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Mixed Riemann-Hilbert boundary value problem with simply connected fibers

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ABSTRACT

We study the existence of solutions of mixed Riemann-Hilbert or Cherepanov boundary value problem with simply connected fibers on the unit disk Δ. Let L be a closed arc on $\partial\Delta$ with the end points ω_{-1}, ω_1 and let a be a smooth function on L with no zeros. Let $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ be a smooth family of smooth Jordan curves in $\mathbb{C}$ which all contain point 0 in their interiors and such that $\gamma_{\omega_{-1}}$, γ_{ω_1} are strongly starshaped with respect to 0. Then under condition that for each $w \in \gamma_{\omega_{\pm 1}}$ the angle between w and the normal to $\gamma_{\omega_{\pm 1}}$ at w is less than $\frac{\pi}{10}$, there exists a Hölder continuous function f on $\overline{\Delta}$, holomorphic on Δ, such that

$$\mathrm{Re}(\overline{a(\xi)}f(\xi)) = 0 \text{ on } L \quad \text{and} \quad f(\xi) \in \gamma_\xi \text{ on } \partial\Delta \setminus \mathring{L}.$$

1. Introduction

Let $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $\partial\Delta = \{\xi \in \mathbb{C}; |\xi| = 1\}$ be the unit circle. Let L be a closed arc on $\partial\Delta$, let $\mathring{L}$ denote its interior with respect to $\partial\Delta$, and let $a : L \to \partial\Delta$ be a smooth function.

Recall that the interior $\mathrm{Int}(\gamma)$ of a Jordan curve $\gamma \subset \mathbb{C}$ is the bounded component of $\mathbb{C} \setminus \gamma$. We orient γ positively with respect to $\mathrm{Int}(\gamma)$. Jordan curve $\gamma \subset \mathbb{C}$ is *starshaped* with respect to 0, if for any point w in the interior of γ the line segment which connects points 0 and w lies in the interior of γ, and it is *strongly starshaped* with respect to 0, [12], if there exists a positive continuous function R on the unit circle such that

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$$\gamma = \left\{ w \in \mathbb{C}; |w| = R\left(\frac{w}{|w|}\right) \right\} \tag{1}$$

and

$$\mathrm{Int}(\gamma) = \left\{ w \in \mathbb{C} \setminus \{0\}; |w| < R\left(\frac{w}{|w|}\right) \right\} \cup \{0\}. \tag{2}$$

Let $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$ be a smooth family of smooth Jordan curves in $\mathbb{C}$ which all contain point 0 in their interiors. In this paper we study the existence and properties of *holomorphic* solutions of the nonlinear mixed Riemann-Hilbert problem, that is, the Cherepanov boundary value problem with simply connected fibers. The problem asks for a continuous function f on $\overline{\Delta}$, holomorphic on Δ, such that

$$\mathrm{Re}(\overline{a(\xi)} f(\xi)) = 0 \text{ for } \xi \in L \tag{3}$$

and

$$f(\xi) \in \gamma_\xi \text{ for } \xi \in \partial\Delta \setminus \mathring{L}. \tag{4}$$

That is, f solves a linear Riemann-Hilbert problem on L and a nonlinear Riemann-Hilbert problem with simply connected fibers on $\partial\Delta \setminus \mathring{L}$. See also [1,2,13,14].

The problem with circular fibers γ_ξ and L a finite union of disjoint arcs was considered by Obnosov and Zulkarnyaev in [14], and by the author in [5]. The structure of the family of solutions of problem (3)-(4) is well known in the cases where either $L = \partial\Delta$ or $L = \emptyset$. If $L = \partial\Delta$, we consider a homogeneous linear Riemann-Hilbert problem. In this case the essential information on the problem is given by the winding number $W(a)$ of function a. It is well known [10,17,18] that if the winding number $W(a)$ is nonnegative, the space of solutions of (3) is a vector subspace of $A^\alpha(\Delta)$, $0 < \alpha < 1$, of real dimension $2W(a) + 1$.

Remark 1.1. The linear Riemann-Hilbert problem can also be considered in the case of a nonorientable line bundle over $\partial\Delta$, that is, in the case where at some point $\xi_0 \in \partial\Delta$ we have $a(\xi_0^-) = -a(\xi_0^+)$. Then the winding number of function a^2 or the Maslov index of the problem is an odd integer. In this case it holds that if $W(a^2) \geq -1$, or, with a little bit of abuse of notation, if $W(a) \geq -\frac{1}{2}$, then the space of solutions of (3) is a vector subspace of $A^\alpha(\Delta)$ of real dimension $2W(a) + 1$, see [3,4,15,18].

If L is empty, we have a nonlinear Riemann-Hilbert problem with smooth simply connected fibers which all contain 0 in their interiors. This problem was considered and solved in [8,16–18]. In particular, it was proved that the family of solutions with exactly m zeros on Δ, $m \in \mathbb{N} \cup \{0\}$, forms a manifold in space $A^\alpha(\Delta)$ of dimension $2m + 1$, and this manifold is compact if and only if $m = 0$. We assume from now on that neither $L = \emptyset$ nor $L = \partial\Delta$.

Theorem 1.2. *Let $k \geq 3$. Let $a : L \to \mathbb{C} \setminus \{0\}$ be a C^{k+1} function and let $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$ be a C^k family of Jordan curves in $\mathbb{C}$ which all contain point 0 in their interiors. Let ω_1 and ω_{-1} be the first and the last point of arc L with respect to the positive orientation of $\partial\Delta$. Let Jordan curves γ_{ω_j}, $j = \pm 1$, be strongly starshaped with respect to 0 and such that for each $w \in \gamma_{\omega_j}$ the angle between w and the normal to γ_{ω_j} at w is less than $\frac{\pi}{10}$. Let w_j, $j = \pm 1$, be the intersection of γ_{ω_j} and the line $\mathrm{Re}(\overline{a(\omega_j)} w) = 0$ of the form $\lambda(-ia(\omega_j))$, $\lambda > 0$, and let $\pi\beta_j$ be the oriented angle of intersection of the line $\mathrm{Re}(\overline{a(\omega_j)} w) = 0$ with the fiber γ_{ω_j} at point w_j, where $\beta_1 \in (0, 1)$ and $\beta_{-1} \in (-1, 0)$. Let*

$$0 < \beta < \min\{\beta_1, 1 - \beta_1, |\beta_{-1}|, 1 - |\beta_{-1}|\}. \tag{5}$$

Then there exists a unique $f \in A^\beta(\Delta)$ *with no zeros on* Δ *which solves (3)-(4) for which* $f(\omega_1) = w_1$ *and* $f(\omega_{-1}) = w_{-1}$.

Remark 1.3. Here $\beta_1 > 0$, if the tangent vector $-ia(\omega_1)$ to $\mathrm{Re}(\overline{a(\omega_1)}w) = 0$ is rotated counterclockwise by angle $\pi\beta_1$ to get a positive tangent vector to γ_{ω_1} at point w_1, and $\beta_{-1} < 0$, if a positive tangent vector to $\gamma_{\omega_{-1}}$ at w_{-1} is rotated clockwise by angle $\pi|\beta_1|$ to get tangent vector $-ia(\omega_{-1})$ to $\mathrm{Re}(\overline{a(\omega_{-1})}w) = 0$.

Remark 1.4. Observe that conditions in Theorem 1.2 imply $||\beta_j| - \frac{1}{2}| < \frac{1}{10}$, $j = \pm 1$, and hence one could choose $\beta = \frac{2}{5}$.

Remark 1.5. In the cases considered in [5,14] all boundary curves were circles with center at point 0. Hence $|\beta_j| = \frac{1}{2}$, $j = \pm 1$, and the maximal regularity we got was $\beta < \frac{1}{2}$.

Corollary 1.6. *Let* $a_1, \dots, a_n \in \Delta$ *be a finite set of points with given multiplicities. Then under the assumptions of Theorem 1.2 there exists* $\beta \in (0, 1)$ *and* $f \in A^\beta(\Delta)$ *which has zeros exactly at points* $a_1, \dots, a_n \in \Delta$ *with the given multiplicites and which solves (3)-(4).*

2. Function spaces, Hilbert transform and defining functions

Let $0 < \alpha < 1$ and let $G \subset \mathbb{C}$ be a compact subset. We denote by $C^\alpha(G)$ the algebra over $\mathbb{C}$ of Hölder continuous complex functions on G and by $C^\alpha_{\mathbb{R}}(G)$ the algebra over $\mathbb{R}$ of real Hölder continuous functions on G. Using the norm

$$\|f\|_\alpha = \max_{z\in G} |f(z)| + \sup_{z,w\in G, z\neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} \tag{6}$$

the algebras $C^\alpha(G)$ and $C^\alpha_{\mathbb{R}}(G)$ become Banach algebras. For $G = \overline{\Delta}$ or $G = \partial\Delta$ and $k \in \mathbb{N} \cup \{0\}$ we also define spaces $C^{k,\alpha}(G)$ and $C^{k,\alpha}_{\mathbb{R}}(G)$ of k times continuously differentiable functions on G, whose all k-th derivatives belong to space $C^\alpha(G)$ or space $C^\alpha_{\mathbb{R}}(G)$.

