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EKSTREMNE KORELACIJE

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Izvleček

V članku je definirana količina, ki meri kvaliteto aproksimacije tabelarično podane funkcije s funkcijo iz neke družine D nekonstantnih funkcij. V ta namen je treba poiskati infimum in supremum množice korelacijskih koeficientov med dano funkcijo in funkcijami iz D. Tu je to storjeno za družino vseh nekonstantnih funkcij in za družino rastnih funkcij.

Ključne besede: aproksimacija, ekstremna korelacija, nelinearni program

EXTREME CORRELATIONS

Anton CEDILNIK*

Abstract

In the article we define a quantity which measures the quality of the approximation of a function, given by a table, with a function from a certain family D of nonconstant functions. For this purpose one has to find the infimum and the supremum of the set of correlation coefficients between the given function and functions from D. We do this for the family of all nonconstant functions and for the family of growth functions.

Key words: approximation, extremal correlation, nonlinear programming

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I. INTRODUCTION

In table 1 and table 2 we have measurings of function values of a function $y(x)$, which we approximate with the function $w_1 = 0.5\,x + 3.0$. Its correlation coefficient with data from table 1 is 0.7048 and with data from table 2 is 0.9986. In the first case the correlation is weak and in the second one very strong. Nevertheless, it can be shown that, contrary to the expectations, in the first case this approximation is the best (in the sense that the correlation cannot be increased), but in the second case we easily find a better approximation, for instance $w_2 = 0.05\,x^2 + 0.35\,x + 3.10$, for which the correlation coefficient is 1. Therefore, the correlation coefficient is not necessarily a good measure of the quality of approximation.

x	y				
1	3.3	3.4	3.5	3.5	3.8
2	3.0	4.0	4.1	4.9	
3	4.1	4.4	4.7	4.8	

Table 1

x	y
1	3.5
2	4.0
3	4.6

Table 2

The purpose of this article is to introduce a better measure of the quality of approximation (definition 1) and to calculate it in two special cases.

But, firstly, some conventions and definitions!

We deal with a function $y(x)$ for which the measurings of function values are in table 3. The table is ordered: $x_1 < x_2 < \ldots < x_n$.

x	y			
x_1	y_{11}	y_{12}	$\cdots$	y_{1s_1}
x_2	y_{21}	y_{22}	$\cdots$	y_{2s_2}
$\vdots$		$\vdots$		
x_n	y_{n1}	y_{n2}	$\cdots$	y_{ns_n}

Table 3

$s_k \geq 1 \quad (k = 1,\dots,n)$. $\sum\limits_k \sim \sum\limits_{k=1}^{n}$, $\sum\limits_l \sim \sum\limits_{l=1}^{s_k}$ (k and l will be the summation indexes only in these two sums).

(1) $\qquad s := \sum\limits_k s_k \geq n$

(2) $\qquad \bar{y}_k := \frac{1}{s_k} \sum\limits_l y_{kl} \qquad (k = 1,\dots,n)$

The values $\bar{y}_k$ form a new function $\bar{y}$.

(3) $\qquad \bar{y} := \frac{1}{s} \sum\limits_k s_k \bar{y}_k$

(4) $\qquad \tau := \left[\sum\limits_k \sum\limits_l (y_{kl} - \bar{y})^2\right]^{1/2} = \left[\sum\limits_k \sum\limits_l y_{kl}^2 - s\bar{y}^2\right]^{1/2}$

(5) $\qquad \omega := \tau \left[\sum\limits_k s_k \bar{y}_k^2 - s\bar{y}^2\right]^{-1/2}$

Presumption:

(P) $\qquad$ There is at least one k with $\bar{y}_k \neq \bar{y}$.

Obviously, $n \geq 2$, $\tau > 0$, $\omega \geq 1$. $\omega = 1$ if and only if $s_1 = \dots = s_n = 1$, hence if $s = n$.

