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PROPER HOLOMORPHIC EMBEDDINGS WITH SMALL LIMIT SETS

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Abstract. Let X be a Stein manifold of dimension $n \geq 1$. Given a continuous positive increasing function h on $\mathbb{R}_+ = [0, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$, we construct a proper holomorphic embedding $f = (z, w) : X \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^n$ satisfying $|w(x)| < h(|z(x)|)$ for all $x \in X$. In particular, f may be chosen such that its limit set at infinity is a linearly embedded copy of $\mathbb{CP}^n$ in $\mathbb{CP}^{2n}$.

1. The main result

A theorem of Remmert [23], Narasimhan [21], and Bishop [4] states that every Stein manifold X of dimension $n \geq 1$ admits a proper holomorphic map to $\mathbb{C}^{n+1}$, a proper holomorphic immersion to $\mathbb{C}^{2n}$, and a proper holomorphic embedding in $\mathbb{C}^{2n+1}$. (See also [18, Chap. VII.C].) We are interested in the question how much space proper holomorphic embeddings or immersions $X \to \mathbb{C}^N$ need, and how small can their limit sets at infinity be.

By Remmert [22], the image $A = f(X) \subset \mathbb{C}^N$ of a proper holomorphic map $f : X \to \mathbb{C}^N$ is a closed complex subvariety of pure dimension $n = \dim X$. Such an A is algebraic if and only if it is contained, after a unitary change of coordinates on $\mathbb{C}^N$, in a domain of the form

$$D = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p = \mathbb{C}^N : |w| < C(1 + |z|)\}$$

for some $C > 0$ (see Chirka [5, Theorem 2, p. 77]). Equivalently, if $H = \mathbb{CP}^N \setminus \mathbb{C}^N \cong \mathbb{CP}^{N-1}$ denotes the hyperplane at infinity and $A_\infty = \overline{A} \cap H$, where $\overline{A}$ is the topological closure of A in $\mathbb{CP}^N$, then A is algebraic if and only if there is a linear subspace $L \cong \mathbb{CP}^{N-n-1}$ of $H \cong \mathbb{CP}^{N-1}$ such that $L \cap A_\infty = \varnothing$. If this holds then $\overline{A}$ and A_∞ are algebraic subvarieties of pure dimension n and $n-1$, respectively. If X is not algebraic then the image of any proper holomorphic immersion $f : X \to \mathbb{C}^N$ is not algebraic either, so its limit set $f(X)_\infty \subset \mathbb{CP}^{N-1}$ has a nonempty intersection with every linear subspace $\mathbb{CP}^{N-n-1} \cong L \subset \mathbb{CP}^{N-1}$.

We construct proper holomorphic embeddings with images in small Hartogs domains.

Theorem 1.1. *Let X be a Stein manifold of dimension $n \geq 1$. Given a continuous increasing function $h : [0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} h(t) = \infty$ there exist a*

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proper holomorphic embedding $(z, w) : X \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^n$ *and a proper holomorphic immersion* $(z, w) : X \to \mathbb{C}^{n+1} \times \mathbb{C}^{n-1}$ *satisfying*

$$|w(x)| < h(|z(x)|) \quad \text{for all } x \in X. \tag{1.1}$$

Furthermore, given a compact $\mathscr{O}(X)$*-convex set* K *in* X*, an open neighbourhood* $U \subset X$ *of* K*, and a holomorphic map* $f_0 = (z_0, w_0) : U \to \mathbb{C}^{n+1} \times \mathbb{C}^p$ *satisfying* (1.1) *for all* $x \in K$*, we can approximate* f_0 *uniformly on* K *by a proper holomorphic embedding* $f = (z, w) : X \to \mathbb{C}^{n+1} \times \mathbb{C}^p$ *if* $p \geq n$*, resp. immersion if* $p = n - 1$*, satisfying* (1.1).

