Saoudi et al. *Boundary Value Problems* (2024) 2024:4
https://doi.org/10.1186/s13661-023-01814-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Nodal solutions for Neumann systems with gradient dependence

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Abstract

We consider the following convective Neumann systems:

$$(\mathrm{S}) \qquad \begin{cases} -\Delta_{p_1} u_1 + \frac{|\nabla u_1|^{p_1}}{u_1+\delta_1} = f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 + \frac{|\nabla u_2|^{p_2}}{u_2+\delta_2} = f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ |\nabla u_1|^{p_1-2}\frac{\partial u_1}{\partial \eta} = 0 = |\nabla u_2|^{p_2-2}\frac{\partial u_2}{\partial \eta} & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a smooth boundary $\partial\Omega$, $\delta_1, \delta_2 > 0$ are small parameters, η is the outward unit vector normal to $\partial\Omega$, $f_1, f_2 : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are Carathéodory functions that satisfy certain growth conditions, and Δ_{p_i} ($1 < p_i < N$, $i = 1,2$) are the p-Laplace operators $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ for $u_i \in W^{1,p_i}(\Omega)$. To prove the existence of solutions to such systems, we use a subsupersolution method. We also obtain nodal solutions by constructing appropriate subsolution and supersolution pairs. To the best of our knowledge, such systems have not been studied yet.

Mathematics Subject Classification: 35J62; 35J92

Keywords: Neumann elliptic system; Gradient dependence; Subsolution and supersolution method; Nodal solution

1 Introduction

We consider the following Neumann systems with gradient dependence:

$$(\mathrm{S}) \qquad \begin{cases} -\Delta_{p_1} u_1 + \frac{|\nabla u_1|^{p_1}}{u_1+\delta_1} = f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 + \frac{|\nabla u_2|^{p_2}}{u_2+\delta_2} = f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) & \text{in } \Omega, \\ |\nabla u_1|^{p_1-2}\frac{\partial u_1}{\partial \eta} = 0 = |\nabla u_2|^{p_2-2}\frac{\partial u_2}{\partial \eta} & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a smooth boundary $\partial\Omega$, $\delta_1, \delta_2 > 0$ are small parameters, η is the outward unit normal vector to $\partial\Omega$, Δ_{p_i} ($1 < p_i < N$, $i = 1,2$) are the p-Laplace operators $\Delta_{p_i} u_i := \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ for $u_i \in W^{1,p_i}(\Omega)$.

In recent years, much has been done regarding the existence of solutions for nonlinear systems with the Dirichlet condition and the reaction term depending on the gradient using different techniques, mainly fixed point theory, variational methods, truncation methods, and subsupersolution methods. We mention for instance, Candito et al. [2], where the authors investigated a quasilinear singular Dirichlet system with gradient dependence. They combined Schauder's fixed point theorem with subsupersolution approach to establish the existence of smooth positive solutions. For more detail, we refer the readers to some recent papers: Carl and Motreanu [5], Infante et al. [10], Miyagaki and Rodrigues [14], Kita and Otani [11], Motreanu et al. [17], Orpel [21], Ou [22], Wang et al. [24], Yang and Yang [25], and the references therein. See also the monograph by Motreanu [16].

On the other hand, the corresponding Neumann system has been much less studied. In this context, the Neumann quasilinear equation involving a connective term equation was studied by Moussaoui et al. [20]. Candito et al. [3] obtained nodal solutions for a (p_1, p_2)-Laplacian Neumann system without gradient terms. Neumann systems involving variable exponent double phase operators and gradient dependence were investigated by Guarnotta et al. [9].

The main interest of the present work is the presence of the gradient term, which constitutes a serious obstacle in the investigation of system (S). Note that system (S) is not in the variational form. Therefore the usual critical point theory cannot be directly applied. This difficulty is overcome by using the theory of pseudomonotone operators. We first introduce an auxiliary system using truncation operators. Then we construct a subsolution $(\underline{u}_1, \underline{u}_2)$ and a supersolution $(\overline{u}_1, \overline{u}_2)$ such that $\underline{u}_1 \leq \overline{u}_1$, $\underline{u}_2 \leq \overline{u}_2$ (see Theorem 5.1). Finally, sub- and supersolutions and truncation techniques provide at least two solutions for system (S) with precise sign properties.

We will assume that the nonlinearities f_i for $i = 1, 2$ are Carathéodory functions $f_1, f_2 : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$, that is, $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ is measurable for every $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$, $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ is continuous for a.e. $x \in \Omega$, and they satisfy the following growth conditions:

$(\mathbf{H_1})$ There exist $\alpha_i, \beta_i, M_i > 0$, $i = 1, 2$, such that $\max\{\alpha_i, \beta_i\} < p_i - 1$ and

$$\bigl|f_i(x, s_1, s_2, \xi_1, \xi_2)\bigr| \leq M_i\bigl(1 + |s_i|^{\alpha_i}\bigr)\bigl(1 + |\xi_i|^{\beta_i}\bigr)$$
$$\text{for } i = 1, 2 \text{ and all } (x, s_1, s_2, \xi_1, \xi_2) \in \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N}.$$

$(\mathbf{H_2})$ With appropriate $m_i > 0$, $i = 1, 2$, we have

$$\liminf_{|s_i| \to 0} \bigl\{f_i(x, s_1, s_2, \xi_1, \xi_2) : (\xi_1, \xi_2) \in \mathbb{R}^{2N}\bigr\} > m_i, \quad \text{uniformly in } x \in \Omega.$$

Our main results are the following theorems.

Theorem 1.1 *Let* $\delta_1, \delta_2 > 0$ *be small enough and suppose that conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are satisfied. Then system* (S) *has a nodal solution* $(u_0, v_0) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ *for some* $\gamma \in (0, 1)$ *such that* $u_0(x)$ *and* $u_0'(x)$ *are negative near* $\partial\Omega$.

Theorem 1.2 *Let* $\delta_1, \delta_2 > 0$ *be small enough and suppose that conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are satisfied. Then system* (S) *has a positive solution* $(u_+, u^+) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ *for some* $\gamma \in (0, 1)$ *such that* $u_+(x)$ *and* $u^+(x)$ *are negative near* $\partial\Omega$.

The paper is organized as follows. In Sect. 2, we collect some needed definitions and results. In Sect. 3, we study auxiliary systems. In Sect. 4, we prove Theorem 3.1. In Sect. 5, we study subsupersolutions. In Sect. 6, we study nodal solutions. In Sect. 7, we prove our main results.

2 Preliminaries

This part is devoted to summarizing the necessary basic definitions, notations, and function spaces. For other necessary material, we refer the reader to the comprehensive monograph by Papageorgiou et al. [23]. The Banach space $W^{1,p}(\Omega)$ is equipped with the usual norm

$$\|u\|_{1,p} := \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p} \quad \text{for } u \in W^{1,p}(\Omega),$$

where

$$\|v\|_p := \begin{cases} (\int_\Omega |v(x)|^p \, dx)^{1/p} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |v(x)| & \text{otherwise.} \end{cases}$$

Moreover, we denote

$$\mathcal{W} = W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega), \qquad W_b^{1,p_i}(\Omega) := W^{1,p_i}(\Omega) \cap L^\infty(\Omega),$$
$$[u_1, u_2] := \{u \in W^{1,p}(\Omega) : u_1 \le u \le u_2\}, \qquad C_0^{1,\gamma}(\overline{\Omega}) := \{u \in C^{1,\gamma}(\overline{\Omega}) : u\backslash_{\partial\Omega} = 0\}.$$

Now we define a weak solution of system (S).

Definition 2.1 We say that $(u_1, u_2) \in \mathcal{W}$ is a weak solutions of system (S) if

$$\begin{aligned} & u_i + \delta_i > 0 \quad \text{a.e. in } \Omega, \qquad \frac{|\nabla u_i|^{p_i}}{u_i + \delta_i} \in L^1(\Omega) \quad \text{for } i = 1, 2, \\ & \int_\Omega |\nabla u_1|^{p_1-2} \nabla u_1 \nabla \varphi_1 \, dx + \int_\Omega \frac{|\nabla u_1|^{p_1}}{u_1 + \delta_1} \varphi_1 \, dx = \int_\Omega f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) \varphi_1 \, dx, \\ & \int_\Omega |\nabla u_2|^{p_2-2} \nabla u_2 \nabla \varphi_2 \, dx + \int_\Omega \frac{|\nabla u_2|^{p_2}}{u_2 + \delta_2} \varphi_2 \, dx = \int_\Omega f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) \varphi_2 \, dx, \end{aligned} \tag{2.1}$$

for every $(\varphi_1, \varphi_2) \in W_b^{1,p_1}(\Omega) \times W_b^{1,p_2}(\Omega)$.

Remark 2.2 Note that the boundedness condition for (φ_1, φ_2) is necessary since $\frac{|\nabla u_i|^p}{u_i+\delta_i}$, $i = 1, 2$, are only in $L^1(\Omega)$.

Next, we state the definition of a sub-solution and a super-solution of system (S).

