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The automorphism group of the zero-divisor digraph of matrices over an antiring

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Abstract

We determine the automorphism group of the zero-divisor digraph of the semiring of matrices over an antinegative commutative semiring with a finite number of zero-divisors.

Keywords: Automorphism group of a graph, zero-divisor graph, semiring.

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1 Introduction

In recent years, the zero-divisor graphs of various algebraic structures have received a lot of attention, since they are a useful tool for revealing the algebraic properties through their graph-theoretical properties. In 1988, Beck [3] first introduced the concept of the zero-divisor graph of a commutative ring. In 1999, Anderson and Livingston [1] made a slightly different definition of the zero-divisor graph in order to be able to investigate the zero-divisor structure of commutative rings. In 2002, Redmond [15] extended this definition to also include non-commutative rings. Different authors then further extended this concept to semigroups [6], nearrings [4] and semirings [8].

Automorphisms of graphs play an important role both in graph theory and in algebra, and finding the automorphism group of certain graphs is often very difficult. Recently, a

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lot of effort has been made to determine the automorphism group of various zero-divisor graphs. In [1], Anderson and Livingston proved that $\mathrm{Aut}(\Gamma(\mathbb{Z}_n))$ is a direct product of symmetric groups for $n \geq 4$ a non-prime integer. In the non-commutative case, the case of matrix rings and semirings is especially interesting. Thus, it was shown in [10] that, when p is a prime, $\mathrm{Aut}(\Gamma(\mathrm{M}_2(\mathbb{Z}_p)))$ is isomorphic to $\mathrm{Sym}(p+1)$, the symmetric group of degree $p+1$. More generally, it was proved in [13], that $\mathrm{Aut}(\Gamma(\mathrm{M}_2(\mathbb{F}_q))) \cong \mathrm{Sym}(q+1)$. In [18], the authors determined the automorphism group of the zero-divisor graph of all rank one upper triangular matrices over a finite field, and in [16] they determined the automorphism group of the zero-divisor graph of the matrix ring of all upper triangular matrices over a finite field. Recently, the automorphism group of the zero-divisor graph of the complete matrix ring of matrices over a finite field has been found independently in [17] and [20].

In this paper, we study the zero-divisor graph of matrices over commutative semirings. The theory of semirings has many applications in optimization theory, automatic control, models of discrete event networks and graph theory (see e.g. [2, 5, 12, 19]) and the zero-divisor graphs of semirings were recently studied in [9, 7, 14]. For an extensive theory of semirings, we refer the reader to [11]. There are many natural examples of commutative semirings, for example, the set of nonnegative integers (or reals) with the usual operations of addition and multiplication. Other examples include distributive lattices, tropical semirings, dioïds, fuzzy algebras, inclines and bottleneck algebras.

The theory of matrices over semirings differs quite substantially from the one over rings, so the methods we use are necessarily distinct from those used in the ring setting. The main result of this paper is the determination of the automorphism group of the zero-divisor digraph of a semiring of matrices over an antinegative commutative semiring with a finite number of zero-divisors (see Theorem 3.12).

2 Definitions and preliminaries

2.1 Digraphs

A *digraph* Γ consists of a set $\mathrm{V}(\Gamma)$ of *vertices*, together with a binary relation $\to$ on $\mathrm{V}(\Gamma)$. An *automorphism* σ of Γ is a permutation of $\mathrm{V}(\Gamma)$ such that $u \to v \iff \sigma(u) \to \sigma(v)$. The automorphisms of Γ form its *automorphism group* $\mathrm{Aut}(\Gamma)$.

Let Γ be a digraph and let $v \in \mathrm{V}(\Gamma)$. We write $\mathrm{N}^-(v) = \{u \in \mathrm{V}(\Gamma) \colon u \to v\}$ and $\mathrm{N}^+(v) = \{u \in \mathrm{V}(\Gamma) \colon v \to u\}$. If, for $u, v \in \mathrm{V}(\Gamma)$, we have $\mathrm{N}^-(u) = \mathrm{N}^-(v)$ and $\mathrm{N}^+(u) = \mathrm{N}^+(v)$, then we say u and v are *twin* vertices. The relation $\sim$ on $\mathrm{V}(\Gamma)$, defined by $u \sim v$ if and only if u and v are twin vertices, is clearly an equivalence relation preserved by $\mathrm{Aut}(\Gamma)$. For $v \in \mathrm{V}(\Gamma)$, we shall denote by $\overline{v}$ the $\sim$-equivalence class of v. Let $\overline{\Gamma}$ be the graph with these equivalence classes as vertices and $\overline{u} \to_{\overline{\Gamma}} \overline{v}$ if and only if $u \to_\Gamma v$. For $\sigma \in \mathrm{Aut}(\Gamma)$ we denote by $\overline{\sigma}$ the induced automorphism of $\overline{\Gamma}$. An automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ is called *regular* if $\overline{\sigma}$ is the identity map.

2.2 Semirings

A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0, and $(S, \cdot)$ is a semigroup. Moreover, the operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S.

A semiring S is *commutative* if $ab = ba$ for all $a, b \in S$, and *antinegative* if, for all $a, b \in S$, $a + b = 0$ implies that $a = 0$ or $b = 0$. Antinegative semirings are also called *zerosum-free semirings* or *antirings*. The smallest nontrivial example of an antiring is the *Boolean antiring* $\mathbb{B} = \{0, 1\}$ with addition and multiplication defined so that $1 + 1 = 1 \cdot 1 = 1$.