Correlation coefficient $r\,(y, u)$ between the function y from table 3 and some other function u with values u_k $(k = 1,\dots,n)$ is the correlation coefficient for table 4:

y	$y_{11},\dots, y_{1s_1}, y_{21},\dots, y_{2s_2},\dots,y_{n1},\dots, y_{ns_n}$
u	$u_1,\dots, u_1, u_2,\dots, u_2,\dots, u_n,\dots, u_n$

Table 4

(6) $$r\,(y, u) := \frac{\sum\limits_k \sum\limits_l (y_{kl} - \bar{y})(u_k - \bar{u})}{\left[\sum\limits_k \sum\limits_l (y_{kl} - \bar{y})^2 \cdot \sum\limits_k \sum\limits_l (u_k - \bar{u})^2\right]^{1/2}} = \frac{\sum\limits_k s_k \bar{y}_k u_k - s\,\bar{y}\,\bar{u}}{\tau \cdot \left[\sum\limits_k s_k u_k^2 - s\bar{u}^2\right]^{1/2}},$$

where

(7) $\qquad \bar{u} := \frac{1}{s} \sum\limits_k s_k u_k .$

Let us approximate the function y from table 3 with a function w from some family D of nonconstant functions, defined on the set $\{x_1,..., x_n\}$. $\{r(y, u) | u \in D\}$ is a subset of the interval $[-1,1]$ with

(8) $\quad A(y) := \inf \{r(y, u) | u \in D\}$,

(9) $\quad B(y) := \sup \{r(y, u) | u \in D\}$,

(10) $\quad A(y) \leq r(y, w) \leq B(y).$

Definition 1. Quality of D-approximation of function y with function w ∈ D is

(11) $$K(y, w \in D) := \frac{2r(y, w) - B(y) - A(y)}{B(y) - A(y)}$$

The definition is based on fig. 1.

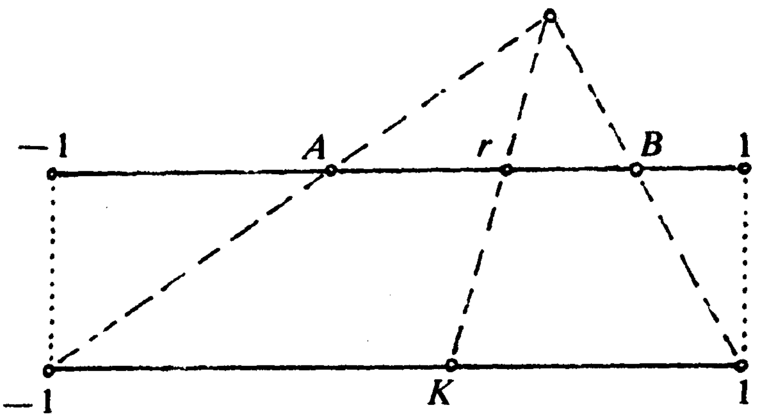

Figure 1

The quality of D-approximation is again a number between -1 and 1, and

(12) $\quad K(y, w \in D) = K(\alpha y + \beta, \gamma w + \delta \in D)$

for all such $\alpha, \beta, \gamma, \delta$ that $\alpha\gamma > 0$ and still is $\gamma w + \delta \in D$.

Obviously, the family D must be at least so extensive that for each y from table 3 and with the presumption (P), the denominator $B(y)-A(y)$ in (11) is > 0. We shall name such family ***sufficient.***

Examples.

1. The family F_n ($n \geq 2$) of all nonconstant functions on the set $\{x_1,...,x_n\}$ is sufficient. For proving this, calculate $r(y, u)$, where $u_k = 0$ $(k \neq i)$, $u_i = \pm 1$, and demand that the result is always the same for all i and both signs.

2. A function u on $\{x_1,\ldots,x_n\}$ is ***growth function*** if $0 \leq u_1 \leq \ldots \leq u_n$ and is nonconstant if $u_1 \neq u_n$. The family R_n^0 $(n \geq 3)$ of such functions is sufficient. To see this, calculate $r(y, u)$, where $u_1 = \ldots = u_{i-1} = 0$, $u_i = \lambda$, $u_{i+1} = \ldots = u_n = 1$ $(2 \leq i \leq n-1)$, and demand that for each i and any $\lambda \in (0,1)$: $\partial r/\partial\lambda = 0$. The family R_2^0 is not sufficient.