Definition 2.3 We say that the pair $(\underline{u}_1, \underline{u}_2) \in \mathcal{W}$ is a sub-solution of system (S) if

$$\underline{u}_i + \delta_i > 0 \quad \text{a.e. in } \Omega, \qquad \frac{|\nabla \underline{u}_i|^{p_i}}{\underline{u}_i + \delta_i} \in L^1(\Omega) \quad \text{for } i = 1, 2,$$

$$\begin{aligned} & \int_\Omega |\nabla \underline{u}_1|^{p_1-2} \nabla \underline{u}_1 \nabla \varphi_1 \, dx + \int_\Omega \frac{|\nabla \underline{u}_1|^{p_1}}{\underline{u}_1 + \delta_1} \varphi_1 \, dx - \int_\Omega f_1(x, \underline{u}_1, w_2, \nabla \underline{u}_1, \nabla w_2) \varphi_1 \, dx \\ & \quad + \int_\Omega |\nabla \underline{u}_2|^{p_2-2} \nabla \underline{u}_2 \nabla \varphi_2 \, dx + \int_\Omega \frac{|\nabla \underline{u}_2|^{p_2}}{\underline{u}_2 + \delta_2} \varphi_2 \, dx \\ & \quad - \int_\Omega f_2(x, w_1, \underline{u}_2, \nabla w_1, \nabla \underline{u}_2) \varphi_2 \, dx \le 0, \end{aligned} \tag{2.2}$$

and we say that the pair $(\overline{u}_1, \overline{u}_2) \in \mathcal{W}$ is a super-solution of system (S) if

$$
\begin{aligned}
&\overline{u}_i + \delta_i > 0 \quad \text{a.e. in } \Omega, \qquad \frac{|\nabla \overline{u}_i|^{p_i}}{\overline{u}_i + \delta_i} \in L^1(\Omega) \quad \text{for } i = 1,2, \\
&\int_\Omega |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1 \nabla \varphi_1 \,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1 + \delta_1} \varphi_1 \,\mathrm{d}x - \int_\Omega f_1(x, \overline{u}_1, w_2, \nabla \overline{u}_1, \nabla w_2) \varphi_1 \,\mathrm{d}x \\
&\quad + \int_\Omega |\nabla \overline{u}_2|^{p_2-2} \nabla \overline{u}_2 \nabla \varphi_2 \,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{u}_2|^{p_2}}{\overline{u}_2 + \delta_2} \varphi_2 \,\mathrm{d}x \\
&\quad - \int_\Omega f_2(x, w_1, \overline{u}_2, \nabla w_1, \nabla \overline{u}_2) \varphi_2 \,\mathrm{d}x \geq 0,
\end{aligned}
\tag{2.3}
$$

for all $(\varphi_1, \varphi_2) \in W_b^{1,p_1}(\Omega) \times W_b^{1,p_2}(\Omega)$ such that $\varphi_1, \varphi_2 \geq 0$ in Ω and for all $(w_1, w_2) \in \mathcal{W}$ such that $\underline{u}_i \leq w_i \leq \overline{u}_i$, $i = 1,2$, with all integrals in (2.2) and (2.3) being finite.

We will use the following conditions:

(H$_3$) Let $0 \leq q_1 \leq p_1 - 1$ and $0 \leq r_1 \leq p_2 - 1$. For every $\rho > 0$, there exists $M_1 := M_1(\rho) > 0$ such that

$$
\bigl|f_1(x, s_1, s_2, \xi_1, \xi_2)\bigr| \leq M_1 \bigl(1 + |\xi_1|^{q_1} + |\xi_2|^{r_1}\bigr) \quad \text{in } \Omega \times [-\rho, \rho]^2 \times \mathbb{R}^{2N}.
$$

(H$_4$) Let $0 \leq q_2 \leq p_1 - 1$ and $0 \leq r_2 \leq p_2 - 1$. For every $\rho > 0$, there exists $M_2 := M_2(\rho) > 0$ such that

$$
\bigl|f_2(x, s_1, s_2, \xi_1, \xi_2)\bigr| \leq M_2 \bigl(1 + |\xi_1|^{q_2} + |\xi_2|^{r_2}\bigr) \quad \text{in } \Omega \times [-\rho, \rho]^2 \times \mathbb{R}^{2N}.
$$

(H$_5$) There are sub- and supersolutions $\underline{u}_1, \overline{u}_1 \in \mathcal{C}^1(\overline{\Omega})$ of system (S), respectively, satisfying

$$
\overline{u}_1 + \delta_1 \geq \underline{u}_1 + \delta_1 > 0 \quad \text{a.e. in } \Omega. \tag{2.4}
$$

(H$_6$) There are sub- and supersolutions $\underline{u}_2, \overline{u}_2 \in \mathcal{C}^1(\overline{\Omega})$ of system (S), respectively, satisfying

$$
\overline{u}_2 + \delta_2 \geq \underline{u}_2 + \delta_2 > 0 \quad \text{a.e. in } \Omega. \tag{2.5}
$$

Via a standard argument, we will prove the following:

Proposition 2.4 *Suppose that conditions* (**H$_3$**), (**H$_4$**), (**H$_5$**), *and* (**H$_6$**) *are satisfied. Let* $(\underline{u}_i, \underline{v}_i), (\overline{u}_i, \overline{v}_i) \in W_b^{1,p_1}(\Omega) \times W_b^{1,p_2}(\Omega)$ *be pairs of sub- and supersolutions of system* (S). *Set*

$$
\begin{aligned}
&\overline{u} = \min\{\overline{u}_1, \overline{u}_2\}, \qquad \underline{u} = \max\{\underline{u}_1, \underline{u}_2\}, \\
&\overline{v} = \min\{\overline{v}_1, \overline{v}_2\}, \qquad \underline{v} = \max\{\underline{v}_1, \underline{v}_2\},
\end{aligned}
$$

and assume that $\overline{u} \leq \overline{v}$ *and* $\underline{u} \leq \underline{v}$ *Then* $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ *is also a pair of sub- and supersolutions of system* (S).

Proof The proof is inspired by the proof of Motreanu et al. [19, Lemma 3]. Fix $\epsilon > 0$ and define the truncation function $\xi_\epsilon(s) = \max\{-\epsilon, \min\{s, \epsilon\}\}$ for $s \in \mathbb{R}$. By Marcus et al. [13] we know that

$$\begin{aligned}
&\xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big), \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big) \in \mathcal{W}, \\
&\nabla\xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big) = \xi'_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big)\nabla(\overline{u}_1 - \overline{u}_2)^-,
\end{aligned}$$

and

$$\nabla\xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big) = \xi'_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\nabla(\underline{u}_1 - \underline{u}_2)^+.$$

Now letting $\varphi \in C_c^1(\Omega)$ be a test function such that $\varphi \geq 0$, we obtain

$$\begin{aligned}
&\left\langle -\Delta_{p_1}\underline{u}_1 + \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1 + \delta_1}, \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\varphi\right\rangle \\
&\quad \leq \int_\Omega f_1(x, \underline{u}_1, w_2, \nabla\underline{u}_1, \nabla w_2)\xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\varphi\,\mathrm{d}x, \qquad (2.6)\\
&\left\langle -\Delta_{p_1}\overline{u}_1 + \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1 + \delta_1}, \xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big)\varphi\right\rangle \\
&\quad \geq \int_\Omega f_1(x, \overline{u}_1, w_2, \nabla\overline{u}_1, \nabla w_2)\xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big)\varphi\,\mathrm{d}x \qquad (2.7)
\end{aligned}$$

for every $w_2 \in W^{1,p_2}(\Omega)$ with $\underline{u}_2 \leq w_2 \leq \overline{u}_2$ and

$$\begin{aligned}
&\left\langle -\Delta_{p_1}\underline{u}_2 + \frac{|\nabla\underline{u}_2|^{p_1}}{\underline{u}_2 + \delta_1}, \big(\epsilon - \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\big)\varphi\right\rangle \\
&\quad \leq \int_\Omega f_1(x, w_1, \underline{u}_2, \nabla w_1, \nabla\underline{u}_2)\big(\epsilon - \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\big)\varphi\,\mathrm{d}x, \qquad (2.8)\\
&\left\langle -\Delta_{p_1}\overline{u}_2 + \frac{|\nabla\overline{u}_2|^{p_1}}{\overline{u}_2 + \delta_1}, \big(\epsilon - \xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big)\big)\varphi\right\rangle \\
&\quad \geq \int_\Omega f_1(x, w_1, \overline{u}_2, \nabla w_1, \nabla\overline{u}_2)\big(\epsilon - \xi_\epsilon\big((\overline{u}_1 - \overline{u}_2)^-\big)\big)\varphi\,\mathrm{d}x \qquad (2.9)
\end{aligned}$$

for every $w_1 \in W^{1,p_1}(\Omega)$ with $\underline{u}_1 \leq w_1 \leq \overline{u}_1$. Therefore by the monotonicity of the $-p$-Laplacian operator we have

$$\begin{aligned}
&\left\langle -\Delta_{p_1}\underline{u}_1 + \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1 + \delta_1}, \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\varphi\right\rangle + \left\langle -\Delta_{p_1}\underline{u}_2 + \frac{|\nabla\underline{u}_2|^{p_1}}{\underline{u}_2 + \delta_1}, \big(\epsilon - \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\big)\varphi\right\rangle \\
&\quad \geq \int_\Omega |\nabla\underline{u}_1|^{p_1-2}(\nabla\underline{u}_1, \nabla\varphi)_{\mathbb{R}^N}\xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\,\mathrm{d}x + \int_\Omega \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1 + \delta_1}\xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\varphi\,\mathrm{d}x \\
&\qquad + \int_\Omega |\nabla\underline{u}_2|^{p_1-2}(\nabla\underline{u}_2, \nabla\varphi)_{\mathbb{R}^N}\big(\epsilon - \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\big)\,\mathrm{d}x \\
&\qquad + \int_\Omega \frac{|\nabla\underline{u}_2|^{p_1}}{\underline{u}_2 + \delta_1}\big(\epsilon - \xi_\epsilon\big((\underline{u}_1 - \underline{u}_2)^+\big)\big)\varphi\,\mathrm{d}x \qquad (2.10)
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle -\Delta_{p_1}\overline{u}_1 + \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}, \xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\varphi\right\rangle + \left\langle -\Delta_{p_1}\overline{u}_2 + \frac{|\nabla \overline{u}_2|^{p_1}}{\overline{u}_2+\delta_1}, \big(\epsilon-\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\big)\varphi\right\rangle \\
&\quad\le \int_\Omega |\nabla \overline{u}_1|^{p_1-2}(\nabla \overline{u}_1,\nabla\varphi)_{\mathbb{R}^N}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\varphi\,\mathrm{d}x \\
&\qquad + \int_\Omega |\nabla \overline{u}_2|^{p_1-2}(\nabla \overline{u}_2,\nabla\varphi)_{\mathbb{R}^N}\big(\epsilon-\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^+\big)\big)\,\mathrm{d}x \\
&\qquad + \int_\Omega \frac{|\nabla \overline{u}_2|^{p_1}}{\overline{u}_2+\delta_1}\big(\epsilon-\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\big)\varphi\,\mathrm{d}x. \qquad (2.11)
\end{aligned}$$

