace of $M \times N$. For any $f \in \mathcal{C}_c^\infty(M)$, we define a π-transversal distribution $[\![E_y, f]\!] \in \mathcal{E}'_\pi(M \times N)$ by

$$[\![E_y, f]\!](F)(x) = f(x)F(x, y),$$

for $F \in \mathcal{C}^\infty(M \times N)$. We think of $[\![E_y, f]\!]$ as a smooth family of Dirac distributions, supported on the constant section E_y and weighted by the function f. In particular, we have

$$[\![E_y, f]\!]_x = f(x)\delta_y.$$

(2) Let $M = \mathbb{R}^l$, $N = \mathbb{R}^k$ and let $\pi : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}^l$ be the projection onto $\mathbb{R}^l$. For $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$, we define a π-transversal distribution $T_\phi \in \mathcal{E}'_\pi(\mathbb{R}^l \times \mathbb{R}^k)$ by

$$T_\phi(F)(x) = \int_{\mathbb{R}^k} \phi(x, y)F(x, y)\,\mathrm{d}y$$

for $F \in \mathcal{C}^\infty(\mathbb{R}^l \times \mathbb{R}^k)$. The distribution T_ϕ corresponds to the family of smooth densities on $\mathbb{R}^k$, parametrized by $\mathbb{R}^l$ and explicitly given by

$$(T_\phi)_x = \phi(x, -)dV,$$

where dV is the Lebesgue measure on $\mathbb{R}^k$.

The map $\phi \mapsto T_\phi$ defines a continuous, injective $\mathcal{C}_c^\infty(\mathbb{R}^l)$-linear map

$$\mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k) \hookrightarrow \mathcal{E}'_\pi(\mathbb{R}^l \times \mathbb{R}^k).$$

Note that the LF-topology on $\mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$ is strictly finer than the subspace topology that is induced from $\mathcal{E}'_\pi(\mathbb{R}^l \times \mathbb{R}^k)$ via the above map. In particular, if $M = \mathbb{R}^0$ is a point, the above construction enables us to consider the space $\mathcal{C}_c^\infty(\mathbb{R}^k)$ as a subspace of the space $\mathcal{E}'(\mathbb{R}^k)$.

Let us now denote for a manifold N by $N^\#$ the set N with the discrete topology. The projection $\pi^\# : M \times N^\# \to M$ is then a local diffeomorphism. Note that the manifold $M \times N^\#$ is paracompact, but not second-countable if $\dim(N) > 0$.

Using the notation from the Example 2.2, we have a decomposition

$$\mathcal{C}_c^\infty(M \times N^\#) = \bigoplus_{y \in N^\#} \mathcal{C}_c^\infty(M \times \{y\}),$$

which enables us to write every element $a \in \mathcal{C}_c^\infty(M \times N^\#)$ uniquely in the form

$$a = \sum_{i=1}^n f_i \cdot \delta_{y_i}$$

for some $f_1, \ldots, f_n \in \mathcal{C}_c^\infty(M)$ and some $y_1, \ldots, y_n \in N$. Now, define an injective $\mathcal{C}_c^\infty(M)$-linear map $\Psi_{M\times N} : \mathcal{C}_c^\infty(M \times N^\#) \to \mathcal{E}'_\pi(M \times N)$ by

$$\Psi_{M\times N}\left(\sum_{i=1}^n f_i \cdot \delta_{y_i}\right) = \sum_{i=1}^n [\![E_{y_i}, f_i]\!].$$

Definition 3.4. Let $M \times N$ be a trivial bundle with projection $\pi : M \times N \to M$. The space of **π-transversal distributions of constant Dirac type** is the space

$$\mathrm{Dirac}_\pi(M \times N) = \Psi_{M\times N}(\mathcal{C}_c^\infty(M \times N^\#)) \subset \mathcal{E}'_\pi(M \times N).$$

The space $\mathrm{Dirac}_\pi(M\times N)$ is equipped with the induced topology from $\mathcal{E}'_\pi(M\times N)$.

If M is a point, we denote by $\mathrm{Dirac}(N)$ the subspace of $\mathcal{E}'(N)$, spanned by Dirac distributions.

We will show in the sequel that $\mathrm{Dirac}_\pi(M \times N)$ is a proper, dense subspace of $\mathcal{E}'_\pi(M \times N)$ if $\dim(N) > 0$. However, in the case, when $N = \Gamma$ is discrete, we have the following description of the space $\mathrm{Dirac}_\pi(M \times \Gamma)$.

Proposition 3.5. *Let $M\times\Gamma$ be a trivial bundle over a second-countable manifold M with a countable discrete fiber Γ and bundle projection $\pi : M\times\Gamma \to M$.*

- (a) *The map $\Psi_{M\times\Gamma} : \mathcal{C}^\infty_c(M \times \Gamma) \to \mathcal{E}'_\pi(M \times \Gamma)$ is an isomorphism of $\mathcal{C}^\infty_c(M)$-modules, so we have* $\mathrm{Dirac}_\pi(M \times \Gamma) = \mathcal{E}'_\pi(M \times \Gamma)$.
- (b) *The map $\Psi_{M\times\Gamma}$ is an isomorphism of locally convex spaces with respect to the LF-topology on $\mathcal{C}^\infty_c(M\times\Gamma)$ and the strong topology on $\mathcal{E}'_\pi(M\times\Gamma)$.*

Proof. (a) First, recall that we have an isomorphism $\mathcal{E}'_\pi(M \times \Gamma) \cong \mathcal{C}^\infty_c(M,\mathcal{E}'(\Gamma))$ of $\mathcal{C}^\infty_c(M)$-modules. It is, therefore, enough to show that every $u \in \mathcal{C}^\infty_c(M,\mathcal{E}'(\Gamma))$ is of the form $u = \Psi_{M\times\Gamma}(a)$ for some $a \in \mathcal{C}^\infty_c(M \times \Gamma)$.

Since Γ is discrete, the space $\mathcal{E}'(\Gamma)$ is isomorphic to the locally convex direct sum $\bigoplus_{y\in\Gamma} \mathrm{Span}(\delta_y)$ of one-dimensional subspaces, spanned by Dirac distributions. Any $u \in \mathcal{C}^\infty_c(M,\mathcal{E}'(\Gamma))$ has compact support, so its image $u(M) \subset \mathcal{E}'(\Gamma)$ is compact and hence bounded. This implies that $u(M)$ lies in some finite-dimensional subspace $\bigoplus_{i=1}^n \mathrm{Span}(\delta_{y_i}) \subset \mathcal{E}'(\Gamma)$ for some $y_1,\dots,y_n \in \Gamma$. We can therefore find functions $f_1,\dots,f_n : M \to \mathbb{C}$, such that

$$u_x = \sum_{i=1}^n f_i(x)\delta_{y_i}$$

for every $x \in M$. If we denote by $1_{y_i} \in \mathcal{C}^\infty(M \times \Gamma)$ the function, which is equal to 1 on $M \times \{y_i\}$ and zero elsewhere, we have $f_i = u(1_{y_i}) \in \mathcal{C}^\infty_c(M)$, and therefore

