Research Article

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On the p-fractional Schrödinger-Kirchhoff equations with electromagnetic fields and the Hardy-Littlewood-Sobolev nonlinearity

https://doi.org/10.1515/dema-2023-0124
received February 10, 2023; accepted October 6, 2023

Abstract: In this article, we deal with the following p-fractional Schrödinger-Kirchhoff equations with electromagnetic fields and the Hardy-Littlewood-Sobolev nonlinearity:

$$M([u]_{s,A}^{p})(-\Delta)_{p,A}^{s}u + V(x)|u|^{p-2}u = \lambda\left(\int_{\mathbb{R}^N}\frac{|u|^{p_{\mu,s}^{*}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{p_{\mu,s}^{*}-2}u + k\,|u|^{q-2}u,\quad x\in\mathbb{R}^N,$$

where $0 < s < 1 < p$, $ps < N$, $p < q < 2p_{s,\mu}^{*}$, $0 < \mu < N$, λ, and k are some positive parameters, $p_{s,\mu}^{*} = \frac{pN - p\frac{\mu}{2}}{N - ps}$ is the critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, and functions V and M satisfy the suitable conditions. By proving the compactness results using the fractional version of concentration compactness principle, we establish the existence of nontrivial solutions to this problem.

Keywords: Hardy-Littlewood-Sobolev nonlinearity, Schrödinger-Kirchhoff equations, variational methods, electromagnetic fields

MSC 2020: 35J10, 35B99, 35J60, 47G20

1 Introduction

In this article, we intend to study the following p-fractional Schrödinger-Kirchhoff equations with electromagnetic fields and the Hardy-Littlewood-Sobolev nonlinearity in $\mathbb{R}^N$:

$$M([u]_{s,A}^{p})(-\Delta)_{p,A}^{s}u + V(x)|u|^{p-2}u = \lambda\left(\int_{\mathbb{R}^N}\frac{|u|^{p_{\mu,s}^{*}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{p_{\mu,s}^{*}-2}u + k\,|u|^{q-2}u,\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1 < p$, $ps < N$, $p < q < 2p_{s,\mu}^{*}$, $0 < \mu < N$, λ, and k are some positive parameters,

$$[u]_{s,A}^{p} := \iint_{\mathbb{R}^{2N}}\frac{\left|u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u(y)\right|^{p}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y,$$

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$p_{s,\mu}^* = \frac{pN - p\frac{\mu}{2}}{N - ps}$ is the critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, $V \in C(\mathbb{R}^N, \mathbb{R}_0^+)$ is an electric potential, $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic potential, and V and M satisfy the following assumptions:

(V) $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and has critical frequency, i.e., $V(0) = \min_{x \in \mathbb{R}^N} V(x) = 0$. Moreover, the set $V^{\tau_0} = \{x \in \mathbb{R}^N : V(x) < \tau_0\}$ has finite Lebesgue measure for some $\tau_0 > 0$.

(M) (m_1) The Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous and nondecreasing. In addition, there exists a positive constant $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \in \mathbb{R}_0^+$;

(m_2) For some $\sigma \in (p/q, 1]$, we have $\widetilde{M}(t) \geq \sigma M(t)t$ for all $t \geq 0$, where $\widetilde{M}(t) = \int_0^t M(s)\mathrm{d}s$.

When $p = 2$, we know that the fractional operator $(-\Delta)_A^s$, which up to normalization constants, can be defined on smooth functions u as:

$$(-\Delta)_A^s u(x) \coloneqq 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x - y|^{N+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

(see d'Avenia and Squassina [1]). There already exist several articles dedicated to the study of the Choquard equation, and this problem can be used to describe many physical models [2,3]. Recently, d'Avenia and Squassina [1] considered the following fractional Choquard equation of the form:

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^p)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N), \quad N \geq 3, \tag{1.2}$$

and the existence of ground-state solutions was obtained by using the Mountain pass theorem and the Ekeland variational principle. For more results on problems with the Hardy-Littlewood-Sobolev nonlinearity without the magnetic operator case, see [4–9].

For the case $p \neq 2$, Iannizzotto et al. [10] investigated the following fractional p-Laplacian equation:

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

The existence and multiple solutions for Problem (1.3) were proved using the Morse theory. Xiang et al. [11] dealt with a class of Kirchhoff-type problems driven by nonlocal elliptic integro-differential operators, and two existence theorems were obtained using the variational method. Souza [12] studied a class of nonhomogeneous fractional quasilinear equations in $\mathbb{R}^N$ with exponential growth of the form:

$$(-\triangle)_p^s u + V(x)|u|^{p-2}u = f(x, u) + \lambda h \quad \text{in } \Omega. \tag{1.4}$$

Using a suitable Trudinger-Moser inequality for fractional Sobolev spaces, they established the existence of weak solutions for Problem (1.4). In particular, Nyamoradi and Razani [13] considered a class of new Kirchhoff-type equations involving the fractional p-Laplacian and Hardy-Littlewood-Sobolev critical nonlinearity. The existence of infinitely many solutions was obtained by using the concentration compactness principle and Krasnoselskii's genus theory. For more recent advances on this kind of problems, we refer the readers to [14–28].

On the other hand, one of the main features of Problem (1.1) is the presence of the magnetic field operator A. When $A \neq 0$, some authors have studied the following equation:

$$-(\nabla u - \mathrm{i}A)^2 u + V(x)u = f(x, |u|)u, \tag{1.5}$$

which has appeared in recent years, where the magnetic operator in equation (1.5) is given by:

$$-(\nabla u - \mathrm{i}A)^2 u = -\Delta u + 2iA(x)\cdot\nabla u + |A(x)|^2 u + iu \operatorname{div} A(x).$$

Squassina and Volzone [29] proved that up to correcting the operator by the factor $(1 - s)$, it follows that $(-\Delta)_A^s u$ converges to $-(\nabla u - \mathrm{i}A)^2 u$ as $s \to 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one.

Recently, many researchers have paid attention to the problems with fractional magnetic operator. In particular, Mingqi et al. [30] proved some existence results for the following Schrödinger-Kirchhoff-type equation involving the magnetic operator:

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x,|u|)u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where f satisfies the subcritical growth condition. For the critical growth case, Binlin et al. [31] considered the following fractional Schrödinger equation with critical frequency and critical growth:

$$\varepsilon^{2s}(-\triangle)_{A_\varepsilon}^s u + V(x)u = f(x,|u|)u + K(x)|u|^{2_\alpha^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

The existence of ground-state solution tending to trivial solution as $\varepsilon \to 0$ was obtained using the variational method. Furthermore, Song and Shi [32] were concerned with a class of the p-fractional Schrödinger-Kirchhoff equations with electromagnetic fields; under suitable additional assumptions, the existence of infinite solutions was obtained using the variational method. More results about fractional equations involving the Hardy-Littlewood-Sobolev and critical nonlinear can be found in [33–36].