Invoking equations (2.6), (2.8), and (2.10), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla \underline{u}_1|^{p_1-2}(\nabla \underline{u}_1,\nabla\varphi)_{\mathbb{R}^N}\frac{1}{\epsilon}\xi_\epsilon\big((\underline{u}_1-\underline{u}_2)^+\big)\,\mathrm{d}x + \int_\Omega \frac{|\nabla \underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1}\frac{1}{\epsilon}\xi_\epsilon((\underline{u}_1-\underline{u}_2)^-\,\mathrm{d}x \\
&\qquad + \int_\Omega |\nabla \underline{u}_2|^{p_1-2}(\nabla \underline{u}_2,\nabla\varphi)_{\mathbb{R}^N}\left(1-\frac{1}{\epsilon}\xi_\epsilon\big((\underline{u}_1-\underline{u}_2)^+\big)\right)\mathrm{d}x \\
&\qquad + \int_\Omega \frac{|\nabla \underline{u}_2|^{p_1}}{\underline{u}_2+\delta_1}(1-\frac{1}{\epsilon}\xi_\epsilon\big((\underline{u}_1-\underline{u}_2)^+\big)\,\mathrm{d}x \\
&\quad\le \int_\Omega f_1(x,\underline{u}_1,w_2,\nabla\underline{u}_1,\nabla w_2)\frac{1}{\epsilon}\xi_\epsilon\big((\underline{u}_1-\underline{u}_2)^+\big)\varphi\,\mathrm{d}x \\
&\qquad + \int_\Omega f_1(x,\underline{u}_1,w_2,\nabla\underline{u}_1,\nabla w_2)\left(1-\frac{1}{\epsilon}\xi_\epsilon\big((\underline{u}_1-\underline{u}_2)^-\big)\right)\varphi\,\mathrm{d}x.
\end{aligned}$$

In a similar manner, invoking equations (2.7), (2.9), and (2.11), we get

$$\begin{aligned}
&\int_\Omega |\nabla \overline{u}_1|^{p_1-2}(\nabla \overline{u}_1,\nabla\varphi)_{\mathbb{R}^N}\frac{1}{\epsilon}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}\frac{1}{\epsilon}\xi_\epsilon((\overline{u}_1-\overline{u}_2)^-\,\mathrm{d}x \\
&\qquad + \int_\Omega |\nabla \overline{u}_2|^{p-2}(\nabla \overline{u}_2,\nabla\varphi)_{\mathbb{R}^N}\left(1-\frac{1}{\epsilon}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\right)\mathrm{d}x \\
&\qquad + \int_\Omega \frac{|\nabla \overline{u}_2|^{p_1}}{\overline{u}_2+\delta_1}(1-\frac{1}{\varepsilon}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\,\mathrm{d}x \\
&\quad\ge \int_\Omega f_2(x,w_1,\overline{u}_2,\nabla w_1,\nabla\overline{u}_2)\frac{1}{\epsilon}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\varphi\,\mathrm{d}x \\
&\qquad + \int_\Omega f_2(x,w_1,\overline{u}_2,\nabla w_1,\nabla\overline{u}_2)\left(1-\frac{1}{\epsilon}\xi_\epsilon\big((\overline{u}_1-\overline{u}_2)^-\big)\right)\varphi\,\mathrm{d}x.
\end{aligned}$$

Letting $\epsilon \to 0$ and observing that

$$\begin{cases}
\frac{1}{\epsilon}\xi_\epsilon((\overline{u}_1-\overline{u}_2)^- \to 1_{\{\overline{u}_1<\overline{u}_2\}}(x), & \text{a.e. in } \Omega \text{ as } \epsilon\to 0, \\
\frac{1}{\epsilon}\xi_\epsilon((\underline{u}_1-\underline{u}_2)^+ \to 1_{\{\underline{u}_1<\underline{u}_2\}}(x), & \text{a.e. in } \Omega \text{ as } \epsilon\to 0,
\end{cases}$$

we see that

$$\int_\Omega |\nabla \underline{u}|^{p_1-2}\nabla\underline{u}\nabla\varphi\,\mathrm{d}x + \int_\Omega \frac{|\nabla\underline{u}|^{p_1}}{\underline{u}+\delta_1}\varphi\,\mathrm{d}x \le \int_\Omega f_1(x,\underline{u}_1,w_2,\nabla\underline{u}_1,w_2)\varphi\,\mathrm{d}x$$

and

$$\int_\Omega |\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\nabla \varphi \,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{u}|^{p_1}}{\overline{u}+\delta_1}\varphi \,\mathrm{d}x \geq \int_\Omega f_1(x,\overline{u}_1,w_2,\nabla \overline{u}_1,\nabla w_2)\varphi \,\mathrm{d}x$$

for every $\varphi \in C_c^1(\Omega)$, $\varphi \geq 0$ a.e. in Ω. By a similar argument we obtain

$$\int_\Omega |\nabla \underline{v}|^{p_2-2}\nabla \underline{v}\nabla \varphi \,\mathrm{d}x + \int_\Omega \frac{|\nabla \underline{v}|^{p_2}}{\underline{v}+\delta_2}\varphi \,\mathrm{d}x \leq \int_\Omega f_2(x,w_1,\underline{v}_2,\nabla w_1,\nabla \underline{v}_2)\varphi \,\mathrm{d}x$$

and

$$\int_\Omega |\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\nabla \varphi \,\mathrm{d}x + \int_\Omega \frac{|\nabla \overline{v}|^{p_2}}{\overline{v}+\delta_2}\varphi \,\mathrm{d}x \geq \int_\Omega f_2(x,w_1,\overline{v}_2,\nabla w_1,\nabla \overline{v}_2)\varphi \,\mathrm{d}x.$$

Finally, in view of the denseness of $C_c^1(\Omega)$ in both $W^{1,p_1}(\Omega)$ and $W^{1,p_2}(\Omega)$, we deduce that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ is also a pair of sub- and supersolutions of system (S). □

3 Auxiliary systems

Let, the pairs $(\underline{u}_1,\underline{u}_2),(\overline{u}_1,\overline{u}_2) \in \mathcal{C}^1(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$ be sub- and supersolutions, respectively, of system (S) as required in conditions $(\mathbf{H_5})$ and $(\mathbf{H_6})$. Now, for a given $(u_1,u_2) \in \mathcal{W}$, we define the truncation operators $\mathcal{T}_i : W^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ by

$$\mathcal{T}_1(u_1) := \begin{cases} \underline{u}_1 & \text{when } u_1 \leq \underline{u}_1, \\ u_1 & \text{if } \underline{u}_1 \leq u_1 \leq \overline{u}_1, \\ \overline{u}_1 & \text{otherwise,} \end{cases} \qquad \mathcal{T}_2(u_2) := \begin{cases} \underline{u}_2 & \text{when } u_2 \leq \underline{u}_2, \\ u_2 & \text{if } \underline{u}_2 \leq u_2 \leq \overline{u}_2, \\ \overline{u}_2 & \text{otherwise.} \end{cases} \tag{3.1}$$

Then by Carl et al. [4, Lemma 2.89], $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous, monotone, and bounded. In view of conditions $(\mathbf{H_3})$ and $(\mathbf{H_4})$, if $\rho > 0$, then

$$-\rho \leq \underline{u}_1 \leq \overline{u}_1 \leq \rho, \qquad -\rho \leq \underline{u}_2 \leq \overline{u}_2 \leq \rho. \tag{3.2}$$

We introduce the Nemitskii operators $\mathcal{N}_{f_1}$ and $\mathcal{N}_{f_2}$ generated by the Carathéodory functions f_1 and f_2, respectively, which are well defined for $i = 1,2$ since the range of $\mathcal{T}_1$ and $\mathcal{T}_2$ lies within the region $[\underline{u}_i,\overline{u}_i]$. So by the Rellich–Kondrachov compactness embedding theorem the maps

$$\mathcal{N}_{f_1} \circ (\mathcal{T}_1,\mathcal{T}_2) : [\underline{u}_1,\overline{u}_1] \subset \mathcal{W} \longrightarrow L^{p_1'}(\Omega) \hookrightarrow W^{-1,p_1}(\Omega), \tag{3.3}$$

$$\mathcal{N}_{f_2} \circ (\mathcal{T}_1,\mathcal{T}_2) : [\underline{u}_2,\overline{u}_2] \subset \mathcal{W} \longrightarrow L^{p_2'}(\Omega) \hookrightarrow W^{-1,p_2}(\Omega) \tag{3.4}$$

are bounded and completely continuous. Furthermore, set

$$\mathcal{F}(u) = \big(\mathcal{N}_{f_1}(\mathcal{T}_1u_1,\mathcal{T}_2u_2,\nabla(\mathcal{T}_1u_1),\nabla(\mathcal{T}_2u_2)),\mathcal{N}_{f_2}(\mathcal{T}_1u_1,\mathcal{T}_2u_2,\nabla(\mathcal{T}_1u_1),\nabla(\mathcal{T}_2u_2))\big).$$

Next, define the cut-off functions $b_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2$, by

$$b_1(x,s) := -\big(\underline{u}_1(x) - s\big)_+^{p_1-1} + \big(s - \overline{u}_1(x)\big)_+^{p_1-1} \quad \text{for } (x,s) \in \Omega \times \mathbb{R}, \tag{3.5}$$

$$b_2(x,s) := -\big(\underline{u}_2(x) - s\big)_+^{p_2-1} + \big(s - \overline{u}_2(x)\big)_+^{p_2-1} \quad \text{for } (x,s) \in \Omega \times \mathbb{R}. \tag{3.6}$$