$$u = \Psi_{M\times\Gamma}\left(\sum_{i=1}^n f_i \cdot \delta_{y_i}\right).$$

(b) To see that $\Psi_{M\times\Gamma}$ is continuous, first, choose a basic neighbourhood $K(B,V)$ of zero in $\mathcal{E}'_\pi(M\times\Gamma)$, where B is a bounded subset of $\mathcal{C}^\infty(M\times\Gamma)$ and V is a neighbourhood of zero in $\mathcal{C}^\infty_c(M)$. From the definition of LF-topology on $\mathcal{C}^\infty_c(M \times \Gamma)$, it follows that we only have to show that the restrictions of $\Psi_{M\times\Gamma}$ onto subspaces of the form $\mathcal{C}^\infty_c(L \times \{y\})$ are continuous for all $y \in \Gamma$ and all compact subsets L of M. Define a multiplication map $\mu : \mathcal{C}^\infty_c(L \times \{y\}) \times \mathcal{C}^\infty(M \times \Gamma) \to \mathcal{C}^\infty_c(M)$ by

$$\mu(f \cdot \delta_y, F)(x) = f(x)F(x,y)$$

for $f \cdot \delta_y \in \mathcal{C}^\infty_c(L \times \{y\})$ and $F \in \mathcal{C}^\infty(M \times \Gamma)$. Note that μ is continuous, so we can find neighbourhoods V_1 and V_2 of zero in $\mathcal{C}^\infty_c(L \times \{y\})$, respectively, $\mathcal{C}^\infty(M \times \Gamma)$, such that $\mu(V_1,V_2) \subset V$. Since $B \subset \mathcal{C}^\infty(M \times \Gamma)$ is bounded, we can assume that $B \subset V_2$ (if necessary, rescale V_1 and V_2 by appropriate inverse factors). Now, observe that

$$\mu(f \cdot \delta_y, F) = \Psi_{M\times\Gamma}(f \cdot \delta_y)(F).$$

For $f \cdot \delta_y \in V_1$ and $F \in B$, we thus have $\Psi_{M\times\Gamma}(f \cdot \delta_y)(F) = \mu(f \cdot \delta_y, F) \in V$, which shows that $\Psi_{M\times\Gamma}(V_1) \subset K(B, V)$.

Finally, we have to show that the map $\Psi^{-1}_{M\times\Gamma}$ is continuous. Let us choose a net $(u_\alpha)_{\alpha\in A}$ that converges to zero in $\mathcal{E}'_\pi(M \times \Gamma)$. The set $\{u_\alpha \mid \alpha \in A\}$ is then a bounded subset of $\mathcal{E}'_\pi(M \times \Gamma)$, so there exists a compact subset of $M \times \Gamma$ which contains all supports $\mathrm{supp}(u_\alpha)$ for $\alpha \in A$. In particular, we can find a compact subset L of M and elements $y_1, y_2, \ldots, y_n \in \Gamma$, such that for every $\alpha \in A$, we can write

$$u_\alpha = \sum_{i=1}^{n} [\![E_{y_i}, f_{\alpha,i}]\!]$$

for some $f_{\alpha,1}, \ldots, f_{\alpha,n} \in \mathcal{C}^\infty_c(L)$. If we evaluate u_α at 1_{y_i} (see part (a) of the proof for the definition of 1_{y_i}), we get that $u_\alpha(1_{y_i}) = f_{\alpha,i}$ converges to zero in $\mathcal{C}^\infty_c(M)$ for $1 \leqslant i \leqslant n$, which implies that $\Psi^{-1}_{M\times\Gamma}(u_\alpha) = \sum_{i=1}^n f_{\alpha,i} \cdot \delta_{y_i}$ converges to zero in $\mathcal{C}^\infty_c(M \times \Gamma)$. □

We will now move on to the study of the space $\mathrm{Dirac}_\pi(M \times N)$ in the case of non-discrete fiber and show that it is a dense subspace of the space $\mathcal{E}'_\pi(M \times N)$.

To do that, we first recall some facts about the convolution of distributions on Euclidean spaces. Usually, the convolution of compactly supported distributions $v, w \in \mathcal{E}'(\mathbb{R}^k)$ is defined as the push-forward of $v \otimes w$ along the multiplication map. However, for the purposes of this paper, we will use the definition of the convolution product on $\mathcal{E}'(\mathbb{R}^k)$ that can be easily generalized to arbitrary Lie groupoids and is useful in concrete computations (see Sect. 5). For any $F \in \mathcal{C}^\infty(\mathbb{R}^k)$, we can define a smooth map $\mathbb{R}^k \to \mathcal{C}^\infty(\mathbb{R}^k)$ by $y \to F \circ L_y$, where the left translation $L_y : \mathbb{R}^k \to \mathbb{R}^k$ is defined by $L_y(y') = y + y'$ for $y \in \mathbb{R}^k$. If we compose this map with an arbitrary distribution $w \in \mathcal{E}'(\mathbb{R}^k)$, we thus get a smooth map $\mathbb{R}^k \to \mathbb{C}$, given by $y \to w(F \circ L_y)$. The convolution product $* : \mathcal{E}'(\mathbb{R}^k) \times \mathcal{E}'(\mathbb{R}^k) \to \mathcal{E}'(\mathbb{R}^k)$ can be then described by the formula

$$(v * w)(F) = v(y \to w(F \circ L_y))$$

for $F \in \mathcal{C}^\infty(\mathbb{R}^k)$ and $v, w \in \mathcal{E}'(\mathbb{R}^k)$. The convolution product is a bilinear, jointly continuous map and it turns $\mathcal{E}'(\mathbb{R}^k)$ into a commutative, locally convex algebra.

As we have seen in Example 3.3, we can consider $\mathcal{C}^\infty_c(\mathbb{R}^k)$ as a subspace of $\mathcal{E}'(\mathbb{R}^k)$. We can explicitly describe the convolution of an arbitrary distribution with a smooth function as follows. Choose any $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^k)$ and consider it as an element $T_\rho \in \mathcal{E}'(\mathbb{R}^k)$, which we will for simplicity denote just by ρ. Let $\tilde{\rho} \in \mathcal{C}^\infty(\mathbb{R}^k)$ be defined by $\tilde{\rho}(z) = \rho(-z)$. For any $v \in \mathcal{E}'(\mathbb{R}^k)$, we then have that $v * \rho \in \mathcal{C}^\infty_c(\mathbb{R}^k) \subset \mathcal{E}'(\mathbb{R}^k)$ is a smooth function, given by

$$(v * \rho)(y) = v(\tilde{\rho} \circ L_{-y})$$

for $y \in \mathbb{R}^k$. This shows that $\mathcal{C}_c^\infty(\mathbb{R}^k)$ is actually an ideal of $\mathcal{E}'(\mathbb{R}^k)$.

We will now use these results in the setting of transversal distributions. For any $u = (u_x)_{x\in\mathbb{R}^l} \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ and any $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^k)$, we define $u * \rho \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ pointwise by

$$(u * \rho)_x = u_x * \rho.$$

Smoothness of $u*\rho$ follows from bilinearity and continuity of the convolution product. One can, moreover, show that $u*\rho$ is actually of the form $u*\rho = T_\phi$ for the smooth function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$, given by

$$\phi(x, y) = u_x(\tilde{\rho} \circ L_{-y}).$$

We will next show that the image of the map $\mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k) \hookrightarrow \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$, given by $\phi \mapsto T_\phi$, is a dense subspace of $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$.