3. A function u on $\{x_1,\ldots,x_n\}$ is nonconstant linear if $u_k = \alpha x_k + \beta$ $(k = 1,\ldots,n)$ and $\alpha \neq 0$. The family L_n $(n \geq 3)$ of these functions is not sufficient, since for the function y from table 5 we have: $A(y) = B(y) = 0$.

x	y
x_1	$x_n - \bar{x}$
x_k	$0\ (2 \leq k \leq n-1)$
x_n	$\bar{x} - x_1$

Table 5

We already said that the family $L_2 = F_2$ is sufficient.

In the next two sections we shall find $A(y)$ and $B(y)$ and so also $K(y, w \in D)$ for the families $D = F_n$ (n $\geq$ 2) and R_n^0 $(n \geq 3)$.

II. F_n — APPROXIMATION

Theorem 2.

(13) $$A(y) = \min\{r(y, u) \mid u \in F_n\} = -\omega^{-1},$$

(14) $$B(y) = \max\{r(y, u) \mid u \in F_n\} = \omega^{-1} = r(y, \bar{y}).$$

For any $w \in F_n$ there is

(15) $$K(y, w \in F_n) = \omega . r(y, w).$$

Proof. It's easy to verify (using (1)—(6)) that

(16) $$r(y, u) = \frac{1}{\omega} \cdot \frac{\sum_k s_k(\bar{y}_k - \bar{y})(u_k - \bar{u})}{\left[\sum_k s_k(\bar{y}_k - \bar{y})^2 \cdot \sum_k s_k(u_k - \bar{u})^2\right]^{1/2}}$$

The numerator in (16) can be ragarded as a scalar product, therefore, by Cauchy--Schwartz inequality, $r(y, u)$ is the biggest for $u_k - \bar{u} = \lambda(\bar{y}_k - \bar{y})$ ($\lambda > 0$, $k = 1,\ldots,n$), and for such u we have: $r(y, u) = \omega^{-1}$.

Obviously, $r(y, u) = -r(y, -u)$, from where we get (13). Then, (15) follows from the definition (11).

QED.

Notes.

1. For $s_1 = \ldots = s_n = 1$ we have: $K(y, w \in F_n) = r(y, w)$.

2. The function $\bar{y}$ has not only the highest correlation with y but also the lowest sum of squares of declinations from y (even if the presumption (P) is not valid). The proof is simple.

3. For $n > 2$ we can continuously deform y to $-y$ not going through the constant. Therefore:

(17) $\{r(y, u) \mid u \in F_n\} = [-\omega^{-1}, \omega^{-1}]$.

III. R_n^0 — APPROXIMATION ($n > 2$)

This time we'll approximate the function y with a function $u \in R_n^0$: $0 \leq u_1 \leq \ldots \leq u_n \neq u_1$. Let us change y with a new function p:

(18) $p_k := \frac{s_k}{\tau}(\bar{y}_k - \bar{y}) \qquad (k = 1,\ldots,n).$

For this function it holds

(19) $\sum_k p_k = 0,$

(20) $\sum_k p_k^2/s_k = \omega^{-2} \leq 1.$

Let us also change u with q:

(21) $q_k := (u_k - u_1)/\delta(u),$

where

(22) $\delta(u) := \frac{1}{s}\sum_k s_k(u_k - u_1) > 0.$

For q it holds:

(23) $\qquad 0 = q_1 \leq q_2 \leq \ldots \leq q_n,$

(24) $\qquad \sum_k s_k q_k = s.$

Because of $q \in R_n^0$ and $q_k = (q_k - q_1)/\delta(q)$ for each k, we conclude that while u having traversed all the set R_n^0, q traverses all that subset defined with (23) and (24). Easy computation shows:

(25) $\qquad r(y, u) = \sum_k p_k q_k \cdot \left[\sum_k s_k q_k^2 - s\right]^{-1/2}.$

The maximum of this function is found in Appendix. If we consider also that $\min r = -\max(-r)$, the equations (26) and (27) of the next theorem follow.