It is easy to see that b_i, $i = 1,2$, are Carathéodory functions fulfilling the following growth condition:

$$\big|b_1(x,s)\big| \le \varphi_1(x) + c_1|s|^{p_1-1} \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}, \tag{3.7}$$

$$\big|b_2(x,s)\big| \le \varphi_2(x) + c_2|s|^{p_2-1} \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}, \tag{3.8}$$

with $\varphi_1, \varphi_2 \in L^\infty(\Omega)$ and $c_1, c_2 > 0$. Moreover, we have the following estimates:

$$\int_\Omega b_1(\cdot,u_1)u_1 \,\mathrm{d}x \ge C_1\|u_1\|_{p_1}^p - C_2 \quad \text{for every } u_1 \in W^{1,p_1}(\Omega), \tag{3.9}$$

$$\int_\Omega b_2(\cdot,u_2)u_2 \,\mathrm{d}x \ge C_1'\|u_1\|_{p_1}^p - C_2' \quad \text{for every } u_1 \in W^{1,p_2}(\Omega), \tag{3.10}$$

where C_1, C_2, C_1', C_2' are some positive constants (for more detail, see, e.g., Carl et al. [4, pp. 95–96]). Let $\mu > 0$ and set

$$\mu\mathcal{B}(u) = \big(\mu\mathcal{B}_1(u_1), \mu\mathcal{B}_2(u_2)\big).$$

Now we introduce the following auxiliary problem:

$$(\mathrm{S}_\mu) \qquad \begin{cases} -\Delta_{p_1}u_1 + \frac{|\nabla(\mathcal{T}u_1)|^{p_1}}{\mathcal{T}u_1+\delta_1} = f_1(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)) - \mu b_1(x,u) & \text{in } \Omega, \\ -\Delta_{p_2}u_2 + \frac{|\nabla(\mathcal{T}u_2)|^{p_2}}{\mathcal{T}u_2+\delta_2} = f_2(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)) - \mu b_2(x,u) & \text{in } \Omega, \\ |\nabla u_1|^{p_1-2}\frac{\partial u_1}{\partial\eta} = 0 = |\nabla u_2|^{p_2-2}\frac{\partial u_2}{\partial\eta} & \text{on } \partial\Omega, \end{cases}$$

where $(u_1,u_2) \in \mathcal{W}$. Our main result in this section concerning system (S_μ) is as follows.

Theorem 3.1 *Suppose that conditions* $(\mathbf{H_3})$, $(\mathbf{H_4})$, $(\mathbf{H_5})$, *and* $(\mathbf{H_6})$ *are satisfied. Then system* (S_μ) *has a pair of weak solutions* $(u_1,u_2) \in \mathcal{W}$.

The following estimates will be a key for the proof of Theorem 3.1 in the next section.

Lemma 3.2 *Suppose that conditions* $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *are satisfied. Then there exist constants* $k_0, k_0' > 0$, *depending only on* p_1, p_2, *and* Ω, *such that*

$$\int_\Omega \big|f_1\big(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\big)\big||u_1| \,\mathrm{d}x \le \frac{1}{2}\big(\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_2\|_{p_2}^{p_2}\big) + k_0\big(1 + \|u_1\|_{p_1} + \|u_1\|_{p_1}^{p_1} + \|u_1\|_{p_2}^{p_2}\big)$$

and

$$\int_\Omega \big|f_2\big(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\big)\big||u_2| \,\mathrm{d}x \le \frac{1}{2}\big(\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_2\|_{p_2}^{p_2}\big) + k_0'\big(1 + \|u_2\|_{p_2} + \|u_2\|_{p_2}^{p_2} + \|u_2\|_{p_1}^{p_1}\big)$$

for every $(u_1,u_2) \in \mathcal{W}$.

Proof We will only prove the first inequality. The second inequality can be verified similarly. First, by condition (**H₃**) we have

$$\begin{aligned}&\int_\Omega \bigl|f_1\bigl(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\bigr)\bigr||u_1|\,\mathrm{d}x\\&\quad\le M_1\int_\Omega\bigl(1+\bigl|\nabla(\mathcal{T}u_1)\bigr|^{q_1}+\bigl|\nabla(\mathcal{T}u_2)\bigr|^{r_1}\bigr)|u_1|\,\mathrm{d}x.\end{aligned}\tag{3.11}$$

Using Young's inequality, we get that for any fixed $\varepsilon\in]0,\frac{1}{2M}[$ and every $u_1\in W^{1,p_1}(\Omega)$,

$$\bigl|\nabla(\mathcal{T}u_1)\bigr|^{q_1}|u_1|\le\varepsilon\bigl|\nabla(\mathcal{T}u_1)\bigr|^{\frac{q_1p_1}{p_1-1}}+c_\varepsilon|u_1|^{p_1}\le\varepsilon\bigl(1+\bigl|\nabla(\mathcal{T}u_1)\bigr|^{p_1}\bigr)+c_\varepsilon|u_1|^{p_1}.\tag{3.12}$$

Similarly, for every $u_2\in W^{1,p_2}(\Omega)$, we have

$$\bigl|\nabla(\mathcal{T}u_2)\bigr|^{r_1}|u_1|\le\varepsilon\bigl|\nabla(\mathcal{T}u_2)\bigr|^{\frac{r_1p_2}{p_2-1}}+c'_\varepsilon|u_1|^{p_2}\le\varepsilon\bigl(1+\bigl|\nabla(\mathcal{T}u_2)\bigr|^{p_2}\bigr)+c'_\varepsilon|u_1|^{p_2}.\tag{3.13}$$

On the other hand, using equation (3.1), we can see that

$$\begin{aligned}\int_\Omega\bigl|\nabla(\mathcal{T}u_1)\bigr|^{p_1}\,\mathrm{d}x&=\int_{\{\underline{u}_1\le u_1\le\overline{u}_1\}}|\nabla u_1|^{p_1}\,\mathrm{d}x+\int_{\{u_1\ge\overline{u}_1\}}|\nabla\overline{u}_1|^{p_1}\,\mathrm{d}x+\int_{\{u_1\le\underline{u}_1\}}|\nabla\underline{u}_1|^{p_1}\,\mathrm{d}x\\&\le\int_\Omega|\nabla u_1|^{p_1}\,\mathrm{d}x+\int_\Omega|\nabla\underline{u}_1|^{p_1}\,\mathrm{d}x+\int_\Omega|\nabla\overline{u}_1|^{p_1}\,\mathrm{d}x\\&\le\|\nabla u_1\|_{p_1}^{p_1}+\bigl(\|\nabla\underline{u}_1\|_\infty^{p_1}+\|\nabla\overline{u}_1\|_\infty^{p_1}\bigr)|\Omega|.\end{aligned}\tag{3.14}$$

Using the same techniques, we get

$$\int_\Omega\bigl|\nabla(\mathcal{T}u_2)\bigr|^{p_2}\,\mathrm{d}x\le\|\nabla u_2\|_{p_2}^{p_2}+\bigl(\|\nabla\underline{u}_2\|_\infty^{p_2}+\|\nabla\overline{u}_2\|_\infty^{p_2}\bigr)|\Omega|.\tag{3.15}$$

Consequently, using equations (3.12)–(3.15), we get

$$\begin{aligned}&\int_\Omega\bigl|f_1\bigl(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\bigr)\bigr||u_1|\,\mathrm{d}x_{p_1}\\&\quad\le M_1\bigl(|\Omega|^{\frac{p_1-1}{p_1}}\|u_1\|+\epsilon|\Omega|\bigl(1+\|\nabla\underline{u}_1\|_\infty^{p_1}+\|\nabla\overline{u}_1\|_\infty^{p_1}\bigr)+\epsilon\|\nabla u_1\|_{p_1}^{p_1}+c_\epsilon\|u_1\|_{p_1}^{p_1}\\&\qquad+\epsilon|\Omega|\bigl(1+\|\nabla\underline{u}_2\|_\infty^{p_2}+\|\nabla\overline{u}_2\|_\infty^{p_2}\bigr)+\epsilon\|\nabla u_2\|_{p_2}^{p_2}+c_\epsilon\|u_1\|_{p_2}^{p_2}\bigr)\\&\quad\le\frac{1}{2}\bigl(\|\nabla u_1\|_{p_1}^{p_1}+\|\nabla u_2\|_{p_2}^{p_2}\bigr)+k_0\bigl(1+\|u_1\|_{p_1}+\|u_1\|_{p_1}^{p_1}+\|u_1\|_{p_2}^{p_2}\bigr)\end{aligned}\tag{3.16}$$

for a suitable $k_0>0$. The proof of the lemma is thus completed. □

The following useful estimates can be verified in a similar way as in Moussaoui et al. [20, Lemma 2.2].