To do that, we first recall from [9] an explicit description of a neighbourhood basis of zero in the space $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$. Denote $K_0 = \emptyset$ and let $K_n = \{x \in \mathbb{R}^l \mid |x| \leqslant n\}$ be the ball with centre at zero and radius $n \in \mathbb{N}$. Choose an increasing sequence of natural numbers $\mathbf{m} = (m_1, m_2, \ldots)$, a decreasing sequence of positive real numbers $\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots)$ and let $\mathbf{B} = (B_1, B_2, \ldots)$ be an increasing sequence of bounded subsets of $\mathcal{C}^\infty(\mathbb{R}^k)$. Now, define a subset $V_{\mathbf{B},\mathbf{m},\mathbf{e}} \subset \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ by

$$\begin{aligned} V_{\mathbf{B},\mathbf{m},\mathbf{e}} = \{u \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k)) \mid p_{B_n}((D^\alpha u)_x) \\ < \epsilon_n \text{ for } x \in K_{n-1}^c \text{ and } |\alpha| \leqslant m_n\}, \end{aligned}$$

where the seminorm p_{B_n} on $\mathcal{E}'(\mathbb{R}^k)$ is given by

$$p_{B_n}(v) = \sup_{F\in B_n} |v(F)|$$

for $v \in \mathcal{E}'(\mathbb{R}^k)$. The family of all such sets $V_{\mathbf{B},\mathbf{m},\mathbf{e}}$, with $\mathbf{B}$, $\mathbf{m}$, and $\mathbf{e}$ as defined above then forms a basis of neighbourhoods of zero for a topology on $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$, for which the natural identification $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k)) \cong \mathcal{E}'_\pi(\mathbb{R}^l \times \mathbb{R}^k)$ becomes an isomorphism of locally convex vector spaces (see [9]).

Proposition 3.6. *The image of the map* $\mathcal{C}_c^\infty(\mathbb{R}^l\times\mathbb{R}^k) \hookrightarrow \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$*, given by* $\phi \mapsto T_\phi$*, is a dense subspace of* $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$*.*

Proof. Let us choose a one-parameter family $(\rho_t) \in \mathcal{C}_c^\infty(\mathbb{R}^k) \subset \mathcal{E}'(\mathbb{R}^k)$, for $t \in (0, 1)$, which converges to the Dirac distribution $\delta_0 \in \mathcal{E}'(\mathbb{R}^k)$ as $t \to 0$.

Now, choose any $u \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ and define $u_t = u * \rho_t \in \mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$ for $t \in (0, 1)$. We will show that $u_t \to u$ as $t \to 0$. To do that, take an arbitrary basic neighbourhood of zero in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ of the form $V_{\mathbf{B},\mathbf{m},\mathbf{e}}$. Since u is compactly supported, we have $\mathrm{supp}(u) = \mathrm{supp}(u_t) \subset K_n$ for some $n \in \mathbb{N}$ and all $t \in (0, 1)$. We need to show that for t small enough, we have $u - u_t \in V_{\mathbf{B},\mathbf{m},\mathbf{e}}$, which by the above observation means that $p_{B_n}(D^\alpha(u - u_t)_x) < \epsilon_n$ for all $x \in K_n$ and all α with $|\alpha| \leqslant m_n$. Equivalently, if we denote $\mathbb{D}(\epsilon_n) = \{z \in \mathbb{C} \mid |z| < \epsilon_n\}$, then for all such x and α, we have

$$D^\alpha(u - u_t)_x \in K(B_n, \mathbb{D}(\epsilon_n)) \subset \mathcal{E}'(\mathbb{R}^k).$$

Now, note that the set $A = \{(D^\alpha u)_x \,|\, x \in K_n, |\alpha| \leqslant m_n\}$ is compact and, hence, a bounded subset of $\mathcal{E}'(\mathbb{R}^k)$. Since the convolution $* : \mathcal{E}'(\mathbb{R}^k) \times \mathcal{E}'(\mathbb{R}^k) \to \mathcal{E}'(\mathbb{R}^k)$ is bilinear and continuous, we can find a neighbourhood V of zero in $\mathcal{E}'(\mathbb{R}^k)$ such that $V * A \subset K(B_n, \mathbb{D}(\epsilon_n))$. Since $\rho_t \to \delta_0$ as $t \to 0$, we have that $\delta_0 - \rho_t \in V$ for t small enough, and hence

$$D^\alpha(u - u_t)_x = (D^\alpha u)_x - (D^\alpha u * \rho_t)_x = (\delta_0 - \rho_t) * (D^\alpha u)_x \in K(B_n, \mathbb{D}(\epsilon_n)).$$

□

We will next show, using ideas from Riemannian integration, that arbitrary transversal distribution of the form $T_\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ can be approximated by elements of $\mathrm{Dirac}_\pi(\mathbb{R}^l \times \mathbb{R}^k)$.

Choose any $L > 0$ and any $n \in \mathbb{N}$ and denote $t_j = -\frac{L}{2} + \frac{jL}{n}$ for $0 \leqslant j \leqslant n-1$. The set $I = \{t_0, t_1, \ldots, t_{n-1}\}^k$ is then a finite subset of the cube $D = [-\frac{L}{2}, \frac{L}{2}]^k$. If we denote for $t \in I$ by $D_t = t + [0, \frac{L}{n}]^k$ the cube with volume $\mathrm{vol}(D_t) = \left(\frac{L}{n}\right)^k$, we can express $D = \bigcup_{t \in I} D_t$ as a union of n^k such small cubes. Now, define a distribution $\Delta_n \in \mathrm{Dirac}(\mathbb{R}^k) \subset \mathcal{E}'(\mathbb{R}^k)$ by

$$\Delta_n = \left(\tfrac{L}{n}\right)^k \sum_{t \in I} \delta_t.$$

Using the fundamental theorem of calculus, one can show that for any $F \in \mathcal{C}^\infty(\mathbb{R}^k)$, we have the following bound:

$$\left| \int_D F(y)\,\mathrm{d}y - \Delta_n(F) \right| \leqslant \frac{kL^{k+1}}{n} p_{D,1}(F),$$

where $p_{D,1}$ measures the size of the gradient of F and is defined as in Sect. 2.1. This bound basically says that the sequence $(\Delta_n)_{n\in\mathbb{N}}$ converges to $\int_D$ in $\mathcal{E}'(\mathbb{R}^k)$.

Proposition 3.7. *The space* $\mathrm{Dirac}_\pi(\mathbb{R}^l \times \mathbb{R}^k)$ *is a dense subspace of* $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$.

Proof. By Proposition 3.6, it is enough to show that for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$, the distribution $T_\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ can be approximated arbitrarily well by elements of $\mathrm{Dirac}_\pi(\mathbb{R}^l \times \mathbb{R}^k)$.

Choose any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^l \times \mathbb{R}^k)$ and suppose $\pi_{\mathbb{R}^k}(\mathrm{supp}(\phi)) \subset D \subset \mathbb{R}^k$ for some $L > 0$ as above. For $n \in \mathbb{N}$, we define a π-transversal distribution $\Delta_{\phi,n} \in \mathrm{Dirac}_\pi(\mathbb{R}^l \times \mathbb{R}^k)$ by the formula

$$\Delta_{\phi,n}(F)(x) = \left(\tfrac{L}{n}\right)^k \sum_{t \in I} \phi(x,t) F(x,t).$$