Inspired by the aforementioned works, in this study, we are interested in the p-fractional Schrödinger-Kirchhoff equations with electromagnetic fields and the Hardy-Littlewood-Sobolev nonlinearity. As far as we know, there have not been any results for Problem (1.1) yet. We note that there are many difficulties in dealing with such problems due to the presence of the electromagnetic field and critical nonlinearity. In order to overcome these difficulties, we shall adopt the concentration-compactness principles and some new techniques to prove the $(PS)_c$ condition. Moreover, we shall use the variational methods in order to establish the existence and multiplicity of solutions for Problem (1.1). Here are our main results.

Theorem 1.1. *Suppose that Conditions* (V) *and* (M) *are satisfied. Then there exists* $\lambda^* > 0$ *such that if* $\lambda > \lambda^* > 0$, *then there exists at least one solution* u_λ *of Problem* (1.1) *and* $u_\lambda \to 0$ *as* $\lambda \to \infty$.

Theorem 1.2. *Suppose that Conditions* (V) *and* (M) *are satisfied. Then, for any* $m \in \mathbb{N}$, *there exists* $\lambda_m^* > 0$ *such that if* $\lambda > \lambda_m^*$, *then Problem* (1.1) *has at least m pairs of solutions* $u_{\lambda,i}$, $u_{\lambda,-i}$, $i = 1, 2, \ldots, m$ *and* $u_{\lambda,\pm i} \to 0$ *as* $\lambda \to \infty$.

This article is organized as follows. In Section 2, we present the working space and the necessary preliminaries. In Section 3, we apply the principle of concentration compactness to prove that the $(PS)_c$ condition holds. In Section 4, we check that the mountain pass geometry is established. In Section 5, we use the critical point theory and some subtle estimates to prove our main results.

2 Preliminaries

In this section, we shall give the relevant notations and some useful auxiliary lemmas. For other background information, we refer to Papageorgiou et al. [37]. Let

$$W_A^{s,p}(\mathbb{R}^N, \mathbb{C}) = \{u \in L^p(\mathbb{R}^N, \mathbb{C}) : [u]_{s,A} < \infty\},$$

where $s \in (0,1)$ and

$$[u]_{s,A} := \left(\iint\limits_{\mathbb{R}^{2N}} \frac{\left| u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)} u(y) \right|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

The norm of the fractional Sobolev space is given by:

$$\|u\|_{W_A^{s,p}(\mathbb{R}^N,\mathbb{C})} = ([u]_{s,A}^p + \|u\|_{L^p}^p)^{1/p}.$$

In order to study Problem (1.1), we shall use the following subspace of $W_A^{s,p}(\mathbb{R}, \mathbb{C})$ defined by:

$$E = \left\{ u \in W_A^{s,p}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_E := \left([u]_{s,A}^p + \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x \right)^{1/p}.$$

Condition (V) implies that $E \hookrightarrow W_A^{s,p}(\mathbb{R}^N, \mathbb{C})$ is continuous.

Next, we state the well-known Hardy-Littlewood-Sobolev inequality and the diamagnetic inequality, which will be used frequently.

Proposition 2.1. (Hardy-Littlewood-Sobolev inequality [38, Theorem 4.3]) *Let* $1 < t, r < \infty$, *and* $0 < \mu < N$ *with* $\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2$, $u \in L^t(\mathbb{R}^N)$, *and* $v \in L^r(\mathbb{R}^N)$. *Then, there exists a sharp constant* $C(N, \mu, t, r) > 0$, *independent of u and v, such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)||v(y)|}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(N, \mu, t, r)\|u\|_t \|v\|_r.$$

By the Hardy-Littlewood-Sobolev inequality, there exists $\widehat{C}(N, \mu) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p^*_{\mu,s}}|u(y)|^{p^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq \widehat{C}(N, \mu)\|u\|_{p^*_s}^{2p_{\mu,s}} \quad \text{for all } u \in E.$$

Also, there exists $C(N, \mu) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p^*_{\mu,s}}|u(y)|^{p^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(N, \mu)\|u\|_E^{2p^*_{\mu,s}} \quad \text{for all } u \in E.$$

Lemma 2.1. (Diamagnetic inequality [1, Lemma 3.1, Remark 3.2]) *For every* $u \in W_A^{s,p}(\mathbb{R}^N, \mathbb{C})$, *we obtain* $|u| \in W^{s,p}(\mathbb{R}^N)$. *More precisely, we have* $[|u|]_s \leq [u]_{s,A}$.

3 The Palais-Smale condition

First, we define the set

$$C_c(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \mathrm{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\}$$

and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\eta|_\infty = \sup_{x \in \mathbb{R}^N} |\eta(x)|$. The measure μ gives the norm:

$$\|\mu\| = \sup_{\eta \in C_0(\mathbb{R}^N), \ |\eta|_\infty = 1} |(\mu, \eta)|,$$

where $(\mu, \eta) = \int_{\mathbb{R}^N} \eta \mathrm{d}\mu$.

In order to prove the compactness condition, we introduce the following fractional version of the concentration compactness principle.

Lemma 3.1. (See Xiang and Zhang [39]) *Assume that there exist bounded non-negative measures ω, ζ, and ν on $\mathbb{R}^N$, and at most countable set $\{x_i\}_{i\in I} \in \Omega\backslash\{0\}$ such that*

$$\begin{aligned} &u_n \to u \text{ weakly in } W^{s,p}(\mathbb{R}^N),\\ &\int_{\mathbb{R}^N} \frac{||u_n(x)|-|u_n(y)||^p}{|x-y|^{N+sp}}\mathrm{d}y \to \omega \text{ weakly * in } \mathcal{M}(\mathbb{R}^N),\\ &\left(\int_{\mathbb{R}^N} \frac{|u_n|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u_n|^{p^*_{\mu,s}} \to \nu \text{ weakly * in } \mathcal{M}(\mathbb{R}^N). \end{aligned}$$

Then, there exist a countable sequence of points $\{x_i\} \subset \mathbb{R}^N$ and families of positive numbers $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$, and $\{\omega_i : i \in I\}$ such that

$$\begin{aligned} &\omega \geq \int_{\mathbb{R}^N} \frac{||u(x)|-|u(y)||^p}{|x-y|^{N+ps}}\mathrm{d}x + \sum_{i\in I}\omega_i\delta_{x_i},\\ &\zeta = |u|^{p^*_s} + \sum_{i\in I}\zeta_i\delta_{x_i},\\ &\nu = \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u|^{p^*_{\mu,s}} + \sum_{i\in I}\nu_i\delta_{x_i}, \end{aligned}$$

where I is at most countable. Furthermore, we have

$$S_{p,H}\nu_i^{\frac{p}{2p^*_{\mu,s}}} \leq \omega_i \quad \text{and} \quad \nu_i \leq C(N,\mu)\zeta_i^{\frac{2N-\mu}{N}}, \tag{3.1}$$

where is the Dirac mass of mass 1 concentrated at $\{x_i\} \subset \mathbb{R}^N$.