The function h in Theorem 1.1 can be chosen to grow arbitrarily slowly, and hence the image $f(X)$ may be arbitrarily close to the subspace $\mathbb{C}^{n+1} \times \{0\}^p$ in the Fubini–Study metric on $\mathbb{CP}^{n+1+p}$. Choosing h such that $\lim_{t\to\infty} h(t)/t = 0$ gives Corollary 1.2.

Corollary 1.2. *Every Stein manifold* X *of dimension* $n \geq 1$ *admits a proper holomorphic embedding* $f : X \hookrightarrow \mathbb{C}^{2n+1}$ *whose limit set* $f(X)_\infty = \overline{f(X)} \cap H$ *is a linearly embedded copy of* $\mathbb{CP}^n$ *in* $H = \mathbb{CP}^{2n+1} \setminus \mathbb{C}^{2n+1} \cong \mathbb{CP}^{2n}$*. In particular, every open Riemann surface* X *admits a proper holomorphic embedding in* $\mathbb{C}^3$ *whose limit set is a projective line* $\mathbb{CP}^1 \subset \mathbb{CP}^2$*. The analogous result holds for proper holomorphic immersions* $X \to \mathbb{C}^{2n}$.

By the preceding discussion, the limit set $f(X)_\infty$ intersects every projective subspace $L \subset \mathbb{CP}^{N-1}$ of dimension $N - n - 1$, unless $f(X)$ is algebraic. Therefore, the nonalgebraic embeddings given by Corollary 1.2 have the smallest possible limit sets.

Given a nonalgebraic complex subvariety X of $\mathbb{C}^N$, its closure $\overline{X} \subset \mathbb{CP}^N$ and the limit set $X_\infty \subset \mathbb{CP}^{N-1}$ need not be analytic subvarieties, and for any pair of integers $1 \leq n < N$ there are n-dimensional closed complex submanifolds $X \subset \mathbb{C}^N$ with $X_\infty = \mathbb{CP}^{N-1}$. (This always holds if $N = n + 1$ and X is nonalgebraic.) Indeed, if X is a closed complex subvariety of $\mathbb{C}^N$ $(N > 1)$ then for any closed discrete set $B = \{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^N$ there exist a domain $\Omega \subset \mathbb{C}^N$ containing X and a biholomorphic map $\Phi : \Omega \to \mathbb{C}^N$ such that $B \subset \Phi(X)$ (see [13, Theorem 6.1] or [14, Theorem 4.17.1 (i)]). Note that $X' = \Phi(X)$ is a closed complex subvariety of $\mathbb{C}^N$. Choosing B such that its closure in $\mathbb{CP}^N$ contains the hyperplane at infinity implies $X'_\infty = \mathbb{CP}^{N-1}$. A characterization of the closed subsets of $\mathbb{CP}^{N-1}$ which are limit sets of closed complex subvarieties of $\mathbb{C}^N$ of a given dimension does not seem to be known.

The corollary is especially interesting in dimension $n = 1$. A long-standing open question (the Forster conjecture [11], also called the Bell–Narasimhan conjecture [2, 3]) asks whether every open Riemann surface, X, admits a proper holomorphic embedding in $\mathbb{C}^2$. Recent surveys of this subject can be found in [14, Secs. 9.10–9.11] and the preprint [1] by Alarcón and López, where the authors constructed a proper harmonic embedding of any open Riemann surface in $\mathbb{C} \times \mathbb{R}^2 \cong \mathbb{C}^2$ with a holomorphic first coordinate function. Note that if $X \to \mathbb{C}^2$ is a proper holomorphic map with nonalgebraic image then $f(X)_\infty = \mathbb{CP}^1$. (There are algebraic open Riemann surfaces which do not embed as smooth proper affine curves in $\mathbb{C}^2$.) Corollary 1.2 gives proper holomorphic embeddings $f : X \hookrightarrow \mathbb{C}^3$ whose images are arbitrarily close to the subspace $\mathbb{C}^2 \times \{0\}$ in the Fubini–Study metric on $\mathbb{CP}^3$, and $f(X)_\infty = \mathbb{CP}^1$.