We will show that $\Delta_{\phi,n} \to T_\phi$ in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ as $n \to \infty$. This means that, for every set of the form $V_{\mathbf{B},\mathbf{m},\mathbf{e}} \subset \mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$, we have $T_\phi - \Delta_{\phi,n} \in V_{\mathbf{B},\mathbf{m},\mathbf{e}}$ for $n \in \mathbb{N}$ big enough. Both T_ϕ and $\Delta_{\phi,n}$ have supports contained in $\pi(\mathrm{supp}(\phi)) \subset K_j$ for some $j \in \mathbb{N}$, so we have to show that

$$|D_x^\alpha(T_\phi(F) - \Delta_{\phi,n}(F))(x)| < \epsilon_j$$

for $x \in K_j$, $F \in B_j$ and $|\alpha| \leqslant m_j$. Since B_j is a bounded subset of $\mathcal{C}^\infty(\mathbb{R}^l \times \mathbb{R}^k)$, the set $\tilde{B}_j = \phi B_j = \{\tilde{F} = \phi F \,|\, F \in B_j\}$ is bounded in $\mathcal{C}^\infty(\mathbb{R}^l \times \mathbb{R}^k)$ as well, so there exists a constant $C < \infty$, such that $\sup\{p_{K_j \times D, m_j+1}(\tilde{F}) \mid \tilde{F} \in \tilde{B}_j\} < C$. For $F \in B_j$, $x \in K_j$ and $|\alpha| \leqslant m_j$, we now compute the following:

$$
\begin{aligned}
&|D_x^\alpha(T_\phi(F) - \Delta_{\phi,n}(F))(x)| \\
&\quad = \left| \int_D D_x^\alpha(\phi(x,y)F(x,y))\,\mathrm{d}y - \left(\tfrac{L}{n}\right)^k \sum_{t \in I} D_x^\alpha(\phi(x,t)F(x,t)) \right|, \\
&\quad = \left| \int_D (D_x^\alpha \tilde{F})(x,y)\,\mathrm{d}y - \left(\tfrac{L}{n}\right)^k \sum_{t \in I} (D_x^\alpha \tilde{F})(x,t) \right|, \\
&\quad \leqslant \frac{kL^{k+1}}{n} p_{K_j \times D, m_j+1}(\tilde{F}) < \frac{CkL^{k+1}}{n}.
\end{aligned}
$$

We conclude that $\Delta_{\phi,n} \to T_\phi$ in $\mathcal{C}_c^\infty(\mathbb{R}^l, \mathcal{E}'(\mathbb{R}^k))$ as $n \to \infty$. □

As a corollary, we get the following result.

Proposition 3.8. *Let $M \times N$ be a trivial bundle over M with fiber N and projection $\pi : M \times N \to M$. The space* $\mathrm{Dirac}_\pi(M \times N)$ *is a dense subspace of $\mathcal{E}'_\pi(M \times N)$.*

Proof. Every transversal distribution $T \in \mathcal{E}'_\pi(M \times N)$ has compact support, so we can write it as a sum

$$T = T_1 + T_2 + \cdots + T_n,$$

where each $T_i \in \mathcal{E}'_\pi(M \times N)$ has support contained in the set of the form $U_i \times U'_i$ for some domains of coordinate charts $U_i \approx \mathbb{R}^l$ on M and $U'_i \approx \mathbb{R}^k$ on N. By Proposition 3.7, we can find for each neighbourhood of zero $V \subset \mathcal{E}'_\pi(M \times N)$ elements $u_i \in \mathrm{Dirac}_{\pi_i}(U_i \times U'_i) \subset \mathrm{Dirac}_\pi(M \times N)$, such that $T_i - u_i \in \frac{1}{n} V$ for $1 \leqslant i \leqslant n$. If we define $u = u_1 + u_2 + \cdots + u_n \in \mathrm{Dirac}_\pi(M \times N)$, we then have $T - u \in V$. □

Let us now denote for simplicity by

$$
\begin{aligned}
\mathrm{Dirac}_\pi(M \times N)' &= \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathrm{Dirac}_\pi(M \times N), \mathcal{C}_c^\infty(M)), \\
\mathcal{E}'_\pi(M \times N)' &= \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{E}'_\pi(M \times N), \mathcal{C}_c^\infty(M))
\end{aligned}
$$

the $\mathcal{C}_c^\infty(M)$-duals of the $\mathcal{C}_c^\infty(M)$-modules $\mathrm{Dirac}_\pi(M \times N)$ and $\mathcal{E}'_\pi(M \times N)$. Define a $\mathcal{C}_c^\infty(M)$-linear map

$$\hat{\ } : \mathcal{C}^\infty(M \times N) \to \mathrm{Dirac}_\pi(M \times N)',$$

by

$$\hat{F}(u) = u(F)$$

for $F \in \mathcal{C}^\infty(M \times N)$ and $u \in \mathrm{Dirac}_\pi(M \times N)$.

Theorem 3.9. *Let $M \times N$ be a trivial bundle over M with fiber N and bundle projection $\pi : M \times N \to M$. The map $\hat{\ } : \mathcal{C}^\infty(M \times N) \to \mathrm{Dirac}_\pi(M \times N)'$ is an isomorphism of locally convex $\mathcal{C}_c^\infty(M)$-modules.*

Proof. We first show that the map $\hat{}: \mathcal{C}^\infty(M \times N) \to \mathrm{Dirac}_\pi(M \times N)'$ is a $\mathcal{C}_c^\infty(M)$-linear isomorphism. It is injective, since $\mathrm{Dirac}_\pi(M \times N)$ separates the points of $\mathcal{C}^\infty(M \times N)$. To see that it is surjective, choose any $\phi \in \mathrm{Dirac}_\pi(M \times N)'$. Since $\mathrm{Dirac}_\pi(M \times N)$ is a dense subspace of $\mathcal{E}'_\pi(M \times N)$ and since $\mathcal{C}_c^\infty(M)$ is complete, there exists a unique continuous extension $\overline{\phi} : \mathcal{E}'_\pi(M \times N) \to \mathcal{C}_c^\infty(M)$ of ϕ to $\mathcal{E}'_\pi(M \times N)$. From Theorem 4.5 in [9], it now follows that $\overline{\phi} = \hat{F}$ for some $F \in \mathcal{C}^\infty(M \times N)$.

It remains to be shown that the map $\hat{}: \mathcal{C}^\infty(M \times N) \to \mathrm{Dirac}_\pi(M \times N)'$ is a homeomorphism. It is continuous as it can be written as a composition

$$\mathcal{C}^\infty(M \times N) \xrightarrow{\hat{}} \mathcal{E}'_\pi(M \times N)' \longrightarrow \mathrm{Dirac}_\pi(M \times N)',$$

where the left map is continuous by Theorem 4.5 in [9] and the right map is the continuous restriction of functionals from $\mathcal{E}'_\pi(M \times N)$ to $\mathrm{Dirac}_\pi(M \times N)$.

In the remainder of the proof, we will show that the above map is open. Let us choose an arbitrary subbasic neighbourhood of zero in $\mathcal{C}^\infty(M \times N)$ of the form

$$\begin{aligned} V_{L\times K,m,\epsilon} = \{F \in \mathcal{C}^\infty(M \times N) \,|\, &|D_x^\alpha D_y^\beta F(x,y)| \\ &< \epsilon \text{ for } (x,y) \in L \times K,\ |\alpha| + |\beta| \leqslant m\}, \end{aligned}$$

where $m \in \mathbb{N}$, $\epsilon > 0$, L is a compact subset of M which lies in some chart $U_M \approx \mathbb{R}^l$, and K is a compact subset of N which lies in some chart $U_N \approx \mathbb{R}^k$. Our goal is to find a bounded subset $B \subset \mathrm{Dirac}_\pi(M \times N)$ and a neighbourhood V of zero in $\mathcal{C}_c^\infty(M)$, such that $K(B,V) \subset \widehat{V_{L\times K,m,\epsilon}}$.