Lemma 3.2. (See Xiang and Zhang [39]) *Let $\{u_n\}_n \subset W^{s,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\begin{aligned} &u_n \to u \textit{ weakly in } W^{s,p}(\mathbb{R}^N),\\ &\int_{\mathbb{R}^N} \frac{||u_n(x)|-|u_n(y)||^p}{|x-y|^{N+sp}}\mathrm{d}y \to \omega \textit{ weakly * in } \mathcal{M}(\mathbb{R}^N),\\ &\left(\int_{\mathbb{R}^N} \frac{|u_n|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}y\right)|u_n|^{p^*_{\mu,s}} \to \nu \textit{ weakly * in } \mathcal{M}(\mathbb{R}^N) \end{aligned}$$

and define

$$\begin{aligned} &\omega_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{||u_n(x)|-|u_n(y)||^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y,\\ &\zeta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*_s}\mathrm{d}x,\\ &\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p^*_{\mu,s}}|u_n(y)|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Then, the quantities ω_∞, ζ_∞, and ν_∞ are well defined and satisfy

$$\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p^*_{\mu,s}}|u_n(y)|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N}\mathrm{d}\nu + \nu_\infty,$$

$$\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{||u_n(x)|-|u_n(y)||^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N}\mathrm{d}\omega + \omega_\infty,$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\mathrm{d}x=\int_{\mathbb{R}^N}d\zeta+\zeta_\infty.$$

In addition, the following inequality holds

$$S_{p,H}\nu_\infty^{\frac{p}{2p_{\mu,s}^*}}\le\omega_\infty\quad\text{and}\quad\nu_\infty\le C(N,\mu)\zeta_\infty^{\frac{2N-\mu}{N}}.\tag{3.2}$$

In order to prove the main results, we define the energy functional of Problem (1.1) as follows

$$J_\lambda(u)\coloneqq\frac{1}{p}\widetilde{M}([u]_{s,A}^p)+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x-\frac{\lambda}{2p_{s,\mu}^*}\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_{s,\mu}^*}|u(x)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x-\frac{k}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x.\tag{3.3}$$

Under hypothetical Conditions (V) and (M), a simple test as in Willem [40], yields that $J_\lambda\in C^1(E,\mathbb{R})$ and its critical points are the weak solutions of Problem (1.1), if

$$M([u]_{s,A}^p)\mathrm{Re}L(u,v)+\mathrm{Re}\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\bar{v}\mathrm{d}x=\mathrm{Re}\int_{\mathbb{R}^N}\left[\lambda\left(\int_{\mathbb{R}^N}\frac{|u|^{p_{\mu,s}^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{p_{\mu,s}^*-2}u+k\,|u|^{q-2}u\right]\bar{v}\mathrm{d}x,\tag{3.4}$$

where

$$L(u,v)=\iint_{\mathbb{R}^{2N}}\frac{\left|u(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u(y)\right|^{p-2}(u(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u(y))\overline{(v(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}v(y))}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\tag{3.5}$$

and $v\in E$.

Next, we state and prove the following lemma.

Lemma 3.3. *Assume that Conditions (V) and (M) hold. Then, any $(\mathrm{PS})_c$ sequence $\{u_n\}_n$ for J_λ is bounded in E and $c\ge0$.*

Proof. Suppose that $\{u_n\}_n\subset E$ is a (PS) sequence for J_λ. Then, we have

$$\begin{aligned}c+o_n(1)=J_\lambda(u_n)=&\frac{1}{p}\widetilde{M}([u_n]_{s,A}^p)+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u_n|^p\mathrm{d}x\\&-\frac{\lambda}{2p_{s,\mu}^*}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_{s,\mu}^*}|u_n(x)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x-\frac{k}{q}\int_{\mathbb{R}^N}|u_n|^q\mathrm{d}x\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\langle J_\lambda'(u_n),v\rangle=\mathrm{Re}\Bigg\{&M([u_n]_{s,A}^p)L(u_n,v)+\int_{\mathbb{R}^N}V(x)|u_n|^{p-2}u_n\bar{v}\mathrm{d}x\\&-\lambda\int_{\mathbb{R}^N}\left[\left(\int_{\mathbb{R}^N}\frac{|u_n|^{p_{\mu,s}^*}}{|x-y|^\mu}\mathrm{d}y\right)|u_n|^{p_{\mu,s}^*-2}u_n(x)\bar{v}\right]\mathrm{d}x-k\int_{\mathbb{R}^N}|u_n|^{q-2}u_n\bar{v}\mathrm{d}x\Bigg\}\\=&\,o(1)\|u_n\|_E.\end{aligned}\tag{3.7}$$

It follows from (3.6), (3.7), and (M) that

$$
\begin{aligned}
c + o(1)\|u_n\|_\lambda &\geq J_\lambda(u_n) - \frac{1}{q}\langle J'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{p}\widetilde{M}([u_n]^p_{s,A}) - \frac{1}{q}M([u_n]^p_{s,A})[u_n]^p_{s,A} + \left(\frac{1}{p} - \frac{1}{q}\right)\int\limits_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x \\
&\quad + \left(\frac{1}{q} - \frac{1}{2p^*_{s,\mu}}\right)\lambda \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p^*_{s,\mu}}|u_n(x)|^{p^*_{s,\mu}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\geq \left(\frac{\sigma}{p} - \frac{1}{q}\right) m_0 [u_n]^p_{s,A} + \left(\frac{1}{p} - \frac{1}{q}\right)\int\limits_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x \\
&\geq \min\left\{\left(\frac{\sigma}{p} - \frac{1}{q}\right) m_0, \left(\frac{1}{p} - \frac{1}{q}\right)\right\}\|u_n\|^p_E.
\end{aligned} \tag{3.8}
$$

This fact implies that $\{u_n\}_n$ is bounded in E. We also obtain $c \geq 0$ from (3.8). □

Now, we can show that the following compactness condition holds.

Lemma 3.4. *Assume that Conditions* (V) *and* (M) *hold. Then,* $J_\lambda(u)$ *satisfies* $(\mathrm{PS})_c$ *condition, for all* $sp < N < sp^2$ *and*

$$
c < \left(\frac{1}{q} - \frac{1}{2p^*_{s,\mu}}\right)\lambda^{-\frac{p}{2p^*_{s,\mu}-p}}(m_0 S_{p,H})^{\frac{2p^*_{s,\mu}}{2p^*_{s,\mu}-p}}.
$$

Proof. Let $\{u_n\}_n$ be a $(\mathrm{PS})_c$ sequence for J_λ. Then, by Lemma 3.3, we know that the sequence $\{u_n\}_n$ is bounded in E. Moreover, we know that there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ weakly in E. Moreover, we have

$$
u_n \to u \ \text{ a.e. in } \mathbb{R}^N, \quad u_n \to u \ \text{ in } L^s(\mathbb{R}^N), \quad 1 \leq s < p^*_s. \tag{3.9}
$$

Now, by the concentration-compactness principle, we may assume that there exist bounded non-negative measures ω, ζ, and ν on $\mathbb{R}^N$, and an at most countable set $\{x_i\}_{i\in I} \in \Omega\backslash\{0\}$ such that

$$
\int\limits_{\mathbb{R}^N} \frac{||u_n(x)|-|u_n(y)||^p}{|x-y|^{N+ps}}\mathrm{d}x \to \omega, \quad |u_n|^{p^*_s} \to \zeta
$$

and

$$
\left(\int\limits_{\mathbb{R}^N} \frac{|u_n|^{p^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u_n|^{p^*_{\mu,s}} \to \nu.
$$