It was recently shown by Drinovec Drnovšek and Forstnerič [8, Theorem 1.3] that, under a mild condition on an unbounded closed convex set $E \subset \mathbb{C}^N$, proper holomorphic embeddings $f : X \hookrightarrow \mathbb{C}^N$ from any Stein manifold X with $2\dim X < N$ such that $f(X) \subset \Omega = \mathbb{C}^N \setminus E$ are dense in the space $\mathscr{O}(X,\Omega)$ of all holomorphic maps $X \to \Omega$. A similar result holds for immersions if $2\dim X \le N$. Their proof relies on the fact, proved by Forstnerič and Wold [17], that such a domain Ω is an Oka domain. (See [14, Definition 5.4.1 and Theorem 5.4.4] for the definition and the main results concerning Oka manifolds.) Note that the domains in Theorem 1.1 are much smaller than those in [8, Theorem 1.3] when the codimension is at least 2. On the other hand, Theorem 1.1 does not pertain to proper maps in codimension 1 (the case $p = 0$). We do not know whether a Hartogs domain of the form

$$\Omega = \{(z,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^p : |w| < h(|z|)\}, \quad n \ge 1,\ p \ge 1, \tag{1.2}$$

which appears in Theorem 1.1, is an Oka domain, except if $p = 1$, the function $h > 0$ grows at least linearly at infinity, and $\log h(|z|)$ is plurisubharmonic on $\mathbb{C}^{n+1}$ (see Forstnerič and Kusakabe [15, Proposition 3.1]). Our proof does not require that Ω be an Oka domain.

We mention that a Stein manifold of dimension $n \ge 2$ admits a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$ and a proper holomorphic immersion with $N = \left[\frac{3n+1}{2}\right]$; see Eliashberg and Gromov [10], Schürmann [24], and [14, Theorem 9.3.1]. The proofs are very delicate and depend on Oka theory. We do not know whether one can expect a similar control of the range of the embedding in these dimensions.

2. Proof of Theorem 1.1

Our proof of Theorem 1.1 relies on the following technical result, which is a special case of [9, Theorem 1.1] by Drinovec Drnovšek and Forstnerič. (See also [16, Theorem 6], which is based on the same result.) Similar results were obtained earlier by Dor [6, 7].

Theorem 2.1. *Assume that X is a Stein manifold of dimension $n \ge 1$, D is a relatively compact, smoothly bounded, strongly pseudoconvex domain in X, K is a compact set contained in D, t_0 is a real number, $\sigma : \mathbb{C}^{n+1} \to \mathbb{R}$ is a strongly plurisubharmonic exhaustion function which has no critical points in the set $\{\sigma \ge t_0\}$, and $g_0 : \overline{D} \to \mathbb{C}^{n+1}$ is a continuous map that is holomorphic in D and satisfies $g_0\big(\overline{D \setminus K}\big) \subset \{\sigma > t_0\}$. Given numbers $t_1 > t_0$ and $\epsilon > 0$, there is a holomorphic map $g : \overline{D} \to \mathbb{C}^{n+1}$ satisfying the following conditions:*

(a) *$g(bD) \subset \{\sigma > t_1\}$.*
(b) *$\sigma(g(x)) > \sigma(g_0(x)) - \epsilon$ for all $x \in \overline{D}$.*
(c) *$|g(x) - g_0(x)| < \epsilon$ for all $x \in K$.*

Note that if $\epsilon > 0$ is small enough then condition (b) implies

$$g\big(\overline{D \setminus K}\big) \subset \{\sigma > t_0\}.$$

The analogous result holds much more generally, and we only stated the case that will be used here. For condition (b), see [9, Lemma 5.3], which is the main inductive step in [9, proof of Theorem 1.1]. We remark that a map from a compact set in a complex manifold is said to be holomorphic if it is holomorphic in an open neighbourhood of the said set.