We also need some algebras of holomorphic functions on Δ. By $A(\Delta)$ we denote the *disc algebra*, that is, the algebra of continuous functions on $\overline{\Delta}$ which are holomorphic on Δ, and by $A^\alpha(\Delta) = A(\Delta) \cap C^\alpha(\overline{\Delta})$ the algebra of Hölder continuous functions on the closed disc which are holomorphic on Δ. Using appropriate norms, that is, the maximum norm $\|\cdot\|_\infty$ for $A(\Delta)$ and the Hölder norm $\|\cdot\|_\alpha$ for $A^\alpha(\Delta)$, these algebras become Banach algebras. Similarly we define $A^{k,\alpha}(\Delta) = A(\Delta) \cap C^{k,\alpha}(\overline{\Delta})$ ($k \in \mathbb{N} \cup \{0\}, 0 < \alpha < 1$).

Recall that Hilbert transform H assigns to a real function u on $\partial\Delta$ a real function Hu on $\partial\Delta$ such that the harmonic extension of $f = u + i\,Hu$ to Δ is holomorphic on Δ and real at 0. It is known that H is a bounded linear operator on $C^{k,\alpha}_{\mathbb{R}}(\partial\Delta)$ ($k \in \mathbb{N} \cup \{0\}, 0 < \alpha < 1$), [18, §1.6.11], and hence the harmonic extension of $f = u + i\,Hu$ to Δ belongs to $A^{k,\alpha}(\Delta)$. Also, [18, §1.6.11], the Hilbert transform is a bounded linear operator on the Sobolev space $W^k_p(\partial\Delta)$ of k times generalized differentiable functions with derivatives in $L^p(\partial\Delta)$ ($k \in \mathbb{N} \cup \{0\}, 1 < p < \infty$) equipped with the norm

$$\|f\|_{W^k_p} = \left(\sum_{j=0}^{k} \|D^j f\|_p\right)^{\frac{1}{p}}. \tag{7}$$

Recall, [18, §1.6.14], that if $\partial\Delta = T_1 \cup T_2$ is a partition of $\partial\Delta$ in two subarcs T_1 and T_2 and if $T_0 \subseteq T_1$ is a compactly contained subarc of T_1, then for $k \in \mathbb{N} \cup \{0\}$, $1 < p < \infty$, $0 < \alpha < 1$ there exists a constant $C = C(k, p, \alpha)$ such that

$$\|Hu\|_{W_p^k(T_0)} \leq C(\|u\|_{W_p^k(T_1)} + \|u\|_{L^1(T_2)}) \tag{8}$$

and

$$\|Hu\|_{C^{k,\alpha}(T_0)} \leq C(\|u\|_{C^{k,\alpha}(T_1)} + \|u\|_{L^1(T_2)}). \tag{9}$$

We will also need compact embedding result, [18, §1.1.8],

$$W_p^1(\partial\Delta) \hookrightarrow C^\beta(\partial\Delta) \hookrightarrow C^\alpha(\partial\Delta) \tag{10}$$

for $0 < \alpha < \beta < 1 - \frac{1}{p}$, $1 < p < \infty$, which holds on arcs in $\partial\Delta$ as well.

Since $L \neq \partial\Delta$ we can extend a to $\partial\Delta$ as a nowhere zero function of class C^{k+1} so that the winding number $W(a) = 0$. Therefore, [18, p. 25], we can write $\overline{a}$ in the form

$$\overline{a} = re^h, \tag{11}$$

where $r > 0$ is a positive $C^{k,\alpha}$ function on $\partial\Delta$ and $h \in A^{k,\alpha}(\Delta)$. Thus the original problem (3)-(4) is equivalent to the problem

$$\operatorname{Im}(f_*(\xi)) = 0 \quad \text{for} \quad \xi \in L \tag{12}$$

and

$$f_*(\xi) \in \gamma_\xi^* \quad \text{for} \quad \xi \in \partial\Delta \setminus \mathring{L}, \tag{13}$$

where $f_* = ie^h f$ and $\gamma_\xi^* = ie^{h(\xi)}\gamma_\xi$. Observe that the number of zeros of f_* and f are the same and that 0 belongs to the interiors of all curves γ_ξ^*, $\xi \in \partial\Delta \setminus \mathring{L}$. Also, since for each $\xi \in \partial\Delta$ the transformation

$$w \longmapsto ie^{h(\xi)}w \tag{14}$$

is a composition of a dilation and a rotation, the angle conditions from Theorem 1.2 stay the same.

Using a holomorphic automorphism of the unit disc we may even assume that $L = \{\xi \in \partial\Delta; \operatorname{Im}(\xi) \leq 0\}$ is the lower semicircle. From now on we will consider problem (12)-(13) with the addition that L is the lower semicircle and instead of f_* and $\{\gamma_\xi^*\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ we will still write f and $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$.

Remark 2.1. One can also create the 'double' of the boundary value problem. Using a biholomorphism one can replace the unit disc Δ with the upper half-disk $\Delta_+ = \{\xi \in \Delta; \operatorname{Im}(\xi) > 0\}$ and L by the interval $[-1, 1]$.

By the reflection principle we see that problem (12)-(13) is equivalent to the nonlinear Riemann-Hilbert problem on Δ, where the boundary curves $\{\gamma_\xi\}_{\xi\in\partial\Delta_+\setminus\mathring{L}}$ are symmetrically extended and defined on the lower semicircle so that we have

$$\gamma_\xi = \overline{\gamma_{\overline{\xi}}} \tag{15}$$

for every $\xi \in \partial\Delta \setminus \{1, -1\}$. In general this symmetrical extension of Jordan curves $\{\gamma_\xi\}_{\xi\in\partial\Delta_+\setminus\mathring{L}}$ to the lower semicircle produces boundary data which are not continuous at points 1 and -1. Because the biholomorphism from Δ to the upper semidisc is in $A^{\frac{1}{2}}(\Delta)$, we get that the regularity of solutions of (12)-(13) is in general a half of the regularity of solutions of the symmetrical Riemann-Hilbert problem.

We will consider smooth families of smooth Jordan curves $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ in $\mathbb{C}$. Let $k \in \mathbb{N}$. The family of Jordan curves $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ is a C^k family parametrized by $\xi \in \partial\Delta \setminus \mathring{L}$ if there exists a function $\rho \in$

$C^k((\partial\Delta \setminus \mathring{L}) \times \mathbb{C})$ such that

$$\gamma_\xi = \{w \in \mathbb{C}; \rho(\xi, w) = 0\} \quad \text{and} \quad \mathrm{Int}(\gamma_\xi) = \{w \in \mathbb{C}; \rho(\xi, w) < 0\}, \tag{16}$$

and the gradient $\frac{\partial \rho}{\partial \overline{w}}(\xi, w) = \rho_{\overline{w}}(\xi, w) \neq 0$ for every $\xi \in \partial\Delta \setminus \mathring{L}$ and $w \in \gamma_\xi$. We call ρ a *defining function* for C^k family of Jordan curves $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$. We will consider only bounded families of Jordan curves which all lie in some fixed disc $\overline{\Delta(0, R)}$, $R > 0$, and the space $C^k((\partial\Delta \setminus \mathring{L}) \times \overline{\Delta(0, R)})$ is equipped with the standard C^k norm.

Since we assume that $\gamma_{\pm 1}$ are strongly starshaped Jordan curves, we also assume that for ρ, the defining function for Jordan curves $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$, and $j = \pm 1$ we have

$$\rho(j, w) = |w|^2 - R_j^2\left(\frac{w}{|w|}\right) \tag{17}$$

for some positive C^k functions $R_j(z)$ on $\mathbb{C}$.

Using parametrization $\theta \mapsto e^{i\theta}$ of the unit circle we will also use the notation γ_θ, $\rho(\theta, w)$ and $\rho_\theta(\theta, w)$ instead of γ_ξ, $\rho(\xi, w)$ and $\rho_\xi(\xi, w)$. Also, for a function h on $\partial\Delta$, we will write either $h(\xi)$ or $h(\theta)$, where $\xi = e^{i\theta}$. Observe that if h is holomorphic on Δ with well defined derivative on $\partial\Delta$, then $\frac{\partial h}{\partial \theta}(\theta) = i\xi h'(\xi)$ for $\xi = e^{i\theta}$.

Remark 2.2. The reflection principle and the symmetric extension to the lower semicircle mentioned in Remark 2.1 is in terms of defining function ρ given as

$$\rho(\xi, w) = \rho(\overline{\xi}, \overline{w}) \tag{18}$$

for every $\xi \in \partial\Delta \setminus \{1, -1\}$ and every $w \in \mathbb{C}$.

3. Regularity of solutions

In this section we prove regularity of continuous solutions of a specific form of problem (12)-(13), where the defining function $\rho \in C^k((\partial\Delta \setminus \mathring{L}) \times \mathbb{C})$ $(k \geq 3)$.

Let $f \in A(\Delta)$ be a solution of (12)-(13). It is well known [6–8,18] that f restricted to $\partial\Delta \setminus \{-1, 1\}$ is in $C^{k-1,\alpha}$ for any $0 < \alpha < 1$. Hence we need information on the regularity of f near points $\xi = \pm 1$. For $j = \pm 1$ we denote $f(j) = w_j \in \mathbb{R} \cap \gamma_j$.

Using Möbius transformation from the unit disc Δ to the upper half-plane $\mathcal{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$ we consider the case where f is bounded and continuous on $\overline{\mathcal{H}}$ and holomorphic on $\mathcal{H}$. Also, point $\xi = 1$ is mapped into $t = 0$ and point $\xi = -1$ into ∞. Now f solves the problem

$$\mathrm{Im}(f(t)) = 0 \text{ for } t \leq 0 \tag{19}$$

and

$$f(t) \in \gamma_t \text{ for } t \geq 0. \tag{20}$$

Also, using translation, we will assume that $f(0) = 0 \in \mathbb{R} \cap \gamma_0$.