Let S be a semiring. For $x \in S$, we define the *left and right annihilators in* S by $\mathrm{Ann}_L(x) = \{y \in S \colon yx = 0\}$ and $\mathrm{Ann}_R(x) = \{y \in S \colon xy = 0\}$. If S is commutative, we simply write $\mathrm{Ann}(x)$ for $\mathrm{Ann}_L(x) = \mathrm{Ann}_R(x)$. We denote by $\mathrm{Z}(S)$ the set of *zero-divisors* of S, that is $\mathrm{Z}(S) = \{x \in S \colon \exists y \in S \setminus \{0\}$ such that $xy = 0$ or $yx = 0\}$. The *zero-divisor digraph* $\Gamma(S)$ of S is the digraph with vertex-set S and $u \to v$ if and only if $uv = 0$.

It is easy to see that if $n \geq 1$ and S is a semiring, then the set $\mathrm{M}_n(S)$ of $n \times n$ matrices forms a semiring with respect to matrix addition and multiplication. If S is antinegative, then so is $\mathrm{M}_n(S)$. If S has an identity 1, let $E_{ij} \in \mathrm{M}_n(S)$ with entry 1 in position (i, j), and 0 elsewhere. For $s \in S$, define $sE_{ij} \in \mathrm{M}_n(S)$ as the matrix with entry s in position (i, j), and 0 elsewhere.

3 The automorphisms of the zero-divisor digraph

The following fact will be used repeatedly.

Lemma 3.1. *Let S be a semiring. If $A, B \in S$ and $\sigma \in \mathrm{Aut}(\Gamma(S))$, then*

$$\sigma(\mathrm{Ann}_L(A)) = \mathrm{Ann}_L(\sigma(A)) \quad \textit{and} \quad \sigma(\mathrm{Ann}_R(A)) = \mathrm{Ann}_R(\sigma(A)).$$

Proof. We have

$$\begin{aligned} X \in \sigma(\mathrm{Ann}_L(A)) &\Longleftrightarrow \sigma^{-1}(X) \in \mathrm{Ann}_L(A) \\ &\Longleftrightarrow \sigma^{-1}(X)A = 0 \\ &\Longleftrightarrow X\sigma(A) = 0 \\ &\Longleftrightarrow X \in \mathrm{Ann}_L(\sigma(A)). \end{aligned}$$

The proof of the second part is analogous. □

Lemma 3.2. *Let S be an antiring and let $\Gamma = \Gamma(S)$. If $A, B \in S$ and $\sigma \in \mathrm{Aut}(\Gamma)$, then $\sigma(A + B)$ and $\sigma(A) + \sigma(B)$ are twin vertices and, in particular, $\overline{\sigma(A + B)} = \overline{\sigma(A) + \sigma(B)}$.*

Proof. Using antinegativity, we have

$$\begin{aligned} X \in \mathrm{Ann}_L(\sigma(A + B)) &\Longleftrightarrow X\sigma(A + B) = 0 \\ &\Longleftrightarrow \sigma^{-1}(X)(A + B) = 0 \\ &\Longleftrightarrow \sigma^{-1}(X)A = \sigma^{-1}(X)B = 0 \\ &\Longleftrightarrow X\sigma(A) = X\sigma(B) = 0 \\ &\Longleftrightarrow X(\sigma(A) + \sigma(B)) = 0 \\ &\Longleftrightarrow X \in \mathrm{Ann}_L(\sigma(A) + \sigma(B)). \end{aligned}$$

We have proved that $\mathrm{Ann}_L(\sigma(A+B)) = \mathrm{Ann}_L(\sigma(A)+\sigma(B))$. An analogous proof yields $\mathrm{Ann}_R(\sigma(A+B)) = \mathrm{Ann}_R(\sigma(A)+\sigma(B))$. This implies that $\sigma(A+B)$ and $\sigma(A)+\sigma(B)$ are twin vertices. □

Definition 3.3. Let S be a commutative semiring, let $n \in \mathbb{N}$ and let $A \in \mathrm{M}_n(S)$ with (i,j) entry a_{ij}. For every $i,j \in \{1,\dots,n\}$, we define $C_i(A) = \bigcap_{k=1}^n \mathrm{Ann}(a_{ki})$ and $R_j(A) = \bigcap_{k=1}^n \mathrm{Ann}(a_{jk})$. Let $\mathcal{A}_R(A) := (C_1(A),\dots,C_n(A)) \in \mathcal{P}(S)^n$ and $\mathcal{A}_L(A) := (R_1(A),\dots,R_n(A)) \in \mathcal{P}(S)^n$, where $\mathcal{P}(S)$ denotes the power set of S.

The next theorem characterizes the twin vertices of $\Gamma(\mathrm{M}_n(S))$.