Proof of Theorem 1.1. We shall construct proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ with $N \geq 2n+1$ satisfying (1.1); the same arguments will yield immersions when $N = 2n$.

Let $\Omega \subset \mathbb{C}^N$ be a domain of the form (1.2) with coordinates $(z,w) \in \mathbb{C}^{n+1} \times \mathbb{C}^p$ where $p \geq n$, $N = n+1+p$, and the function $h : [0,\infty) \to (0,\infty)$ is as in the theorem. We shall use Theorem 2.1 with the exhaustion function $\sigma(z) = |z|$ on $\mathbb{C}^{n+1}$; the nonsmooth point at the origin will not matter. We denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^{n+1}$.

Since the set $K \subset X$ is compact and $\mathscr{O}(X)$-convex, there exist a smooth strongly plurisubharmonic Morse exhaustion function $\rho : X \to \mathbb{R}_+$ and a sequence $0 < c_0 < c_1 < \cdots$ with $\lim_{i\to\infty} c_i = +\infty$ such that every c_i is a regular value of ρ and, setting

$$D_i = \{x \in X : \rho(x) < c_i\} \quad \text{for } i = 0,1,2,\ldots,$$

we have that $K \subset D_0 \subset \bar{D}_0 \subset U$, where $U \subset X$ is a neighbourhood of K as in the theorem (see [19, Theorem 5.1.6, p. 117]). We may assume that the given holomorphic map $f_0 = (z_0, w_0) : U \to \Omega$ satisfies condition (1.1) for all $x \in \bar{D}_0$ and $z_0(x) \neq 0$ for all $x \in bD_0$. (We shall use the subscript in z_i and w_i as an index in the induction process; a notation for the components of these maps will not be needed.) Pick a number $t_0 \in \mathbb{R}$ with

$$0 < t_0 < \min_{x \in bD_0} |z_0(x)|.$$

Choose a number t_{-1} with $0 < t_{-1} < t_0$ and close enough to t_0 such that the sublevel set $D_{-1} = \{\rho < t_{-1}\}$ satisfies $K \subset D_{-1} \subset \bar{D}_{-1} \subset D_0$ and

$$z_0(\overline{D_0 \setminus D_{-1}}) \subset \mathbb{C}^{n+1} \setminus t_0\overline{\mathbb{B}}.$$

Note that the set $\bar{D}_i$ is $\mathscr{O}(X)$-convex for every $i = -1, 0, 1, \ldots$.

By the Oka–Weil theorem, we can approximate the map $w_0 : U \to \mathbb{C}^p$ uniformly on $\bar{D}_0$ by a holomorphic map $w_1 : X \to \mathbb{C}^n$ such that $(z_0, w_1)(\bar{D}_0) \subset \Omega$. We shall now construct a holomorphic map $z_1 : \overline{D}_1 \to \mathbb{C}^{n+1}$ such that the holomorphic map $f_1 = (z_1, w_1) : \overline{D}_1 \to \Omega$ enjoys suitable properties to be explained. This will be the first step of an induction procedure.

Pick a number $t_1 \geq t_0 + 1$ so big that

$$h(t_1) > \max\{|w_1(z)| : z \in \bar{D}_1\}. \tag{2.1}$$

(Such a number exists since $\lim_{t\to\infty} h(t) = +\infty$.) Fix $\epsilon > 0$ whose precise value will be determined later. Let $\tilde{z}_0 : \bar{D}_0 \to \mathbb{C}^{n+1}$ be a holomorphic map given by Theorem 2.1 (with $\tilde{z}_0 = g$ in the notation of that theorem, applied to the map $g_0 = z_0$, the compact set $\bar{D}_{-1} \subset D_0$, and the numbers ϵ and $t_0 < t_1$). Condition (b) in Theorem 2.1 gives

$$|\tilde{z}_0(x)| > |z_0(x)| - \epsilon \quad \text{for all } x \in \bar{D}_0.$$