Now, there exists a countable sequence of points $\{x_i\} \subset \mathbb{R}^N$ and families of positive numbers $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$, and $\{\omega_i : i \in I\}$ such that

$$
\begin{aligned}
&\omega \geq \int\limits_{\mathbb{R}^N} \frac{||u(x)|-|u(y)||^p}{|x-y|^{N+ps}}\mathrm{d}x + \sum_{i\in I}\omega_i\delta_{x_i}, \\
&\zeta = |u|^{p^*_s} + \sum_{i\in I}\zeta_i\delta_{x_i}, \\
&\nu = \left(\int\limits_{\mathbb{R}^N} \frac{|u|^{p^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{p^*_{\mu,s}} + \sum_{i\in I}\nu_i\delta_{x_i}.
\end{aligned}
$$

We can also obtain

$$
S_{p,H}\nu_i^{\frac{p}{2p^*_{\mu,s}}} \leq \omega_i \quad \text{and} \quad \nu_i \leq C(N,\mu)\zeta_i^{\frac{2N-\mu}{N}}. \tag{3.10}
$$

In the sequel, we shall prove that

$$I = \varnothing. \tag{3.11}$$

Suppose, to the contrary, that $I \neq \varnothing$. Then, we can define a smooth cut-off function such that $\phi \in C_0^\infty(\mathbb{R}^N)$ and $0 \le \phi \le 1$; $\phi \equiv 1$ in $B(x_i, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \backslash B(x_i, 2\varepsilon)$. Let $\varepsilon > 0$ and $\phi_\varepsilon^i = \phi\left(\frac{x - x_i}{\varepsilon}\right)$, where $i \in I$. It is not difficult to see that $\{u_n\phi_\varepsilon^i\}_n$ is bounded in E. Then, $\langle J'(u_n), u_n\phi_\varepsilon^i\rangle \to 0$, which implies

$$\begin{aligned} &M([u_n]_{s,A}^p)\iint\limits_{\mathbb{R}^{2N}} \frac{\left| u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u_n(y)\right|^p \phi_\varepsilon^i(y)}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x)|u_n|^p\phi_\varepsilon^i(x)\mathrm{d}x \\ &= -\operatorname{Re}\{M([u_n]_{s,A}^p)\mathcal{L}(u_n, u_n\phi_\varepsilon^i)\} + \lambda\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_{s,\mu}^*}|u_n(x)|^{p_{s,\mu}^*}\phi_\varepsilon^i(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + k\int\limits_{\mathbb{R}^N}|u_n|^q\phi_\varepsilon^i(x)\mathrm{d}x + o_n(1), \end{aligned} \tag{3.12}$$

where

$$\mathcal{L}(u_n, u_n\phi_\varepsilon^i) = \iint\limits_{\mathbb{R}^{2N}} \frac{\left| u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u_n(y)\right|^{p-2}(u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u_n(y))\overline{u_n(x)(\phi_\varepsilon^i(x) - \phi_\varepsilon^i(y))}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.$$

By the Hölder inequality, we know that

$$\begin{aligned} &|\operatorname{Re}\{M([u_n]_{s,A}^p)\mathcal{L}(u_n, u_n\phi_\varepsilon^i)\}| \\ &\le C\left(\iint\limits_{\mathbb{R}^{2N}} \frac{\left| u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u_n(y)\right|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{(p-1)/p}\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_\varepsilon^i(x) - \phi_\varepsilon^i(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p} \\ &\le C\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_\varepsilon^i(x) - \phi_\varepsilon^i(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{1/p}. \end{aligned} \tag{3.13}$$

On the other hand, as in the proof of Lemma 3.4 in Zhang et al. [24], we can obtain that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_\varepsilon^i(x) - \phi_\varepsilon^i(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = 0. \tag{3.14}$$

It follows from (3.12)–(3.14) and the diamagnetic inequality that

$$m_0\omega_i \le \lambda\nu_i. \tag{3.15}$$

This fact together with (3.8) implies that

$$\text{(I)} \quad \nu_i = 0 \quad \text{or} \qquad \text{(II)} \quad \nu_i \ge (\lambda^{-1}m_0S_{p,H})^{\frac{2p_\mu^*}{2p_\mu^*-p}}.$$

If (II) occurs for some $i_0 \in I$, then

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{q}\langle J_\lambda'(u_n), u_n\rangle\right) \\ &\ge \left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right)\lambda\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_{s,\mu}^*}|u_n(x)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &\ge \left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right)\lambda\nu_i \ge \left(\frac{1}{q} - \frac{1}{2p_{s,\mu}^*}\right)\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}(m_0S_{p,H})^{\frac{2p_{s,\mu}^*}{2p_{s,\mu}^*-p}}. \end{aligned} \tag{3.16}$$

This is an obvious contradiction to the choice of c. This completes the proof of (3.11).

Next, we shall prove the concentration at infinity. To this end, set $\phi_R \in C_0^\infty(\mathbb{R}^N)$ for $R > 0$, and satisfies $\phi_R(x) = 0$ for $|x| < R$, $\phi_R(x) = 1$ for $|x| > 2R$, $0 \le \phi_R \le 1$, and $|\nabla \phi_R| \le \frac{2}{R}$. Invoking Theorem 2.4 of Xiang and Zhang [39], we define

$$\omega_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{||u_n(x)|-|u_n(y)||^p \phi_R(x)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y,$$

$$\zeta_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} \phi_R \mathrm{d}x$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_{\mu,s}^*} |u_n(y)|^{p_{\mu,s}^*} \phi_R(x)}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y.$$

By Lemma 3.2, we have

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_{\mu,s}^*} |u_n(y)|^{p_{\mu,s}^*} \phi_R(x)}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} \mathrm{d}\nu + \nu_\infty,$$

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{||u_n(x)|-|u_n(y)||^p \phi_R(x)}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} \mathrm{d}\omega + \omega_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} \mathrm{d}x = \int_{\mathbb{R}^N} d\zeta + \zeta_\infty.$$

Moreover,

$$S_{p,H} \nu_\infty^{\frac{p}{2p_{\mu,s}^*}} \le \omega_\infty \quad \text{and} \quad \nu_\infty \le C(N,\mu) \zeta_\infty^{\frac{2N-\mu}{N}}. \tag{3.17}$$

Similar discussion as earlier yields

$$\text{(III)} \quad \nu_\infty = 0 \quad \text{or} \qquad \text{(IV)} \quad \nu_\infty \ge (\lambda^{-1} m_0 S_{p,H})^{\frac{2p_\mu^*}{2p_\mu^* - p}}.$$

Furthermore, proceeding as in the proof of (3.14), we can obtain $\nu_\infty = 0$. Thus,

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_{s,\mu}^*} |u_n(x)|^{p_{s,\mu}^*}}{|x-y|^{\mu}} \mathrm{d}y\mathrm{d}x \to \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_{s,\mu}^*} |u(x)|^{p_{s,\mu}^*}}{|x-y|^{\mu}} \mathrm{d}y\mathrm{d}x \quad \text{as } n \to \infty. \tag{3.18}$$