Lemma 3.3 *Suppose that conditions* (**H₃**), (**H₄**), (**H₅**), *and* (**H₆**) *are satisfied. Then for every* $u=(u_1,u_2)\in\mathcal{W}$, *there exist constants* k_1 *and* k_2, *independent of* u_1 *and* u_2, *respectively, such that*

$$\frac{|\nabla(\mathcal{T}u_1)|^{p_1}}{\mathcal{T}u_1+\delta_1}|u_1|\in L^1(\Omega)\quad\textit{and}\quad\int_\Omega\frac{|\nabla(\mathcal{T}u_1)|^{p_1}}{\mathcal{T}u_1+\delta_1}|u_1|\,\mathrm{d}x\le k_1\bigl(1+\|u_1\|_{p_1}\bigr),\tag{3.17}$$

$$\frac{|\nabla(\mathcal{T}u_2)|^{p_2}}{\mathcal{T}u_2+\delta_2}|u_2| \in L^1(\Omega) \quad \textit{and} \quad \int_\Omega \frac{|\nabla(\mathcal{T}u_2)|^{p_2}}{\mathcal{T}u_2+\delta_1}|u_2|\,\mathrm{d}x \le k_2\big(1+\|u_2\|_{p_2}\big). \tag{3.18}$$

4 Proof of Theorem 3.1

First, by the growth conditions (3.7) and (3.8) we know that the Nemytskii operators $\mathcal{B}_i : W^{1,p_i}(\Omega) \longrightarrow W^{-1,p_i'}(\Omega)$ given by $\mathcal{B}_i u_i(x) = b(\cdot,u_i)$ are well defined, continuous, and bounded for $i = 1,2$. Also, the operator $\mathcal{B}(u) = (\mathcal{B}_1(u_1),\mathcal{B}_2(u_2))$ is well defined. Moreover, using the compact embedding $W^{1,p_i}(\Omega) \hookrightarrow L^{p_i}(\Omega)$, we have that the operator $\mathcal{B}$ is completely continuous. Next, using conditions (H.3) and (H.4), we can introduce the functions $\pi_{p_i,\delta_i} : (-\delta_i,+\infty)\times\mathbb{R}^N \longrightarrow \mathbb{R}$, $i = 1,2$, defined by

$$\pi_{p_i,\delta_i}(s_i,\xi_i) = \frac{|\xi_i|^{p_i}}{s_i+\delta_i}$$

having the growth

$$\big|\pi_{p_i,\delta_i}(s_i,\xi_i)\big| \le \delta_0|\xi_i|^{p_i}$$

for all $s_i > -\delta_i$ and $\xi_i \in \mathbb{R}^N$, where $\delta_0 > 0$ is a constant such that $\overline{u}_i + \delta_i \ge \underline{u}_i + \delta_i > \delta_0$ a.e. in Ω for $i = 1,2$.

By Motreanu et al. [18, Theorem 2.76] and Gasinski et al. [8, Theorem 3.4.4]) we know that the corresponding Nemytskii operators

$$\Pi_{p_i,\delta_i} : [\underline{u}_i,\overline{u}_i] \subset W^{1,p_i}(\Omega) \longrightarrow L^1(\Omega) \subset W^{-1,p_i'}(\Omega)$$

are bounded and continuous for $i = 1,2$. By virtue of the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we know that $\Pi_{p,\delta}(u) = (\Pi_{p_1,\delta_1}(u_1),\Pi_{p_2,\delta_2}(u_2))$ is completely continuous. Finally, $\mathcal{A}(u) = (A_1(u_1),A_2(u_2))$, where $\mathcal{A} : \mathcal{W} \to \mathcal{W}^*$ is defined in equation (2.1), is well defined, bounded, continuous, strictly monotone, and of type (S_+). Therefore, for every u and $\varphi \in \mathcal{W}$, we have the following representations:

$$\big\langle \mathcal{A}(u),\varphi\big\rangle_{\mathcal{W}} = \sum_{i=1}^{2}\int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i \nabla\varphi_i\,\mathrm{d}x,$$

$$\big\langle \Pi_{p,\delta}(u),\varphi\big\rangle_{\mathcal{W}} = \sum_{i=1}^{2}\int_\Omega \Pi_{p_i,\delta_i}(u_1,u_2,\nabla u_1,\nabla u_2)\varphi_i\,\mathrm{d}x,$$

$$\big\langle \mathcal{B}(u),\varphi\big\rangle_{\mathcal{W}} = \sum_{i=1}^{2}\int_\Omega \mathcal{B}_i(u_1,u_2,\nabla u_1,\nabla u_2)\varphi_i\,\mathrm{d}x,$$

$$\big\langle \mathcal{F}(u),\varphi\big\rangle_{\mathcal{W}} = \sum_{i=1}^{2}\int_\Omega \mathcal{N}_{f_i}(\mathcal{T}_1u_1,\mathcal{T}_2u_2,\nabla(\mathcal{T}_1u_1),\nabla(\mathcal{T}_2u_2)\varphi_i\,\mathrm{d}x.$$

Now for every u and $\varphi \in \mathcal{W}$, system (P_μ) can be given in the form

$$\big\langle \mathcal{A}(u)+\mu\mathcal{B}u+\Pi_{p,\delta}(u),\varphi\big\rangle_{\mathcal{W}} = \big\langle \mathcal{F}(u),\varphi\big\rangle_{\mathcal{W}}. \tag{4.1}$$

Set

$$\chi_\mu := \mathcal{A}(u)+\mu\mathcal{B}u+\Pi_{p,\delta}(u)-\mathcal{F}(u).$$

First, by conditions (H.1) and (H.2), χ_μ is well defined, continuous, and bounded. The next step in the proof is showing that the operator χ_μ is pseudo-monotone. To this end, using the $(\mathrm{S})_+$-property of $\mathcal{A}$, in view of the compactness of the operators $\Pi_{p,\delta}, \mathcal{B}, \mathcal{F}$, we can use Gambera et al. [7, Lemma 2.2] to deduce that the operator χ_μ also has the $(\mathrm{S})_+$-property. Furthermore, we can apply Zeidler [26, Proposition 26.2] to see that the operator χ_μ is pseudo-monotone.

Let us show that the operator $\chi_\mu : \mathcal{W} \to \mathcal{W}^*$ is coercive. To this end, using equation (4.1), we get

$$\begin{aligned}
\langle \chi_\mu(u), u\rangle &= \sum_{i=1}^{2}\int_\Omega |\nabla u_i|^{p_i}\,\mathrm{d}x + \mu\sum_{i=1}^{2}\int_\Omega b_i(x,u_i)u_i\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega \frac{|\nabla(\mathcal{T}u_i)|^{p_i}}{\mathcal{T}u_i+\delta_i}u_i\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{2}\int_\Omega f_i\big(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\big)u_i\,\mathrm{d}x \\
&\geq \sum_{i=1}^{2}\int_\Omega |\nabla u_i|^{p_i}\,\mathrm{d}x + \mu\sum_{i=1}^{2}\int_\Omega b_i(x,u_i)u_i\,\mathrm{d}x - \sum_{i=1}^{2}\int_\Omega \frac{|\nabla(\mathcal{T}u_i)|^{p_i}}{\mathcal{T}u_i+\delta_i}u_i\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{2}\int_\Omega f_i\big(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\big)u_i\,\mathrm{d}x.
\end{aligned} \tag{4.2}$$

Now using equation (4.2) and combining equations (3.9) and (3.10) with Lemmas 3.2 and 3.3, we obtain

$$\begin{aligned}
\langle \chi_\mu(u), u\rangle &\geq \sum_{i=1}^{2}\|\nabla u_i\|_{p_i}^{p_i} + \mu\big(C_1\|u_1\|_{p_1}^{p_1} - C_2\big) + \mu\big(C_1'\|u_2\|_{p_2}^{p_2} - C_2'\big) \\
&\quad - \sum_{i=1}^{2} k_i\big(1+\|u_i\|_{p_i}\big) - \big(\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_2\|_{p_2}^{p_2}\big) \\
&\quad - k_0\big(1+\|u_1\|_{p_1} + \|u_1\|_{p_1}^{p_1} + \|u_1\|_{p_2}^{p_2}\big) \\
&\quad - k_0'\big(1+\|u_2\|_{p_2} + \|u_2\|_{p_2}^{p_2} + \|u_2\|_{p_1}^{p_1}\big) \\
&\geq \big(\|\nabla u_1\|_{p_1}^{p_1} + \|\nabla u_2\|_{p_2}^{p_2}\big) + \mu C_1^*\big(\|u_1\|_{p_1}^{p_1} + \|u_2\|_{p_2}^{p_2}\big) - \mu\big(C_2 + C_2'\big) \\
&\quad - \sum_{i=1}^{2} k_i\big(1+\|u_i\|_{p_i}\big) - k_0^*\big(1+\|u_1\|_{p_1} + \|u_1\|_{p_1}^{p_1} + \|u_1\|_{p_2}^{p_2}\big),
\end{aligned} \tag{4.3}$$

where $k_0^* := \max\{k_0, k_0'\}$ and $C_1^* := \min\{C_1, C_1'\}$. Then invoking Moussaoui and Saoudi [20, Lemma 2.2], we can deduce that

$$\big\|\nabla(u_i\mathbb{1}_{\{\underline{u}_i<u_i<\overline{u}_i\}})\big\|_{p_i} \leq \tilde{C}_i \quad \text{for some } \tilde{C}_i > 0 \text{ independent of } u_i, i = 1,2.$$

Furthermore, for sufficiently large $\mu > 0$ such that $\mu C_1^* - k_0^* > 0$ and for every sequence $(u_n)_n$ in $\mathcal{W}$, inequality (4.3) implies

$$\langle \chi_\mu(u_n), u_n\rangle \to +\infty \quad \text{as } \|u_n\|_{\mathcal{W}} \to +\infty.$$

Therefore, since χ_μ is continuous, bounded, coercive, and pseudomonotone, invoking the pseudo-monotone operator theorem (see, e.g., Carl et al. [4, Theorem 2.99]), we get the existence of $u \in \mathcal{W}$ such that

$$\langle \chi_\mu(u_1,u_2),(\varphi_1,\varphi_2)\rangle = 0 \quad \text{for every } (\varphi_1,\varphi_2) \in \mathcal{W}. \tag{4.4}$$

Moreover, using Casas et al. [6, Theorem 3], we have

$$|\nabla u_1|^{p_1-2}\frac{\partial u_1}{\partial \eta} = 0 = |\nabla u_2|^{p_2-2}\frac{\partial u_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega.$$

Therefore we conclude that $u = (u_1,u_2) \in \mathcal{W}$ is a weak solution of (S_μ). This completes the proof of the theorem.

5 Subsupersolutions

The aim of this section is to construct pairs of sub- and supersolutions of system (S).

Theorem 5.1 *Assume that conditions* $(\mathbf{H_3})$, $(\mathbf{H_4})$, $(\mathbf{H_5})$, *and* $(\mathbf{H_6})$ *are satisfied. Then system* (S) *has a solution* $u = (u_1,u_2) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega}) \cap [\underline{u}_1,\overline{u}_1] \times [\underline{u}_2,\overline{u}_2]$ *for some* $\gamma \in (0,1)$.