Since the function h in (1.2) is continuous, it follows that if $\epsilon > 0$ is small enough then the map $(\tilde{z}_0, w_1) : \bar{D}_0 \to \mathbb{C}^N$ has range in Ω, and we have that

$$\tilde{z}_0(bD_0) \subset \mathbb{C}^{n+1} \setminus t_1\overline{\mathbb{B}}, \quad \tilde{z}_0(\overline{D_0 \setminus D_{-1}}) \subset \mathbb{C}^{n+1} \setminus t_0\overline{\mathbb{B}}, \quad |\tilde{z}_0 - z_0| < \epsilon \ \text{ on } \bar{D}_{-1}. \tag{2.2}$$

We now use the fact that $\mathbb{C}^{n+1} \setminus t_1\overline{\mathbb{B}}$ is an Oka domain (see Kusakabe [20, Corollary 1.3]). Hence, the main result of Oka theory gives a holomorphic map $z_1 : \overline{D}_1 \to \mathbb{C}^{n+1}$ satisfying

$$z_1(\overline{D_1 \setminus D_0}) \subset \mathbb{C}^{n+1} \setminus t_1\overline{\mathbb{B}} \quad \text{and} \quad |z_1 - \tilde{z}_0| < \epsilon \ \text{ on } \bar{D}_0. \tag{2.3}$$

(See [12, Theorem 1.3] for a precise statement of a more general result. In the special case at hand, the existence of a map z_1 satisfying (2.3) was proved by a more involved argument in the paper [16] by Forstnerič and Ritter, predating Kusakabe's work [20].) If the number $\epsilon > 0$ is chosen small enough, it follows from (2.1)–(2.3) and the definition of Ω (1.2) that

$$z_1(\overline{D_0 \setminus D_{-1}}) \subset \mathbb{C}^{n+1} \setminus t_0\mathbb{B} \quad \text{and} \quad (z_1, w_1)(\overline{D}_1) \subset \Omega. \tag{2.4}$$

Since the dimension of the target space $\mathbb{C}^N$ is at least $2\dim X + 1$, we may assume after a small perturbation that the map $f_1 = (z_1, w_1) : \overline{D}_1 \hookrightarrow \Omega$ is an embedding satisfying the above conditions (see [14, Corollary 8.9.3]). Assuming as we may that all approximations are close enough, we also have that $|f_1 - f_0| < \epsilon_0$ on $\bar{D}_{-1}$ for a given $\epsilon_0 > 0$.

Continuing inductively, we obtain an increasing sequence $t_0 < t_1 < t_2 < \cdots$ with $\lim_{i\to\infty} t_i = \infty$, a decreasing sequence $\epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots > 0$ with $\lim_{i\to\infty} \epsilon_i = 0$, and a sequence of holomorphic embeddings $f_i = (z_i, w_i) : \bar{D}_i \hookrightarrow \mathbb{C}^{2n+1}$ satisfying the following conditions for $i = 1, 2, \ldots$.

(i) $f_i(\overline{D_i}) \subset \Omega$.
(ii) $z_i(\overline{D_i \setminus D_{i-1}}) \subset \mathbb{C}^{n+1} \setminus t_i\overline{\mathbb{B}}$.
(iii) $z_i(\overline{D_{i-1} \setminus D_{i-2}}) \subset \mathbb{C}^{n+1} \setminus t_{i-1}\overline{\mathbb{B}}$.
(iv) $|f_i - f_{i-1}| < \epsilon_{i-1}$ on $\bar{D}_{i-2}$.
(v) $t_i \ge t_{i-1} + 1$.
(vi) $0 < \epsilon_i < \epsilon_{i-1}/2$.
(vii) Every holomorphic map $f : \bar{D}_i \to \mathbb{C}^N$ with $|f - f_i| < 2\epsilon_i$ on $\overline{D}_{i-1}$ is an embedding on $\bar{D}_{i-2}$ and satisfies $f(\bar{D}_{i-1}) \subset \Omega$.