Let $\pi\beta_1$ $(\beta_1 \in (-1, 1) \setminus \{0\})$ be the oriented angle of intersection of the real axis $\mathrm{Im}(w) = 0$ and γ_0 at $w = f(0) = 0$. The orientation of the real axis is positive with respect to the upper half-plane and the orientation of γ_0 is positive with respect to the interior of γ_0. Hence $\beta_1 > 0$, if the tangent vector to the real axis is rotated counterclockwise by angle $\pi\beta_1$ to get a tangent vector to γ_0 at point 0, and $\beta_1 < 0$, if the tangent vector to the real axis is rotated clockwise by angle $\pi|\beta_1|$ to get a tangent vector to γ_0 at 0.

The defining function ρ can near $(0,0)$ for $t \geq 0$ be written as

$$\rho(t,w) = \rho(0,0) + \rho_t(0,0)t + 2\mathrm{Re}(\rho_w(0,0)w) + \frac{1}{2}\rho_{tt}(0,0)t^2 + \tag{21}$$

$$+ \rho_{w\overline{w}}(0,0)|w|^2 + \mathrm{Re}(\rho_{ww}(0,0)w^2 + \rho_{tw}(0,0)tw) + \sqrt{t^2+|w|^2}\, g(t,w), \tag{22}$$

where $g \in C^1(\mathbb{R}\times\mathbb{C})$ such that $g(0,0) = g_t(0,0) = g_w(0,0) = g_{\overline{w}}(0,0) = 0$.

Recall that $\rho(0,0) = 0$ and that $\rho_{\overline{w}}(0,0)$ represents an outer normal to γ_0 at point $w = 0$. So we have

$$\rho_{\overline{w}}(0,0) = -i\lambda e^{i\pi\beta_1} \tag{23}$$

for some real $\lambda > 0$. We may assume $\lambda = \frac{1}{2}$.

Because

$$\mathrm{Re}(ie^{-i\pi\beta_1}w) = -\mathrm{Im}(e^{-i\pi\beta_1}w) = \mathrm{Im}(e^{i\pi(1-\beta_1)}w) \tag{24}$$

we have

$$\rho(t,w) = At + \mathrm{Im}(e^{i\pi(1-\beta_1)}w) + Bt^2 + C|w|^2 + \tag{25}$$

$$+ \mathrm{Re}(Dw^2) + t\,\mathrm{Re}(Ew) + \sqrt{t^2+|w|^2}\, g(t,w) \tag{26}$$

for some $A, B, C \in \mathbb{R}$ and $D, E \in \mathbb{C}$.

Let us assume that we have a solution f of the problem (19)-(20) of the form

$$f(t) = t^s\kappa(t), \tag{27}$$

where κ is bounded and continuous on $\overline{\mathcal{H}}$, holomorphic on $\mathcal{H}$, and $0 < s < 1$ to be determined.

For $t \leq 0$ we have $t = (-1)|t|$ and from (19) we get

$$\mathrm{Im}(e^{i\pi s}\kappa(t)) = -\mathrm{Im}(e^{i\pi(1+s)}\kappa(t)) = 0. \tag{28}$$

On the other hand for $t > 0$ we have

$$\frac{1}{t^s}\rho(t, t^s\kappa(s)) = At^{1-s} + \mathrm{Im}(e^{i\pi(1-\beta_1)}\kappa(t)) + Bt^{2-s} + Ct^s|\kappa(t)|^2 + \tag{29}$$

$$+ t^s\mathrm{Re}(D\kappa(t)^2) + t\mathrm{Re}(E\kappa(t)) + \sqrt{t^{2-2s}+|\kappa(t)|^2}\, g(t, t^s\kappa(t)) = 0. \tag{30}$$

We choose $0 < s < 1$ so that κ solves boundary value problem with continuous boundary data. That is, we choose $s = 1-\beta_1$, if $\beta_1 > 0$, and $s = -\beta_1 = |\beta_1|$, if $\beta_1 < 0$.

Thus κ solves the following Riemann-Hilbert problem

$$\mathrm{Im}(e^{i\pi(1-\beta_1)}\kappa(t)) = 0 \text{ for } t \leq 0 \tag{31}$$

and

$$\widetilde{\rho}(t, \kappa(t)) = 0 \text{ for } t \geq 0, \tag{32}$$

where, if $\beta_1 > 0$,

$$\widetilde{\rho}(t,w) = At^{\beta_1} + \mathrm{Im}(e^{i\pi(1-\beta_1)}w) + Bt^{1+\beta_1} + Ct^{1-\beta_1}|w|^2 + \tag{33}$$

$$+ t^{1-\beta_1}\mathrm{Re}(Dw^2) + t\mathrm{Re}(Ew) + \sqrt{t^{2\beta_1} + |w|^2}\, g(t, t^{1-\beta_1}w), \tag{34}$$

and, if $\beta_1 < 0$,

$$\widetilde{\rho}(t,w) = At^{1-|\beta_1|} + \mathrm{Im}(e^{i\pi(1-\beta_1)}w) + Bt^{2-|\beta_1|} + Ct^{|\beta_1|}|w|^2 + \tag{35}$$

$$+ t^{|\beta_1|}\mathrm{Re}(Dw^2) + t\mathrm{Re}(Ew) + \sqrt{t^{2-2|\beta_1|} + |w|^2}\, g(t, t^{|\beta_1|}w). \tag{36}$$

For such choice of s are the defining function for problem (31)-(32)

$$(t,w) \longmapsto \begin{cases} \widetilde{\rho}(t,w) = \frac{1}{t^s}\rho(t,t^s w); & t \geq 0, w \in \mathbb{C} \\ \mathrm{Im}(e^{i\pi(1-\beta_1)}w); & t \leq 0, w \in \mathbb{C} \end{cases} \tag{37}$$

and its partial w-derivative

$$(t,w) \longmapsto \begin{cases} \widetilde{\rho}_w(t,w) = \rho_w(t,t^s w); & t \geq 0, w \in \mathbb{C} \\ \frac{1}{2i}e^{i\pi(1-\beta_1)}; & t \leq 0, w \in \mathbb{C} \end{cases} \tag{38}$$

continuous on $\mathbb{R} \times \mathbb{C}$.

On the other hand, the partial derivative of defining function (37) with respect to the t variable is not continuous at $t = 0$, but, as we will see, it still has certain L^p regularity properties, which will imply regularity conditions on κ and f.

We know that κ is $C^{k-1,\alpha}$ on $\mathbb{R} \setminus \{0\}$ and we can differentiate (31)-(32) on $\mathbb{R} \setminus \{0\}$ to get

$$\mathrm{Im}(e^{i\pi(1-\beta_1)}\kappa'(t)) = 0 \text{ for } t < 0 \tag{39}$$

and

$$\widetilde{\rho}_t(t,\kappa(t)) + 2\mathrm{Re}(\widetilde{\rho}_w(t,\kappa(t))\kappa'(t)) = 0 \text{ for } t > 0. \tag{40}$$

For $t > 0$ and $\beta_1 > 0$ we have

$$\widetilde{\rho}_t(t,w) = A\beta_1 t^{\beta_1 - 1} + B(1+\beta_1)t^{\beta_1} + (1-\beta_1)Ct^{-\beta_1}|w|^2 + \tag{41}$$

$$+(1-\beta_1)t^{-\beta_1}\mathrm{Re}(Dw^2) + \mathrm{Re}(Ew) + \frac{\beta_1\, t^{2\beta_1 - 1}}{\sqrt{t^{2\beta_1} + |w|^2}}\, g(t, t^{1-\beta_1}w) + \tag{42}$$

$$+\sqrt{t^{2\beta_1} + |w|^2}\,(g_t(t, t^{1-\beta_1}w) + 2\mathrm{Re}(g_w(t, t^{1-\beta_1}w)(1-\beta_1)t^{-\beta_1}w)) \tag{43}$$

and for $t > 0$ and $\beta_1 < 0$ we have

$$\widetilde{\rho}_t(t,w) = A(1-|\beta_1|)t^{-|\beta_1|} + B(2-|\beta_1|)t^{1-|\beta_1|} + |\beta_1|Ct^{|\beta_1|-1}|w|^2 \tag{44}$$

$$+|\beta_1|t^{|\beta_1|-1}\mathrm{Re}(Dw^2) + \mathrm{Re}(Ew) + \frac{(1-|\beta_1|)\, t^{1-2|\beta_1|}}{\sqrt{t^{2-2|\beta_1|} + |w|^2}}\, g(t, t^{|\beta_1|}w) + \tag{45}$$

$$+\sqrt{t^{2-2|\beta_1|} + |w|^2}\,(g_t(t, t^{|\beta_1|}w) + 2\mathrm{Re}(g_w(t, t^{|\beta_1|}w)|\beta_1|t^{|\beta_1|-1}w)). \tag{46}$$

The t-derivative of defining function (37) is 0 for $t < 0$.

Since $\beta_1 \in (-1,1)\setminus\{0\}$ and κ is bounded, we have that $\widetilde{\rho}_t(t,\kappa(t))$ is in $L^p_{\mathrm{loc}}(\mathbb{R})$ for

$$1 \le p < \min\left\{\frac{1}{|\beta_1|}, \frac{1}{1-|\beta_1|}\right\}. \tag{47}$$

A similar argument can be used for point $\xi = -1 \in \partial\Delta$. Let $\pi\beta_{-1}$ ($\beta_{-1} \in (-1,1)\setminus\{0\}$) be the orientied angle of intersection of γ_{-1} and the real axis $\mathrm{Im}(w) = 0$ at point $f(-1)$. Now β_{-1} is positive, if a positive tangent vector to γ_{-1} at $f(-1)$ is rotated counterclockwise to get a positive tangent vector to the real axis and negative otherwise. For $j = \pm 1$ we define $\delta_j = 1 - \beta_j$, if $\beta_j \in (0,1)$, and $\delta_j = |\beta_j|$, if $\beta_j \in (-1,0)$.