For $n \in \mathbb{N}$, $t \in (0,\infty)$ and $y \in \mathbb{R}$, we define a distribution $\Delta_t^n(y) \in \mathrm{Dirac}(\mathbb{R})$ by

$$\Delta_t^n(y) = \frac{1}{(2t)^n} \sum_{k=0}^n (-1)^k \binom{n}{k} \delta_{y+(n-2k)t}.$$

Using the Taylor's theorem, one can show that $\Delta_t^n(y)$ converges in $\mathcal{E}'(\mathbb{R})$ to $D_y^n|_y$ as $t \to 0$, where $D_y^n|_y$ is the distribution which computes the nth derivative at the point y. More generally, denote $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{N}_0^k$, $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$ and define

$$\Delta_t^\beta(y) = \Delta_t^{\beta_1}(y_1) \otimes \Delta_t^{\beta_2}(y_2) \otimes \cdots \otimes \Delta_t^{\beta_k}(y_k) \in \mathcal{E}'(\mathbb{R}^k).$$

Again, we have that $\Delta_t^\beta(y) \in \mathrm{Dirac}(\mathbb{R}^k)$ converges to $D_y^\beta|_y$ in $\mathcal{E}'(\mathbb{R}^k)$ as $t \to 0$. Using K and m from the definition of $V_{L\times K,m,\epsilon}$, we now define the subset

$$B_{K,m} = \{\Delta_t^\beta(y) \,|\, y \in K, t \in (0,1), |\beta| \leqslant m\} \subset \mathrm{Dirac}(\mathbb{R}^k) \subset \mathcal{E}'(\mathbb{R}^k).$$

Using estimates from the Taylor's theorem, one can show that $B_{K,m}$ is a bounded subset of $\mathcal{E}'(\mathbb{R}^k)$. Now, note that the bilinear map $\mathcal{C}_c^\infty(M) \times \mathcal{E}'(N) \to \mathcal{C}_c^\infty(M, \mathcal{E}'(N))$, given by $(f,v) \mapsto fv$ for $(fv)_x = f(x)v$, is continuous. If we choose a function $\eta \in \mathcal{C}_c^\infty(U_M) \subset \mathcal{C}_c^\infty(M)$, such that $\eta \equiv 1$ on some neighbourhood of L, it now follows from the above observation that:

$$B = \eta B_{K,m} = \{\eta v \,|\, v \in B_{K,m}\}$$

is a bounded subset of $\mathrm{Dirac}_\pi(M \times N)$. Finally, let us define an open neighbourhood V of zero in $\mathcal{C}_c^\infty(M)$ by

$$V = \{f \in \mathcal{C}_c^\infty(M) \,|\, |D_x^\alpha f(x)| < \tfrac{\epsilon}{2} \text{ for } x \in L,\, |\alpha| \leqslant m\}.$$

Now, choose any $\phi = \hat{F} \in K(B, V)$, so that $\phi(u) = u(F) \in V$ for $u \in B$. If we write $u = \eta v = \eta \sum_{i=1}^n a_i \delta_{y_i}$ for some $a_1, \ldots, a_n \in \mathbb{C}$ and some $y_1, \ldots, y_n \in U_N$, we have for $x \in L$ and $|\alpha| \leqslant m$ the following estimate:

$$|u(D_x^\alpha F)(x)| = \left| \eta(x) \sum_{i=1}^n a_i (D_x^\alpha F)(x, y_i) \right| = |D_x^\alpha(u(F))(x)| < \tfrac{\epsilon}{2}.$$

Here, we have used the fact that $\eta \equiv 1$ on some neighbourhood of L and denoted by $D_x^\alpha F$ the α-partial derivative of F in the horizontal direction. For any $y \in K$ and any β with $|\beta| \leqslant m$, the net $\Delta_t^\beta(y) \in B_{K,m}$ converges to $D_y^\beta|_y$ in $\mathcal{E}'(\mathbb{R}^k)$ as $t \to 0$. If we define $u_t = \eta \Delta_t^\beta(y) \in B$, we then have for $x \in L$ the estimate

$$|D_y^\beta D_x^\alpha F(x, y)| = \lim_{t \to 0} |u_t(D_x^\alpha F)(x)| \leqslant \tfrac{\epsilon}{2} < \epsilon.$$

To sum it up, for $(x, y) \in L \times K$ and $|\alpha|, |\beta| \leqslant m$, we have $|D_y^\beta D_x^\alpha F(x, y)| < \epsilon$, which implies that $F \in V_{L \times K, m, \epsilon}$ and, consequently, $\phi = \hat{F} \in \widehat{V_{L \times K, m, \epsilon}}$. □

4. Spectral Bundle of the Coalgebra of Transversal Distributions of Constant Dirac Type

From Theorem 3.9, it follows that $\mathcal{C}^\infty(M \times N)$ is isomorphic to the strong $\mathcal{C}_c^\infty(M)$-dual of $\mathrm{Dirac}_\pi(M \times N)$. We will now equip the space $\mathrm{Dirac}_\pi(M \times N)$ with a structure of a locally convex coalgebra over $\mathcal{C}_c^\infty(M)$, such that its strong $\mathcal{C}_c^\infty(M)$-dual $\mathrm{Dirac}_\pi(M \times N)'$ is a Fréchet algebra, isomorphic to $\mathcal{C}^\infty(M \times N)$.

We will use the isomorphism $\Psi_{M \times N} : \mathcal{C}_c^\infty(M \times N^\#) \to \mathrm{Dirac}_\pi(M \times N)$ to transfer coalgebra structure from $\mathcal{C}_c^\infty(M \times N^\#)$ to $\mathrm{Dirac}_\pi(M \times N)$. Explicitly, using the notation from Example 2.2, we define on $\mathrm{Dirac}_\pi(M \times N)$ a structure of a coalgebra over $\mathcal{C}_c^\infty(M)$ with structure maps

$$\Delta : \mathrm{Dirac}_\pi(M \times N) \to \mathrm{Dirac}_\pi(M \times N) \otimes_{\mathcal{C}_c^\infty(M)} \mathrm{Dirac}_\pi(M \times N),$$
$$\epsilon : \mathrm{Dirac}_\pi(M \times N) \to \mathcal{C}_c^\infty(M),$$

explicitly given by

$$\Delta\left(\sum_{i=1}^n [\![E_{y_i}, f_i]\!]\right) = \sum_{i=1}^n [\![E_{y_i}, f_i]\!] \otimes [\![E_{y_i}, 1_{f_i}]\!],$$
$$\epsilon\left(\sum_{i=1}^n [\![E_{y_i}, f_i]\!]\right) = \sum_{i=1}^n f_i.$$

Since $\mathcal{C}^\infty(M \times N)$ is isomorphic to the strong dual of $\mathrm{Dirac}_\pi(M \times N)$, we can use it to define the $\mathcal{C}_c^\infty(M)$-injective topology on $\mathrm{Dirac}_\pi(M \times N) \otimes_{\mathcal{C}_c^\infty(M)}$

$\mathrm{Dirac}_\pi(M \times N)$. For any pair of functions $F, G \in \mathcal{C}^\infty(M \times N)$, we define a $\mathcal{C}_c^\infty(M)$-linear map

$$F \otimes G : \mathrm{Dirac}_\pi(M \times N) \otimes_{\mathcal{C}_c^\infty(M)} \mathrm{Dirac}_\pi(M \times N) \to \mathcal{C}_c^\infty(M)$$

by

$$(F \otimes G)\left(\sum_{i=1}^n u_i' \otimes u_i''\right) = \sum_{i=1}^n u_i'(F) u_i''(G).$$

The $\mathcal{C}_c^\infty(M)$-injective topology on $\mathrm{Dirac}_\pi(M \times N) \otimes_{\mathcal{C}_c^\infty(M)} \mathrm{Dirac}_\pi(M \times N)$ is now defined by specifying basic neighbourhoods of zero of the form

$$K(A, B, V) = \{\tilde{u} \in \mathrm{Dirac}_\pi(M \times N)^{\otimes 2} \,|\, (F \otimes G)(\tilde{u}) \in V, \text{ for } F \in A, G \in B\},$$

where $A, B \subset \mathcal{C}^\infty(M \times N) \cong \mathrm{Dirac}_\pi(M \times N)'$ are bounded subsets and V is a neighbourhood of zero in $\mathcal{C}_c^\infty(M)$.