Theorem 3.4. *Let S be a commutative antiring, let $n \in \mathbb{N}$ and let $A, B \in \mathrm{M}_n(S)$. Then A and B are twin vertices of $\Gamma(\mathrm{M}_n(S))$ if and only if $\mathcal{A}_L(A) = \mathcal{A}_L(B)$ and $\mathcal{A}_R(A) = \mathcal{A}_R(B)$.*

Proof. Let a_{ij} and b_{ij} be the (i,j) entry of A and B, respectively. Suppose first that A and B are twin vertices of $\Gamma(\mathrm{M}_n(S))$ and assume that $\mathcal{A}_R(A) \neq \mathcal{A}_R(B)$. This implies that, for some $i \in \{1,\dots,n\}$, we have $C_i(A) \neq C_i(B)$. Swapping the role of A and B if necessary, there exists $s \in S$ such that $s \in C_i(A)$ and $s \notin C_i(B)$. Therefore, there exists $k \in \{1,\dots,n\}$ such that $s \notin \mathrm{Ann}(b_{ki})$. Now, let $C = sE_{ik} \in \mathrm{M}_n(S)$ and observe that $AC = 0$ but $BC \neq 0$, so $\mathrm{N}^+(A) \neq \mathrm{N}^+(B)$, which is a contradiction with the fact that A and B are twin vertices. We have thus proved that $\mathcal{A}_R(A) = \mathcal{A}_R(B)$. A similar argument yields that $\mathcal{A}_L(A) = \mathcal{A}_L(B)$.

Conversely, assume now that $\mathcal{A}_L(A) = \mathcal{A}_L(B)$ and $\mathcal{A}_R(A) = \mathcal{A}_R(B)$. Suppose there exists $X \in \mathrm{M}_n(S)$ such that $AX = 0$. Therefore, for all $i,j \in \{1,\dots,n\}$ we have $\sum_{k=1}^n a_{ik}x_{kj} = 0$. Since S is an antiring, this further implies that $a_{ik}x_{kj} = 0$ for all $i,j,k \in \{1,\dots,n\}$. So, $x_{kj} \in \mathrm{Ann}(a_{ik})$ and therefore $x_{kj} \in C_k(A) = C_k(B)$ for all $k \in \{1,\dots,n\}$. Thus, for all $i,j,k \in \{1,\dots,n\}$, we have $x_{kj} \in \mathrm{Ann}(b_{ik})$. This yields $b_{ik}x_{kj} = 0$ for all $i,j,k \in \{1,\dots,n\}$, so $BX = 0$. Thus, we have proved that $\mathrm{N}^+(A) \subseteq \mathrm{N}^+(B)$. By swapping the roles of A in B we also get $\mathrm{N}^+(B) \subseteq \mathrm{N}^+(A)$, so $\mathrm{N}^+(A) = \mathrm{N}^+(B)$. A similar argument yields that $\mathrm{N}^-(A) = \mathrm{N}^-(B)$, thus A and B are twin vertices. □

Definition 3.5. Let S be a commutative semiring and let $\alpha \in S \setminus \mathrm{Z}(S)$. We say that $\alpha = e_1 + e_2 + \cdots + e_s$ such that $e_i \neq 0$ for all i and $e_ie_j = 0$ for all $i \neq j$ is *a decomposition of α of length s*. The *length* $\ell(\alpha)$ of α is the supremum of the length of a decomposition of α (note that $\ell(\alpha)$ can be infinite). We say that α is of *maximal length* if $\ell(\alpha) \geq \ell(\beta)$ for all $\beta \in S \setminus \mathrm{Z}(S)$.

A semiring S is *decomposable* if $S \setminus Z(S)$ contains an element of length at least 2, otherwise it is *indecomposable*.

Lemma 3.6. *Let S be a commutative antiring and let $\alpha \in S \setminus \mathrm{Z}(S)$ be of finite maximal length s with decomposition $\alpha = e_1 + e_2 + \cdots + e_s$. Then, for every $i \in \{1,\dots,s\}$, the subsemiring e_iS is indecomposable.*

Proof. Suppose that e_iS is decomposable for some $i \in \{1,\dots,s\}$, say $i = 1$ without loss of generality. By definition, there exists $e_1w \in e_1S \setminus \mathrm{Z}(e_1S)$ such that $e_1w = f_1 + f_2$, where $f_1, f_2 \in e_1S \setminus \{0\}$ and $f_1f_2 = 0$. For all $j \neq 1$, we have $e_je_1w = 0$ and thus $e_jf_1 = e_jf_2 = 0$ by antinegativity. Let $\beta = e_1w + e_2 + \cdots + e_s$.

Suppose that $\beta x = 0$ for some $x \in S$. By antinegativity, we have $(e_1w)(e_1x) = 0$ and $e_2x = \cdots = e_sx = 0$. Since e_1w is not a zero-divisor in e_1S this implies that $e_1x = 0$ and therefore also $\alpha x = 0$. However, α is not a zero-divisor, so we can conclude that $x = 0$.

This shows that $\beta = f_1 + f_2 + e_2 + \cdots + e_s$ is not a zero-divisor in S, which is a contradiction with the maximal length of α. □

We shall investigate commutative antirings with identity where 1 is an element of finite maximal length. The next lemma shows that in this case, we can study the automorphisms of the zero-divisor digraph of the matrix ring componentwise.

