



Resolvability and Convexity Properties in the Sierpiński Product of Graphs

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Abstract. Let G and H be graphs and let $f: V(G) \rightarrow V(H)$ be a function. The Sierpiński product of G and H with respect to f , denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$, consisting of $|V(G)|$ copies of H ; for every edge gg' of G there is an edge between copies gH and $g'H$ of H associated with the vertices g and g' of G , respectively, of the form $(g, f(g'))(g', f(g))$. The Sierpiński metric dimension and the upper Sierpiński metric dimension of two graphs are determined. Closed formulas are determined for Sierpiński products of trees, and for Sierpiński products of two cycles where the second factor is a triangle. We also prove that the layers with respect to the second factor in a Sierpiński product graph are convex.

Mathematics Subject Classification. 05C12, 05C76.

Keywords. Sierpiński product of graphs, metric dimension, tree, convex subgraph.

1. Introduction

Sierpiński graphs represent a very interesting and widely studied family of graphs. They were introduced in 1997 in the paper [21], where the primary motivation for their introduction was the intrinsic link to the Tower of Hanoi problem, for the latter problem see the book [16]. Intensive research of Sierpiński graphs led to a review article [15] in which state of the art up to 2017 is summarized and unified approach to Sierpiński-type graph families is also proposed. Later research on Sierpiński graphs includes [3, 7–9, 19, 25, 32].

In this paper, we study a recent generalization of Sierpiński graphs proposed by Kovič, Pisanski, Zemljčič, and Žitnik in [23]. Let G and H be graphs and let $f: V(G) \rightarrow V(H)$ be an arbitrary function. The *Sierpiński product of graphs G and H with respect to f* , denoted by $G \otimes_f H$, is defined as the graph on the vertex set $V(G) \times V(H)$ with edges of two types:

- *Type-1 edge:* $(g, h)(g, h')$ is an edge of $G \otimes_f H$ for every vertex $g \in V(G)$ and every edge $hh' \in E(H)$,

- *Type-2 edge*: $(g, f(g'))(g', f(g))$ is an edge of $G \otimes_f H$ for every edge $gg' \in E(G)$.

We observe that the edges of Type-1 induce $n(G) = |V(G)|$ copies of the graph H in the Sierpiński product $G \otimes_f H$. For each vertex $g \in V(G)$, we let gH be the copy of H corresponding to the vertex g . A Type-2 edge joins vertices from different copies of H in $G \otimes_f H$, and is called a *connecting edge* of $G \otimes_f H$. A vertex incident with a connecting edge is called a *connecting vertex*. We observe that two different copies of H in $G \otimes_f H$ are joined by at most one edge. We denote by H^G the family of functions from $V(G)$ to $V(H)$.

It might be readily observed that the Sierpiński product is closely related to other graph products. For instance, by considering a constant function f in the product, we obtain graphs which are indeed the same as the so-called rooted product graphs (see [12] for its definition). Also, selecting the identity function $\text{id} \in G^G$, the Sierpiński product $G \otimes_{\text{id}} G$ is the (first iteration of the) generalized Sierpiński graph in the sense of [13]. Moreover, a Sierpiński product can also be considered as a subgraph of the (Cartesian, strong or lexicographic) product. Consequently, any contribution to the study of the Sierpiński product could give some more knowledge on these related products.

In the next two subsections, we give motivation, basic terminology, and notation concerning the classical metric dimension of graphs, and introduce the study of the Sierpiński metric dimension and the upper Sierpiński metric dimension. Thereafter in Sect. 2, we determine the upper Sierpiński metric dimension for Sierpiński products of arbitrary trees. A general lower bound is established for the Sierpiński metric dimension for products of two trees, and an exact formula when the first factor is a path. In Sect. 3, a closed formula is determined for both dimensions when the first factor in the product is an arbitrary cycle and the second factor a triangle. In Sect. 4, we prove that the layers with respect to the second factor in a Sierpiński product graph are convex. In Sect. 5, we pose several open problems.

1.1. The Metric Dimension of Graphs

The *distance* between two vertices u and v in a connected graph G , denoted by $d_G(u, v)$, is the number of edges in a shortest path from u to v , that is, $d_G(u, v)$ is the minimum length of a u, v -path in G . Given an ordered subset $S = \{v_1, \dots, v_k\}$ of vertices in G , the *metric S -representation* of a vertex v in G is the k -tuple vector

$$r_G(v|S) = (d_G(v, v_1), \dots, d_G(v, v_k)).$$

If every two distinct vertices of G have different metric S -representations, then the set S is called a *resolving set* of G (also called a *metric generator*). The *metric dimension* of G , denoted by $\dim(G)$, is the cardinality of a smallest possible resolving set in G . A *metric basis* of G is a resolving set of cardinality $\dim(G)$. A vertex v in a graph G is said to *distinguish* (or *resolve*) two vertices x and y if $d_G(v, x) \neq d_G(v, y)$.

The concept of the metric dimension of a graph was birthed independently by Harary and Melter [14] in 1976 and by Slater [29] in 1975, and is

now well studied in graph theory. To date, MathSciNet lists over 380 papers on metric dimension in graphs, covering a large number of different investigations dealing with theoretical and applied results on such parameter.

According to the structural properties of resolving sets in graphs, they can easily be used to model several practical situations in which uniquely recognizing points or locations is required. That was precisely one of the motivations of the seminal works [14, 29], where resolving sets appeared to be used for the location of intruders in networks. Further on, some other related models and applications have appeared here and there. Among them, we remark the recent work [30], where the authors presented a connection between some metric dimension parameter and the representation of genomic sequences. Among the theoretical studies on this topic, the literature contains a wide range of different contributions, some recent and remarkable articles are for instance [6, 11, 28]. For more information on investigations on the classical version, we suggest the fairly complete survey [31].