To transfer our observations to the boundary value problem (12)-(13) on the unit disc, let $\Psi \in A^{\frac{1}{2}}(\Delta)$ be a biholomorphic map from Δ to the upper half-disc Δ_+, which maps the lower semicircle L on $[-1,1]$ so that $\Psi(\pm 1) = \pm 1$. Let $F(x) = \frac{1}{2}x(3-x^2)$. Then $F(x)-1 = -\frac{1}{2}(x-1)^2(x+2)$ and $F(x)+1 = -\frac{1}{2}(x+1)^2(x-2)$. Hence function $\psi(\xi) = F(\Psi(\xi))$ is real on L, $\psi(\pm 1) = \pm 1$, and C^1 on $\partial\Delta$.

Recall that w_j is the positive intersection of γ_j and the real axis, $j = \pm 1$. Now we consider only those solutions f of the Cherepanov problem (12)-(13), which are of the form

$$f(\xi) = (\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi) + w_1\frac{1+\psi(\xi)}{2} + w_{-1}\frac{1-\psi(\xi)}{2}, \tag{48}$$

where κ is in $A(\Delta)$.

We will define two (local) defining functions $\widetilde{\rho}_1(\xi,w)$ for $\xi \neq -1$ and $\widetilde{\rho}_{-1}(\xi,w)$ for $\xi \neq 1$. Let

$$T_1(\xi) = \frac{(\xi-1)}{i(\xi+1)} \quad \text{and} \quad T_{-1}(\xi) = \frac{1}{T_1(\xi)} = \frac{i(\xi+1)}{(\xi-1)}. \tag{49}$$

Then $T_1(-i) = T_{-1}(-i) = -1$, and T_1, T_{-1} map the upper semicircle to the positive real axis and the lower semicircle to the negative real axis. For $j = \pm 1$ and $\mathrm{Im}(\xi) > 0$ we define

$$\widetilde{\rho}_j(\xi,w) = \frac{1}{T_j(\xi)^{\delta_j}}\rho\left(\xi, (\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}w + w_1\frac{1+\psi(\xi)}{2} + w_{-1}\frac{1-\psi(\xi)}{2}\right) \tag{50}$$

and for $\mathrm{Im}(\xi) < 0$ we set

$$\widetilde{\rho}_j(\xi,w) = \mathrm{Im}\left(e^{i\pi(1-\beta_j)}\frac{(\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}}{T_j(\xi)^{\delta_j}}w\right). \tag{51}$$

As before one can check that $\widetilde{\rho}_j$ and $\widetilde{\rho}_{jw}$ are continuous on $\partial\Delta\setminus\{-j\}$, $j = \pm 1$. Since f solves the original boundary value problem, we have that $\widetilde{\rho}_j(\xi,\kappa(\xi)) = 0$, $j = \pm 1$.

Let $\chi : \partial\Delta\setminus\{-i\} \to [0,1]$ be a smooth function such that $\chi(\xi) = 1$ for $\xi = e^{i\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{3}$, and $\chi(\xi) = 0$ for $\xi = e^{i\theta}$, $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{2}$.

We define a new (global) defining function as $\widetilde{\rho}(\xi,w) = \chi(\xi)\widetilde{\rho}_1(\xi,w) + (1-\chi(\xi))\widetilde{\rho}_{-1}(\xi,w)$. Then $\widetilde{\rho}$ and $\widetilde{\rho}_w$ are well defined continuous function on $(\partial\Delta\setminus\{-i\})\times\mathbb{C}$. If β_1, β_{-1} have the same sign, then both functions are also continuous at $\xi = -i$, but if β_1, β_{-1} have the opposite signs, then

$$\widetilde{\rho}(-i^-,w) = -\widetilde{\rho}(-i^+,w) \ \text{ and } \ \widetilde{\rho}_w(-i^-,w) = -\widetilde{\rho}_w(-i^+,w), \tag{52}$$

which means that we have a nonorientable bundle as the boundary value data for κ.

Now locally considered problem (39)-(40) for $\kappa(t)$ and $\kappa'(t)$ becomes global boundary value problem for $\kappa(\theta)$ and $\frac{\partial\kappa}{\partial\theta}$ ($\xi = e^{i\theta}$). Hence $\frac{\partial\kappa}{\partial\theta}$ solves the linear Riemann-Hilbert problem

$$2\mathrm{Re}\left(\widetilde{\rho}_w(\theta,\kappa(\theta))\frac{\partial\kappa}{\partial\theta}\right) = -\widetilde{\rho}_\theta(\theta,\kappa(\theta)), \tag{53}$$

where $\widetilde{\rho}_w(\theta, \kappa(\theta))$ is either a nonzero continuous function on $\partial\Delta$ or

$$\widetilde{\rho}_w(-i^-, \kappa(-i)) = -\widetilde{\rho}_w(-i^+, \kappa(-i)) \tag{54}$$

and $\widetilde{\rho}_\theta(\theta, \kappa(\theta))$ belongs to the appropriate $L^p(\partial\Delta)$ space

$$1 \leq p < \min\left\{\frac{1}{|\beta_1|}, \frac{1}{1-|\beta_1|}, \frac{1}{|\beta_{-1}|}, \frac{1}{1-|\beta_{-1}|}\right\}. \tag{55}$$

Remark 3.1. In fact $\widetilde{\rho}_\theta(\theta, \kappa(\theta))$ belongs to L^p_{loc} for

$$1 \leq p < \min\left\{\frac{1}{|\beta_1|}, \frac{1}{1-|\beta_1|}\right\} \tag{56}$$

near $\xi = 1$ and to L^p_{loc} near $\xi = -1$ for

$$1 \leq p < \min\left\{\frac{1}{|\beta_{-1}|}, \frac{1}{1-|\beta_{-1}|}\right\}. \tag{57}$$

Let N be the winding number of function $\widetilde{\rho}_w(\theta, \kappa(\theta))$, that is, $2N$ is the Maslov index of the associated linear Riemann-Hilbert problem. If $\widetilde{\rho}_w(\theta, \kappa(\theta))$ is a continuous function on $\partial\Delta$, Maslov index is an even integer and hence N is an integer. On the other hand, if $\widetilde{\rho}_w(-i^-, \kappa(-i)) = -\widetilde{\rho}_w(-i^+, \kappa(-i))$, Maslov index is an odd integer and N is a half of an odd integer.

Let $r(\xi)$ be the square root function, where we take the branch where $\mathbb{C}$ is cut along the negative imaginary axis. Then function $\widetilde{\rho}_w(\theta, \kappa(\theta))$ can be written in the form

$$\widetilde{\rho}_w(\theta, \kappa(\theta)) = \xi^{-N} e^{u+iv}(\theta), \tag{58}$$

where u and v are real continuous functions on $\partial\Delta$, [18, p. 25]. In the case $N = \frac{2M+1}{2}$, $M \in \mathbb{Z}$, is a half of an odd integer, we define $\xi^N = \xi^M r(\xi)$, which corresponds to the sign changing of $\widetilde{\rho}_w$ at $\xi = -i$. See also [3,4,15]. Hence $e^{\pm Hv}$ belongs to $L^{p'}(\partial\Delta)$ for any $p' \geq 1$, [18, p. 23] and thus

$$e^{\pm i(v+iHv)} \tag{59}$$

belongs to $L^{p'}(\partial\Delta)$ for any $p' \geq 1$.

Therefore

$$\mathrm{Re}\left(\xi^{-N} e^{i(v+iHv)} \frac{\partial\kappa}{\partial\theta}\right) = -e^{-u} e^{-(Hv)} \widetilde{\rho}_\theta(\theta, \kappa). \tag{60}$$

We conclude that the right-hand side belongs to the same $L^p(\partial\Delta)$ space as function $\widetilde{\rho}_\theta(\theta, \kappa)$. Since Hilbert transform is bounded in $L^p(\partial\Delta)$ spaces, $1 < p < \infty$, [18, p. 23], we get that $\frac{\partial\kappa}{\partial\theta}$ is in $L^p(\partial\Delta)$ for the same set (55) of values of p as function $\widetilde{\rho}_\theta(\theta, \kappa)$. Therefore κ belongs to $L^{1,p}(\partial\Delta)$ for all such values of p and this implies that $\kappa \in C^\beta(\partial\Delta)$, [18, p. 10], where

$$0 < \beta < \min\{|\beta_1|, 1-|\beta_1|, |\beta_{-1}|, 1-|\beta_{-1}|\}. \tag{61}$$

Remark 3.2. Observe that regularity of κ and f could also be expressed locally, that is, near $j = \pm 1$ functions κ and f belong to Hölder space C^β, where $0 < \beta < \min\{|\beta_j|, 1-|\beta_j|\}$.

Proposition 3.3. *Let $k \geq 3$. Let $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$ be a C^k family of Jordan curves in $\mathbb{C}$. Let w_j, $j = \pm 1$, be an intersection of γ_j and the real axis and let $\pi\beta_j$, $\beta_j \in (-1,1) \setminus \{0\}$, be the oriented angle of intersection of γ_j with the real axis at point w_j. Let*

$$0 < \beta < \min\{|\beta_1|, 1 - |\beta_1|, |\beta_{-1}|, 1 - |\beta_{-1}|\}. \tag{62}$$

Then for every solution f of (12)-(13) of the form

$$f(\xi) = (\xi - 1)^{\delta_1}(\xi + 1)^{\delta_{-1}}\kappa(\xi) + w_1 \frac{1 + \psi(\xi)}{2} + w_{-1}\frac{1 - \psi(\xi)}{2}, \tag{63}$$

where $\kappa \in A(\Delta)$, we have $f, \kappa \in A^\beta(\Delta)$.