Lemma 3.7. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \dots + e_s$. Let $n \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\Gamma(\mathrm{M}_n(S)))$. Then there exists $\omega \in \mathrm{Sym}(s)$ such that, for every $r \in \{1, \dots, s\}$, we have $\sigma(e_r \mathrm{M}_n(S)) = e_{\omega(r)} \mathrm{M}_n(S)$.*

Proof. Let $r \in \{1, \dots, s\}$, let $i, j \in \{1, \dots, n\}$ and let $B = \sigma(e_r E_{ij})$. So, $\overline{\sigma(e_r E_{ij})} = \overline{\sum_{k=1}^s e_k B}$. By Lemma 3.2, we have $\overline{e_r E_{ij}} = \overline{\sum_{k=1}^s \sigma^{-1}(e_k B)}$. Since S is antinegative, for every $k \in \{1, \dots, s\}$, there exists $f_k \in S$ such that $\sigma^{-1}(e_k B) = f_k E_{ij}$. Since $f_k = f_k(e_1 + e_2 + \dots + e_s) = f_k e_r \in e_r S$, we have $\sum_{k=1}^s f_k = e_r z$ for $z = \sum_{k=1}^s f_k$. Observe that $\overline{e_r E_{ij}} = \overline{z E_{ij}}$ yields $z \in S \setminus \mathrm{Z}(S)$.

Let $k, k' \in \{1, \dots, s\}$. We have $\mathrm{Ann}_L(f_k E_{ij}), \mathrm{Ann}_L(f_{k'} E_{ij}) \subseteq \mathrm{Ann}_L(f_k f_{k'} E_{ij})$ hence $\mathrm{Ann}_L(e_k B), \mathrm{Ann}_L(e_{k'} B) \subseteq \mathrm{Ann}_L(\sigma(f_k f_{k'} E_{ij}))$. If $k \neq k'$, then $\mathrm{M}_n(S) \subseteq \mathrm{Ann}_L(e_k B) + \mathrm{Ann}_L(e_{k'} B)$, which is possible only if $f_k f_{k'} = 0$. We have shown that $f_k f_{k'} = 0$ for every $k, k' \in \{1, \dots, s\}$ with $k \neq k'$.

It follows that

$$z = (e_1 + e_2 + \dots + e_s) z = \sum_{i \neq r} e_i z + \sum_{k=1}^s f_k$$

is a decomposition of z. Since $z \notin \mathrm{Z}(S)$, $e_i z \neq 0$ and, since $\ell(z) \leq \ell(1) = s$, it follows that all but exactly one of the f_k's are 0. This implies that all but one of the $e_k B$'s are 0 and there exists $k \in \{1, \dots, s\}$ such that $\overline{\sigma(e_r E_{ij})} = \overline{e_k B}$. This shows the existence of a permutation $\omega \in \mathrm{Sym}(s)$ such that $\overline{\sigma(e_r E_{ij})} = \overline{e_{\omega(r)} B}$.

Let $t \in \{1, \dots, n\}$. We have $e_r^2 = e_r \neq 0$, so $e_r E_{ij} e_r E_{jt} \neq 0$ and $\overline{(e_{\omega(r)} B) \sigma(e_r E_{jt})} \neq 0$. This implies $\overline{\sigma(e_r E_{jt})} \in \overline{e_{\omega(r)} \mathrm{M}_n(S)}$.

As this holds for all $j, t \in \{1, \dots, n\}$ and for any $A \in \mathrm{M}_n(S)$, we have $A = (e_1 + e_2 + \dots + e_s) A$, we have $\overline{\sigma(e_r \mathrm{M}_n(S))} \subseteq \overline{e_{\omega(r)} \mathrm{M}_n(S)}$. By the same token, we can conclude that a twin vertex to a vertex from $e_r \mathrm{M}_n(S)$ is itself in $e_r \mathrm{M}_n(S)$, therefore also $\sigma(e_r \mathrm{M}_n(S)) \subseteq e_{\omega(r)} \mathrm{M}_n(S)$. Since σ is a bijection, $\sigma(e_r \mathrm{M}_n(S)) = e_{\omega(r)} \mathrm{M}_n(S)$. □

We next focus on the automorphisms restricted to the matrices over indecomposable subsemirings.

Proposition 3.8. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \dots + e_s$. Let $u, v \in \{1, \dots, s\}$, $S_1 = e_u S$ and $S_2 = e_v S$. Let $n \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\Gamma(\mathrm{M}_n(S)))$ such that $\sigma(\mathrm{M}_n(S_1)) = \mathrm{M}_n(S_2)$. If $i, j \in \{1, \dots, n\}$, then there exist $y \in S_2 \setminus \mathrm{Z}(S_2)$ and $k, \ell \in \{1, \dots, n\}$ such that $\sigma(e_u E_{ij}) = y E_{k\ell}$.*

Proof. Write $\sigma(e_u E_{ij}) = \sum_{k,\ell} \beta_{k\ell} E_{k\ell}$. Let $A_{k\ell} = \sigma^{-1}(\beta_{k\ell} E_{k\ell})$. By Lemma 3.2, $\sigma(e_u E_{ij})$ and $\sigma\left(\sum_{k,\ell} A_{k\ell}\right)$ are twin vertices, therefore $e_u E_{ij}$ and $\sum_{k,\ell} A_{k\ell}$ are twin vertices as well. Now, twin vertices of $e_u E_{ij}$ must be of the form $z E_{ij}$, so $\sum_{k,\ell} A_{k\ell} = z E_{ij}$ for some $z \in S$. Since $z = z(e_1 + e_2 + \dots + e_s)$, we conclude that $z \in S_1$. Note also that z is not a zero-divisor in S_1, since e_uE_{ij} and zE_{ij} are twin vertices. Since S is antinegative, we can conclude that, for all $k,\ell \in \{1,\dots,n\}$, there exist $\alpha_{k\ell} \in S_1$ such that $A_{k\ell} = \alpha_{k\ell}E_{ij}$ and $\sum_{k,\ell}\alpha_{k\ell} = z$.