With respect to the theoretical studies, the metric dimension of graph products and graph operations has attracted the attention of several investigations. In this sense, we mention a few interesting contributions related with this exposition due to the relationship between the Sierpiński product and some other products previously mentioned. The metric dimension of Cartesian product graphs has been considered in several works like [4], for the general case, and among other ones, in [5, 18, 20] for some particular examples of Cartesian products. The lexicographic product of graphs has been studied with respect to its metric dimension in [17, 27], while the strong product has been considered in [1, 26]. On the other hand, the metric dimension of the rooted product has been dealt with in [10, 24].

1.2. Sierpiński Metric Dimension

Let G and H be graphs and H^G be the family of functions from $V(G)$ to $V(H)$. We introduce new types of metric dimension, the *Sierpiński metric dimension*, denoted by $\dim_S(G, H)$, as the minimum over all functions f from H^G of the metric dimension of the Sierpiński product with respect to f , and *upper Sierpiński metric dimension*, denoted by $\text{Dim}_S(G, H)$, as the maximum over all functions $f \in H^G$ of the metric dimension of the Sierpiński product with respect to f . That is,

$$\dim_S(G, H) =: \min_{f \in H^G} \{\dim(G \otimes_f H)\}$$

and

$$\text{Dim}_S(G, H) =: \max_{f \in H^G} \{\dim(G \otimes_f H)\}.$$

We might remark that the classical metric dimension of Sierpiński graphs was already studied in [22], as well as, that of the generalized Sierpiński graphs over stars was considered in [2].

2. Sierpiński Products of Trees

A vertex of degree at least 3 in a tree T is called a *branch vertex* (also called a *major vertex* in the literature). A leaf u of T is called a *terminal leaf* of a branch vertex v of T if $d_T(u, v) < d_T(u, w)$ for every other branch vertex w of T . The *terminal degree* of a branch vertex v is the number of terminal leaves associated with v . A branch vertex v of T is an *exterior branch vertex* of T if it has positive terminal degree. The path from a terminal leaf to the vertex immediately preceding the branch vertex that it is closest to is called a *terminal path*. Thus, every vertex on a terminal path in T is either a leaf of T or has degree 2 in T . A vertex on a terminal path that has degree 2 in T is called an *internal terminal vertex*. Equivalently, every vertex on a terminal path that is not a terminal leaf, is an internal terminal vertex. Thus, if u is an internal terminal vertex in T , then the vertex u is an internal vertex of a path P that joins a leaf and a branch vertex closest to that leaf in T where every internal vertex of P has degree 2 in T .

Let $n_1(T)$ denote the number of leaves of T , and let $\text{ex}(T)$ denote the number of exterior branch vertices of T . The formula for the metric dimension of a tree reads as follows.

Theorem 2.1. [14, 29] *If T is a tree that is not a path, then*

$$\dim(T) = n_1(T) - \text{ex}(T). \quad (1)$$

It is clear that $\dim(P_n) = 1$. Combining this fact with Theorem 2.1 yields the following consequence.

Corollary 2.2. *If T is a tree, then $\dim(T) = 1$ if T is a path and $\dim(T) \geq 2$ if T is not a path.*

Let T be a tree that is not a path, and let v_1, \dots, v_k be the exterior branch vertices in T that have terminal degree at least 2. If the exterior branch vertex v_i has terminal degree $\ell_i \geq 2$ and if L_i is a set consisting of all terminal leaves but one associated with v_i for all $i \in [k] = \{1, \dots, k\}$, then (1) can be equivalently stated as:

$$\dim(T) = \sum_{i=1}^k (\ell_i - 1),$$

and the set

$$B(T) = \bigcup_{i=1}^k L_i$$

is a metric basis of T (of cardinality $\dim(T)$). We call the basis $B(T)$ a *standard metric basis* of T . Thus, every vertex in a standard metric basis of a tree T is a leaf, and such a basis contains all but one selected fixed leaf associated with the exterior branch vertex of terminal degree at least 2 in T .

2.1. Upper Sierpiński Metric Dimension in Trees

In this section, we determine the upper Sierpiński metric dimension of the Sierpiński product of trees. Notice that for arbitrary trees T_1 and T_2 and any function $f \in H^G$, the Sierpiński product $T_1 \otimes_f T_2$ is a tree.

Theorem 2.3. *If T_1 and T_2 are trees with $n(T_2) \geq 3$, then*

$$\text{Dim}_S(T_1, T_2) = n(T_1) \dim(T_2).$$

Proof. Let w be a branch vertex of T_2 , and let $f_w: V(T_1) \rightarrow V(T_2)$ be the constant function defined by $f_w(v) = w$ for every vertex $v \in V(T_1)$. The exterior branch vertices in $T_1 \otimes_{f_w} T_2$ are precisely the exterior branch vertices in each of the copies of T_2 , and so $\text{ex}(T_1 \otimes_{f_w} T_2) = n(T_1)\text{ex}(T_2)$. Moreover, the leaves in $T_1 \otimes_{f_w} T_2$ are precisely the leaves in each of the copies of T_2 , and so $n_1(T_1 \otimes_{f_w} T_2) = n(T_1)n_1(T_2)$. Therefore, as $T_1 \otimes_{f_w} T_2$ is a tree, by (1) we have

$$\begin{aligned} \dim(T_1 \otimes_{f_w} T_2) &= n_1(T_1 \otimes_{f_w} T_2) - \text{ex}(T_1 \otimes_{f_w} T_2) \\ &= n(T_1)n_1(T_2) - n(T_1)\text{ex}(T_2) \\ &= n(T_1)(n_1(T_2) - \text{ex}(T_2)) \\ &= n(T_1) \dim(T_2), \end{aligned}$$

implying that

$$\text{Dim}_S(T_1, T_2) \geq \dim(T_1 \otimes_{f_w} T_2) = n(T_1) \dim(T_2). \tag{2}$$