Remark 3.4. Observe that in cases where $\beta_1, \beta_{-1} \in (0,1)$, the regularity conditions we get for solutions of the Cherepanov/mixed Riemann-Hilbert problem (3)-(4) are consistent with results on the regularity of Riemann maps from the unit disc into simply connected domains bounded by Jordan curves which satisfy so called wedge condition, [11]. If the defining function ρ is independent of ξ and $\beta_j \in (0,1)$, we get $(1 - \beta_j)$-regularity. The β_j-regularity comes from ξ-dependence.

Similarly, the expected regularity and the 'order' of zeros of Riemann maps in the cases where $\beta_j \in (-1,0)$ and which are ξ independent, would be $1+|\beta_j|$, but ξ-dependence of the defining function ρ changes regularity conditions.

On the other hand, results in [9] show that in the case of nontransversal intersection of the real axis with either γ_1 or γ_{-1} solutions might not be of the form $(\xi - 1)^{\delta_1}\kappa(\xi)$ or $(\xi + 1)^{\delta_{-1}}\kappa(\xi)$ for some function $\kappa \in A(\Delta)$.

4. Linear Cherepanov boundary value problem

In this section we consider the linear version of problem (12)-(13), that is, a linear Riemann-Hilbert problem with piecewise continuous boundary data, [19, p. 169], and L the lower semicircle. First we consider homogeneous linear problem with piecewise continuous boundary data

$$\mathrm{Im}(f(\xi)) = 0 \text{ for } \xi \in L \tag{64}$$

and

$$\mathrm{Re}(\overline{B(\xi)}f(\xi)) = 0 \text{ for } \xi \in \partial\Delta \setminus \mathring{L}, \tag{65}$$

where B is a complex nonzero function of class C^β on the upper semicircle. The regularity exponent $\beta \in (0,1)$ is bounded by conditions given in Proposition 3.3. We may assume without loss of generality that $|B(\xi)| = 1$ for all $\xi \in \partial\Delta \setminus \mathring{L}$.

Let $\pi\beta_1$, $\beta_1 \in (-1,1) \setminus \{0\}$, be the oriented angle of intersection of the real axis $\mathrm{Im}(w) = 0$ and $\mathrm{Re}(\overline{B(1)}w) = 0$ at point 0, that is, $B(1) = -ie^{i\pi\beta_1}$. Similarly, let $\pi\beta_{-1}$, $\beta_{-1} \in (-1,1) \setminus \{0\}$, be the oriented angle of intersection of $\mathrm{Re}(\overline{B(-1)}w) = 0$ and the real axis $\mathrm{Im}(w) = 0$ at point 0, that is, $B(-1) = -ie^{-i\pi\beta_{-1}}$.

We search for solutions $f \in A(\Delta)$ of (64)-(65) of the form $f(\xi) = (\xi - 1)^{\delta_1}(\xi + 1)^{\delta_{-1}}\kappa(\xi)$ for some $\kappa \in A^\beta(\Delta)$. Recall that for $j = \pm 1$ we defined $\delta_j = 1 - \beta_j$, if $\beta_j \in (0,1)$, and $\delta_j = |\beta_j|$, if $\beta_j \in (-1,0)$. Hence we also have $f \in A^\beta(\Delta)$.

To define noninteger powers of $(\xi - 1)$ and $(\xi + 1)$ we take appropriate branches of the complex logarithm. For $(\xi - 1)^{\delta_1}$ the complex plane is cut along positive real numbers so that the argument of $(\xi - 1)$ for $\xi \in \partial\Delta$ lies on interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, and for $(\xi + 1)^{\delta_{-1}}$ the complex plane is cut along negative real numbers and the argument of $(\xi + 1)$ for $\xi \in \partial\Delta$ lies on interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

An argument similar to the argument in Section 3 shows that κ solves homogeneous linear Riemann-Hilbert problem

$$\operatorname{Re}(\overline{\widetilde{B}(\xi)}\kappa(\xi)) = 0 \text{ for all } \xi \in \partial\Delta, \tag{66}$$

where $\overline{\widetilde{B}} \in C^{\beta}(\partial D \setminus \{1\})$ is defined as

$$\overline{\widetilde{B}(\xi)} = \begin{cases} \overline{B(\xi)} \left(\frac{\xi-1}{|\xi-1|}\right)^{\delta_1} \left(\frac{\xi+1}{|\xi+1|}\right)^{\delta_{-1}}, & \text{if } \operatorname{Im}(\xi) > 0 \\ \pm i \left(\frac{\xi-1}{|\xi-1|}\right)^{\delta_1} \left(\frac{\xi+1}{|\xi+1|}\right)^{\delta_{-1}}, & \text{if } \operatorname{Im}(\xi) < 0 \end{cases} \tag{67}$$

with the left and the right limits at $\xi = \pm 1$. The sign for $\operatorname{Im}(\xi) < 0$ is chosen so that $\widetilde{B}$ is continuous at -1, that is, we have plus sign, if $\beta_{-1} < 0$, and minus sign, if $\beta_{-1} > 0$. At point $\xi = 1$ function B might not be continuous. In general we have $\widetilde{B}(1^+) = \pm\widetilde{B}(1^-)$. See [19, p. 169-170] for more.

Each factor

$$\frac{\xi - 1}{|\xi - 1|}, \frac{\xi + 1}{|\xi + 1|} \tag{68}$$

changes the argument by π when ξ passes $\partial\Delta$ once in the positive direction. Hence possible widing number of $\widetilde{B}$ is either an integer (Maslov index of problem (66) is even) or a half of an odd integer (Maslov index of problem (66) is odd).

Example 4.1. Consider the case $B(e^{i\theta}) = e^{i\theta}$ for $\theta \in [0, \pi]$. In particular we have $\beta_1 = \beta_{-1} = \frac{1}{2}$. Then we get

$$\widetilde{B}(\xi) = \begin{cases} e^{-i\frac{\pi}{4}} B(\xi)\,\overline{\xi}^{\frac{1}{2}} = e^{i\frac{2\theta-\pi}{4}}, & \text{if } 0 \le \theta \le \pi \\ e^{-i\frac{5\pi}{4}}\,\overline{\xi}^{\frac{1}{2}} = e^{i\frac{3\pi-2\theta}{4}}, & \text{if } \pi < \theta < 2\pi. \end{cases} \tag{69}$$

Hence the winding number $W(\widetilde{B}) = 0$. Using identification of the boundary problem (64)-(65) with the problem on the unit disc with reflected boundary conditions (15), this example corresponds to the linearization of the boundary value problem, where all boundary curves are unit circles and we linearize at $f(z) = z$. The family of (nearby) solutions which are real on the real axis is one-dimensional $f_a(z) = \frac{z-a}{1-az}$, where $a \in (-1, 1)$ is a real number.

Example 4.2. Consider the case $B(e^{i\theta}) = 1$ for $\theta \in [0, \pi]$. In particular we have $\beta_1 = -\beta_{-1} = \frac{1}{2}$. Then we get

$$\widetilde{B}(\xi) = e^{-i\frac{\pi+2\theta}{4}} \tag{70}$$

and the winding number $W(\widetilde{B}) = -\frac{1}{2}$. Using identification of the boundary problem (64)-(65) with the problem on the unit disc with reflected boundary conditions (15), this example corresponds to the linearization of the problem where all boundary curves are unit circles and we linearize at function $f(z) = 1$. The family of (nearby) solutions which are real on the real axis is zero-dimensional.

The dimension of the space of solutions in $A^{\beta}(\Delta)$ depends on the winding number $W(\widetilde{B})$ of function $\widetilde{B}$. It equals $2W(\widetilde{B}) + 1$ if $W(\widetilde{B}) \ge -\frac{1}{2}$, see [18, p. 25, p. 59] and [3,4,15]. We define the winding number of $B \in C^{\alpha}(\partial\Delta \setminus \mathring{L})$ as the winding number of $\widetilde{B}$.

Now we can solve appropriate nonhomogeneous linear Riemann-Hilbert problem with piecewise continuous boundary data

$$\mathrm{Im}(f(\xi)) = 0 \text{ for } \xi \in L \tag{71}$$

and

$$\mathrm{Re}(\overline{B(\xi)}f(\xi)) = b(\xi) \text{ for } \xi \in \partial\Delta \setminus \mathring{L}, \tag{72}$$

where B is as above and b a real function on $\partial\Delta$ of the form

$$b(\xi) = |\xi - 1|^{\delta_1}|\xi + 1|^{\delta_{-1}}\widetilde{b}(\xi) \tag{73}$$

for some function $\widetilde{b} \in C^\beta_{\mathbb{R}}(\partial\Delta)$ which equals 0 on L.

To solve (71)-(72) in the space of functions $f \in A^\beta(D)$ of the form $f(\xi) = (\xi - 1)^{\delta_1}(\xi + 1)^{\delta_{-1}}\kappa(\xi)$ for some $\kappa \in A^\beta(\Delta)$ is equivalent to solve the problem

$$\mathrm{Re}(\overline{\widetilde{B}}(\xi)\kappa(\xi)) = \widetilde{b}(\xi) \text{ for all } \xi \in \partial\Delta. \tag{74}$$

It is well known that if $W(B) = W(\widetilde{B}) \geq -\frac{1}{2}$, then the equation is solvable for any $\widetilde{b} \in C^\beta_{\mathbb{R}}(\partial\Delta)$, see [18, p. 25, p. 59] and [3,4,15].