By the Brézis-Lieb lemma [41], we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_{\mu,s}^*} |u_n(y)|^{p_{\mu,s}^*}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_{\mu,s}^*} |u(y)|^{p_{\mu,s}^*}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u(x)|^{p_{\mu,s}^*} |u_n(y)-u(y)|^{p_{\mu,s}^*}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y + o(1). \tag{3.19}$$

Hence, (3.18) and (3.19) imply that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u(x)|^{p_{\mu,s}^*} |u_n(y)-u(y)|^{p_{\mu,s}^*}}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y + o(1) \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

Moreover, it is easy to see that

$$\int_{\mathbb{R}^N} (|u_n(x)|^{q-2} u_n(x) - |u(x)|^{q-2} u(x))(u_n(x) - u(x)) \mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.21}$$

By (3.20), (3.21), and the Hölder inequality, we have

$$\begin{aligned}
\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle &= \mathrm{Re}\Bigg\{M([u_n]_{s,A}^p)L(u_n, u_n - u) - M([u]_{s,A}^p)L(u, u_n - u) \\
&\quad + \int_{\mathbb{R}^N} V(x)(|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x))(u_n(x) - u(x))\mathrm{d}x \\
&\quad - \lambda\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(x)|^{p_{\mu,s}^*}|u_n(y) - u(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\quad - k\int_{\mathbb{R}^N} (|u_n(x)|^{q-2}u_n(x) - |u(x)|^{q-2}u(x))(u_n(x) - u(x))\mathrm{d}x\Bigg\} \\
&\geq \mathrm{Re}\Bigg\{M([u_n]_{s,A}^p)([u_n]_{s,A}^p)^{\frac{p-1}{p}}\left[([u_n]_{s,A}^p)^{\frac{1}{p}} - ([u]_{s,A}^p)^{\frac{1}{p}}\right] \\
&\quad + M([u]_{s,A}^p)([u]_{s,A}^p)^{\frac{p-1}{p}}\left[([u]_{s,A}^p)^{\frac{1}{p}} - ([u_n]_{s,A}^p)^{\frac{1}{p}}\right] \\
&\quad + \left(\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left[\left(\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right] \\
&\quad + \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left[\left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\Bigg\} \\
&= \mathrm{Re}\Bigg\{\left[([u_n]_{s,A}^p)^{\frac{1}{p}} - ([u]_{s,A}^p)^{\frac{1}{p}}\right]\left[M([u_n]_{s,A}^p)([u_n]_{s,A}^p)^{\frac{p-1}{p}} - M([u]_{s,A}^p)([u]_{s,A}^p)^{\frac{p-1}{p}}\right] \\
&\quad + \left[\left(\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right] \\
&\quad \times \left[\left(\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right]\Bigg\}.
\end{aligned} \tag{3.22}$$

Since $u_n \rightharpoonup u$ in E and $J_\lambda'(u_n) \to 0$ as $n \to \infty$ in E^*, we can conclude that

$$\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

It follows from $u_n \to u$ a.e in $\mathbb{R}^N$ and the Fatou lemma that

$$[u]_{s,A}^p \leq \liminf_{n\to\infty}[u_n]_{s,A}^p = d_1 \tag{3.23}$$

and

$$\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x = d_2. \tag{3.24}$$

We note that

$$\left[(d_1)^{\frac{1}{p}} - ([u]_{s,A}^p)^{\frac{1}{p}}\right]\left[M(d_1)d_1^{\frac{p-1}{p}} - M([u]_{s,A}^p)([u]_{s,A}^p)^{\frac{p-1}{p}}\right] \geq 0 \tag{3.25}$$

and

$$\left[(d_2)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right]\left[(d_2)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\right] \geq 0, \tag{3.26}$$

since $g(t) = M(t)t^{\frac{p-1}{p}}$ is nondecreasing for $t \geq 0$. Thus, by

$$\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty$$

and (3.20)–(3.26), we obtain

$$\begin{aligned} 0 &\geq \liminf_{n\to\infty} \mathrm{Re}\Bigg\{\Big[([u_n]_{s,A}^p)^{\frac{1}{p}} - ([u]_{s,A}^p)^{\frac{1}{p}}\Big]\Big[M([u_n]_{s,A}^p)([u_n]_{s,A}^p)^{\frac{p-1}{p}} - M([u]_{s,A}^p)([u]_{s,A}^p)^{\frac{p-1}{p}}\Big] \\ &\quad + \left[\left(\int_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x\right)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x\right)^{\frac{1}{p}}\right]\left[\left(\int_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x\right)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x\right)^{\frac{p-1}{p}}\right]\Bigg\} \\ &\geq \mathrm{Re}\Bigg\{\Big[(d_1)^{\frac{1}{p}} - ([u]_{s,A}^p)^{\frac{1}{p}}\Big]\Big[M(d_1)d_1^{\frac{p-1}{p}} - M([u]_{s,A}^p)([u]_{s,A}^p)^{\frac{p-1}{p}}\Big] \\ &\quad + \left[(d_2)^{\frac{1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x\right)^{\frac{1}{p}}\right]\left[(d_2)^{\frac{p-1}{p}} - \left(\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x\right)^{\frac{p-1}{p}}\right]\Bigg\}. \end{aligned} \tag{3.27}$$

It follows from (3.25)–(3.27) that

$$\iint_{\mathbb{R}^{2N}} \frac{\left|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{p}\right)}u(y)\right|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = d_1 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x = d_2.$$

Then, $\|u_n\|_E \to \|u\|_E$. We note that E is a reflexive Banach space; thus, $u_n \to u$ strongly converges in E. This completes the proof of Lemma 3.4. □

4 Auxiliary results

First, we shall prove that functional J_λ has a mountain path structure.

Lemma 4.1. *Let Conditions (V) and (M) hold. Then,*

(C1) *There exist some constants $\alpha_\lambda, \beta_\lambda > 0$ such that $J_\lambda(u) > 0$ if $u \in B_{\beta_\lambda}\backslash\{0\}$ and $J_\lambda(u) \geq \alpha_\lambda$ if $u \in \partial B_{\beta_\lambda}$, where $B_{\beta_\lambda} = \{u \in E : \|u\|_E \leq \beta_\lambda\}$;*

(C2) *We have*

$$J_\lambda(u) \to -\infty \quad \text{as } u \in F \subset E, \quad \|u\|_E \to \infty,$$

where F is a finite-dimensional subspace of E.

Proof. It follows from the Hardy-Littlewood-Sobolev inequality that there exists $C(N,\mu) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p^*_{\mu,s}}|u_n(y)|^{p^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(N,\mu)\|u\|_E^{2p^*_{\mu,s}} \quad \text{for all } u \in E.$$

By virtue of (V) and (M), we obtain

$$J_\lambda(u) \geq \min\left\{\frac{\sigma\alpha_0}{p}, \frac{1}{p}\right\}\|u\|_E^p - \frac{\lambda}{2p^*_{s,\mu}}C(N,\mu)\|u\|_E^{2p^*_{s,\mu}} - Ck\|u\|^q. \tag{4.1}$$

Since $p^*_{s,\mu}, q > p$, we know that Conclusion (C_1) of Lemma 4.1 holds.