Proof of Theorem 5.1 First, using Theorem 3.1, we can fix $\mu > 0$ sufficiently large such that system (S_μ) admits a pair of weak solutions $u = (u_1,u_2) \in \mathcal{W}$. It remains to verify that $u = (u_1,u_2) \in [\underline{u}_1,\overline{u}_1] \times [\underline{u}_2,\overline{u}_2]$. Here we give just the proof for $u_1 \in [\underline{u}_1,\overline{u}_1]$. A similar reasoning yields the second inequality. To this end, we set $(\varphi_1,\varphi_2) = ((u_1-\overline{u}_1)_+,0)$. By Lemma 3.3 and condition $(\mathbf{H_5})$, combined with equation (4.4), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_1|^{p_1-2}\nabla u_1 \nabla(u_1-\overline{u}_1)_+\,\mathrm{d}x + \int_\Omega \frac{|\nabla(\mathcal{T}u_1)|^{p_1}}{\mathcal{T}u_1+\delta_1}(u_1-\overline{u}_1)_+\,\mathrm{d}x \\
&\quad = \int_\Omega f\bigl(x,\mathcal{T}u_1,\mathcal{T}u_2,\nabla(\mathcal{T}u_1),\nabla(\mathcal{T}u_2)\bigr)(u_1-\overline{u}_1)_+\,\mathrm{d}x - \mu\int_\Omega b(x,u_1)(u_1-\overline{u}_1)_+\,\mathrm{d}x \\
&\quad = \int_\Omega f\bigl(x,\overline{u}_1,,\mathcal{T}u_2,\nabla\overline{u}_1,\nabla(\mathcal{T}u_2)\bigr)(u_1-\overline{u}_1)_+\,\mathrm{d}x - \mu\int_\Omega (u_1-\overline{u}_1)_+^{p_1}\,\mathrm{d}x \\
&\quad \le \int_\Omega |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1\nabla(u_1-\overline{u}_1)_+\,\mathrm{d}x + \int_\Omega \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}(u_1-\overline{u}_1)_+\,\mathrm{d}x - \mu\int_\Omega (u_1-\overline{u}_1)_+^{p_1}\,\mathrm{d}x.
\end{aligned}$$

Now, according to equation (3.1),

$$\int_\Omega \frac{|\nabla(\mathcal{T}u_1)|^{p_1}}{\mathcal{T}u_1+\delta_1}(u_1-\overline{u}_1)_+\,\mathrm{d}x = \int_\Omega \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}(u_1-\overline{u}_1)_+\,\mathrm{d}x,$$

so it follows that

$$\int_\Omega \bigl(|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1\bigr)\nabla(u_1-\overline{u}_1)_+\,\mathrm{d}x \le -\mu\int_\Omega (u_1-\overline{u}_1)_+^{p_1}\,\mathrm{d}x \le 0. \tag{5.1}$$

Hence it follows from equation (5.1), combined with the monotonicity of A_1, that $u_1 \le \overline{u}_1$. In the same way, to see that $\underline{u}_1 \le u_1$, we set $(\varphi_1,\varphi_2) = ((\underline{u}_1 - u_1)_+,0)$. So, $u = (u_1,u_2) \in [\underline{u}_1,\overline{u}_1] \times [\underline{u}_2,\overline{u}_2]$. Moreover, according to Miyajima et al. [15, Remark 8], we obtain that $u = (u_1,u_2) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and $\frac{\partial u_1}{\partial\eta} = \frac{\partial u_2}{\partial\eta} = 0$ on $\partial\Omega$. Therefore we have shown that $u = (u_1,u_2) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ is a solution of the system (S) within $[\underline{u}_1,\overline{u}_1] \times [\underline{u}_2,\overline{u}_2]$. $\square$

6 Nodal solutions

The objective of this section is to show the existence of nodal solutions of system (S). The proof is mostly based on finding pairs of sub- and supersolutions of system (S). To this end, first, recall from Candito et al. [3, Lemma 2] that $z_i \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$, $i = 1, 2$, for some $\gamma \in (0, 1)$ are the unique solutions of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta_{p_i} u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

which satisfies

$$\|z_i\|_{\mathcal{C}^{1,\gamma}(\overline{\Omega})} \leq L \quad \text{and} \quad \|\nabla z_i\|_\infty \leq \hat{L}, \tag{6.2}$$

$$ld(x) \leq z_i \leq Ld(x) \quad \text{in } \Omega, \qquad \frac{\partial z_i}{\partial \eta} < 0 \quad \text{on } \partial\Omega, \tag{6.3}$$

for certain constants $\hat{L}$, l, and L. Moreover, by the Minty–Browder theorem (see Brezis [1]), combined with the Lieberman regularity Theorem [12], we know that the Dirichlet problem

$$-\Delta_{p_i} u = \begin{cases} 1 & \text{if } x \in \Omega \backslash \overline{\Omega}_\tau, \\ -1 & \text{otherwise}, \end{cases} \qquad u = 0 \quad \text{on } \partial\Omega, \tag{6.4}$$

has a unique solution, denoted by $z_{i,\tau} \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$ for a given $0 < \tau < \operatorname{diam}(\Omega)$, satisfying

$$z_{i,\tau} \leq z_i \quad \text{in } \Omega, \tag{6.5}$$

$$\frac{\partial z_{i,\tau}}{\partial \eta} < \frac{1}{2} \frac{\partial z_i}{\partial \eta} < 0 \quad \text{on } \partial\Omega, \quad \text{and} \quad z_{i,\tau} \geq \frac{1}{2} z_i \quad \text{in } \Omega. \tag{6.6}$$

Now for a given $\tau > 0$, we define

$$\underline{u}_1 := \tau^{\frac{1}{p_1}} z_{1,\tau}^{\omega_1} - \tau, \qquad \underline{u}_2 := \tau^{\frac{1}{p_2}} z_{2,\tau}^{\omega_2} - \tau, \tag{6.7}$$

$$\overline{u}_1 := \tau^{-p_1} z_1^{\overline{\omega}_1} - \tau, \qquad \overline{u}_2 := \tau^{-p_2} z_2^{\overline{\omega}_2} - \tau, \tag{6.8}$$

where

$$\frac{\omega_i - 1}{\omega_i} > \frac{1}{p_i - 1} > \frac{\overline{\omega}_i - 1}{\overline{\omega}_i} \quad \text{with } \omega_i > \overline{\omega}_i > 1 \tag{6.9}$$

and

$$\overline{\omega_i} < 1 + p_i \left(1 - \frac{\max\{\alpha_i, \beta_i\}}{p_i - 1} \right). \tag{6.10}$$

According to equations (6.2)–(6.3), we have

$$\overline{u}_1 \leq \tau^{-p_1} (Ld)^{\overline{\omega}_1}, \qquad \overline{u}_2 \leq \tau^{-p_2} (Ld)^{\overline{\omega}_2} \tag{6.11}$$

$$\|\nabla\overline{u}_1\|_\infty \le \tau^{-p_1}\hat{L}_1, \qquad \|\nabla\overline{u}_2\|_\infty \le \tau^{-p_2}\hat{L}_2, \tag{6.12}$$

with $\hat{L}_i := \overline{\omega}_i L^{\overline{\omega}_i}$ for $i = 1,2$. Furthermore, on $\partial\Omega$, we have

$$\begin{cases} \frac{\partial\overline{u}_1}{\partial\eta} = \tau^{-p_1}\frac{\partial(z_1^{\overline{\omega}_1})}{\partial\eta} = \tau^{-p_1}\overline{\omega}_1 z_1^{\overline{\omega}_1-1}\frac{\partial z_1}{\partial\eta} = 0, \\ \frac{\partial\overline{u}_2}{\partial\eta} = \tau^{-p_2}\frac{\partial(z_2^{\overline{\omega}_2})}{\partial\eta} = \tau^{-p_2}\overline{\omega}_2 z_2^{\overline{\omega}_2-1}\frac{\partial z_2}{\partial\eta} = 0, \end{cases} \tag{6.13}$$

since z_i is a solution of the Dirichlet problem (6.1) for $\omega_i, \overline{\omega}_i > 1$, $i = 1,2$.

Now we will prove the following result.

Lemma 6.1 *For a sufficiently small $\tau > 0$, we have $\underline{u}_1 \le \overline{u}_1$ and $\underline{u}_2 \le \overline{u}_2$.*

Proof First, we show that $\underline{u}_1 \le \overline{u}_1$ in Ω. By a direct computation we obtain

$$\begin{aligned} \overline{u}_1(x) - \underline{u}_1(x) &= \left(\tau^{-p_1} z_1^{\bar{\omega}_1} - \tau\right) - \left(\tau^{\frac{1}{p_1}} z_{i,\tau}^{\omega_i} - \tau\right) \\ &\ge \tau^{-p_1} z_1^{\bar{\omega}_1} - \tau^{\frac{1}{p_1}} z_1^{\omega_1} = z^{\omega_1}\left(\tau^{-p_1} z_1^{\bar{\omega}_1-\omega_1} - \tau^{\frac{1}{p_1}}\right) \\ &\ge z_1^{\omega_1}\left(\tau^{-p_1}\left(cd(x)\right)^{\bar{\omega}_1-\omega_1} - \tau^{\frac{1}{p_1}}\right) \ge 0, \end{aligned}$$

since $\omega_1 > \overline{\omega}_1$ and $z_{1,\tau} \le z_1$ for every small enough $\tau < \operatorname{diam}(\Omega)$. Therefore $\underline{u}_1 \le \overline{u}_1$ in Ω. Finally, using a similar argument, we can obtain that $\underline{u}_2 \le \overline{u}_2$ in Ω. □