We next show that $\text{Dim}_S(T_1, T_2) \leq n(T_1) \dim(T_2)$. Let $f: V(T_1) \rightarrow V(T_2)$ be an arbitrary function. It suffices for us to show that

$$\dim(T_1 \otimes_f T_2) \leq n(T_1) \dim(T_2).$$

Let us first consider that T_2 is a path P_n with $n \geq 3$. Hence, we here indeed need to prove that $\dim(T_1 \otimes_f P_n) \leq n(T_1)$ since $\dim(P_n) = 1$. Suppose to the contrary that $\dim(T_1 \otimes_f P_n) > n(T_1)$. Thus, from (1), we have that $n_1(T_1 \otimes_f P_n) - \text{ex}(T_1 \otimes_f P_n) = \dim(T_1 \otimes_f P_n) > n(T_1)$. Hence,

$$\text{ex}(T_1 \otimes_f P_n) < n_1(T_1 \otimes_f P_n) - n(T_1) \leq 2n(T_1) - n(T_1) = n(T_1),$$

which means there is a positive integer k such that $\text{ex}(T_1 \otimes_f P_n) = n(T_1) - k$. Now, notice that if $T_1 \otimes_f P_n$ has $n(T_1) - k$ exterior branch vertices, then there must be at least k copies of P_n in $T_1 \otimes_f P_n$ not containing any exterior branch vertex. This situation can only happen when the connecting edges of $T_1 \otimes_f P_n$ in such copies of P_n are incident with at least one leaf of each of these copies. Consequently, we deduce that $n_1(T_1 \otimes_f P_n) \leq 2n(T_1) - k$. Therefore, by (1), we have

$$\begin{aligned} \dim(T_1 \otimes_f P_n) &= n_1(T_1 \otimes_f P_n) - \text{ex}(T_1 \otimes_f P_n) \\ &\leq 2n(T_1) - k - (n(T_1) - k) \\ &= n(T_1), \end{aligned}$$

which is a contradiction with our assumption, and so $\dim(T_1 \otimes_f P_n) \leq n(T_1)$ as required.

We next consider the case when T_2 is not a path. Let $E_c = \{e_1, e_2, \dots, e_{m(T_1)}\}$ be the set of connecting edges in $T_1 \otimes_f T_2$. We order these connecting edges and define e_i as the i th connecting edge of $T_1 \otimes_f T_2$ for $i \in [m(T_1)]$. We

next define forests $X_0, X_1, \dots, X_{m(T_1)}$ as follows. Let X_0 be obtained from the tree $T_1 \otimes_f T_2$ by removing the connecting edges in E_c . We note that X_0 is the disjoint union of $n(T_1)$ copies of the tree T_2 . Applying (1) to each component of the forest X_0 , we have

$$\begin{aligned} \dim(X_0) &= n_1(X_0) - \text{ex}(X_0) \\ &= n(T_1)n_1(T_2) - n(T_1)\text{ex}(T_2) \\ &= n(T_1)(n_1(T_2) - \text{ex}(T_2)) \\ &= n(T_1) \dim(T_2). \end{aligned}$$

We now define the forests $X_1, \dots, X_{m(T_1)}$ as follows. For $i \in [m(T_1)]$, let X_i be the forest obtained from X_{i-1} by adding the i th connecting edge, that is, $X_i = X_{i-1} \cup \{e_i\}$. We note that

$$T_1 \otimes_f T_2 = X_{m(T_1)}.$$

Applying (1) to each component of the forest X_i , we have

$$\dim(X_i) = n_1(X_i) - \text{ex}(X_i)$$

for all $i \in \{0, 1, \dots, m(T_1)\}$. Let

$$\Phi_1(X_i) = n_1(X_{i-1}) - n_1(X_i),$$

which represents the number of vertices of degree 1 in X_{i-1} which are of degree at least 2 in X_i . Roughly speaking, $\Phi_1(X_i)$ is the number of degree 1 vertices in X_{i-1} “destroyed” by adding the i th connecting edges e_i to X_{i-1} when constructing X_i . Set further

$$\Phi_2(X_i) = -[\text{ex}(X_{i-1}) - \text{ex}(X_i)],$$

where $\text{ex}(X_{i-1}) - \text{ex}(X_i)$ is the difference between the number of exterior branch vertices in X_{i-1} and the number of exterior branch vertices in X_i . We note that

$$\begin{aligned} \dim(X_i) &= n_1(X_i) - \text{ex}(X_i) \\ &= (n_1(X_{i-1}) - \Phi_1(X_i)) - (\Phi_2(X_i) + \text{ex}(X_{i-1})) \\ &= \dim(X_{i-1}) - (\Phi_1(X_i) + \Phi_2(X_i)) \end{aligned}$$

for all $i \in [m(T_1)]$. We would like to show that

$$\Phi_1(X_i) + \Phi_2(X_i) \geq 0 \tag{3}$$

for all $i \in [m(T_1)]$, which would imply that $\dim(X_i) \leq \dim(X_{i-1}) \leq \dim(X_0) = n(T_1) \dim(T_2)$, from which we deduce that

$$\dim(T_1 \otimes_f T_2) = \dim(X_{m(T_1)}) \leq n(T_1) \dim(T_2).$$

Hence, to prove the theorem, it remains to show that (3) holds. For this purpose, let the i th connecting edge e_i joins two vertices x_i and y_i in X_{i-1} when constructing X_i .

Suppose that x_i is neither an internal terminal vertex nor a leaf in X_{i-1} . In this case, the vertex x_i contributes 0 to both terms $\Phi_1(X_i)$ and $\Phi_2(X_i)$.