***Theorem* 3.** Let $a := [p_2, \ldots, p_n]$ and $b := [s_2, \ldots, s_n]$. Then:

(26) $\qquad A(y) = -H_{n-1}(-a, b, s, -s),$

(27) $\qquad B(y) = H_{n-1}(a, b, s, -s),$

(28) $\qquad \{r(y, u) \mid u \in R_n^0\} = [A(y), B(y)].$

Proof. We have to prove (28) else. The family R_n^0 is convex. If $\min r(y, u) = r(y, u')$ and $\max r(y, u) = r(y, u'')$, then $\lambda \to r(y, \lambda u'' + (1-\lambda)u')$ is a continuous function, which maps the interval $[0,1]$ onto the interval $[A(y), B(y)]$. QED.

For the illustration we'll examine a special case: R_n^0 — approximation of a nonconstant increasing onevalued function. Instead of table 3 we have much more special table 6, for which we demand:

x	$x_1, x_2, \ldots, x_n$
y	$y_1, y_2, \ldots, y_n$

Table 6

1. $n > 2$,
2. $x_1 < x_2 < \ldots < x_n$,
3. $y_1 \leq y_2 \leq \ldots \leq y_n \neq y_1$.

The function $u := y - y_1$ ($u_k := y_k - y_1$ ($k = 1,\ldots,n$)) is from R_n^0.

Because of $r(y, u) = 1$ we already know:

(29) $\quad B(y) = 1.$

According to (26) we also have:

(30) $\quad A(y) = -H_{n-1}(-a, b, n, -n)$

where

(31) $\quad a_i = (y_{i+1} - \bar{y}) / \tau \quad (i = 1,\ldots,n-1),$

(32) $\quad b_i = 1 \ (i = 1,\ldots,n-1),$

(33) $\quad \bar{y} = \frac{1}{n} \sum_k y_k,$

(34) $\quad \tau = \left[\sum_k y_k^2 - n y^2\right]^{1/2}.$

With the extension of (31) to index $i = 0$ we have:

(35) $\quad a_0 \leq a_1 \leq \ldots \leq a_{n-1},$

(36) $\quad \sum_{i=0}^{n-1} a_i = 0.$

In the notation from Appendix there is

(37) $\quad E_j = \frac{1}{j} \sum_{i=0}^{j-1} a_i \quad (j = 1,\ldots,n-1),$

(38) $\quad E_1 \leq E_2 \leq \ldots \leq E_{n-1} < 0.$

Therefore, $\beta = n - 1$ and

(39) $\quad H_{n-1}(-a, b, n, -n) = \max_{0<\alpha<n} \sqrt{\frac{n}{\alpha(n-\alpha)}} \sum_{i=0}^{\alpha-1} a_i.$

From (30) we endly get

$$A(y) = \frac{1}{\tau} \min_{0<\alpha<n} \sqrt{\frac{n}{\alpha(n-\alpha)}} (\alpha\bar{y} - \sum_{i=1}^{\alpha} y_i). \tag{40}$$

It is not difficult to see that $A(y) = r(y, u)$, where u is a function such as: $u_1 = \ldots = u_\varkappa = 0$, $u_{\varkappa+1} = \ldots = u_n = 1$ (for certain $\varkappa \in \{1, \ldots, n-1\}$). Hence: $A(y) \geq A(u)$. From (40) we find out:

$$A(u) = \min \left\{ \sqrt{\frac{n-\varkappa}{\varkappa(n-1)}}, \sqrt{\frac{\varkappa}{(n-\varkappa)(n-1)}} \right\}. \tag{41}$$

Minimum is got for $\varkappa = 1$ and $\varkappa = n-1$ and its value is $\frac{1}{n-1}$ · So we have found found the infimum of the numbers $A(y)$:

$$A(y) \geq \frac{1}{n-1}. \tag{42}$$

Note. The results of this section are because of (12) valid also for the R_n—approximations, where R_n is the family of nonconstant increasing functions.