Proposition 4.1. *The triple* $(\mathrm{Dirac}_\pi(M \times N), \Delta, \epsilon)$ *is a cocommutative, locally convex coalgebra over* $\mathcal{C}_c^\infty(M)$*, in the sense that* Δ *and* ϵ *are continuous maps.*

Proof. Let us denote by 1 the unit of the algebra $\mathcal{C}^\infty(M \times N)$. We then have $\epsilon = \hat{1}$, which shows that ϵ is a continuous map.

To see that Δ is continuous, we choose any basic neighbourhood of zero in $\mathrm{Dirac}_\pi(M \times N) \otimes_{\mathcal{C}_c^\infty(M)} \mathrm{Dirac}_\pi(M \times N)$ of the form $K(A, B, V)$ as above. The set

$$A \cdot B = \{FG \,|\, F \in A, G \in B\}$$

is then a bounded subset of $\mathcal{C}^\infty(M \times N)$. For any $u = \sum_{i=1}^n [\![E_{y_i}, f_i]\!] \in K(A \cdot B, V)$ any $F \in A$ and any $G \in B$, we now have

$$\begin{aligned}(F \otimes G)(\Delta(u))(x) &= (F \otimes G)\left(\sum_{i=1}^n [\![E_{y_i}, f_i]\!] \otimes [\![E_{y_i}, 1_{f_i}]\!]\right)(x), \\ &= \sum_{i=1}^n f_i(x) 1_{f_i}(x) F(x, y_i) G(x, y_i), \\ &= u(FG)(x).\end{aligned}$$

This implies that $\Delta(K(A \cdot B, V)) \subset K(A, B, V)$, and hence, Δ is continuous. □

Since $\mathrm{Dirac}_\pi(M \times N)$ is a cocommutative, counital coalgebra over $\mathcal{C}_c^\infty(M)$, its dual $\mathrm{Dirac}_\pi(M \times N)'$ naturally becomes a commutative algebra with unit ϵ^* over $\mathcal{C}_c^\infty(M)$, if we define

$$(\phi \cdot \psi)(u) = (\phi \otimes \psi)(\Delta(u))$$

for $\phi, \psi \in \mathrm{Dirac}_\pi(M \times N)'$ and $u \in \mathrm{Dirac}_\pi(M \times N)$. Continuity of $\phi \cdot \psi$ follows from continuity of $\phi \otimes \psi$ and Δ.

On both $\mathrm{Dirac}_\pi(M \times N)$ and $\mathrm{Dirac}_\pi(M \times N)'$, we can naturally define conjugation as follows. For any $u \in \mathrm{Dirac}_\pi(M \times N)$, we define conjugation by

$$u = \sum_{i=1}^n [\![E_{y_i}, f_i]\!] \Longrightarrow \overline{u} = \sum_{i=1}^n [\![E_{y_i}, \overline{f_i}]\!].$$

Using the above formula and complex conjugation on $\mathcal{C}_c^\infty(M)$, we now define for any $\phi \in \mathrm{Dirac}_\pi(M \times N)'$ the element $\overline{\phi} \in \mathrm{Dirac}_\pi(M \times N)'$ by

$$\overline{\phi}(u) = \overline{\phi(\overline{u})}$$

for $u \in \mathrm{Dirac}_\pi(M \times N)$. It is now a straightforward calculation to extend Theorem 3.9 in the following way.

Proposition 4.2. *Let $M \times N$ be a trivial bundle over M with fiber N and bundle projection $\pi : M \times N \to M$. The map* $\hat{}: \mathcal{C}^\infty(M \times N) \to \mathrm{Dirac}_\pi(M \times N)'$ *is an isomorphism of locally convex algebras with involutions.*

Using the definitions and notations from the Subsection 2.4, we now define for any $x \in M$ the local $\mathcal{C}_c^\infty(M)_x$-coalgebra

$$\mathrm{Dirac}_\pi(M \times N)_x = \mathrm{Dirac}_\pi(M \times N)/I_x \mathrm{Dirac}_\pi(M \times N).$$

It follows from [22] that the space $\mathrm{Dirac}_\pi(M \times N)_x$ is a free $\mathcal{C}_c^\infty(M)_x$-module, generated by the set $G(\mathrm{Dirac}_\pi(M \times N)_x)$ of grouplike elements. The spectral sheaf of the $\mathcal{C}_c^\infty(M)$-coalgebra $\mathrm{Dirac}_\pi(M \times N)$ is the sheaf

$$\pi_{sp} : \mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N)) \to M$$

with the stalk at the point $x \in M$ given by

$$\mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N))_x = G(\mathrm{Dirac}_\pi(M \times N)_x).$$

Note that the sheaves $M \times N^\#$ and $\mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N))$ over M are isomorphic via the map

$$(x, y) \mapsto [\![E_y, f]\!]|_x,$$

where $f \in \mathcal{C}_c^\infty(M)$ is any function with $f|_x = 1 \in \mathcal{C}_c^\infty(M)_x$.

Let us now define the real part of $\mathrm{Dirac}_\pi(M \times N)'$ by

$$\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}} = \{\phi \in \mathrm{Dirac}_\pi(M \times N)' \,|\, \overline{\phi} = \phi\},$$

and note that it corresponds to the algebra $\mathcal{C}^\infty(M \times N, \mathbb{R})$ via the isomorphism from Proposition 4.2. This implies that $\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}}$ satisfies the conditions of the main theorem in [18], so it can be used to define a smooth structure on the space $\mathrm{Spec}(\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}})$. Furthermore, we have a natural bijection

$$\Theta_{M \times N} : \mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N)) \to \mathrm{Spec}(\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}}),$$

defined by

$$\Theta_{M \times N}([\![E_y, f]\!]|_x)(\phi) = \phi([\![E_y, f]\!])(x)$$

for $\phi \in \mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}}$. We will now use this bijection to transfer the smooth structure from $\mathrm{Spec}(\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}})$ to $\mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N))$.

Definition 4.3. Let $\pi : M \times N \to M$ be a trivial bundle over M with fiber N. **The spectral bundle**

$$\mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N))$$

of the coalgebra $\mathrm{Dirac}_\pi(M \times N)$ is the set $\mathcal{E}_{sp}(\mathrm{Dirac}_\pi(M \times N))$, equipped with the bundle projection $\pi_{sp} : \mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N)) \to M$ and the topology and smooth structure, such that $\Theta_{M \times N}$ is a diffeomorphism.