Suppose that x_i is an internal terminal vertex in X_{i-1} . Let w_i be the exterior branch vertex associated with the vertex x_i in X_{i-1} . In this case, when e_i is added to X_{i-1} , the vertex x_i becomes an exterior branch vertex in X_i , while the vertex w_i may no longer be an exterior branch vertex, implying

that the vertex x_i contributes 0 to the term $\Phi_1(X_i)$, and contributes at least $1 - 1 = 0$ to the term $\Phi_2(X_i)$.

Suppose that x_i is a leaf in X_{i-1} . As before, let w_i be the exterior branch vertex associated with the vertex x_i in X_{i-1} . In this case, when e_i is added to X_{i-1} , the leaf x_i in X_{i-1} is not a leaf in X_i , and therefore the vertex x_i contributes 1 to the term $\Phi_1(X_i)$. Moreover, the effect of adding e_i is that the vertex w_i may no longer be an exterior branch vertex, implying that the vertex x_i contributes at least -1 to the term $\Phi_2(X_i)$. In all of the above three cases, the contribution of x_i to $\Phi_1(X_i) + \Phi_2(X_i)$ is at least 0. Analogous arguments hold for the vertex y_i , showing that the contribution of y_i to $\Phi_1(X_i) + \Phi_2(X_i)$ is at least 0. Therefore, (3) holds. This completes the proof of Theorem 2.3. □

2.2. Sierpiński Metric Dimension in Trees

In this section, we study the Sierpiński metric dimension of the Sierpiński product of trees. The Sierpiński metric dimension of two paths is given by the following result.

Proposition 2.4. *If T_1 and T_2 are both paths, then $\dim_S(T_1, T_2) = 1$.*

Proof. Let $T_1 = P_n$ and let $T_2 = P_m$. If $n = 1$ or $m = 1$, then $T_1 \otimes_f T_2$ is a path, and so by Corollary 2.2, $\dim_S(T_1, T_2) = 1$. Hence, we may assume that $n \geq 2$ and $m \geq 2$, for otherwise the result is immediate. Let the path T_2 be an x, y -path that starts at vertex x and ends at vertex y , and let T_1 be the path $v_1 v_2 \dots v_n$. Let $f: V(T_1) \rightarrow V(T_2)$ be the function defined by

$$f(v_i) = \begin{cases} x; & i \pmod 4 \in \{1, 2\}, \\ y; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$. In this case, the Sierpiński product $T_1 \otimes_f T_2$ is a path P_{nm} , and so by Corollary 2.2, $\dim_S(T_1, T_2) \leq \dim(T_1 \otimes_f T_2) = 1$. Consequently, $\dim_S(T_1, T_2) = 1$. □

In view of Proposition 2.4, it is only of interest to study the Sierpiński product of two trees which at least one of them is not a path. In this case, we shall establish the following lower bound on the Sierpiński metric dimension, where we use the notation $d_T(v)$ to represent the *degree* of a vertex v in T .

Lemma 2.5. *If T_1 and T_2 are trees, where T_2 is not a path, then*

$$\dim_S(T_1, T_2) \geq \sum_{v \in V(T_1)} \max\{0, \dim(T_2) - d_{T_1}(v)\}.$$

Proof. Let $f: V(T_1) \rightarrow V(T_2)$ be an arbitrary function. Recall that for each vertex $v \in V(T_1)$, vT_2 denotes the copy of the tree T_2 in $T_1 \otimes_f T_2$ corresponding to the vertex v . We let C_v be the set of vertices in vT_2 that are connecting vertices in $T_1 \otimes_f T_2$. Thus, each vertex in C_v is incident with a connecting edge in $T_1 \otimes_f T_2$ that joins that vertex to a vertex in a copy of T_2 different from vT_2 . We note that

$$|C_v| \leq d_{T_1}(v)$$

since every edge incident with v in the tree T_1 is associated with a connecting edge in $T_1 \otimes_f T_2$ that is incident with a vertex in vT_2 . Let B be a standard metric basis of $T_1 \otimes_f T_2$, and so $\dim(T_1 \otimes_f T_2) = |B|$ and the basis B contains all but one leaf associated with the exterior branch vertices of degree at least 2 in $T_1 \otimes_f T_2$. Let B_v be the restriction of B to vT_2 , that is,

$$B_v = B \cap V(vT_2)$$

for every vertex $v \in V(T_1)$. The set $B_v \cup C_v$ is a resolving set in the tree vT_2 , and so

$$\dim(T_2) = \dim(vT_2) \leq |B_v| + |C_v| \leq |B_v| + d_{T_1}(v),$$

and so $|B_v| \geq \dim(T_2) - d_{T_1}(v)$. As clearly $|B_v| \geq 0$, we get

$$|B_v| \geq \max\{0, \dim(T_2) - d_{T_1}(v)\}$$

for every vertex $v \in V(T_1)$. Therefore,

$$\dim_S(T_1, T_2) = |B| = \sum_{v \in V(T_1)} |B_v| \geq \sum_{v \in V(T_1)} \max\{0, \dim(T_2) - d_{T_1}(v)\}.$$

This establishes the desired lower bound in the statement of the theorem. □

Using Lemma 2.5, we have the following result.