Remark 4.3. If the winding number $W(B) = W(\widetilde{B})$ is an odd integer, the function on the right-hand side of (74) needs to belong to a special space of Hölder continuous real functions on $\partial\Delta \setminus \{1\}$ of the form $b_0(r(\xi))$, where $r(\xi)$ is the principal branch of the square root and $b_0 \in C^\beta_{\mathbb{R}}(\partial\Delta)$ is an odd function. Hence we need condition $\widetilde{b}(1^-) + \widetilde{b}(1^+) = 0$, which is satisfied because in our case we have $\widetilde{b}(1^-) = \widetilde{b}(1^+) = 0$. See [3,4] for more information.

Proposition 4.4. *Let $0 < \beta < 1$. Let $B : \partial\Delta \setminus \mathring{L} \to \mathbb{C} \setminus \{0\}$ be a non-vanishing complex function in $C^\beta(\partial\Delta \setminus \mathring{L})$ and let $W(B) \geq -\frac{1}{2}$. Then for every real function b on $\partial\Delta$ of the form*

$$b(\xi) = |\xi - 1|^{\delta_1}|\xi + 1|^{\delta_{-1}}\widetilde{b}(\xi) \tag{75}$$

for some $\widetilde{b} \in C^\beta_{\mathbb{R}}(\partial\Delta)$ which equals 0 on L, there exists a solution f of the linear Cherepanov problem

$$\mathrm{Im}(f(\xi)) = 0 \text{ for } \xi \in L \tag{76}$$

and

$$\mathrm{Re}(\overline{B(\xi)}f(\xi)) = b(\xi) \text{ for } \xi \in \partial\Delta \setminus \mathring{L} \tag{77}$$

of the form

$$f(\xi) = (\xi - 1)^{\delta_1}(\xi + 1)^{\delta_{-1}}\kappa(\xi), \tag{78}$$

where $\kappa \in A^\beta(\Delta)$. Moreover, the space of solutions of this form is $2W(B) + 1$ dimensional real subspace of $A^\beta(\Delta)$.

Proposition 4.5. *Let $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ be a C^k $(k\geq 3)$ family of Jordan curves in $\mathbb{C}$ and let $\rho_0\in C^k((\partial\Delta\setminus\mathring{L})\times\mathbb{C})$ be its defining function. Let β_1,β_{-1} and β be as in Proposition 3.3. Let f_0 be a solution of the Cherepanov problem (12), (13) of the form*

$$f_0(\xi)=(\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa_0(\xi)+w_1\frac{1+\psi(\xi)}{2}+w_{-1}\frac{1-\psi(\xi)}{2}, \tag{79}$$

where $\kappa_0\in A^\beta(\Delta)$. Then the mapping $\Phi(\kappa):A^\beta(\Delta)\to C^\beta_{\mathbb{R}}(\partial\Delta)$, for each κ evaluated at point $\xi\in\partial\Delta$ as

$$\begin{cases}\rho_0(\xi,(\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi)+w_1\frac{1+\psi(\xi)}{2}+w_{-1}\frac{1-\psi(\xi)}{2}), & \text{if } \mathrm{Im}(\xi)\geq 0,\\ \pm\mathrm{Im}((\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi)+w_1\frac{1+\psi(\xi)}{2}+w_{-1}\frac{1-\psi(\xi)}{2}), & \text{if } \mathrm{Im}(\xi)<0\end{cases} \tag{80}$$

is differentiable at κ_0 with the derivative $(D\Phi)(\kappa_0)$ acting on $\kappa\in A^\beta(\Delta)$ as

$$\begin{cases}2\mathrm{Re}(\partial\rho_{0w}(\xi,f_0(\xi))(\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi)), & \text{if } \mathrm{Im}(\xi)\geq 0,\\ \pm\mathrm{Im}((\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi)), & \text{if } \mathrm{Im}(\xi)<0.\end{cases} \tag{81}$$

The sign for $\mathrm{Im}(\xi)<0$ is chosen as in (66)-(67).

Remark 4.6. Let $\Omega\subset C^{k+1}((\partial\Delta\setminus\mathring{L})\times\mathbb{C})$ be an open subset of defining functions ρ of the families of Jordan curves over $\partial\Delta\setminus\mathring{L}$ such that the intersection of the corresponding γ_1 and γ_{-1} with the real axis at some points $w_1\in\gamma_1$ and $w_{-1}\in\gamma_{-1}$ are transversal with the oriented angles of intersection given by $\beta_1,\beta_{-1}\in(-1,1)\setminus\{0\}$. Then, at least locally, w_1,w_{-1} and β_1,β_{-1} smoothly depend on ρ. Let

$$X=\{(\kappa,\rho)\in A^\beta(\Delta)\times\Omega;\mathrm{Im}((\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi))=0,\ \text{if } \mathrm{Im}(\xi)<0\} \tag{82}$$

which is a Banach submanifold of $A^\beta(\Delta)\times\Omega$. Also, let

$$Y=\{b(\xi)=|\xi-1|^{\delta_1}|\xi+1|^{\delta_{-1}}\widetilde{b}(\xi);\widetilde{b}\in C^\beta_{\mathbb{R}}(\Delta),\widetilde{b}(\xi)=0,\ \text{if } \mathrm{Im}(\xi)<0\}. \tag{83}$$

The mapping $\Phi:X\to Y$ defined as in (80) has partial derivative with respect to κ as a map from

$$X_\rho=\{\kappa\in A^\beta(\Delta);\mathrm{Im}((\xi-1)^{\delta_1}(\xi+1)^{\delta_{-1}}\kappa(\xi))=0,\ \text{if } \mathrm{Im}(\xi)<0\} \tag{84}$$

to Y of the form (81). If the winding number $W(B)$ of the Cherepanov problem defined by (81) is greater or equal to $-\frac{1}{2}$, then the partial derivative is surjective with $2W(B)+1$ dimensional kernel. Hence implicit function theorem applies and there is a neighborhood of ρ_0 in Ω and a neighborhood of κ_0 in $A^\beta(\Delta)$ such that for every $\rho\in\Omega$ close to ρ_0 there is a $2W(B)+1$ dimensional family of solutions of (12)-(13) near κ_0.

5. A priori estimates

5.1. A priori estimates on function f

To get existence results using continuity method we need a priori estimates on solutions of (3)-(4). It is well known that such a priori estimates can only be achieved for the family of solutions with no zeros on Δ, [8,16,18]. We follow the approach in [8].

By assumption all Jordan curves $\{\gamma_\xi\}_{\xi\in\partial\Delta\setminus\mathring{L}}$ contain point 0 in their interiors. Hence the function

$$(\theta,w)\longmapsto w\rho_w(\theta,w), \tag{85}$$

defined for (θ, w) such that $w \in \gamma_\theta$, is homotopic to 0 in $\mathbb{C} \setminus \{0\}$ and it can be written in the form

$$w\rho_w(\theta, w) = e^{c(\theta,w)+id(\theta,w)} \tag{86}$$

for some C^{k-1} functions c and d, defined for (θ, w) such that $\theta \in [0, \pi]$ and $w \in \gamma_\theta$. Observe that for each $w \in \gamma_\theta$ function $d(\theta, w)$ represents the angle between w and the normal to γ_θ at point w.

Remark 5.1. There exists a C^k isotopy ρ^t, $t \in [0, 1]$, where $\rho^0 = \rho$ and $\rho^1(\xi, w) = |w|^2 - R^2$ for $R > 0$ large enough, such that the gradient $\rho^t_{\overline{w}}$ is nonzero on $\rho^t = 0$ for each t, [8]. Then one can find C^{k-1} functions $c(t, \theta, w)$ and $d(t, \theta, w)$ such that (86) holds for each $t \in [0, 1]$ and $w \in \gamma^t_\theta$. In addition, the isotopy can be made such that for every $t \in [0, 1]$, $j = \pm 1$, Jordan curves γ^t_j are strongly starshaped with respect to 0 and that for each $w \in \gamma^t_{\omega_j}$ the angle between w and the normal to $\gamma^t_{\omega_j}$ at w is less than $\frac{\pi}{10}$.

Instead of solving (12)-(13) on the unit disc we consider equivalent problem on the upper semidisc $\Delta^+ = \{z \in \Delta; \mathrm{Im}(z) > 0\}$, where the role of the lower semicircle L is replaced by the interval $[-1, 1]$. Using the reflection principle $f(\xi) = \overline{f(\overline{\xi})}$ we can holomorphically extend every solution f of (12)-(13) to the unit disc such that it solves nonlinear Riemann-Hilbert problem defined by the function ρ which we get as an extension of the original function ρ using the reflection to the lower semicircle as

$$\rho(\xi, w) = \rho(\overline{\xi}, \overline{w}) \text{ for } \xi \neq \pm 1. \tag{87}$$

For $\xi = \pm 1$ function $\rho(\xi, w)$ has well defined limits as ξ approaches ± 1 from above and below. Then we have

$$\rho_w(\xi, w) = \rho_{\overline{w}}(\overline{\xi}, \overline{w}) = \overline{\rho_w(\overline{\xi}, \overline{w})} \text{ for } \xi \neq \pm 1 \tag{88}$$

and hence

$$\rho_w(\xi, w)w = \rho_{\overline{w}}(\overline{\xi}, \overline{w})w = \overline{\rho_w(\overline{\xi}, \overline{w})\overline{w}}. \tag{89}$$

Therefore $c(\overline{\xi}, \overline{w}) = c(\xi, w)$ and $d(\overline{\xi}, \overline{w}) = -d(\xi, w)$. Also, observe that for w, an intersection of γ_1 with the real axis, we have

$$\rho_w(1+, w) = \overline{\rho_w(1-, \overline{w})} = \overline{\rho_w(1-, w)} \tag{90}$$

and similarly for an intersection of γ_{-1} with the real axis.