Let $k,k',\ell,\ell' \in \{1,\dots,n\}$ with $(k,\ell) \neq (k',\ell')$. Now, we either have $k \neq k'$ or $\ell \neq \ell'$. Suppose first that $k \neq k'$. Since S is commutative, we have $\operatorname{Ann}_L(A_{k\ell}) = \operatorname{Ann}_L(\alpha_{k\ell}E_{ij}) \subseteq \operatorname{Ann}_L(\alpha_{k\ell}\alpha_{k'\ell'}E_{ij})$. By Lemma 3.1, this implies $\operatorname{Ann}_L(\beta_{k\ell}E_{k\ell}) \subseteq \operatorname{Ann}_L(\sigma(\alpha_{k\ell}\alpha_{k'\ell'}E_{ij}))$. Similarly, we have $\operatorname{Ann}_L(\beta_{k'\ell'}E_{k'\ell'}) \subseteq \operatorname{Ann}_L(\sigma(\alpha_{k\ell}\alpha_{k'\ell'}E_{ij}))$. Since $k \neq k'$, $\mathrm{M}_n(S) = \operatorname{Ann}_L(\beta_{k\ell}E_{k\ell}) + \operatorname{Ann}_L(\beta_{k'\ell'}E_{k'\ell'}) \subseteq \operatorname{Ann}_L(\sigma(\alpha_{k\ell}\alpha_{k'\ell'}E_{ij}))$, which implies $\sigma(\alpha_{k\ell}\alpha_{k'\ell'}E_{ij}) = 0$ and thus $\alpha_{k\ell}\alpha_{k'\ell'} = 0$. If $\ell \neq \ell'$, we arrive at the same conclusion by using right annihilators, namely that distinct $\alpha_{k\ell}$'s annihilate each other. Since S_1 is indecomposable by Lemma 3.6, the sum $\sum_{k,\ell}\alpha_{k\ell} = z$ has at most one non-zero summand. It follows that there is at most one non-zero $A_{k\ell}$ and at most one non-zero $\beta_{k\ell}E_{k\ell}$. This concludes the proof of the first part, with $y = \beta_{k\ell}$.

It remains to show that $y \notin \mathrm{Z}(S_2)$. Suppose, on the contrary, that $y \in \mathrm{Z}(S_2)$. By the first part of the result, there exist $y' \in S_1$ and $i',j' \in \{1,\dots,n\}$ such that $\sigma^{-1}(e_vE_{k\ell}) = y'E_{i'j'}$. Since $y \in \mathrm{Z}(S_2)$, we have $\operatorname{Ann}_L(e_vE_{k\ell}) \subsetneq \operatorname{Ann}_L(yE_{k\ell})$ and $\operatorname{Ann}_R(e_vE_{k\ell}) \subsetneq \operatorname{Ann}_R(yE_{k\ell})$. By Lemma 3.1, it follows that $\operatorname{Ann}_L(y'E_{i'j'}) \subsetneq \operatorname{Ann}_L(e_uE_{ij})$ and of course also $\operatorname{Ann}_R(y'E_{i'j'}) \subsetneq \operatorname{Ann}_R(e_uE_{ij})$. This is only possible if $i = i'$ and $j = j'$ which implies $\operatorname{Ann}_L(y'E_{ij}) \subsetneq \operatorname{Ann}_L(e_uE_{ij})$, a contradiction. □

Lemma 3.9. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \dots + e_s$. Let $u,v \in \{1,\dots,s\}$, $S_1 = e_uS$ and $S_2 = e_vS$. Let $n \in \mathbb{N}$ and $\sigma \in \operatorname{Aut}(\Gamma(\mathrm{M}_n(S)))$ such that $\sigma(\mathrm{M}_n(S_1)) = \mathrm{M}_n(S_2)$. If $x \in \mathrm{Z}(S_1)$ and $i,j \in \{1,\dots,n\}$, then there exist $z \in \mathrm{Z}(S_2)$ and $k,\ell \in \{1,\dots,n\}$ such that $\sigma(xE_{ij}) = zE_{k\ell}$.*

Proof. By Proposition 3.8, we know that $\sigma(e_uE_{ij}) = yE_{k\ell}$ for some $y \notin \mathrm{Z}(S_2)$ and $k,\ell \in \{1,\dots,n\}$. Since $x \in \mathrm{Z}(S_1)$, we have $\operatorname{Ann}_L(e_uE_{ij}) \subsetneq \operatorname{Ann}_L(xE_{ij})$ and $\operatorname{Ann}_R(e_uE_{ij}) \subsetneq \operatorname{Ann}_R(xE_{ij})$. By Lemma 3.1, it follows that $\operatorname{Ann}_L(yE_{k\ell}) \subsetneq \operatorname{Ann}_L(\sigma(xE_{ij}))$ and also $\operatorname{Ann}_R(yE_{k\ell}) \subsetneq \operatorname{Ann}_R(\sigma(xE_{ij}))$. This implies that all entries of $\sigma(xE_{ij})$ are zeros except entry (k,ℓ), so $\sigma(xE_{ij}) = zE_{k\ell}$ for some $z \in S_2$. Because $\operatorname{Ann}_L(yE_{k\ell}) \neq \operatorname{Ann}_L(\sigma(xE_{ij})) = \operatorname{Ann}_L(zE_{k\ell})$ and $\operatorname{Ann}_R(yE_{k\ell}) \neq \operatorname{Ann}_R(\sigma(xE_{ij})) = \operatorname{Ann}_R(zE_{k\ell})$, we must have $z \in \mathrm{Z}(S_2)$. □