Theorem 2.6. *For $n \geq 2$, if $T_1 = P_n$ and T_2 is a tree that is not a path, then*

$$\dim_S(T_1, T_2) = n(\dim(T_2) - 2) + 2.$$

Proof. Let $T_1 = P_n$ and let T_2 be a tree that is not a path. By Corollary 2.2, $\dim(T_2) \geq 2$. By Lemma 2.5,

$$\dim_S(T_1, T_2) \geq n(\dim(T_2) - 2) + 2$$

noting that T_1 contains two vertices of degree 1 and $n - 2$ vertices of degree 2. Hence, it suffices for us to show that

$$\dim_S(T_1, T_2) \leq n(\dim(T_2) - 2) + 2.$$

By assumption, T_2 is not a path. Hence, T_2 contains at least one exterior branch vertex with terminal degree at least 2. If T_2 contains two distinct exterior branch vertices both with terminal degree at least 2, then let u_1 and u_2 be two selected (terminal) leaves associated with these two exterior branch vertices. If T_2 contains only one exterior branch vertex, then T_2 is a star or a subdivided star with terminal degree at least 3. In this case, let u_1 and u_2 be two arbitrary leaves in T_2 . Let T_1 be the path $v_1 v_2 \dots v_n$, and define the function $f: V(T_1) \rightarrow V(T_2)$ by

$$f(v_i) = \begin{cases} u_1; & i \pmod 4 \in \{1, 2\}, \\ u_2; & \text{otherwise.} \end{cases}$$

for all $i \in [n]$. For notational simplicity, instead of $v_i T_2$ we simply write iT_2 for the copy of T_2 in the Sierpiński product $T_1 \otimes_f T_2$ that corresponds to the vertex v_i for all $i \in [n]$. We note that the copy $1T_2$ and nT_2 both contain one

less leaf in the product $T_1 \otimes_f T_2$, while every copy iT_2 where $i \in [n - 1] \setminus \{1\}$ contains two fewer leaves in the product $T_1 \otimes_f T_2$, implying that

$$n_1(T_1 \otimes_f T_2) = n(T_1)(n_1(T_2) - 2) + 2 = n(n_1(T_2) - 2) + 2.$$

On the other hand, by our choice of the vertices u_1 and u_2 , the number of exterior branch vertices in each copy iT_2 where $i \in [n]$ remains unchanged in the product $T_1 \otimes_f T_2$, that is,

$$\text{ex}(T_1 \otimes_f T_2) = n(T_1)\text{ex}(T_2) = n \times \text{ex}(T_2).$$

Therefore,

$$\begin{aligned} \dim(T_1 \otimes_f T_2) &= n_1(T_1 \otimes_f T_2) - \text{ex}(T_1 \otimes_f T_2) \\ &= (n(n_1(T_2) - 2) + 2) - n \times \text{ex}(T_2) \\ &= n(n_1(T_2) - \text{ex}(T_2) - 2) + 2 \\ &= n(\dim(T_2) - 2) + 2, \end{aligned}$$

completing the proof of Theorem 2.6. □

3. Sierpiński Products of Cycles

In this section, we study the Sierpiński metric dimension of the Sierpiński product of cycles, and proved closed formulas for the cases in which at least one of the factors is a triangle.

Theorem 3.1. *If $n \geq 3$, then $\text{Dim}_S(C_n, C_3) = n$.*

Proof. Let $H = C_3$ and $G = C_n$. Let $V(H) = [3]$, and let the vertices of the cycle G be g_1, g_2, \dots, g_n in the natural order of adjacencies. Let $f : V(G) \rightarrow V(H)$ be an arbitrary function and consider $G \otimes_f H$.

For notational simplicity, let iH denote g_iH for all $i \in [n]$, that is, iH is the i th copy of H corresponding to the vertex g_i of G . Let $x_i y_{i+1}$ be the connecting edge between iH and $(i + 1)H$, $i \in [n]$, where the sum is made mod n . Thus, $x_i = (g_i, f(g_{i+1}))$ and $y_{i+1} = (g_{i+1}, f(g_i))$. Set further w_i be a vertex in iH different from x_i and y_i . Note that if $x_i \neq y_i$, then $V(iH) = \{x_i, y_i, w_i\}$. In case $x_i = y_i$, then let w'_i be the third vertex of $V(iH)$, that is, in this case we have $V(iH) = \{x_i, w_i, w'_i\}$.

Let $Z = \{z_1, z_2, \dots, z_n\} \subseteq V(G \otimes_f H)$ be defined as follows. Consider an arbitrary iH , $i \in [n]$. If $x_i \neq y_i$, then let $z_i = y_i$, and if $x_i = y_i$, then let $z_i = w_i$. Since the vertices z_i are pairwise different, $|Z| = n$. We claim that Z is a resolving set of $G \otimes_f H$.

Consider first two vertices from a given iH . If $z_i = w_i$, then the vertices x_i and w'_i are distinguished by all the vertices of $Z \setminus \{z_i\}$. If $z_i = y_i$, then $d(w_i, z_{i+1}) = d(x_i, z_{i+1}) + 1$ and, therefore, x_i and w_i are distinguished. Consider now two vertices $u \in V(iH)$ and $v \in V(jH)$, where $i \neq j$. Assume first that $z_i = w_i$. Then, $d(u, z_i) \leq 1$ while $d(v, z_i) > 1$ and so u and v are distinguished by z_i . The case when $z_j = w_j$ is treated exactly the same. Hence, it remains to consider the case when $z_i = y_i$ and $z_j = y_j$. If $|i - j| > 1$, then as before $d(u, z_i) < d(v, z_i)$. Assume finally that $j = i + 1$. But then $d(v, z_i) > d(u, z_i)$ and so again u and v are distinguished.

We have thus proved that Z is a resolving set and hence $\text{Dim}_S(C_n, C_3) \leq |Z| = n$. To prove that $\text{Dim}_S(C_n, C_3) \geq n$, consider the constant function $f_1(g_i) = 1$ for every $i \in [n]$. Then, in every iH , the vertices corresponding to 2 and 3 are twins and hence at least one of them must be included in every resolving set. Then, $\dim(G \otimes_{f_1} H) \geq n$ and, therefore, $\text{Dim}_S(C_n, C_3) \geq n$. □

We remark that, in the proof above, the situation in which we consider the constant function $f_1(g_i) = 1$ for every $i \in [n]$, leads to a graph which indeed represents a rooted product graph, and so, the value of its metric dimension can be also deduced from results appearing in [10, 24].