Thus for every solution f of (12)-(13) the absolute value of $f(\theta)\rho_w(\theta, f(\theta))$ is well defined and continuous on $\partial\Delta$, whereas

$$d(0+, f(0+)) = -d(2\pi-, f(2\pi-)) \tag{91}$$

and similarly at $\theta = \pi$.

Let f be a solution of the symmetrized boundary value problem with no zeros. Hence f can be written in the exponential form

$$f = e^g. \tag{92}$$

Remark 5.2. Since the biholomorphic map ψ from Δ to the upper half-disc Δ_+ is of class $C^{\frac{1}{2}}$, a C^β estimate on solutions of the symmetrized boundary value problem gives $C^{\frac{\beta}{2}}$ estimate on solutions of (12)-(13).

Let us differentiate function $\rho(\theta, f(\theta))$ to get

$$\rho_\theta(\theta, f(\theta)) + 2\mathrm{Re}(\rho_w(\theta, f(\theta))\frac{\partial f}{\partial \theta}(\theta)) = 0. \tag{93}$$

Since $f = e^g$, we get

$$\rho_\theta(\theta, f(\theta)) + 2\mathrm{Re}(\rho_w(\theta, f(\theta))f(\theta)\frac{\partial g}{\partial \theta}(\theta)) = 0 \tag{94}$$

and so

$$\rho_\theta(\theta, f(\theta)) + 2\mathrm{Re}(e^{c(\theta,f(\theta))+id(\theta,f(\theta))}\frac{\partial g}{\partial \theta}(\theta)) = 0. \tag{95}$$

From here we get

$$2\mathrm{Re}(e^{i(d(\theta,f(\theta))+iHd(\theta,f(\theta)))}\frac{\partial g}{\partial \theta}(\theta)) = -\rho_\theta(\theta, f(\theta))e^{-c(\theta,f(\theta))-Hd(\theta,f(\theta))}. \tag{96}$$

Observe that function

$$\theta \longmapsto e^{i(d(\theta,f(\theta))+iHd(\theta,f(\theta)))}\frac{\partial g}{\partial \theta}(\theta) \tag{97}$$

extends holomorphically to the unit disc with value 0 at 0.

We will get C^β a priori estimates on g and hence on f by getting C^β a priori estimates on function (97). Using Hilbert transform it is enough to get C^β a priori estimates on its real part. Hence we need a priori estimates on the right hand side of (96).

Function

$$\theta \longmapsto -\rho_\theta(\theta, f(\theta))e^{-c(\theta,f(\theta))} \tag{98}$$

is bounded with the bound which does not depend on function f but only on the data $\gamma_{\xi\in\partial\Delta\setminus\mathring{L}}$ and defining function ρ. The bound can also be found to be independent of the C^k isotopy ρ^t, $t \in [0, 1]$. Hence one needs a priori bound on function

$$\theta \longmapsto e^{\pm Hd(\theta,f(\theta))}. \tag{99}$$

Recall that, [18, p. 23], for $u \in L^\infty(\partial\Delta)$, such that $\|u\|_\infty < \frac{\pi}{2p}$ $(1 \le p < \infty)$ we have the estimate

$$\|e^{Hu}\|_p \le \left(\frac{2\pi}{\cos(p\|u\|_\infty)}\right)^{\frac{1}{p}}. \tag{100}$$

Let $a \in (0, \frac{\pi}{5})$ and let χ_0, χ_π be smooth functions on $[0, \pi]$ with values in $[0, 1]$ such that $\chi_0(t) = 1$ on $[0, a]$, $\chi_\pi(t) = 1$ on $[\pi - a, \pi]$, $\chi_0(t) = 0$ on $[2a, \pi]$, and $\chi_\pi(t) = 0$ on $[0, \pi - 2a]$.

Let us consider the function

$$\widetilde{d}(\theta, w) = d(\theta, w) - \chi_0(\theta)d_0(w) - \chi_\pi(\theta)d_\pi(w) \tag{101}$$

for $\theta \in [0, \pi]$ and $\widetilde{d}(\theta, w) = -\widetilde{d}(2\pi - \theta, \overline{w})$ for $\theta \in [\pi, 2\pi]$. Here we used notation $d_0(w) = d(0+, w)$ and $d_\pi(w) = d(\pi-, w)$.

We see that $\widetilde{d}(0, w) = \widetilde{d}(\pi, w) = 0$ and so $\widetilde{d}(\theta, w)$ is a continuous function on $\partial\Delta \times \mathbb{C}$. Let $1 < \widetilde{p} < \infty$ be given. By results from [8, p. 881] we can write $\widetilde{d} = \mathrm{Re}(q) + \widetilde{e}$, where $\widetilde{p}\|\widetilde{e}\|_\infty < \frac{\pi}{2}$ and q is a finite sum of terms of the form $e^{ij\theta}w^m$, $j \in \mathbb{Z}$, $m \in \mathbb{N} \cup \{0\}$, on which Hilbert transform acts as a bounded nonlinear operator from $A(\partial\Delta)$ into $C(\partial\Delta)$.

Therefore for a given solution f of (12)-(13) with no zeros we can write continuous function $\widetilde{d}(\theta, f(\theta))$ on $\partial\Delta$ in the form

$$\widetilde{d}(\theta, f(\theta)) = \mathrm{Re}(q(\theta, f(\theta))) + \widetilde{e}(\theta, f(\theta)) \tag{102}$$

and so

$$H(\widetilde{d}) = H(\mathrm{Re}(q)) + H(\widetilde{e}) \tag{103}$$

where the first term is uniformly bounded and for the second we have $\|\widetilde{e}\|_\infty < \frac{\pi}{2\widetilde{p}}$. Hence

$$e^{\pm H\widetilde{d}} = e^{\pm H\mathrm{Re}(q)} e^{\pm H\widetilde{e}} \tag{104}$$

where the first factor is uniformly bounded and the second factor is bounded in $L^{\widetilde{p}}(\partial\Delta)$ for a given $1 < \widetilde{p} < \infty$.

Since for a given $\widetilde{p} \in (1, \infty)$ we can get $L^{\widetilde{p}}(\partial\Delta)$ bounds on (104), the boundedness of $e^{\pm Hd}$ in some $L^p(\partial\Delta)$ is determined by Hilbert transform of the extension of function $\chi_0(\theta)d_0(f(\theta)) + \chi_\pi(\theta)d_\pi(f(\theta))$ to $[0, 2\pi]$.

Recall that $\gamma_{\pm 1}$ are strongly starshaped Jordan curves with respect to 0 and we may assume that for $j = \pm 1$ we have

$$\rho(j, w) = |w|^2 - R_j^2\left(\frac{w}{|w|}\right) \tag{105}$$

for some positive C^k function $R_j(z)$ on $\mathbb{C}$. A short calculation gives

$$\rho_w(j, w) = \overline{w} - 2R_j\left(-\frac{1}{2}\frac{\overline{w}^2}{|w|^3}(R_j)_{\overline{z}} + \frac{1}{2}\frac{1}{|w|}(R_j)_z\right) \tag{106}$$

and so

$$w\rho_w(j, w) = |w|^2 - 2i\frac{R_j}{|w|}\mathrm{Im}(w(R_j)_z) \tag{107}$$

which has strictly positive real part on $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$. Functions d_0 and d_π represent the argument of (107). By compactness it follows that there exists $0 < \beta_0 < 1$, such that $|d_0(w)| \leq \frac{\pi}{2}\beta_0$ and $|d_\pi(w)| \leq \frac{\pi}{2}\beta_0$ on $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$ and therefore

$$|\chi_0(\theta)d_0(f(\theta)) + \chi_\pi(\theta)d_\pi(f(\theta))| \leq \frac{\pi}{2}\beta_0 \tag{108}$$

for every θ. Also, if there is an open condition on the size of $d_{j\pi}$ on γ_j, such as $|d_{i\pi}(w)| < \frac{\pi}{2}\beta_0$, $j = \pm 1$, we can, by choosing the supports of functions χ_0 and χ_π small enough, that is, by choosing $a > 0$ small enough, assume that the same condition on the size holds for function $\chi_0(\theta)d_0(f(\theta)) + \chi_\pi(\theta)d_\pi(f(\theta))$ for all θ. Observe also that $|d_0(f(0))| = \pi|\beta_1 - \frac{1}{2}|$ and $|d_\pi(f(\pi))| = \pi||\beta_{-1}| - \frac{1}{2}|$. and so

$$\left||\beta_j| - \frac{1}{2}\right| < \frac{\beta_0}{2}, \quad j = \pm 1. \tag{109}$$

By (100) and (104) we get that for every fixed $1 < p < \infty$ such that $p\beta_0 < 1$ the estimate

$$\|e^{\pm Hd}\|_p \leq C \tag{110}$$

holds. Hence we also have a priori L^p estimate on function (97).