Lemma 3.10. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \dots + e_s$. Let $u,v \in \{1,\dots,s\}$, $S_1 = e_uS$ and $S_2 = e_vS$. Let $n \in \mathbb{N}$ and $\sigma \in \operatorname{Aut}(\Gamma(\mathrm{M}_n(S)))$ such that $\sigma(\mathrm{M}_n(S_1)) = \mathrm{M}_n(S_2)$. Then there exists $\pi \in \operatorname{Sym}(n)$ such that $\overline{\sigma(e_uE_{ij})} = \overline{e_vE_{\pi(i)\pi(j)}}$ for all $i,j \in \{1,\dots,n\}$.*

Proof. Let $i,j,j' \in \{1,\dots,n\}$ with $j \neq j'$. By Proposition 3.8, there exist $k,k',\ell,\ell' \in \{1,\dots,n\}$ such that $\overline{\sigma(e_uE_{ij})} = \overline{e_vE_{k\ell}}$ and $\overline{\sigma(e_uE_{ij'})} = \overline{e_vE_{k'\ell'}}$. For all $r,s \in \{1,\dots,n\}$ with $s \neq i$, we have $e_uE_{rs}(e_uE_{ij} + e_uE_{ij'}) = 0$. By Lemma 3.2, this implies that $\overline{\sigma(e_uE_{rs})(e_uE_{k\ell} + e_uE_{k'\ell'})} = 0$ and thus $\overline{\left(\sum_{r,s\neq i}\sigma(e_uE_{rs})\right)(e_uE_{k\ell} + e_uE_{k'\ell'})} = 0$. By Proposition 3.8, $\overline{\sigma(e_uE_{rs})} = \overline{e_vE_{r's'}}$ for some $r',s' \in \{1,\dots,n\}$. Since σ is a permutation, $\sum_{r,s\neq i}\sigma(e_uE_{rs})$ is a matrix with exactly n entries equal to 0. It follows that $k = k'$.

By the paragraph above, there exists $\pi \in \operatorname{Sym}(n)$ such that $\overline{\sigma(e_uE_{ab})} = \overline{e_vE_{\pi(a)c}}$, for some c. A similar argument yields that there exists a permutation such that $\overline{\sigma(e_uE_{ab})} =$

$\overline{e_v E_{c\pi'(b)}}$, for some c. However, for every $j,k \in \{1,\dots,n\}$ with $j \neq k$, we have $E_{jj}E_{kk} = 0$ and thus $E_{\pi(j)\pi'(j)}E_{\pi(k)\pi'(k)} = 0$. This implies that $\pi(k) \neq \pi'(j)$ for every $k \neq j$, so $\pi(j) = \pi'(j)$. Therefore $\pi' = \pi$. □

For $\pi \in \mathrm{Sym}(n)$ and $A \in \mathrm{M}_n(S)$, let $\theta_\pi(A)$ be the matrix obtained from A by applying the permutation π to its rows and columns. Note that θ_π induces a permutation of $\mathrm{M}_n(S)$.

Corollary 3.11. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \cdots + e_s$. Let $u, v \in \{1, 2, \dots, s\}$, $S_1 = e_u S$ and $S_2 = e_v S$. Let $n \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\Gamma(\mathrm{M}_n(S)))$ such that $\sigma(\mathrm{M}_n(S_1)) = \mathrm{M}_n(S_2)$. Then there exist $\pi \in \mathrm{Sym}(n)$ and τ an isomorphism from $\Gamma(S_1)$ to $\Gamma(S_2)$ such that, if we extend τ entry-wise to a mapping $\mathrm{M}_n(S_1) \to \mathrm{M}_n(S_2)$ and restrict σ to $\mathrm{M}_n(S_1)$, then $\overline{\sigma} = \overline{\theta_\pi \circ \tau}$.*

Proof. By Lemma 3.10, there exists $\pi \in \mathrm{Sym}(n)$ such that $\overline{\sigma(e_u E_{ij})} = \overline{\theta_\pi(e_v E_{ij})}$ for all $i,j \in \{1,\dots,n\}$. Let $\rho = \theta_\pi^{-1} \circ \sigma$ and note that $\rho \in \mathrm{Aut}(\Gamma(\mathrm{M}_n(S)))$ and we have $\overline{\rho(e_u E_{ij})} = \overline{e_v E_{ij}}$ for all $i,j \in \{1,\dots,n\}$.

Let $x \in \mathrm{Z}(S_1)$ and $i,j,j' \in \{1,\dots,n\}$. Clearly, $\rho(\mathrm{M}_n(S_1)) = \mathrm{M}_n(S_2)$ so, by Lemma 3.9, there exist $z, z' \in \mathrm{Z}(S_2)$ such that $\rho(xE_{ij}) = zE_{ij}$ and $\rho(xE_{ij'}) = z'E_{ij'}$.