For a given integer $k \geq 3$, let F_k be the graph obtained as follows. We begin with a cycle $v_0v_1 \dots v_{2k-1}v_0$ and do all computations modulo $2k$. Next, we add k isolated vertices $u_0, u_2, u_4, \dots, u_{2(k-1)}$ and the edges $u_{2i}v_{2i-1}, u_{2i}v_{2i}$ for every $0 \leq i \leq k - 1$.

Lemma 3.2. *If $k \geq 3$, then $\dim(F_k) = 2$.*

Proof. We claim that the set $S = \{u_0, u_{2(\lceil k/2 \rceil - 1)}\}$ is a metric basis for F_k . Let x, y be any two distinct vertices of F_k with $x, y \notin S$ (if $x \in S$ or $y \in S$, then they are clearly identified by S). If $d(x, u_0) = d(y, u_0)$, then we have either one of the following situations.

Case 1: $x, y \in A_1 = \{v_0, v_1, \dots, v_{k-1}\} \cup \{u_0, u_2, \dots, u_{2\lceil k/2 \rceil}\}$.

In such situation, it must happen that (w.l.g.) $x = v_{2i}$ and $y = u_{2i}$ for some $i \in \{1, 2, \dots, \lceil k/2 \rceil\}$. If $i = \lceil k/2 \rceil - 1$, then $y \in S$ and the conclusion is clear. In all the other cases we notice that v_{2i} belongs to the $u_{2i}, u_{2(\lceil k/2 \rceil - 1)}$ -geodesic, which means $u_{2(\lceil k/2 \rceil - 1)}$ identifies $x = v_{2i}$ and $y = u_{2i}$.

Case 2: $x, y \in A_2 = V(F_k) \setminus A_1$.

A similar conclusion as in Case 1 can be deduced, but taking into account that $x = v_{2i-1}$ and $y = u_{2i}$ for some $i \in \{\lceil k/2 \rceil + 1, \dots, k - 1\}$.

Case 3: $x \in A_1$ and $y \in A_2$.

Hence, $x = u_i$ or $x = v_i$, and $y = v_{2k-i-1}$ or $y = u_{2k-i}$, for some $i \in \{0, 1, \dots, k-1\}$, where if $x = u_i$, then i is even. We consider now the distances between x, y and $u_{2(\lceil k/2 \rceil - 1)}$. That is:

$$\begin{aligned} d(v_i, u_{2(\lceil k/2 \rceil - 1)}) &= 2(\lceil k/2 \rceil - 1) - i = 2\lceil k/2 \rceil - i - 2, \\ d(u_i, u_{2(\lceil k/2 \rceil - 1)}) &= 2(\lceil k/2 \rceil - 1) - i + 1 = 2\lceil k/2 \rceil - i - 1, \\ d(v_{2k-i-1}, u_{2(\lceil k/2 \rceil - 1)}) &= 2k - i - 2(\lceil k/2 \rceil - 1) = 2\lceil k/2 \rceil - i + 2, \\ d(u_{2k-i}, u_{2(\lceil k/2 \rceil - 1)}) &= 2k - i - 2(\lceil k/2 \rceil - 1) + 1 = 2\lceil k/2 \rceil - i + 3. \end{aligned}$$

Consequently, we now obtain a contradiction from each situation in which we would suppose that $d(x, u_{2(\lceil k/2 \rceil - 1)}) = d(y, u_{2(\lceil k/2 \rceil - 1)})$, for any possible assumption taken for x, y as considered before. Therefore, x, y are identified by $u_{2(\lceil k/2 \rceil - 1)}$, and so, S is a resolving set. Since F_k is not a path, then S it is indeed a metric basis, as claimed. □

Theorem 3.3. *If $n \geq 3$, then $\text{dim}_S(C_n, C_3) = 2$.*

Proof. Clearly, $\text{dim}_S(C_n, C_3) \geq 2$, hence we only need to prove that $\text{dim}_S(C_n, C_3) \leq 2$. We use the notation from the first two paragraphs of the proof of Theorem 3.1. In particular, $G = C_n$ and $H = C_3$.

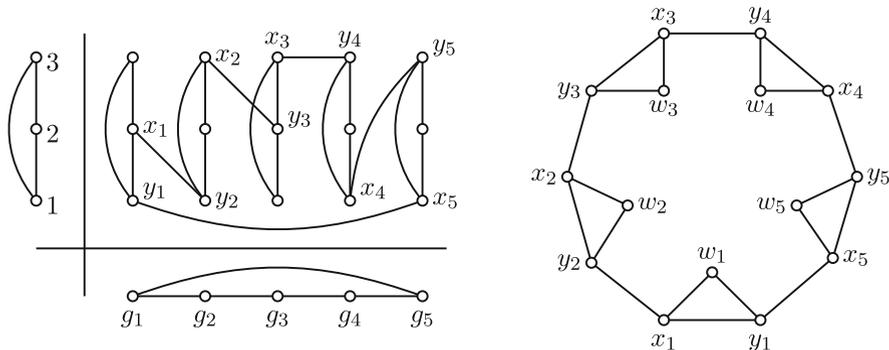


Figure 1. $C_5 \otimes_{f_5} C_3$, where $f_5 = (1, 2, 3, 3, 1)$

For each $n \geq 3$ we define a function $f_n : V(G) \rightarrow V(H)$ as follows. To do so, let us represent a function $f : V(G) \rightarrow V(H)$ as the vector $(f(g_1), f(g_2), \dots, f(g_n))$. Then, set $f_3 = (1, 2, 3)$ and $f_4 = (1, 1, 2, 2)$. Let then $n \geq 5$.