We will show in the next theorem that $\mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N))$ is a trivial bundle over M, naturally isomorphic to the bundle $M \times N$. Define a map

$$\Phi^{\mathrm{bun}}_{M\times N} : M \times N \to \mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N)),$$

by

$$\Phi^{\mathrm{bun}}_{M\times N}(x, y) = [\![E_y, f]\!]|_x.$$

Theorem 4.4. *Let $\pi : M \times N \to M$ be a trivial bundle over M with fiber N. The map $\Phi^{bun}_{M\times N} : M \times N \to \mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N))$ is an isomorphism of trivial bundles over M.*

Proof. Let us denote by $\mathrm{Sp} : \mathrm{Spec}(\mathcal{C}^\infty(M \times N, \mathbb{R})) \to \mathrm{Spec}(\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}})$ the diffeomorphism, induced by the inverse of $\hat{\ }: \mathcal{C}^\infty(M\times N, \mathbb{R}) \to \mathrm{Dirac}_\pi(M\times N)'_{\mathbb{R}}$. We then have the commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\Phi^{\mathrm{bun}}_{M\times N}} & \mathcal{B}_{sp}(\mathrm{Dirac}_\pi(M \times N)) \\ {\scriptstyle \Phi^{\mathrm{man}}_{M\times N}}\big\downarrow & & \big\downarrow {\scriptstyle \Theta_{M\times N}} \\ \mathrm{Spec}(\mathcal{C}^\infty(M \times N, \mathbb{R})) & \xrightarrow{\mathrm{Sp}} & \mathrm{Spec}(\mathrm{Dirac}_\pi(M \times N)'_{\mathbb{R}}) \end{array}$$

Since $\Phi^{\mathrm{man}}_{M\times N}$, $\Theta_{M\times N}$ and Sp are diffeomorphisms, $\Phi^{\mathrm{bun}}_{M\times N}$ is a diffeomorphism, as well. The equality $\pi_{sp} \circ \Phi^{\mathrm{bun}}_{M\times N} = \pi$ follows from the equality $\pi_{sp} \circ \Phi^{\mathrm{shv}}_{M\times N\#} = \pi$. □

Let us now take a look at this construction in the case of a single manifold.

Example 4.5. Let M be a single point and consider the manifold N as a trivial bundle over a point. In this case, we have

$$\mathrm{Dirac}_\pi(M \times N) = \mathrm{Dirac}(N) = \mathrm{Span}\{\delta_y \,|\, y \in N\}.$$

Every element $u \in \mathrm{Dirac}(N)$ can be expressed as a finite sum $u = \sum_{i=1}^n \lambda_i \delta_{y_i}$ for unique $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $y_1, \ldots, y_n \in N$. The space $\mathrm{Dirac}(N)$ is a coalgebra over $\mathbb{C}$ with structure maps

$$\Delta\left(\sum_{i=1}^n \lambda_i \delta_{y_i}\right) = \sum_{i=1}^n \lambda_i \delta_{y_i} \otimes \delta_{y_i},$$
$$\epsilon\left(\sum_{i=1}^n \lambda_i \delta_{y_i}\right) = \sum_{i=1}^n \lambda_i.$$

Grouplike elements of $\mathrm{Dirac}(N)$ are precisely Dirac distributions, so we have

$$G(\mathrm{Dirac}(N)) = \{\delta_y \,|\, y \in N\}.$$

The spectral sheaf $\mathcal{E}_{sp}(\mathrm{Dirac}(N))$ is the set $G(\mathrm{Dirac}(N))$ with the discrete topology and projection onto the point. The map $\Theta_N : \mathcal{E}_{sp}(\mathrm{Dirac}(N)) \to \mathrm{Spec}(\mathrm{Dirac}(N)'_{\mathbb{R}})$ is defined by $\Theta_N(\delta_y)(\phi) = \phi(\delta_y)$, which means that $\Theta_N(\delta_y) = \hat{\delta}_y$. The topology on $\mathcal{B}_{sp}(\mathrm{Dirac}(N)) = G(\mathrm{Dirac}(N))$, which is induced by

Θ_N, coincides with the subspace topology on $G(\mathrm{Dirac}(N))$, induced from $\mathcal{E}'(N)$. Finally, the diffeomorphism

$$\Phi_N^{\mathrm{bun}} : N \to \mathcal{B}_{sp}(\mathrm{Dirac}(N))$$

is given by $\Phi_N^{\mathrm{bun}}(y) = \delta_y$.

5. Locally Convex Bialgebroid of an Action Lie Groupoid

In this section, we will assign to each action groupoid $M \rtimes H$ a locally convex bialgebroid with antipode $\mathrm{Dirac}(M \rtimes H)$ over $\mathcal{C}_c^\infty(M)$, from which the Lie groupoid $M \rtimes H$ can be reconstructed.

Let M be a second-countable manifold and let H be a second-countable Lie group, which acts on M from the right. If we denote by $H^\#$ the group H with the discrete topology, the group $H^\#$ acts on M from the right as well, so we obtain two action groupoids $M \rtimes H$ and $M \rtimes H^\#$. These two groupoids are isomorphic as groupoids but not as Lie groupoids if $\dim(H) > 0$.

Groupoid $M \rtimes H^\#$ is étale, so we can construct its Hopf algebroid

$$\mathcal{C}_c^\infty(M \rtimes H^\#) = \bigoplus_{h \in H^\#} \mathcal{C}_c^\infty(M \times \{h\}).$$

Moreover, since $M \rtimes H$ is a Lie groupoid, we also have a convolution product, as defined in [14], on the space

$$\mathcal{E}'_{\mathrm{t}}(M \rtimes H) = \mathrm{Hom}_{\mathcal{C}_c^\infty(M)}(\mathcal{C}^\infty(M \times H), \mathcal{C}_c^\infty(M))$$

of t-transversal distributions on $M \rtimes H$. It can be described explicitly as follows. The left translation by $g \in M \rtimes H$ is the diffeomorphism $\mathrm{L}_g : \mathrm{t}^{-1}(\mathrm{s}(g)) \to \mathrm{t}^{-1}(\mathrm{t}(g))$, defined by $\mathrm{L}_g(h) = gh$. For any $F \in \mathcal{C}^\infty(M \times H)$ and any $g \in M \rtimes H$, it follows that $F \circ \mathrm{L}_g \in \mathcal{C}^\infty(\mathrm{t}^{-1}(\mathrm{s}(g)))$ and one can show that the function $M \times H \to \mathbb{R}$, $g \mapsto T_{\mathrm{s}(g)}(F \circ \mathrm{L}_g)$, is smooth for any $T \in \mathcal{E}'_{\mathrm{t}}(M \rtimes H)$. For any $T', T'' \in \mathcal{E}'_{\mathrm{t}}(M \rtimes H)$, the convolution $T' * T'' \in \mathcal{E}'_{\mathrm{t}}(M \rtimes H)$ is then defined by

$$(T' * T'')(F)(x) = T' \left(g \mapsto T''_{\mathrm{s}(g)}(F \circ \mathrm{L}_g) \right)(x),$$

for any $F \in \mathcal{C}^\infty(M \times H)$ and any $x \in M$.