In order to prove Conclusion (C_2) of Lemma 4.1, we note that it follows from Condition (m_2) that

$$\widetilde{M}(t) \le \frac{\widetilde{M}(t_0)}{t_0^{\frac{1}{\sigma}}} t^{\frac{1}{\sigma}} = C_0 t^{\frac{1}{\sigma}}, \quad \text{for all } t \ge t_0 > 0. \tag{4.2}$$

Let $\omega \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with $\|\omega\| = 1$. Thus,

$$J_\lambda(t\omega) \le \frac{C_0}{p} t^{\frac{p}{\sigma}} + \frac{1}{p} t^p - \frac{\lambda}{2p_{s,\mu}^*} t^{2p_{s,\mu}^*} \iint_{\mathbb{R}^{2N}} \frac{|\omega(y)|^{p_{s,\mu}^*} |\omega(x)|^{p_{s,\mu}^*}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x - \frac{k}{q} t^q \, |\omega|_q^q.$$

Note that all norms are equivalent in a finite-dimensional space. Then, the aforementioned fact together with $p < \frac{p}{\sigma} < 2p_{s,\mu}^*$ implies that Conclusion (C_2) of Lemma 4.1 holds. □

Invoking Binlin et al. [31, Theorem 3.2], we have

$$\inf\left\{\iint_{\mathbb{R}^{2N}} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y : \phi \in \overset{\infty}{\underset{0}{C}}(\mathbb{R}^N), \, |\phi\,|_q = 1\right\} = 0.$$

For any $1 > \zeta > 0$, let $\phi_\zeta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\zeta|_q = 1$ and supp $\phi_\zeta \subset B_{r_\zeta}(0)$ be such that

$$\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(x) - \phi_\zeta(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \le C\zeta^{(pN-(N-ps)q)/q}$$

and define

$$\psi_\zeta(x) = e^{iA(0)x}\phi_\zeta(x), \quad \psi_{\lambda,\zeta}(x) = \psi_\zeta(\lambda^{-\tau}x) \tag{4.3}$$

and

$$\tau = \frac{1}{(N-ps)}\left(-\frac{p}{2p_{s,\mu}^* - p}\right). \tag{4.4}$$

So, we have

$$\begin{aligned} J_\lambda(t\psi_{\lambda,\zeta}) &\le \frac{C_0}{p} t^{p/\sigma} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_{\lambda,\zeta}(x) - e^{i(x-y)\cdot A((x+y)/p)}\psi_{\lambda,\zeta}(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)^{1/\sigma} + \frac{t^p}{p} \int_{\mathbb{R}^N} V(x)|\psi_{\lambda,\zeta}|^p \mathrm{d}x - t^q \frac{k}{q} \int_{\mathbb{R}^N} |\psi_{\lambda,\zeta}|^q \mathrm{d}x \\ &\le \lambda^{\tau(N-ps)} \left[\frac{C^0}{p} t^{p/\sigma} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(x-y)\cdot A((\lambda^\tau x + \lambda^\tau y)/p)}\psi_\zeta(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)^{1/\sigma} + \frac{t^p}{p} \int_{\mathbb{R}^N} V(\lambda^\tau x)|\psi_\zeta|^p \mathrm{d}x - t^q \frac{k}{q} \int_{\mathbb{R}^N} |\psi_\zeta|^q \mathrm{d}x\right] \\ &= \lambda^{-\frac{p}{2p_{s,\mu}^* - p}} \Psi_\lambda(t\psi_\zeta), \end{aligned}$$

where

$$\Psi_\lambda(u) := \frac{C_0}{p} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A((\lambda^\tau x + \lambda^\tau y)/p)} u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)^{1/\sigma} + \frac{1}{p} \int_{\mathbb{R}^N} V(\lambda^\tau x)|u|^p \mathrm{d}x - \frac{k}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x.$$

Since $q > p/\sigma$, we can find $t_0 \in [0, +\infty)$ such that

$$\max_{t \ge 0} \Psi_\lambda(t\psi_\zeta) \le \frac{C^0}{p} t_0^{p/\sigma} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(x-y)\cdot A((\lambda^\tau x + \lambda^\tau y)/p)}\psi_\zeta(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)^{1/\sigma} + \frac{t_0^p}{p} \int_{\mathbb{R}^N} V(\lambda^\tau x)|\psi_\zeta|^p \mathrm{d}x.$$

Using the aforementioned analysis, we can prove the following conclusions.

Lemma 4.2. *For each $\zeta > 0$, there exists $\lambda_0 = \lambda_0(\zeta) > 0$ such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(x-y)\cdot A((\lambda^\tau x+\lambda^\tau y)/p)}\psi_\zeta(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \le C\zeta^{(pN-(N-ps)q)/q} + \frac{2^{p-1}}{p-ps}\zeta^{ps} + \frac{2^{2p-1}}{ps}\zeta^{ps}$$

for all $0 < \lambda_0 < \lambda$ *and some constant* $C > 0$ *depending only on* $[\phi]_{s,0}$.

Proof. For each $\zeta > 0$, we know that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(x-y)\cdot A((\lambda^\tau x+\lambda^\tau y)/p)}\psi_\zeta(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{|e^{iA(0)\cdot x}\phi_\zeta(x) - e^{i(x-y)\cdot A((\lambda^\tau x+\lambda^\tau y)/p)}e^{iA(0)\cdot y}\phi_\zeta(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&\le 2^{p-1}\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(x) - \phi_\zeta(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + 2^{p-1}\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(y)|^p|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Note that

$$|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p = 2^p\sin^p\left[\frac{(x-y)\cdot(A(0)-A(\frac{\lambda^\tau x+\lambda^\tau y}{p}))}{p}\right]. \tag{4.5}$$

Let $y \in B_{r_\zeta}$ and take $|x-y| \le 1/\zeta\ |\phi_\zeta|_{L^p}^{1/s}$ such that $|x| \le r_\zeta + 1/\zeta\ |\phi_\zeta|_{L^p}^{1/s}$. Then, we have

$$\left|\frac{\lambda^\tau x + \lambda^\tau y}{p}\right| \le \frac{\lambda^\tau}{p}\left(2r_\zeta + \frac{1}{\zeta}\ |\phi_\zeta|_{L^p}^{1/s}\right).$$

By the continuity of the function A, there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, one has

$$\left|A(0) - A\left(\frac{\lambda^\tau x+\lambda^\tau y}{p}\right)\right| \le \zeta\ |\phi_\zeta|_{L^p}^{-1/s} \quad \text{for} \quad |y| \le r_\zeta \quad \text{and} \quad |x| \le r_\zeta + \frac{1}{\zeta}\ |\phi_\zeta|_{L^p}^{1/s},$$

which means

$$|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p \le |x-y|^p\zeta^p\ |\phi_\zeta|_{L^p}^{-p/s}.$$

Let $\zeta > 0$ and $y \in B_{r_\zeta}$, and define

$$N_{\zeta,y} \coloneqq \left\{x \in \mathbb{R}^N : |x-y| \le \frac{1}{\zeta}\ |\phi_\zeta|_{L^p}^{1/s}\right\}.$$