Since

$$\frac{\partial f}{\partial \theta} = f\frac{\partial g}{\partial \theta}, \tag{111}$$

an estimate on $\frac{\partial g}{\partial \theta}$ will give an estimate on $\frac{\partial f}{\partial \theta}$. We can write

$$\frac{\partial g}{\partial \theta} = \left(e^{-i(d+iHd)}\right)\left(e^{i(d+iHd)}\frac{\partial g}{\partial \theta}\right). \tag{112}$$

By assumptions of Theorem 1.2 we have $\beta_0 \leq \frac{1}{5} < \frac{1}{2}$ and we can choose $p > 2$. By Cauchy-Schwarz inequality we then have

$$\left\|\frac{\partial g}{\partial \theta}\right\|_{\frac{p}{2}} \leq \left\|e^{-i(d+iHd)}\right\|_p \left\|e^{i(d+iHd)}\frac{\partial g}{\partial \theta}\right\|_p. \tag{113}$$

From here we get $L^{\frac{p}{2}}$ a priori estimates on $\frac{\partial g}{\partial \theta}$ which imply a priori estimates on g and f in Hölder space $C^\beta(\partial\Delta)$ for $0 < \beta < 1 - \frac{2}{p} < 2(\frac{1}{2} - \beta_0)$. Recall (Remark 5.2) that this gives Hölder space a priori estimates on solutions with no zeros of the nonsymmetrical problem (12)-(13) for $\beta \in (0, \frac{1}{2} - \beta_0)$.

5.2. A priori estimates on function κ

We also need a priori estimates on function κ for which it holds

$$f(\xi) = (\xi - 1)^{\delta_1}(\xi + 1)^{\delta_{-1}}\kappa(\xi) + f(1)\frac{1 + \psi(\xi)}{2} + f(-1)\frac{1 - \psi(\xi)}{2}. \tag{114}$$

In this subsection we again consider the nonsymmetrical case (12)-(13). We denote by C a universal constant, which depends on the data but does not depend on the particular function we consider.

We know that f and hence κ are $C^{k-1,\alpha}$ smooth on $\partial\Delta \setminus \{-1, 1\}$ and on compact subsets of $\partial\Delta \setminus \{1, -1\}$ we get a priori estimates on κ by expressing it in terms of f. Hence we need a priori estimates on κ near points ± 1. Also, we know from Section 3 that if κ is continuous on $\overline{\Delta}$, then both functions belong to $A^\beta(\Delta)$ for

$$0 < \beta < \min\{|\beta_1|, 1 - |\beta_1|, |\beta_{-1}|, 1 - |\beta_{-1}|\}. \tag{115}$$

Let us fix $0 < \beta < \frac{1}{2} - \beta_0$ that we have a priori estimates on function f.

Recall (38) and that $t^s\kappa(t) = f(t)$. Hence $\widetilde{\rho}_w(\theta, \kappa)$ is a C^β function with a priori bounds. As in (60) we can globally write

$$\operatorname{Re}\left(re^{i(v+iHv)}\frac{\partial \kappa}{\partial \theta}\right) = -e^{-u}e^{-(Hv)}\widetilde{\rho}_\theta(\theta, \kappa), \tag{116}$$

where u and v are real C^β functions with a priori bounds. To get $L^{p'}$ a priori bounds on $\frac{\partial \kappa}{\partial \theta}$ for some $p' > 1$ we will get $L^{p'}$ bounds on the right-hand side function $\widetilde{\rho}_\theta(\theta, \kappa(\theta))$, that it, on $\widetilde{\rho}_t(t, \kappa(t))$ near $t = 0$.

Considering (41)-(42)-(43) termwise we get that t^{β_1-1}, $t^{-\beta_1}$, $\kappa(t) = t^{\beta_1-1}f(t)$, and all terms with function g are $L^{p'}$ bounded for any $p' > 1$ such that

$$p'(1-\beta_1) < 1 \quad \text{and} \quad p'\beta_1 < 1. \tag{117}$$

Let us consider terms which are bounded by $t^{-\beta_1}|\kappa(t)^2| = t^{\beta_1-2}|f(t)^2|$. Since we have $\beta \in (0, \frac{1}{2} - \beta_0)$ a priori bounds on f, we have

$$|f(t)| \leq C|t|^\beta \tag{118}$$

for some universal constant C. Hence

$$t^{-\beta_1}|\kappa(t)^2| \leq C|t|^{2\beta+\beta_1-2} \tag{119}$$

and this function is in some $L^{p'}$, $p' > 1$, if $1 < 2\beta + \beta_1$. The bound $0 < \beta < \frac{1}{2} - \beta_0$ implies that this will be the case for some such β if $2\beta_0 < \beta_1$. Similar argument near $\xi = -1$ gives $2\beta_0 < 1 - |\beta_{-1}|$.

If these two conditions are satisfied, we get $L^{p'}$ a priori estimates on $\widetilde{\rho}_\theta(\theta, \kappa(\theta))$ for some $p' > 1$. This implies $C^{\beta'}$ a priori estimate on κ for $\beta' < 1 - \frac{1}{p'}$.

There are natural bounds on β_j, $j = \pm 1$, in terms of β_0, that is,

$$\frac{1}{2} - \frac{\beta_0}{2} < |\beta_j| < \frac{1}{2} + \frac{\beta_0}{2}. \tag{120}$$

Hence, if $2\beta_0 \leq \frac{1}{2} - \frac{\beta_0}{2}$ and $\frac{1}{2} + \frac{\beta_0}{2} \leq 1 - 2\beta_0$ both inequalities needed for $L^{p'}$ a priori estimates will be satisfied. These two inequalities are equivalent to the condition $\beta_0 \leq \frac{1}{5}$, that is, the angle between w and the normal to γ_{ω_j} at w is less than $\frac{\pi}{10}$.

6. Final remarks

If arc L is the lower semicircle, we can state Theorem 1.2 in an equivalent simplified form.

Theorem 6.1. *Let $\{\gamma_\xi\}_{\xi \in \partial\Delta \setminus \mathring{L}}$ be a C^k $(k \geq 3)$ family of Jordan curves in $\mathbb{C}$ which all contain point 0 in their interiors. Let Jordan curves γ_j, $j = \pm 1$, be strongly starshaped with respect to 0 and such that for each $w \in \gamma_{\omega_j}$ the angle between w and the normal to γ_{ω_j} at w is less than $\frac{\pi}{10}$. Let w_j, $j = \pm 1$, be the positive intersection of γ_j and the real axis with the oriented angle of intersection $\pi\beta_j$, where $\beta_1 \in (0,1)$ and $\beta_{-1} \in (-1, 0)$. Let*

$$0 < \beta < \min\{\beta_1, 1 - \beta_1, |\beta_{-1}|, 1 - |\beta_{-1}|\}. \tag{121}$$

Then there exists a unique $f \in A^\beta(\Delta)$ with no zeros on Δ which solves (3)-(4) for which $f(1) = w_1$ and $f(-1) = w_{-1}$.

To prove Theorem 6.1 one uses continuity method (see also [8]). The starting boundary value problem (3)-(4) can be, using an isotopy from Jordan curves $\{\gamma_\xi\}_{\xi\partial\Delta\setminus\mathring{L}}$ to circles with center at 0 and fixed radius $R > 0$, embedded in a one parameter family of boundary value problems which all satisfy assumptions of Theorem 6.1. Here, for $t = 0$ we have the starting boundary value problem and for $t = 1$ circles as the boundary data.

Results in Section 4 (Proposition 4.4, Proposition 4.5) imply that a solution of the boundary value problem (3)-(4) for curves $\{\gamma_\xi^t\}_{\xi\partial\Delta\setminus\mathring{L}}$ can be locally perturbed into a solution for the nearby perturbed boundary data. Hence the set of parameters t for which there is a solution of (3)-(4) is open. On the other hand, a priori estimates from Section 5 together with compact embeddings (10) imply that the set of parameters $t \in [0,1]$ for which there is a solution of (3)-(4) is closed. Since there is an obvious solution for the case $t = 1$, where all Jordan curves are circles with center at 0 and fixed radius $R > 0$, we get that there is a solution of (3)-(4) for $t = 0$.

Corollary 6.2. *Let $a_1, \ldots, a_n \in \Delta$ be a finite set of points with given multiplicities. Then under the assumptions of Theorem 6.1 there exists $\beta \in (0,1)$ and $f \in A^{\beta}(\Delta)$ which has zeros exactly at points $a_1, \ldots, a_n \in \Delta$ with the given multiplicites and which solves (12)-(13).*

To prove the corollary we search for solutions f of the symmetric problem on the unit disc of the form

$$f(z) = \frac{z-a}{1-\overline{a}z}\,\frac{z-\overline{a}}{1-az}\,\widetilde{f}(z), \tag{122}$$

where a is a point in the upper half-disc. Then $\widetilde{f}$ has to solve a modified problem, where the boundary curves are given by

$$\widetilde{\gamma}_\xi = \frac{1-\overline{a}\xi}{\xi-a}\,\frac{1-a\xi}{\xi-\overline{a}}\,\gamma_\xi. \tag{123}$$

Observe that $\widetilde{\gamma}_\xi = \gamma_\xi$ for $\xi = \pm 1$.

Remark 6.3. In a similar way one can create a zero at a point $a \in (-1,1)$, that is, on $\mathring{L}$ in the original problem. Jordan curves for the modified problem are

$$\widetilde{\gamma}_\xi = \frac{1-a\xi}{\xi-a}\,\gamma_\xi. \tag{124}$$

Then $\widetilde{\gamma}_1 = \gamma_1$ and $\widetilde{\gamma}_{-1} = -\gamma_{-1}$ but conditions of Theorem 1.2 are still satisfied.

Declaration of competing interest

No potential conflict of interest was reported by the author.

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