Using the notation from Example 2.2, we define an injective $\mathcal{C}_c^\infty(M)$-linear map $\Psi_{M \rtimes H} : \mathcal{C}_c^\infty(M \rtimes H^\#) \to \mathcal{E}'_{\mathrm{t}}(M \rtimes H)$ by

$$\Psi_{M \rtimes H} \left(\sum_{i=1}^n f_i \cdot \delta_{h_i} \right) = \sum_{i=1}^n [\![E_{h_i}, f_i]\!].$$

Proposition 5.1. *The map* $\Psi_{M \rtimes H} : \mathcal{C}_c^\infty(M \rtimes H^\#) \to \mathcal{E}'_{\mathrm{t}}(M \rtimes H)$ *is multiplicative.*

Proof. Let us first prove that $\Psi_{M\rtimes H}$ is multiplicative on the set of basis elements. Choose $f_1, f_2 \in \mathcal{C}_c^\infty(M)$, $h_1, h_2 \in H$ and denote $a_1 = f_1 \cdot \delta_{h_1}$, respectively, $a_2 = f_2 \cdot \delta_{h_2}$. We then have

$$\begin{aligned}
&(\Psi_{M\rtimes H}(a_1) * \Psi_{M\rtimes H}(a_2))(F)(x) \\
&\quad = \Psi_{M\rtimes H}(a_1)\left((x,h) \mapsto [\![E_{h_2}, f_2]\!]_{xh}(F \circ \mathrm{L}_{(x,h)})\right)(x), \\
&\quad = [\![E_{h_1}, f_1]\!]\left((x,h) \mapsto f_2(xh)F(x,hh_2)\right)(x), \\
&\quad = f_1(x)f_2(xh_1)F(x,h_1h_2).
\end{aligned}$$

On the other hand (see Example 2.3), we have $a_1 * a_2 = (f_1(h_1 f_2)) \cdot \delta_{h_1h_2}$, and hence

$$\Psi_{M\rtimes H}(a_1 * a_2)(F)(x) = [\![E_{h_1h_2}, f_1(h_1f_2)]\!](F)(x) = f_1(x)f_2(xh_1)F(x,h_1h_2).$$

Multiplicativity of the map $\Psi_{M\rtimes H}$ now follows from linearity of $\Psi_{M\rtimes H}$ and bilinearity of both convolution products. □

Definition 5.2. Let $M \rtimes H$ be an action groupoid of an action of a second-countable Lie group H on a second-countable manifold M and let $M \rtimes H^\#$ be the associated étale groupoid. The **Dirac bialgebroid** of $M \rtimes H$ is the space

$$\mathrm{Dirac}(M \rtimes H) = \Psi_{M\rtimes H}(\mathcal{C}_c^\infty(M \rtimes H^\#)).$$

The Dirac bialgebroid $\mathrm{Dirac}(M \rtimes H)$ inherits from $\mathcal{C}_c^\infty(M \rtimes H^\#)$ a structure of a locally grouplike Hopf algebroid over $\mathcal{C}_c^\infty(M)$. Moreover, by Proposition 4.1, we obtain on $\mathrm{Dirac}(M \rtimes H)$ a structure of a locally convex coalgebra. Finally, as shown in [14], the multiplication on $\mathcal{E}'_{\mathrm{t}}(M \rtimes H)$ and hence on $\mathrm{Dirac}(M \rtimes H)$ is separately continuous. We sum up these observations in the following proposition.

Proposition 5.3. *The Dirac bialgebroid* $\mathrm{Dirac}(M \rtimes H)$ *of any action Lie groupoid* $M \rtimes H$ *is a locally convex bialgebroid with an antipode over* $\mathcal{C}_c^\infty(M)$.

Remark 5.4. A locally convex bialgebroid is a bialgebroid $(A, \Delta, \epsilon, \mu)$, equipped with a locally convex structure, such that Δ and ϵ are continuous maps and μ is separately continuous. We do not know if the antipode S on $\mathrm{Dirac}(M \rtimes H)$ is continuous in general, which would mean that it is a locally convex Hopf algebroid.

Example 5.5. Let us take a look at the case when the group H acts trivially on M. The associated action groupoid $M \rtimes H$ is in this case just the trivial bundle of Lie groups over M with fiber H, which will be denoted by $M \times H$. The multiplication and antipode can be expressed on generators by the formulas

$$\begin{aligned}
[\![E_{h_1}, f_1]\!] * [\![E_{h_2}, f_2]\!] &= [\![E_{h_1h_2}, f_1f_2]\!], \\
S([\![E_h, f]\!]) &= [\![E_{h^{-1}}, f]\!].
\end{aligned}$$

In this case, $\mathcal{C}_c^\infty(M)$ is a central subalgebra of $\mathrm{Dirac}(M \times H)$. Moreover, from the equality $\mathrm{t} \circ \mathrm{inv} = t$, it follows that S is continuous. As a result, we see that $\mathrm{Dirac}(M \times H)$ is a locally convex Hopf algebra over $\mathcal{C}_c^\infty(M)$.

Now, take any action Lie groupoid $M \rtimes H$. The spectral étale Lie groupoid $\mathcal{G}_{sp}(\mathrm{Dirac}(M \rtimes H))$ of $\mathrm{Dirac}(M \rtimes H)$ is then isomorphic to the étale groupoid $M \rtimes H^{\#}$. Moreover, since $\mathrm{Dirac}(M \rtimes H)$ is a locally convex coalgebra over $\mathcal{C}_c^\infty(M)$, we also have the bijection $\Theta_{M\rtimes H} : \mathcal{G}_{sp}(\mathrm{Dirac}(M \rtimes H)) \to \mathrm{Spec}(\mathrm{Dirac}(M \rtimes H))'_{\mathbb{R}})$.

Definition 5.6. The spectral action Lie groupoid

$$\mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H))$$

of the Dirac bialgebroid $\mathrm{Dirac}(M \rtimes H)$ is the groupoid $\mathcal{G}_{sp}(\mathrm{Dirac}(M \rtimes H))$, equipped with the smooth structure, such that the map $\Theta_{M\rtimes H}$ is a diffeomorphism.

Define a map

$$\Phi^{\mathrm{agr}}_{M\rtimes H} : M \rtimes H \to \mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H)),$$

by

$$\Phi^{\mathrm{agr}}_{M\rtimes H}(x, h) = [\![E_h, f]\!]|_x,$$

where $f \in \mathcal{C}_c^\infty(M)$ is such that $f|_x = 1 \in \mathcal{C}_c^\infty(M)_x$.

Theorem 5.7. *Let $M \rtimes H$ be an action groupoid of an action of a second-countable Lie group H on a second-countable manifold M. The map*

$$\Phi^{agr}_{M\rtimes H} : M \rtimes H \to \mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H))$$

is an isomorphism of Lie groupoids.

Proof. Since $M \rtimes H$ is isomorphic to $M \rtimes H^{\#}$ and $\mathcal{AG}_{sp}(\mathrm{Dirac}(M \rtimes H))$ is isomorphic to $\mathcal{G}_{sp}(\mathrm{Dirac}(M \rtimes H))$, the map $\Phi^{\mathrm{agr}}_{M\rtimes H}$ is an isomorphism of groupoids. By Theorem 4.4, it is also a diffeomorphism, which implies that it is an isomorphism of Lie groupoids. □

Example 5.8. Let H be a Lie group, so that $\mathrm{Dirac}(H)$ is a locally convex Hopf algebra over $\mathbb{C}$. Example 4.5 shows that $\mathcal{AG}_{sp}(\mathrm{Dirac}(H)) = \{\delta_h \,|\, h \in H\}$ is naturally diffeomorphic to H. The multiplication and inverse maps on $\mathcal{AG}_{sp}(\mathrm{Dirac}(H))$ are induced by the multiplication and the antipode on $\mathrm{Dirac}(H)$. Namely, for any $h, h' \in H$, we have

$$\delta_h \delta_{h'} = \delta_h * \delta_{h'} = \delta_{hh'},$$
$$\delta_h^{-1} = S(\delta_h) = \delta_{h^{-1}}.$$

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