Let $X = \{s \in S_2; sz = 0\}$. We show that $X \subseteq \mathrm{Ann}(z')$. Let $a \in S_2$ such that $az = 0$. Note that $(aE_{ii})(zE_{ij}) = 0$. Since $a \in \mathrm{Z}(S_2)$, Lemma 3.9 implies that there exists $b \in \mathrm{Z}(S_1)$ such that $aE_{ii} = \rho(bE_{ii})$, hence $\rho(bE_{ii})\rho(xE_{ij}) = 0$ and therefore also $(bE_{ii})(xE_{ij}) = 0$ which implies $bx = 0$. It follows that $(bE_{ii})(xE_{ij'}) = 0$, $\rho(bE_{ii})\rho(xE_{ij'}) = 0$ and $(aE_{ii})(z'E_{ij'}) = 0$ which yields $az' = 0$.

We have shown that $X \subseteq \mathrm{Ann}(z')$. A symmetrical argument yields $\mathrm{Ann}(z') \subseteq X$ hence $X = \mathrm{Ann}(z')$ which implies that $\overline{\rho(xE_{ij'})} = \overline{z'E_{ij'}} = \overline{zE_{ij'}}$ (where for $A \in \mathrm{M}_n(S_2)$, by a slight abuse of notation, $\overline{A}$ now refers to the image in $\overline{\Gamma}(\mathrm{M}_n(S_2))$ and not in $\overline{\Gamma}(\mathrm{M}_n(S))$). A similar argument shows that $\overline{\rho(xE_{i'j})} = \overline{zE_{i'j}}$ for all $i' \in \{1,\dots,n\}$. This implies that $\overline{\rho(xE_{k\ell})} = \overline{zE_{k\ell}}$ for all $k,\ell \in \{1,\dots,n\}$.

Let τ denote the mapping $S_1 \to S_2$ that satisfies $\rho(xE_{11}) = \tau(x)E_{11}$. Since ρ is a bijection from $\mathrm{M}_n(S_1)$ to $\mathrm{M}_n(S_2)$, τ is a bijection from S_1 to S_2. If $x,y \in S_1$, then $xy = 0$ if and only if $(xE_{11})(yE_{11}) = 0$ if and only if $\tau(x)\tau(y) = 0$, therefore τ is an isomorphism from $\Gamma(S_1)$ to $\Gamma(S_2)$. Now, extend τ to an entry-wise mapping $\mathrm{M}_n(S_1) \to \mathrm{M}_n(S_2)$. Let $A = [a_{ij}], B = [b_{ij}] \in \mathrm{M}_n(S_1)$. Note that $AB = 0$ if and only if $\sum_{k=1}^n a_{ik}b_{kj} = 0$ for every $1 \le i,j \le n$ if and only if $a_{ik}b_{kj} = 0$ for every $1 \le i,j,k \le n$, so τ induces an isomorphism from $\Gamma(\mathrm{M}_n(S_1))$ to $\Gamma(\mathrm{M}_n(S_2))$. Observe that, restricted to $\mathrm{V}(\Gamma(\mathrm{M}_n(S_1)))$, we have $\overline{\rho} = \overline{\tau}$. As $\sigma = \theta_\pi \circ \rho$, this concludes the proof. □

We can now join these findings into the following theorem.

Theorem 3.12. *Let S be a commutative antiring and suppose $1 \in S$ is of finite maximal length s with decomposition $1 = e_1 + e_2 + \cdots + e_s$. Let $n \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\Gamma(\mathrm{M}_n(S)))$. Then there exist $\omega \in \mathrm{Sym}(s)$ and, for every $i \in \{1,\dots,s\}$, there exist $\pi_i \in \mathrm{Sym}(n)$ and an isomorphism $\tau_i \colon \Gamma(e_i S) \to \Gamma(e_{\omega(i)}S)$ such that, if we extend τ_i entry-wise to a mapping $\mathrm{M}_n(e_i S) \to \mathrm{M}_n(e_{\omega(i)}S)$, then*

$$\overline{\sigma(A)} = \overline{\left(\sum_{i=1}^{s} (\theta_{\pi_i} \circ \tau_i)(e_i A)\right)} \text{ for all } A \in \mathrm{M}_n(S).$$

Conversely, if $\omega \in \mathrm{Sym}(s)$ *has the property that, for every* $i \in \{1, \ldots, s\}$, *we have* $\Gamma(e_i S) \cong \Gamma(e_{\omega(i)} S)$, τ_i *is an isomorphism from* $\Gamma(e_i S)$ *to* $\Gamma(e_{\omega(i)} S)$ *and* $\pi_i \in \mathrm{Sym}(n)$, *then* σ *defined with* $\sigma(A) = \sum_{i=1}^{s} (\theta_{\pi_i} \circ \tau_i)(e_i A)$ *is an automorphism of* $\Gamma(\mathrm{M}_n(S))$.

Proof. By Lemma 3.7, there exists $\omega \in \mathrm{Sym}(s)$ such that, for every $i \in \{1, \ldots, s\}$, we have $\sigma(e_i \mathrm{M}_n(S)) = e_{\omega(i)} \mathrm{M}_n(S)$.

By Corollary 3.11, there exist $\pi_i \in \mathrm{Sym}(n)$ and τ_i an isomorphism from $\Gamma(e_i S)$ to $\Gamma(e_{\omega(i)} S)$ such that, if we extend τ_i entry-wise to a mapping $\mathrm{M}_n(e_i S) \to \mathrm{M}_n(e_{\omega(i)} S)$ and restrict σ to $\mathrm{M}_n(e_i S)$, then $\overline{\sigma} = \overline{\theta_{\pi_i} \circ \tau_i}$.