Case 1: n is odd.

Let B be the sequence 3, 1, 2, 3. Let f_n be defined as follows.

- If $n = 4(k + 1) + 1, k \geq 0$, then $f_n = (1, 2, 3, B, \dots, B, 3, 1)$, where B appears k times.
- If $n = 4(k + 1) + 2, k \geq 0$, then $f_n = (1, 2, 3, B, \dots, B)$, where B appears $k + 1$ times.

Case 2: n is even.

Let C be the sequence 2, 2, 3, 3. Let f_n be defined as follows.

- If $n = 4(k + 1), k \geq 0$, then $f_n = (1, 1, C, \dots, C, 2, 2)$, where C appears k times.
- If $n = 4(k + 1) + 2, k \geq 0$, then $f_n = (1, 1, C, \dots, C)$, where C appears $k + 1$ times.

It is straightforward to verify that for every $n \geq 3$, the Sierpiński product $G \otimes_{f_n} H$ has the following structure. For each i we have $x_i \neq y_i$, and hence w_i is the third vertex from $V(iH)$, see Fig. 1 where $G \otimes_{f_5} H$ is drawn in two different ways. It is now clear that $G \otimes_{f_n} H \cong F_n$ and hence Lemma 3.2 completes the argument. □

4. Convexity Property of Sierpiński Products

In this section, we establish a distance convex property of the Sierpiński product of two graphs. Recall that a subgraph H of a graph G is *convex* if whenever $u, v \in V(H)$ and P is a shortest u, v -path in G , then P lies completely in H .

Theorem 4.1. *If G and H be connected graphs, $f : V(G) \rightarrow V(H)$, and $g \in V(G)$, then gH is a convex subgraph of $G \otimes_f H$.*

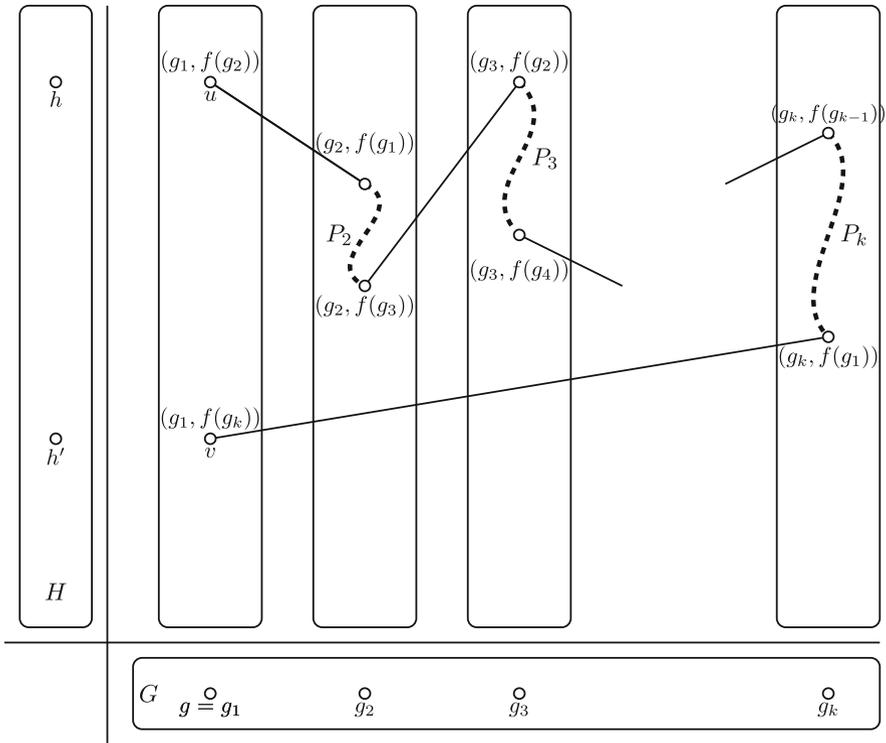


Figure 2. The shape of P

Proof. Throughout the proof, let $X = G \otimes_f H$. Suppose on the contrary that there exists vertices $u, v \in V(gH)$ such that $d_X(u, v) < d_{gH}(u, v)$. Note that this does not happen in trees, hence, in the rest, we may assume that G contains cycles. Suppose now that u, v , and gH are selected such that $d_X(u, v)$ is as small as possible among all such counterexamples. Let $u = (g, h)$, $v = (g, h')$, and let P be a shortest u, v -path in X . Set further $g_1 = g$.

Claim. The shape of the path P is as follows. Let g_1, \dots, g_k , $k \geq 2$, be the vertices of G ordered such that P passes through g_1H, \dots, g_kH in that order. Then P starts with the connecting edge $(g_1, f(g_2))(g_2, f(g_1))$, proceeds with a geodesic P_2 in g_2H between $(g_2, f(g_1))$ and $(g_2, f(g_3))$, then continuing with the connecting edge $(g_2, f(g_3))(g_3, f(g_2))$, and so on. Finally, P arrives at g_kH , proceeds along a geodesic in G_k between $(g_k, f(g_{k-1}))$ and $(g_k, f(g_1))$, and ends with the connecting edge $(g_k, f(g_1))(g_1, f(g_k))$, where $f(g_k) = h'$. See Fig. 2.

Note first that $k \geq 3$ because $k = 2$ would imply that there are two connecting edges between g_1H and g_2H .