7 Proofs of main results

Proof of Theorem 1.1 First, we claim that equation (2.3) is satisfied by the pair of functions $(\overline{u}_1, \overline{u}_2)$ given by equation (6.8). To see this, pick $(u_1, u_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ within $[\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$ such that $\underline{u}_2 \le u_2 \le \overline{u}_2$, $\underline{u}_1 \le u_1 \le \overline{u}_1$. Now, in view of condition $(\mathbf{H_1})$, combined with equations (6.11) and (6.12), we have

$$\begin{aligned} \left|f_1(\cdot, \overline{u}_1, u_2, \nabla\overline{u}_1, \nabla u_2)\right| &\le M_1\left(1 + |\overline{u}_1|^{\alpha_1}\right)\left(1 + |\nabla\overline{u}_1|^{\beta_1}\right) \\ &\le M_1\left(1 + \left(\tau^{-p_1}(Ld)^{\overline{\omega}_1}\right)^{\alpha_1}\right)\left(1 + \left(\tau^{-p_1}\hat{L}\right)^{\beta_1}\right) \\ &\le 2M_1(C_1C_2)^{\alpha_1\beta_1}\tau^{-p_1\max\{\alpha_1,\beta_1\}} \\ &\le C\tau^{-p_1\max\{\alpha_1,\beta_1\}}, \end{aligned} \tag{7.1}$$

where $C := 2M_1(C_1C_2)^{\alpha_1\beta_1}$, and $\tau > 0$ is small enough. Using the same argument as in equation (7.1), we obtain

$$\left|f_2(\cdot, u_1, \overline{u}_2, \nabla u_1, \nabla\overline{u}_2)\right| \le C'\tau^{-p_2\max\{\alpha_2,\beta_2\}} \tag{7.2}$$

for some constant $C' > 0$ and for $\tau > 0$ small enough. Now, in view of equations (6.8) and (6.9), we have

$$-\Delta_{p_1}\overline{u}_1 + \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1 + \delta_1} = \tau^{-p_1(p_1-1)}\left(-\Delta_{p_1} z_1^{\bar{\omega}_1} + \frac{|\nabla z_1^{\bar{\omega}_1}|_1^p}{z_1^{\bar{\omega}_1}}\right). \tag{7.3}$$

On the other hand, by a direct computation we get

$$\begin{aligned}
-\Delta_{p_1} z_1^{\bar{\omega}_1} + \frac{|\nabla z_1^{\bar{\omega}_1}|^{p_1}}{z_1^{\bar{\omega}_1}} &= \bar{\omega}_1^{p_1-1}\left(1-(\bar{\omega}_1-1)(p_1-1)\frac{|\nabla z_1|^{p_1}}{z_1}\right)z_1^{(\bar{\omega}_1-1)(p_1-1)} \\
&\quad + \bar{\omega_1}^{p_1}\frac{z_1^{(\bar{\omega}_1-1)p_1}|\nabla z_1|^{p_1}}{z_1^{\bar{\omega}_1}} \\
&= \bar{\omega}^{p_1-1}\left(1-(\bar{\omega}_1-1)(p_1-1)\frac{|\nabla z_1|^{p_1}}{z_1}\right)z_1^{(\bar{\omega}_1-1)(p_1-1)} \\
&\quad + \bar{\omega}_1^{p_1} z_1^{(\bar{\omega}_1-1)(p_1-1)}\frac{z_1^{\bar{\omega}_1-1}|\nabla z_1|^{p_1}}{z_1^{\bar{\omega}_1}} \\
&= \bar{\omega}_1^{p_1-1}\left[1+\bar{\omega}_1\left(1-\frac{(\bar{\omega}_1-1)(p_1-1)}{\bar{\omega}_1}\right)\frac{|\nabla z_1|^{p_1}}{z_1}\right]z_1^{(\bar{\omega}_1-1)(p_1-1)}. \quad (7.4)
\end{aligned}$$

Invoking equations (7.3) and (7.4), it follows that

$$\begin{aligned}
&-\Delta_{p_1}\overline{u}_1 + \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1} \\
&= \tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1-1}\left[1+\bar{\omega}_1\left(1-\frac{(\bar{\omega}_1-1)(p_1-1)}{\bar{\omega}_1}\right)\frac{|\nabla z_1|^{p_1}}{z_1}\right]z_1^{(\bar{\omega}_1-1)(p_1-1)} \\
&\geq \tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1-1}\begin{cases} z_1^{(\bar{\omega}_1-1)(p_1-1)} & \text{in } \Omega\backslash\overline{\Omega}_\tau, \\ \bar{\omega}_1(1-\frac{(\bar{\omega}_1-1)(p_1-1)}{\bar{\omega}_1})z_1^{(\bar{\omega}_1-1)(p_1-1)-1}|\nabla z_1|^{p_1} & \text{in } \Omega_\tau. \end{cases}
\end{aligned}$$

Moreover, using equation (6.10) and decreasing τ if necessary, we have

$$\begin{aligned}
\tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1-1}z_1^{(\bar{\omega}_1-1)(p_1-1)} &\geq \tau^{-p_1(p_1-1)}\bar{\omega}^{p_1-1}\left(c^{-1}d(x)\right)^{(\bar{\omega}_1-1)(p_1-1)} \\
&\geq \tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1-1}\left(c^{-1}\tau\right)^{(\bar{\omega}-1)(p_1-1)} \\
&= \tau^{(\bar{\omega}_1-1-p_1)(p_1-1)}\bar{\omega}_1^{p_1-1}c^{-(\bar{\omega}_1-1)(p_1-1)} \\
&\geq \tau^{-p_1\max\{\alpha_1,\beta_1\}} \quad \text{in } \Omega\backslash\overline{\Omega}_\tau. \quad (7.5)
\end{aligned}$$

Finally, combining equations (7.1) and (7.5), we obtain

$$-\Delta_{p_1}\overline{u}_1 + \frac{|\nabla \overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1} \geq f_1(\cdot,\overline{u}_1,u_2,\nabla\overline{u}_1,\nabla u_2) \quad \text{in } \Omega\backslash\overline{\Omega}_\tau. \quad (7.6)$$

Now pick any $x\in\Omega_\tau$. By equations (6.3) and (6.9) we can find a constant $\beta>0$ such that

$$\left(1-\frac{(\bar{\omega}_1-1)(p_1-1)}{\bar{\omega}_1}\right)|\nabla z_1|>\beta \quad \text{in } \Omega_\tau.$$

By equations (6.3) and (6.9) we have

$$\begin{aligned}
&\tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1}\left(1-\frac{(\bar{\omega}_1-1)(p_1-1)}{\bar{\omega}_1}\right)z_1^{(\bar{\omega}_1-1)(p_1-1)-1}|\nabla z_1|^{p_1} \\
&\quad \geq \tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1}\left(Ld(x)\right)^{(\bar{\omega}_1-1)(p_1-1)-1}\bar{\mu}^{p}
\end{aligned}$$

$$\geq \tau^{-p_1(p_1-1)}\bar{\omega}_1^{p_1}(L\tau)^{(\bar{\omega}_1-1)(p_1-1)-1}\beta^{p_1}$$
$$\geq \tau^{-p_1\max\{\alpha_1,\beta_1\}} \quad \text{in } \Omega_\tau.$$

Therefore, for $\tau > 0$ sufficiently small, we obtain

$$-\Delta_{p_1}\overline{u}_1 + \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1 + \delta_1} \geq f_1(\cdot,\overline{u}_1,u_2,\nabla\overline{u}_1,\nabla u_2) \quad \text{in } \Omega_\tau. \tag{7.7}$$

Combining equations (7.6) and (7.7), we get

$$-\Delta_{p_1}\overline{u}_1 + \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1 + \delta_1} \geq f_1(\cdot,\overline{u}_1,u_2,\nabla\overline{u}_1,\nabla u_2) \quad \text{in } \Omega. \tag{7.8}$$

A similar argument yields

$$-\Delta_{p_2}\overline{u}_2 + \frac{|\nabla\overline{u}_2|^{p_2}}{\overline{u}_2 + \delta_2} \geq f_2(\cdot,u_1,\overline{u}_2,\nabla u_1,\nabla\overline{u}_2) \quad \text{in } \Omega. \tag{7.9}$$

Now test equation (7.8), equation (7.9) with $(\varphi_1,\varphi_2) \in W_b^{1,p_1}(\Omega) \times W_b^{1,p_2}(\Omega)$, $\varphi \geq 0$ a.e. in Ω, and equation (6.13) yield

$$\int_\Omega |\nabla\overline{u}_1|^{p_1-2}\nabla\overline{u}_1\nabla\varphi_1\,dx + \int_\Omega \frac{|\nabla\overline{u}_1|^{p_1}}{\overline{u}_1+\delta_1}\varphi_1\,dx - \left\langle \frac{\partial\overline{u}}{\partial\eta_{p_1}},\gamma_0(\varphi_1)\right\rangle_{\partial\Omega}$$
$$\geq \int_\Omega f_1(\cdot,\overline{u}_1,u_2,\nabla\overline{u}_1,\nabla u_2)\varphi_1\,dx,$$
$$\int_\Omega |\nabla\overline{u}_2|^{p_2-2}\nabla\overline{u}_2\nabla\varphi_2\,dx + \int_\Omega \frac{|\nabla\overline{u}_2|^{p_2}}{\overline{u}_2+\delta_2}\varphi_2\,dx - \left\langle \frac{\partial\overline{u}}{\partial\eta_{p_2}},\gamma_0(\varphi_2)\right\rangle_{\partial\Omega}$$
$$\geq \int_\Omega f_2(\cdot,u_1,\overline{u}_2,\nabla u_1,\nabla\overline{u}_2)\varphi_2\,dx,$$

where γ_0 is the trace operator on $\partial\Omega$,

$$\frac{\partial w}{\partial\eta_{p_i}} := |\nabla w|^{p_i-2}\frac{\partial w}{\partial\eta} \quad \text{for every } w \in W^{1,p_i}(\Omega)\cap C^1(\overline{\Omega}), \tag{7.10}$$

and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ is the duality brackets for the pair

$$\big(W^{1/p_i',p_i}(\partial\Omega), W^{-1/p_i',p_i'}(\partial\Omega)\big).$$

The proof of the claim is now completed.