Now, let $A \in \mathrm{M}_n(S)$. We have $\overline{A} = \overline{e_1 A + e_2 A + \cdots + e_s A}$ and the result follows by Lemma 3.2. □

Remark 3.13. Throughout the paper, we restricted ourselves to studying semirings with the property that no non-zero-divisor element can be written as a sum of infinitely many mutually orthogonal zero-divisors. Obviously, any semiring with a finite set of zero-divisors satisfies this condition.

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References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434–447, doi:10.1006/jabr.1998.7840, https://doi.org/10.1006/jabr.1998.7840.

[2] F. Baccelli and J. Mairesse, Ergodic theorems for stochastic operators and discrete event networks, in: *Idempotency*, Cambridge University Press, Cambridge, pp. 171–208, 1998.

[3] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208–226, doi:10.1016/0021-8693(88)90202-5, https://doi.org/10.1016/0021-8693(88)90202-5.

[4] G. A. Cannon, K. M. Neuerburg and S. P. Redmond, Zero-divisor graphs of nearrings and semigroups, in: H. Kiechle, A. Kreuzer and M. Thomsen (eds.), *Nearrings and Nearfields*, Springer, Dordrecht, pp. 189–200, 2005.

[5] R. Cuninghame-Green, *Minimax Algebra*, volume 166, Springer Science & Business Media, 2012.

[6] F. R. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** (2002), 206–214, doi:10.1007/s002330010128, https://doi.org/10.1007/s002330010128.

[7] D. Dolžan and P. Oblak, The zero-divisor graphs of rings and semirings, *Int. J. Algebra Comput.* **22** (2012), 1250033, 20, doi:10.1142/s0218196712500336, https://doi.org/10.1142/s0218196712500336.

[8] S. Ebrahimi Atani, The zero-divisor graph with respect to ideals of a commutative semiring., *Glas. Mat., III. Ser.* **43** (2008), 309–320, doi:10.3336/gm.43.2.06, https://doi.org/10.3336/gm.43.2.06.

[9] S. Ebrahimi Atani, An ideal based zero-divisor graph of a commutative semiring., *Glas. Mat., III. Ser.* **44** (2009), 141–153, doi:10.3336/gm.44.1.07, https://doi.org/10.3336/gm.44.1.07.

[10] J. Han, The zero-divisor graph under group actions in a noncommutative ring, *J. Korean Math. Soc.* **45** (2008), 1647–1659, doi:10.4134/jkms.2008.45.6.1647, https://doi.org/10.4134/jkms.2008.45.6.1647.

[11] U. Hebisch and H. J. Weinert, *Semirings: Algebraic theory and Applications in Computer Science*, volume 5 of *Ser. Algebra*, World Scientific Publishing, Singapore, 1998.

[12] P. Li, A heuristic method to compute the approximate postinverses of a fuzzy matrix, *IEEE Trans. Fuzzy Syst.* **22** (2014), 1347–1351, doi:10.1109/tfuzz.2013.2282231, https://doi.org/10.1109/tfuzz.2013.2282231.

[13] S. Park and J. Han, The group of graph automorphisms over a matrix ring, *J. Korean Math. Soc.* **48** (2011), 301–309, doi:10.4134/jkms.2011.48.2.301, https://doi.org/10.4134/jkms.2011.48.2.301.

[14] A. Patil, B. N. Waphare and V. Joshi, Perfect zero-divisor graphs, *Discrete Math.* **340** (2017), 740–745, doi:10.1016/j.disc.2016.11.027, https://doi.org/10.1016/j.disc.2016.11.027.

[15] S. P. Redmond, The zero-divisor graph of a non-commutative ring, *Int. J. Commut. Rings* **1** (2002), 203–211.

[16] L. Wang, A note on automorphisms of the zero-divisor graph of upper triangular matrices., *Linear Algebra Appl.* **465** (2015), 214–220, doi:10.1016/j.laa.2014.09.035, https://doi.org/10.1016/j.laa.2014.09.035.

[17] L. Wang, Automorphisms of the zero-divisor graph of the ring of all $n \times n$ matrices over a finite field, *Discrete Math.* **339** (2016), 2036–2041, doi:10.1016/j.disc.2016.02.021, https://doi.org/10.1016/j.disc.2016.02.021.

[18] D. Wong, X. Ma and J. Zhou, The group of automorphisms of a zero-divisor graph based on rank one upper triangular matrices, *Linear Algebra Appl.* **460** (2014), 242–258, doi:10.1016/j.laa.2014.07.041, https://doi.org/10.1016/j.laa.2014.07.041.

[19] S. Zhao and X.-p. Wang, Invertible matrices and semilinear spaces over commutative semirings, *Inf. Sci.* **180** (2010), 5115–5124, doi:10.1016/j.ins.2010.08.033, https://doi.org/10.1016/j.ins.2010.08.033.

[20] J. Zhou, D. Wong and X. Ma, Automorphisms of the zero-divisor graph of the full matrix ring, *Linear Multilinear Algebra* **65** (2017), 991–1002, doi:10.1080/03081087.2016.1219302, https://doi.org/10.1080/03081087.2016.1219302.