Then, for all $\lambda > \lambda_0 > 0$, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(y)|^p|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&= \int_{B_{r_\zeta}} |\phi_\zeta(y)|^p\mathrm{d}y \int_{N_{\zeta,y}} \frac{|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p}{|x-y|^{N+ps}}\mathrm{d}x \\
&\quad + \int_{B_{r_\zeta}} |\phi_\zeta(y)|^p\mathrm{d}y \int_{\mathbb{R}^N\setminus N_{\zeta,y}} \frac{|e^{i(x-y)\cdot(A(0)-A((\lambda^\tau x+\lambda^\tau y)/p))}-1|^p}{|x-y|^{N+ps}}\mathrm{d}x \\
&\le \int_{B_{r_\zeta}} |\phi_\zeta(y)|^p\mathrm{d}y \int_{N_{\zeta,y}} \frac{|x-y|^p}{|x-y|^{N+ps}}\zeta^p\ |\phi_\zeta|_{L^p}^{-p/s}\mathrm{d}x + \int_{B_{r_\zeta}} |\phi_\zeta(y)|^p\mathrm{d}y \int_{\mathbb{R}^N\setminus N_{\zeta,y}} \frac{2p}{|x-y|^{N+ps}}\mathrm{d}x \\
&\le \frac{1}{p-ps}\zeta^{ps} + \frac{2^p}{ps}\zeta^{ps}.
\end{aligned}$$

This completes the proof of Lemma 4.2. □

It follows from $V(0)=0$ and $\mathrm{supp}\phi_\varsigma \subset B_{r_\varsigma}(0)$ that

$$V(\lambda^\tau x) \le \frac{\zeta}{|\phi_\zeta|_p^p} \quad \text{for all} \quad |x| \le r_\zeta \quad \text{and} \quad \lambda > \lambda^*.$$

Thus,

$$\max_{t\ge0}\Psi_\lambda(t\phi_\delta) \le \frac{C_0}{p} t_0^{p/\sigma}(C\zeta^{(pN-(N-ps)q)/q} + \frac{2^{p-1}}{p-ps}\zeta^{ps} + \frac{2^{2p-1}}{ps}\zeta^{ps})^{1/\sigma} + \frac{t_0^p}{p}\zeta, \tag{4.6}$$

where $C>0$ and $C_0>0$. So, for any $\lambda > \max\{\lambda_0, \lambda^*\}$, we can obtain

$$\max_{t\ge0} J_\lambda(t\psi_{\lambda,\zeta}) \le \left[\frac{C_0}{p} t_0^{p/\sigma}\left(C\zeta^{(pN-(N-ps)q)/q} + \frac{2^{p-1}}{p-ps}\zeta^{ps} + \frac{2^{p-1}}{ps}\zeta^{ps}\right)^{1/\sigma} + \frac{t_0^p}{p}\zeta\right]\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}. \tag{4.7}$$

So we have the following conclusion.

Lemma 4.3. *Let Conditions (V) and (M) hold. Then, for each $\kappa>0$, there exists $\lambda_\kappa>0$ such that for any $0<\lambda_\kappa<\lambda$, and $\widetilde{e}_\lambda \in E$, we have that $\|\widetilde{e}_\lambda\| > \varrho_\lambda, J_\lambda(t\widetilde{e}_\lambda) \le 0$, and*

$$\max_{t\in[0,1]} J_\lambda(t\widetilde{e}_\lambda) \le \kappa\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}. \tag{4.8}$$

Proof. Select $\zeta>0$ so small that

$$\frac{C_0}{p} t_0^{p/\sigma}\left(C\zeta^{(pN-(N-ps)q)/q} + \frac{2^{p-1}}{p-ps}\zeta^{ps} + \frac{2^{p-1}}{ps}\zeta^{ps}\right)^{1/\sigma} + \frac{t_0^p}{p}\zeta \le \kappa.$$

Let $\psi_{\lambda,\zeta}\in E$ be the function defined by (4.3). Let $\lambda_\kappa = \min\{\lambda_0,\lambda^*\}$ and choose $\widetilde{t}_\lambda>0$ such that $\widetilde{t}_\lambda\|\psi_{\lambda,\zeta}\| > \varrho_\lambda$ and $J_\lambda(t\psi_{\lambda,\zeta}) \le 0$ for all $t \ge \widetilde{t}_\lambda$. By (4.7), setting $\widetilde{e}_\lambda = \widetilde{t}_\lambda\psi_{\lambda,\zeta}$, we can obtain the conclusion of Lemma 4.3. □

Now, fix $m^* \in N$. Then, we can select m^* functions $\phi_\zeta^i \in C_0^\infty(\mathbb{R}^N)$ such that supp $\phi_\zeta^i \cap$ supp $\phi_\zeta^k = \emptyset, i \ne k, |\phi_\zeta^i|_s = 1$ and

$$\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta^i(x)-\phi_\zeta^i(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \le C\zeta^{(pN-(N-ps)q)/q}.$$

Let $r_\zeta^{m^*}>0$ be such that supp $\phi_\zeta^i \subset B_{r\zeta}^i(0)$ for $i=1,2,\dots,m^*$. Define

$$\psi_\zeta^i(x) = e^{iA(0)x}\phi_\zeta^i(x) \tag{4.9}$$

and

$$\psi_{\lambda,\zeta}^i(x) = \psi_\zeta^i(\lambda^{-\tau}x). \tag{4.10}$$

Let

$$H_{\lambda\zeta}^{m^*} = \mathrm{span}\left\{\psi_{\lambda,\zeta}^1, \psi_{\lambda,\zeta}^2, \dots, \psi_{\lambda,\zeta}^{m^*}\right\}.$$

Since for each $u = \Sigma_{i=1}^{m^*} c_i\psi_{\lambda,\zeta}^i \in H_{\lambda\zeta}^{m^*}$, we have

$$[u]_{s,A}^p \le C\sum_{i=1}^{m^*}|c_i|^p\left[\psi_{\lambda,\zeta}^i\right]_{s,A}^p,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x = \sum_{i=1}^{m^*}|c_i|^p\int_{\mathbb{R}^N}V(x)|\psi_{\lambda,\zeta}^i|^p\mathrm{d}x$$

and

$$\begin{aligned}&\frac{1}{2p^*_{s,\mu}}\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p^*_{s,\mu}}|u(x)|^{p^*_{s,\mu}}}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x\\&=\sum_{i=1}^{m^*}\left(\frac{1}{2p^*_{s,\mu}}\iint_{\mathbb{R}^{2N}}\frac{|c_i\psi^i_{\lambda,\zeta}(y)|^{p^*_{s,\mu}}|c_i\psi^i_{\lambda,\zeta}(x)|^{p^*_{s,\mu}}}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|c_i\psi^i_{\lambda,\zeta}|^q\right)\mathrm{d}x.\end{aligned}$$

Hence,

$$J_\lambda(u)\leq C\sum_{i=1}^{m^*}J_\lambda(c_i\psi^i_{\lambda,\zeta})$$

for $C>0$. Similar to the previous discussion, we have

$$J_\lambda(c_i\psi^i_{\lambda,\zeta})\leq\lambda^{-\frac{p}{2p^*_{s,\mu}-p}}\Psi(|c_i|\psi^i_\zeta)$$

and we can obtain the following estimate:

$$\max_{u\in H^{m^*}_{\lambda\delta}}J_\lambda(u)\leq Cm^*\left[\frac{C_0}{p}t_0^{p/\sigma}(C\zeta^{(pN-(N-ps)q)/q}+\frac{2^{p-1}}{p-ps}\zeta^{ps}+\frac{2^{2p-1}}{ps}\zeta^{ps})^{1/\sigma}+\frac{t_0^p}{p}\zeta\right]\lambda^{-\frac{p}{2p^*_{s,\mu}-p}}\tag{4.11}$$

for any $\zeta\to0$ and $C>0$. From (4.11), we obtain the following lemma.