Note that conditions (i) and (ii) also holds for $i = 0$ by the assumptions on f_0, and conditions (i)–(v) hold for $i = 1$ by the construction of the map f_1.

The inductive step is similar to the one from $i = 0$ to $i = 1$, which was explained above. Assume inductively that conditions (i)–(v) hold for some $i \in \{1, 2, \ldots\}$. Pick a number ϵ_i satisfying conditions (vi) and (vii). Also, fix a number $\epsilon > 0$ whose precise value will be determined during this induction step. By the Oka–Weil theorem, there is a holomorphic map $w_{i+1} : X \to \mathbb{C}^p$ with $|w_{i+1} - w_i| < \epsilon$ on $\bar{D}_i$. Choose a number $t_{i+1} \ge t_i + 1$ so big that

$$h(t_{i+1}) > \max\{|w_{i+1}(x)| : x \in \bar{D}_{i+1}\}. \tag{2.5}$$

If $\epsilon > 0$ is chosen small enough then Theorem 2.1, applied to the map $g_0 = z_i : \bar{D}_i \to \mathbb{C}^{n+1}$, the compact set $\bar{D}_{i-1} \subset D_i$, and the numbers $t_i < t_{i+1}$ furnishes a holomorphic map $\tilde{z}_i : \bar{D}_i \to \mathbb{C}^{n+1}$ such that the map $(\tilde{z}_i, w_{i+1}) : \bar{D}_0 \to \mathbb{C}^N$ has range in Ω and the following conditions hold:

$$\tilde{z}_i(bD_i) \subset \mathbb{C}^{n+1} \setminus t_{i+1}\overline{\mathbb{B}}, \quad \tilde{z}_i(\overline{D_i \setminus D_{i-1}}) \subset \mathbb{C}^{n+1} \setminus t_i\overline{\mathbb{B}}, \quad |\tilde{z}_i - z_i| < \epsilon \ \text{ on } \bar{D}_{i-1}.$$

(For $i = 0$ these are conditions (2.2).) Since $\mathbb{C}^{n+1} \setminus t_{i+1}\overline{\mathbb{B}}$ is an Oka domain (see [20, Corollary 1.3]), there is a holomorphic map $z_{i+1} : \bar{D}_{i+1} \to \mathbb{C}^{n+1}$ satisfying

$$z_{i+1}(\overline{D_{i+1} \setminus D_i}) \subset \mathbb{C}^{n+1} \setminus t_{i+1}\overline{\mathbb{B}} \quad \text{and} \quad |z_{i+1} - \tilde{z}_i| < \epsilon \ \text{ on } \bar{D}_i.$$

(This is an analogue of condition (2.3).) Finally, we perturb the holomorphic map

$$f_{i+1} = (z_{i+1}, w_{i+1}) : \bar{D}_{i+1} \to \mathbb{C}^N$$

slightly to make it an embedding. If all approximations are close enough then f_{i+1} satisfies conditions (i)–(iv), and (v) holds by the choice of t_{i+1}. This completes the induction step.

Conditions (iv) and (vi) imply that the sequence f_i converges to the limit map

$$f = (z, w) = \lim_{i\to\infty} f_i : X \to \mathbb{C}^N$$

satisfying $|f - f_i| < 2\epsilon_i$ on $\bar{D}_{i-1}$ for every $i = 0, 1, \ldots$. (In particular, $|f - f_0| < 2\epsilon_0$ on K.) Conditions (i), (vi), and (vii) then imply that f is a holomorphic embedding with $f(X) \subset \Omega$. Finally, conditions (ii)–(vi) imply that the map $z : X \to \mathbb{C}^{n+1}$ is proper, and hence f is proper as map to $\mathbb{C}^N$. $\square$

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