We next show that the vertices g_1, \dots, g_k are pairwise different. Suppose on the contrary that there exist i and j , such that $2 \leq i < j < k$ and $g_{j+1} = g_i$. Let P' be the subpath of P between the vertices $u' = (g_i, f(g_{i+1}))$

and $v' = (g_{j+1}, f(g_j)) = (g_i, f(g_j))$. As P is a geodesic in X , Bellman's principle of optimality implies that P' is also a geodesic. In addition to its connecting edges, P' contains geodesics P_i, P_{i+1}, \dots, P_j , which are, respectively, projected onto H , geodesics

- between $f(g_{i-1})$ and $f(g_{i+1})$,
- between $f(g_i)$ and $f(g_{i+2})$,
- \vdots
- between $f(g_{j-2})$ and $f(g_j)$, and
- between $f(g_{j-1})$ and $f(g_i)$.

Suppose first that $j - i$ is odd. Then, we have

$$\begin{aligned} |P'| &> d_H(f(g_{i-1}), f(g_{i+1})) + d_H(f(g_{i+1}), f(g_{i+3})) + \dots + d_H(f(g_{j-2}), f(g_j)) \\ &\geq d_H(f(g_{i-1}), f(g_j)) \\ &= d_{g_i H}(u', v'). \end{aligned}$$

This is a contradiction with the selection of u and v as a minimal counterexample. Suppose second that $j - i$ is even. Then, we have

$$\begin{aligned} |P'| &> [d_H(f(g_{i-1}), f(g_{i+1})) + d_H(f(g_{i+1}), f(g_{i+3})) + \dots + d_H(f(g_{j-1}), f(g_i))] \\ &\quad + [d_H(f(g_i), f(g_{i+2})) + d_H(f(g_{i+2}), f(g_{i+4})) + \dots + d_H(f(g_{j-2}), f(g_j))] \\ &\geq d_H(f(g_{i-1}), f(g_i)) + d_H(f(g_i), f(g_j)) \\ &\geq d_H(f(g_{i-1}), f(g_j)) \\ &= d_{g_i H}(u', v'). \end{aligned}$$

Hence, we get the same contradiction as in the previous case.

We have thus proved that the vertices g_1, \dots, g_k are pairwise different. To complete the proof of the claim, we need to verify that P starts and ends with a connecting edge. Suppose on the contrary that P starts with a subpath in g_1H from u to w and then proceed along the connecting edge between g_1H and g_2H . By the minimality assumption on u and v , we have $d_X(u, w) = d_{g_1H}(u, w)$. We now have:

$$\begin{aligned} d_X(u, w) + d_X(w, v) &= d_X(u, v) \\ &< d_{g_1H}(u, v) \\ &\leq d_{g_1H}(u, w) + d_{g_1H}(w, v) \\ &= d_X(u, w) + d_{g_1H}(w, v) \end{aligned}$$

which yields $d_X(w, v) < d_{g_1H}(w, v)$. This contradiction proves that P indeed starts with a connecting edge. A parallel argument yields that P also ends with a connecting edge. This proves the claim.

To conclude the proof, let P_2, P_3, \dots, P_k be the sections of P restricted to g_2H, g_3H, \dots, g_kH , respectively. Then, we proceed similarly as we did for

the subpath P' above. More precisely, if $k - 1$ is odd, then

$$\begin{aligned} |P| &> d_H(f(g_2), f(g_4)) + d_H(f(g_4), f(g_6)) + \cdots + d_H(f(g_{k-2}), f(g_k)) \\ &\geq d_H(f(g_2), f(g_k)) \\ &= d_{g_1 H}(u, v). \end{aligned}$$

And if $k - 1$ is even, then

$$\begin{aligned} |P'| &> [d_H(f(g_1), f(g_3)) + d_H(f(g_3), f(g_5)) + \cdots + d_H(f(g_{k-2}), f(g_k))] \\ &\quad + [d_H(f(g_2), f(g_4)) + d_H(f(g_4), f(g_6)) + \cdots + d_H(f(g_{k-1}), f(g_1))] \\ &\geq d_H(f(g_1), f(g_k)) + d_H(f(g_2), f(g_1)) \\ &\geq d_H(f(g_2), f(g_k)) \\ &= d_{g_1 H}(u, v). \end{aligned}$$

This final contradiction proves the theorem. \square

5. Concluding Remarks

In Theorem 2.6, we determined $\dim_S(P_n, T)$ where T is an arbitrary tree different from a path. It remains to determine $\dim_S(T', T)$ where T' and T are arbitrary trees.

In Theorems 3.1 and 3.3, we determined $\text{Dim}_S(C_n, C_3)$ and $\dim_S(C_n, C_3)$ for all $n \geq 3$. It remains to determine $\text{Dim}_S(C_n, C_m)$ and $\dim_S(C_n, C_m)$ for all $n \geq 3$ and $m \geq 4$.

It would be interesting to determine $\text{Dim}_S(G, H)$ and $\dim_S(G, H)$ for other classes of graphs G and H .

Acknowledgements

Research of Michael Henning was supported in part by the South African National Research Foundation (grant number 129265) and the University of Johannesburg, and obtained during his sabbatical visit at the University of Ljubljana. He thanks them for their kindness in hosting him. Sandi Klavžar was supported by the Slovenian Research Agency (ARRS) under the Grants P1-0297, J1-2452, and N1-0285. Ismael G. Yero has been partially supported by the Spanish Ministry of Science and Innovation through the Grant PID2019-105824GB-I00. Moreover, this investigation was developed while this author (Ismael G. Yero) was visiting the University of Ljubljana, Slovenia, supported by “Ministerio de Educación, Cultura y Deporte”, Spain, under the “José Castillejo” program for young researchers (reference number: CAS21/00100).

Author Contribution All authors wrote and reviewed the manuscript.

Data Availability Our manuscript has no associated data.

Declarations

Conflict of Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Received: July 27, 2022.

Revised: July 29, 2022.

Accepted: October 25, 2023.