APPENDIX: A NONLINEAR PROGRAM

We shall solve the next nonlinear program.

There are given $a = [a_k]_{m\times1}$, $b = [b_k]_{m\times1}$ and numbers c and d.

These parameters answer the following four conditions:

$m \in \{1, 2, 3, \ldots\}$,

$b_k > (k = 1, \ldots, m)$.

$c > 0$,

(1) $\varepsilon := c^2 + d \sum_k b_k > 0$,

where, $\sum_k \sim \sum_{k=1}^{m}$ and — avoiding the triviality — at least one a_k *is* $\neq 0$.

Maximize the objective function

(2) $$z = [z_k]_{mx1} \to h_m(z; a, b, c\, d) := \sum_k a_k z_k \cdot \left[\sum_k b_k z_k^2 + d\right]^{-1/2}.$$

subject to the constraints

(3) $$0 \leq z_1 \leq \ldots \leq z_m,$$

(4) $$\sum_k b_k z_k = c.$$

The function h_m is well defined:

(5) $$\sum_k b_k z_k^2 + d = \sum_k b_k \left(z_k - \frac{c + \sqrt{\varepsilon}}{\sum_k b_k}\right)^2 > 0.$$

The constraints (3) and (4) determine a nonvoid convex compact set S, so the function h_m, which is contionuous, for certain has the maximum

(6) $$\mathrm{H}_m(a, b, c, d) := \max\{h_m(z; a, b, c, d) \mid z \in S\}.$$

For $m = 1$ there is only one point z: $z_1 = c/b_1$ in S and so:

(7) $$\mathrm{H}_1(a, b, c, d) = \frac{ca_1}{\sqrt{\varepsilon b_1}}.$$

The case $m = 2$ demands much more work — though quite elementary — than the previous one. If the condition

(8) $$\frac{d(a_1 + a_2)}{\varepsilon} \leq \frac{a_1}{b_1} \leq \frac{a_1 + a_2}{b_1 + b_2}$$

is fulfilled, then

(9) $$\mathrm{H}_2(a, b, c, d) = \left[\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} - \frac{d}{\varepsilon}(a_1 + a_2)^2\right]^{1/2}$$

for $\underline{z}$:

(10) $$z_k = \frac{a_k\, \varepsilon}{c(a_1 + a_2)b_k} - \frac{d}{c} \quad (k = 1,2).$$

If the condition (8) is not true, then

(11) $$H_2(a, b, c, d) = \max\left\{ a_2 \left[\frac{1}{b_2} - \frac{d}{\varepsilon - db_1}\right]^{1/2}, (a_1 + a_2)\left[\frac{1}{b_1 + b_2} - \frac{d}{\varepsilon}\right]^{1/2} \right\}$$

for z:

(12) $$z_1 = 0,\ z_2 = c/b_2$$

(if the first term in (11) is greater) or

(13) $$z_1 = z_2 = c/(b_1 + b_2)$$

(if the second term in (11) is greater).

From now on we shall suppose that $m > 2$. The Lagrangian function:

(14) $$L = h_m + \sum_{k=1}^{m-1} \lambda_k (z_{k+1} - z_k),$$

where λ-s are the Lagrange multipliers. The corresponding Kahn-Tucker system ($j = 1,\ldots,m-1$):

(15) $$z_j \geq 0,$$

(16) $$\partial L / \partial z_j \leq 0,$$

(17) $$z_j \cdot (\partial L / \partial z_j) = 0,$$

(18) $$\lambda_j \geq 0,$$

(19) $$\partial L / \partial \lambda_j = z_{j+1} - z_j \geq 0,$$

(20) $$\lambda_j \cdot (\partial L / \partial \lambda_j) = \lambda_j (z_{j+1} - z_j) = 0,$$

considering (4) and

(21) $$\partial z_m / \partial z_j = -b_j / b_m.$$

Let us sum up the equations (17), using (4) and (20):