Lemma 4.4. *Let Conditions* (V) *and* (M) *hold. Then, for each* $m^*\in\mathbb{N}$, *there exists* $\lambda_{m^*}>0$ *such that for each* $0<\lambda_{m^*}<\lambda$ *and* m^**-dimensional subspace* $F_{\lambda_{m^*}}$ *the following holds*

$$\max_{u\in F_{\lambda_{m^*}}}J_\lambda(u)\leq\kappa\lambda^{-\frac{p}{2p^*_{s,\mu}-p}}.$$

Proof. Let $\zeta>0$ be small enough so that

$$Cm^*\left[\frac{C_0}{p}t_0^{p/\sigma}(C\zeta^{(pN-(N-ps)q)/q}+\frac{2^{p-1}}{p-ps}\zeta^{ps}+\frac{2^{2p-1}}{ps}\zeta^{ps})^{1/\sigma}+\frac{t_0^p}{p}\zeta\right]\leq\kappa.$$

Set $F_{\lambda,m^*}=H^{m^*}_{\lambda\zeta}=\mathrm{span}\left\{\psi^1_{\lambda,\zeta},\psi^2_{\lambda,\zeta},\ldots,\psi^{m^*}_{\lambda,\zeta}\right\}$. Thus, the conclusion of Lemma 4.4 follows from (4.11). □

5 Proofs of main results

In the section, we shall prove the existence and multiplicity of solutions for Problem (1.1).

Proof of Theorem 1.1. Let $0<\kappa<\sigma_0$. By Lemma 3.4, we can select $\lambda_k>0$ and for $0<\lambda<\lambda_k$, and define the minimax value as follows:

$$c_\lambda\coloneqq\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}J_\lambda(t\hat{e}_\lambda),$$

where

$$\Gamma_\lambda\coloneqq\{\gamma\in C([0,1],E):\gamma(0)=0\quad\text{and}\quad\gamma(1)=\hat{e}_\lambda\}.$$

By Lemma 4.1, we know that

$$\alpha_\lambda\leq c_\lambda\leq\kappa\lambda^{-\frac{p}{2p^*_{s,\mu}-p}}.$$

By virtue of Lemma 3.4, we can see that J_λ satisfies the $(\mathrm{PS})_c$ condition, and there is $u_\lambda \in E$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$. Then, u_λ is a nontrivial solution of Problem (1.1). Moreover, since u_λ is a critical point of J_λ, by (M) and $\gamma \in [p, p_s^*]$, we have

$$\begin{aligned}\kappa\lambda^{-\frac{p}{2p_{s,\mu}^*-p}} \geq J_\lambda(u_\lambda) &= J_\lambda(u_\lambda) - \frac{1}{\gamma}J'_\lambda(u_\lambda)u_\lambda \\ &= \frac{1}{p}\widetilde{M}([u_\lambda]_{s,A}^p) - \frac{1}{\gamma}M([u_\lambda]_{s,A}^p)[u_\lambda]_{s,A}^p + \left(\frac{1}{p} - \frac{1}{\gamma}\right)\int_{R^N} V(x)|u_\lambda|^p \mathrm{d}x \\ &\quad + \left(\frac{1}{\gamma} - \frac{1}{2p_{s,\mu}^*}\right)\lambda\iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(y)|^{p_{s,\mu}^*}|u_\lambda(x)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + k\int_{R^N}\left[\frac{1}{\tau}|u_\lambda|^q - \frac{1}{q}|u_\lambda|^q\right]\mathrm{d}x \\ &\geq \left(\frac{\sigma}{p} - \frac{1}{\gamma}\right)m_0[u_\lambda]_{s,A}^p + \left(\frac{1}{p} - \frac{1}{\gamma}\right)\int_{R^N} V(x)|u_\lambda|^p\mathrm{d}x \\ &\quad + \left(\frac{1}{\gamma} - \frac{1}{2p_{s,\mu}^*}\right)\lambda\iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(y)|^{p_{s,\mu}^*}|u_\lambda(x)|^{p_{s,\mu}^*}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + \left(\frac{1}{\gamma} - \frac{1}{q}\right)k\int_{R^N}|u_\lambda|^q\mathrm{d}x.\end{aligned} \tag{5.1}$$

So, we have $u_\lambda \to 0$ as $\lambda \to \infty$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by $\sum$. For each $Z \in \sum$, define gen (Z) to be the Krasnoselski genus and

$$j(Z) := \min_{\iota \in \Gamma_{m^*}} gen(\iota(Z) \cap \partial B_{\varrho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $\iota \in C(E, E)$, and ϱ_λ is the number from Lemma 4.1. Then j is a version of Benci's pseudoindex [42]. Let

$$c_{\lambda i} := \inf_{j(Z)\geq i}\sup_{u\in Z} J_\lambda(u), 1 \leq i \leq m^*.$$

Since $J_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B_{\varrho_\lambda}^+$ and since $j(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$, we have

$$\alpha_\lambda \leq c_{\lambda 1} \leq \cdots \leq c_{\lambda m^*} \leq \sup_{u\in H_{\lambda m^*}} J_\lambda(u) \leq \kappa\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}.$$

Lemma 3.4 implies that J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition at all levels $c < \sigma_0\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}$. By the critical point theory, we know that all $c_{\lambda i}$ are critical levels, i.e., J_λ has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\lambda \leq J_\lambda(u_\lambda) \leq \kappa\lambda^{-\frac{p}{2p_{s,\mu}^*-p}}.$$

Therefore, Problem (1.1) has at least m^* pairs of solutions and $u_{\lambda,\pm i} \to 0$ as $\lambda \to \infty$. □

Acknowledgement: The authors thank the reviewers for their constructive remarks on their work.

Funding information: Song was supported by the Science and Technology Development Plan Project of Jilin Province, China (Grant Nos. 20230101287JC and YDZJ202201ZYTS582), the National Natural Science Foundation of China (Grant No. 12001061) and Innovation and Entrepreneurship Talent Funding Project of Jilin Province (No. 2023QN21). Repovš was supported by the Slovenian Research and Innovation Agency grants P1-0292, N1-0278, N1-0114, N1-0083, J1-4031, and J1-4001.

Author contributions: All authors contributed to the study conception, design, material preparation, data collection, and analysis. All authors read and approved the final manuscript.

Conflict of interest: Prof. Dušan D. Repovš is a member of the Editorial Board in Demonstratio Mathematica but was not involved in the review process of this article.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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