Next, we show that equation (2.2) is satisfied by the pair of functions $(\underline{u}_1,\underline{u}_2)$ given by equation (6.7). A direct computation yields

$$-\Delta_{p_1}z_{1,\tau}^{\omega_1} + \frac{|\nabla z1,\tau^{\omega_1}|^{p_1}}{z_{1,\tau}^{\omega_1}} = \omega_1^{p_1-1}\left(1-(\omega_1-1)(p_1-1)\frac{|\nabla z_{1,\tau}|^{p_1}}{z_{1,\tau}}\right)z_{1,\tau}^{(\omega_1-1)(p_1-1)}$$
$$+ \omega_1^{p_1}\frac{z_{1,\tau}^{(\omega_1-1)p_1}|\nabla z_{1,\tau}|^{p_1}}{z_{1,\tau}^{\omega_1}}$$
$$= \omega_1^{p_1-1}\left[1+\omega_1\left(1-\frac{(\omega_1-1)(p_1-1)}{\omega_1}\right)\frac{|\nabla z_{1,\tau}|^{p_1}}{z_{1,\tau}}\right]z_{1,\tau}^{(\omega_1-1)(p_1-1)}$$

in $\Omega\backslash\overline{\Omega}_\tau$. Similarly, it follows that

$$-\Delta_{p_1} z_{1,\tau}^{\omega_1} + \frac{|\nabla z_{1,\tau}^{\omega_1}|^{p_1}}{z_{1,\tau}^{\omega_1}} = \omega_1^{p_1-1}\left[-1+\omega_1\left(1-\frac{(\omega_1-1)(p_1-1)}{\omega}\right)\frac{|\nabla z_{1,\tau}|^{p_1}}{z_{1,\tau}}\right] z_{1,\tau}^{(\omega_1-1)(p_1-1)}$$

in Ω_τ. In fact, by equations (6.7) and (6.9) we have

$$\begin{aligned} -\Delta_{p_1}\underline{u}_1 + \frac{|\nabla \underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1} &= \tau^{\frac{1}{p_1'}}\left(-\Delta_{p_1} z_{1,\tau}^{\omega_1} + \frac{|\nabla z_{1,\tau}^{\omega_1}|^{p_1}}{z_{1,\tau}^{\omega_1}}\right) \\ &\le \begin{cases} \tau^{\frac{1}{p_1'}}\omega_1^{p_1-1} z_{1,\tau}^{(\omega_1-1)(p_1-1)} & \text{in } \Omega\backslash\overline{\Omega}_\tau, \\ 0 & \text{in } \Omega_\tau. \end{cases} \end{aligned} \tag{7.11}$$

In view of equations (6.2)–(6.5), choosing an appropriate constant m_1 in $(\mathbf{H_2})$, we have

$$m_1 > \tau^{\frac{1}{p_1'}}\omega_1^{p_1-1} L^{(\omega_1-1)(p_1-1)} \quad \text{for } \tau>0 \text{ small enough.} \tag{7.12}$$

Combining equations (7.11) and (7.12), we arrive at

$$-\Delta_{p_1}\underline{u}_1 + \frac{|\nabla \underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1} \le f_1(\cdot,\underline{u}_1,u_2,\nabla\underline{u}_1,\nabla u_2). \tag{7.13}$$

Using a similar argument, we obtain

$$-\Delta_{p_2}\underline{u}_2 + \frac{|\nabla \underline{u}_2|^{p_2}}{\underline{u}_2+\delta_2} \le f_2(\cdot,u_1,\underline{u}_2,\nabla u_1,\underline{u}_2). \tag{7.14}$$

Finally, by test equations (7.13) and (7.14) with $(\varphi_1,\varphi_2)\in W_b^{1,p_1}(\Omega)\times W_b^{1,p_2}(\Omega)$, where $\varphi_1,\varphi_2\ge 0$ a.e. in Ω, equation (6.13), and the Green formula [6] we obtain

$$\begin{aligned} &\int_\Omega |\nabla\underline{u}_1|^{p_1-2}\nabla\underline{u}_1\nabla\varphi_1\,\mathrm{d}x + \int_\Omega \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1}\varphi_1\,\mathrm{d}x \\ &\quad\le \int_\Omega |\nabla\underline{u}_1|^{p_1-2}\nabla\underline{u}_1\nabla\varphi_1\,\mathrm{d}x - \left\langle\frac{\partial\underline{u}_1}{\partial\eta_{p_1}},\gamma_0(\varphi_1)\right\rangle_{\partial\Omega} + \int_\Omega \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1}\varphi_1\,\mathrm{d}x \\ &\quad= \int_\Omega\left(-\Delta_{p_1}\underline{u}_1 + \frac{|\nabla\underline{u}_1|^{p_1}}{\underline{u}_1+\delta_1}\right)\varphi_1\,\mathrm{d}x \\ &\quad\le \int_\Omega f_1(\cdot,\underline{u}_1,u_2,\nabla\underline{u}_1,\nabla u_2)\varphi_1\,\mathrm{d}x, \\ &\int_\Omega |\nabla\underline{u}_2|^{p_2-2}\nabla\underline{u}_2\nabla\varphi_2\,\mathrm{d}x + \int_\Omega \frac{|\nabla\underline{u}_2|^{p_2}}{\underline{u}_2+\delta_2}\varphi_2\,\mathrm{d}x \\ &\quad\le \int_\Omega |\nabla\underline{u}_2|^{p_2-2}\nabla\underline{u}_2\nabla\varphi_2\,\mathrm{d}x - \left\langle\frac{\partial\underline{u}_2}{\partial\eta_{p_2}},\gamma_0(\varphi_2)\right\rangle_{\partial\Omega} + \int_\Omega \frac{|\nabla\underline{u}_2|^{p_2}}{\underline{u}_2+\delta_2}\varphi_2\,\mathrm{d}x \\ &\quad= \int_\Omega\left(-\Delta_{p_2}\underline{u}_2 + \frac{|\nabla\underline{u}_2|^{p_2}}{\underline{u}_2+\delta_2}\right)\varphi_2\,\mathrm{d}x \\ &\quad\le \int_\Omega f_2(\cdot,u_1\underline{u}_2,\nabla u_1,\nabla\underline{u}_2)\varphi_2\,\mathrm{d}x, \end{aligned}$$

since $\gamma_0(\varphi_1), \gamma_0(\varphi_2) \geq 0$ whenever $(\varphi_1, \varphi_2) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega)$ (for more detail, see Carl et al. [4, p. 35]).

Consequently, $(\underline{u}_1, \underline{u}_2)$ and $(\overline{u}_1, \overline{u}_2)$ satisfy equations (2.4) and (2.5). Therefore we can apply Theorem 5.1 to obtain the existence of a solution $(u_0, u_0') \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ of system (S) satisfying

$$\underline{u}_1 \leq u_0 \leq \overline{u}_1, \qquad \underline{u}_2 \leq u_0' \leq \overline{u}_2. \tag{7.15}$$

Furthermore, (u_0, u_0') is a nodal solution. Indeed, combining equations (6.3), (6.7), and (6.8), we arrive at

$$\begin{aligned} \overline{u}_1 &= \tau^{-p_1} z^{\overline{\omega}_1} - \tau \leq \tau^{-p_1} \big(Ld(x)\big)^{\overline{\omega}_1} - \tau, \\ \overline{u}_2 &= \tau^{-p_2} z^{\overline{\omega}_2} - \tau \leq \tau^{-p_2} \big(Ld(x)\big)^{\overline{\omega}_2} - \tau, \end{aligned}$$

which implies that

$$\max\big\{\overline{u}_1(x), \overline{u}_2(x)\big\} < 0, \quad \text{provided that } d(x) < L^{-1} \tau^{\frac{p_i+1}{\overline{\omega}_i}} \tag{7.16}$$

for $i = 1, 2$. Combining equations (6.3), (6.7), and (6.8) yields

$$\begin{aligned} \underline{u}_1 &= \tau^{\frac{1}{p_1}} z_{1,\tau}^{\omega_1} - \tau \geq \tau^{\frac{1}{p_1}} \big(ld(x)\big)^{\omega_1} - \tau, \\ \underline{u}_2 &= \tau^{\frac{1}{p_2}} z_{2,\tau}^{\omega_2} - \tau \geq \tau^{\frac{1}{p_2}} \big(ld(x)\big)^{\omega_2} - \tau, \end{aligned}$$

and hence

$$\min\big\{\underline{u}_1(x), \underline{u}_2(x)\big\} > 0 \quad \text{when } d(x) > l\tau^{\frac{1}{\omega_i p_i'}} \tag{7.17}$$

for $i = 1, 2$. The conclusion now follows from equations (7.16) and (7.17). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 First, using the same notation as in equations (6.7) and (6.8) and applying the same argument as in the proof of Theorem 1.1, we can ensure that $(\underline{u}_+, \underline{u}^+)$ and $(\overline{u}_+, \overline{u}^+)$ satisfy equations (2.4) and (2.5). Therefore, invoking Theorem 5.1, we obtain the existence of a solution $(u_+, u^+) \in \mathcal{C}^{1,\gamma}(\overline{\Omega}) \times \mathcal{C}^{1,\gamma}(\overline{\Omega})$ with the following properties:

$$\begin{aligned} &u_0 \leq u_+ \leq \overline{u}_+ \quad \text{and} \quad u_+ \geq 0 \quad \text{on } \Omega, \\ &u_0' \leq u^+ \leq \overline{u}^+ \quad \text{and} \quad u^+ \geq 0 \quad \text{on } \Omega. \end{aligned}$$

Finally, using equation (6.8), we can easily deduce that $u_+(x)$ and $u^+(x)$ are zero as $d(x) \to 0$. This completes the proof of Theorem 1.2. □

Acknowledgements
The authors gratefully acknowledge comments and suggestions by the reviewers.

Funding
Repovš was supported by the Slovenian Research and Innovation Agency grants P1-0292, J1-4031, J1-4001, N1-0278, N1-0114, and N1-0083.

Data availability
Not applicable. Moreover, all of the material is owned by the authors and/or no permissions are required.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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Received: 31 October 2023 Accepted: 18 December 2023 Published online: 02 January 2024

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