(22) $$\lambda_{m-1} = -\left[a_m - b_m \left(z_m + \frac{d}{c}\right) \frac{\sum_k a_k z_k}{\sum_k b_k z_k^2 + d}\right] \cdot \left(\sum_k a_k z_k^2 + d\right)^{-1/2}.$$

We shall change λ-s:

(23) $$\mu_j := \lambda_j \sqrt{\sum_k b_k z_k^2 + d} \qquad (j = 1,\ldots,m-1).$$

Putting (22) and (23) in (16) and (17) we get:

$$(24) \qquad a_j - b_j\,(z_j + \frac{d}{c}) \quad \frac{\sum_k a_k z_k}{\sum_k b_k z_k^2 + d} + \left\{ \begin{array}{lcr} & & -\mu_1 \; (j = 1) \\ \mu_{j-1} & - & \mu_j \; (1 < j < m) \\ \mu_{m-1} & & (j = \mathrm{m}) \end{array} \right\} \leq 0,$$

$$(25) \qquad z_j.\ [\text{left side of (24)}] = 0.$$

We now have to solve the system (3), (4), (24), (25), (26) and (27):

$$(26) \qquad \mu_j \geq 0,$$

$$(27) \qquad \mu_j\,(z_{j+1} - z_j) = 0$$

(for $j = 1,\ldots.,m-1$).

One more definiton ($k = 1,\ldots.,m$):

$$(28) \qquad E_k := \sum_{j=k}^{m} a_j \cdot \left[c + \frac{d}{c} \sum_{j=k}^{m} b_j \right]^{-1}.$$

Let β be the index of the greatest E_k:

$(k > \beta \Longrightarrow E_k < E_\beta) \wedge (k \leq \beta \Longrightarrow E_k \leq E_\beta)$.

Suppose that α is the adjacent index, for which $z_\alpha > 0$, $z_{\alpha-1} = 0$ (or $\alpha = 1$ if $z_1 > 0$). α could be any number in $\{1,\ldots,m\}$. Because of (27) it is for $\alpha \neq 1$:

$$(29) \qquad \mu_{\alpha-1} = 0.$$

The inequalities (24) are because of (25) the equations, as well, for $\alpha \leq j \leq m$. Summing up these equations we get:

$$(30) \qquad E_\alpha = \sum_k a_k z_k \,/\, (\sum_k b_k z_k^2 + d),$$

and (24) can be written in this way:

$$\begin{array}{l} (31) \\ \\ (32) \end{array} \qquad a_j - b_j\,(z_j + \frac{\mathrm{d}}{c})\,E_\alpha + \left\{ \begin{array}{lr} & -\mu_1 \; (j=1) \\ \mu_{j-1} & -\mu_j \; (1<j<m) \\ \mu_{m-1} & (j=m) \end{array} \right\} \begin{array}{ll} < 0 & (0<j<\alpha) \\ & \\ = 0 & (\alpha \leq j \leq m) \end{array}$$

Suppose that $\alpha > 1$ and $i < \alpha$. Summing up the inequalities (31) for $i \leq j < \alpha$ we find out

$$(33) \qquad (E_\alpha - E_i) \cdot (c + \frac{d}{c} \sum_{j=i}^{m} b_j) - \left\{ \begin{matrix} 0 & (i=1) \\ \mu_{j-1} & (i>1) \end{matrix} \right\} \geq 0,$$

and therefore

$$(34) \qquad E_i \leq E_\alpha \ (i < \alpha),$$

hence

$$(35) \qquad \alpha \leq \beta.$$

For $\alpha > 1$ and

$$(36) \qquad \mu_i = (E_\alpha - E_{i+1}) \cdot (c + \frac{d}{c} \sum_{j=i+1}^{m} b_j) \qquad (1 \leq i < \alpha)$$

the inequalities (31) are fulfilled, so we can forget them. But from the equations (32) we can calculate all other μ-s: from the first equation we get μ_α, from the second one $\mu_{\alpha+1}$, and so on. Summing up all this equations, we get the identity, which means that these equations are dependent and the system is surely solvable. We find out:

$$(37) \qquad \mu_i = (E_\alpha - E_{i+1}) \ (c + \frac{d}{c} \sum_{j=i+1}^{m} b_j) - E_\alpha \sum_{j=\alpha}^{i} b_j z_j \qquad (\alpha \leq i < m).$$

Now we shall prove that the system (27) is already fulfilled too. This system is equivalent with

$$(38) \qquad \sum_{i=\alpha}^{m-1} \mu_i (z_{i+1} - z_i) = 0,$$

if, of course, $\alpha < n$; for $\alpha = n$ there is no equations (27) at all. Firstly we shall write (37) in another way:

$$(39) \qquad \mu_i = \sum_{j=\alpha}^{i} a_j - \frac{d}{c} E_\alpha \sum_{j=\alpha}^{i} b_j - E_\alpha \sum_{j=\alpha}^{i} b_j z_j \ .$$

An auxiliary formula:

$$\sum_{i=\alpha}^{m-1} (z_{i+1}-z_i) \cdot \sum_{j=\alpha}^{i} \sigma_j = (z_{\alpha+1}-z_\alpha)\sigma_\alpha + (z_{\alpha+2} - z_{\alpha+1})(\sigma_\alpha + \sigma_{\alpha+1}) + \dots +$$

$$+ (z_m - z_{m-1})(\sigma_\alpha + \dots + \sigma_{m-1}) =$$

$$= \sigma_\alpha (z_m - z_\alpha) + \sigma_{\alpha+1}(z_m - z_{\alpha+1}) + \dots + \sigma_{m-1}(z_m - z_{m-1}) =$$

$$= z_m \sum_{i=\alpha}^{m-1} \sigma_i - \sum_{i=\alpha}^{m-1} \sigma_i z_i.$$

(40) $$\sum_{j=\alpha}^{m-1} (z_{i+1} - z_i) \sum_{j=\alpha}^{i} \sigma_j = z_m \sum_{i=\alpha}^{m} \sigma_i - \sum_{i=\alpha}^{m} \sigma_i z_i$$

Putting (39) in (38) with the help of (4) and (40) we rediscover (30).

The previous Kuhn-Tucker system is now reduced to the following one:

(41) $$H_m(a, b, c, d) = \frac{\sum_{i=\alpha}^{m} a_i z_j}{\sqrt{\sum_{i=\alpha}^{m} b_j z_i^2 + d}} = E_\alpha \sqrt{\sum_{i=\alpha}^{m} b_i z_{ik}^2 + d},$$

(42) $$0 < z_\alpha \leq \dots \leq z_m,$$

(43) $$\sum_{i=\alpha}^{m} b_i z_i = c,$$

(44) $$(E_\alpha - E_{i+1}) \left(c + \frac{d}{c} \sum_{j=i+1}^{m} b_j \right) \geq E_\alpha \sum_{j=\alpha}^{i} b_j z_j \quad (\alpha \leq j < m).$$

Now we shall consider the sign of E_β.

Case **1.** $E_\beta = 0$.

For the function $z' : z_i = 0\ (i<\beta),\ z_i = c/\sum_{j=\beta}^{m} b_j\ (i \geq \beta)$,

we easily find: $h_m(z'; a, b, c, d) = 0$. From (41) it follows then: $E_\alpha \geq 0$.

But since $E_\beta \geq E_\alpha$, there must be $E_\alpha = 0$ and

(45) $$H_m(a, b, c, d) = h_m(z'; a, b, c, d) = 0.$$

Case 2. $E_\beta < 0$.

For the function z'' : $z_i = 0 \quad (i<\alpha), \quad z_i = c/\sum_{j=\alpha}^{m} b_j \quad (i\geq\alpha)$,

we get

(46) $$\mathrm{H}m\,(a, b, c, d) \geq h_m\,(z''; a, b, c, d) = E_\alpha \left[d + c^2/\sum_{i=\alpha}^{m} b_i\right]^{1/2}$$

On the other side, by (41):

(47) $$\mathrm{H}_m\,(a, b, c, d) \leq \max E_\alpha \left(\sum_{i=\alpha}^{m} b_i z_i^2 + d\right)^{1/2} = E_\alpha \cdot \left(\min \sum_{i=\alpha}^{m} b_i z_i^2 + d\right)^{1/2},$$

where we determine the extreme for all z -*s* satisfying (43) only. It is easy to see that the extreme is gotten for $z = z''$ and so:

(48) $$\mathrm{H}_m\,(a, b, c, d) = h_m\,(z''; a, b, c, d).$$

The question remains, how to find α. With some effort we transform (44) into

(49) $$\sum_{j=\alpha}^{m} a_j/\sum_{j=\alpha}^{m} b_j \geq \sum_{j=i}^{m} a_j/\sum_{j=i}^{m} b_j \qquad (\alpha<i\leq m)$$

But this condition is complicated and perhaps it would be simplier to say:

(50) $$\mathrm{H}_m(a, b, c, d) = \max_{1\leq\alpha\leq\beta} E_\alpha \sqrt{d + c^2/\sum_{j=\alpha}^{m} b_j}$$

Case 3. $E_\beta > 0$.

For the function z' from case 1 there is $h_m\,(z'; a, b, c, d) > 0$. Hence

$\mathrm{H}_m\,(a, b, c, d) > 0$ and $E_\alpha > 0$ because of (41). From (44) it follows then that

(51) $$\alpha = \beta.$$

If $\beta = m$, we have

(52) $$\mathrm{H}_m\,(a, b, c, d) = \frac{c a_m}{\sqrt{b_m(c^2 + d b_m)}}\,.$$

From now on let $\beta < m$. It is easy to verify that

(53) $$\frac{1}{2}\sum_{k=\beta}^{m} b_k z_k^2 - \frac{1}{E_\beta}\sum_{k=\beta}^{m} a_k z_k = \frac{1}{2E_\beta^2}\left[\left(E_\beta\sqrt{\sum_{k=\beta}^{m} b_k z_k^2 + d} - h_m\right)^2 - h_m^2\right] - \frac{d}{2}$$

for all z which correspond the conditions (42) and (43). The right side of this equation is minimal exactly for that z for which the function h_m is maximal. Hence, if:

$$(54) \qquad M = \min_{(42),(43)} \left[\frac{1}{2} \sum_{k=\beta}^{m} b_k z_k^2 - \frac{1}{E_\beta} \sum_{k=\beta}^{m} a_k z_k\right].$$

then (53) is equivalent to

$$(55) \qquad M = \frac{1}{2E_\beta^2} [-H_m^2] - \frac{d}{2}$$

or, explicitely,

$$(56) \qquad H_m(a, b, c, d) = E_\beta \sqrt{-d - 2M}.$$

The determination of M is a problem of quadratic programming and there is no need to discuss it here.

POVZETEK

Če je tabelarično podana neka funkcija in jo aproksimiramo s funkcijo iz neke izbrane družine nekonstantnih funkcij, ustrezni korelacijski koeficient v splošnem ne more biti katerokoli število med —1 in 1. V tem sestavku zato s pomočjo ekstremnih korelacijskih koeficientov definiramo mero za kvaliteto aproksimacije (definicija 1). Ekstremne korelacije izračunamo najprej za družino vseh nekonstantnih funkcij, nato pa še za družino rastnih funkcij. Medtem ko je prva naloga preprosta, pa pri rastnih funkcijah zahteva po monotonosti povzroči, da je iskanje ekstremnih korelacij reševanje nelinearnega programa. Tega eksaktno rešimo v Dodatku.

SUMMARY

If a function given by a table is approximated by a function from a certain family of non-constant functions, then, in general, the correlation coefficient cannot be any number between —1 and 1. The aim of the study was to define a quantity that measures the quality of approximation of a function by calculating extreme correlation coefficients (Definition 1). Extreme correlations were calculated first for the family of all non-constant functions and then for the family of growth functions. The former is an easy task while the latter, due to the demand for monotonicity, becomes in fact a solving of a non-linear program, the exact solution of which is given in the Appendix.

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